VISCOSITY

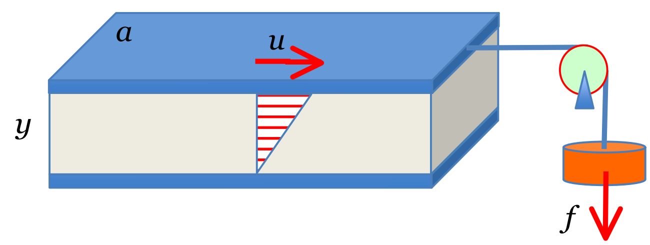
These notes develop the mechanics of viscous deformation. We begin with viscosity in shear and in dilation. We then consider a fluid in a homogeneous state, define stress and velocity gradient, develop thermodynamics of deformation, and construct the model of linear, isotropic, viscous fluids. We end with basic equations that govern inhomogeneous state.

SHEAR

Newton's law of viscosity. A layer of a fluid lies between two rigid plates. Subject to an applied force f, one plate slides relative to the other at a velocity u, and the fluid shears. In many cases a fluid deforms whenever the force is applied, no matter how small the force is. As an idealized model, Newton's law of viscosity says that the force is linearly proportional to the velocity, $f \propto u$. Write

$$f = cu,$$

where c is called the coefficient of viscous damping.



The coefficient of viscous damping depends on both the fluid and the size of the sample. We next define viscosity, a property that is specific to the fluid but independent of the size of the sample.

Define shear stress by the force per unit area:

$$\tau = \frac{f}{a},$$

where a is the area of the fluid. Define shear rate by the velocity gradient:

$$\dot{\gamma} = \frac{u}{y},$$

where y is the thickness of the fluid. Define viscosity η by the relation

$$\tau = \eta\dot{\gamma},$$

We adopt the idealization that the viscosity is independent of the shear rate, so that the shear stress is linear in the shear rate.

The viscosity is specific to the fluid, but is independent of the size of the sample. For the layer of fluid, area a and thickness y, the coefficient of viscous damping relates to the viscosity as

$$c = \frac{\eta a}{y}.$$

The coefficient of viscous damping increases when the area increases or when the thickness decreases.

The viscosity of a given fluid also depends on temperature and pressure. At the atmospheric pressure, the viscosity of water is around 10^{-3} Pa·s at room temperature, 1.8×10^{-3} Pa·s at the freezing point, and 0.28×10^{-3} Pa·s at the boiling point.

Potential energy of an applied force. We can represent the applied force by a hanging weight. By definition, the potential energy of the hanging weight is the weight times its height. When the plate moves at the velocity *u*, the hanging weight lowers its height at the same velocity and changes its potential energy at the rate $-fu$. Recall $f = \tau a$ and $u = \dot{\gamma} y$. Denote the volume of the fluid by $V = ay$. The potential energy of the hanging weight changes at the rate $-fu = -V\tau\dot{\gamma}$. Thus, $-\tau\dot{\gamma}$ is the change in the potential energy of the applied force, per unit volume of the fluid, per unit time.

Helmholtz free energy of a fluid. The sample of fluid has a fixed number of molecules, and the Helmholtz free energy of the sample is a function of temperature and volume, $F(T,V)$. The function is specific to each fluid, and has been tabulated online for many common fluids.

Recall a thermodynamic identity:

$$dF = -SdT - pdV,$$

where *S* is the entropy and *p* is the pressure. Also recall an identity in calculus:

$$dF = \frac{\partial F(T,V)}{\partial T}dT + \frac{\partial F(T,V)}{\partial V}dV.$$

A comparison of the two expressions gives more identities:

$$S = -\frac{\partial F(T,V)}{\partial T}, \quad p = -\frac{\partial F(T,V)}{\partial V}.$$

The functions $S(T,V)$ and $p(T,V)$ are also tabulated online for many common fluids.

Thermodynamic condition. The fluid and the hanging weight together constitute a composite thermodynamic system. The composite system exchanges energy, but not molecules, with a heat bath of a fixed temperature. We restrict our analysis to isothermal processes, and do not vary the temperature *T*. As an idealization, we assume that the Helmholtz free energy of the fluid, $F(T,V)$, remains valid even when the fluid undergoes viscous flow. The Helmholtz free energy of the composite system is the sum over the parts (i.e., the fluid and the hanging weight). Thermodynamics requires that the Helmholtz free energy of the composite system should never increase:

$$\frac{dF}{dt} - V\tau\dot{\gamma} \le 0 .$$

The equality holds when the fluid and the force are in thermodynamic equilibrium, whereas the inequality holds when the fluid and the force are not in thermodynamic equilibrium.

The above thermodynamic condition is also commonly phrased in other words. The quantity $V\tau\dot{\gamma}$ is the work done by the external forces per unit time. The thermodynamics says that the work done by the external forces can be no less than the change in the Helmholtz free energy of the fluid. The excess is the energy dissipated into the reservoir of the fixed temperature.

We have set the temperature to be a constant. We further assume that the fluid during the shear flow does not change volume. Consequently, the fluid during the shear flow does not change its thermodynamic state, so that $dF / dt = 0$. The above thermodynamic condition becomes that

$$\tau\dot{\gamma} \ge 0 .$$

The quantity $\tau\dot{\gamma}$ is the energy dissipated by the fluid into the heat bath.

Thermodynamic equilibrium. When the equality in the thermodynamic condition holds, $\tau\dot{\gamma} = 0$, the composite system is in thermodynamic equilibrium. We satisfy this condition of equilibrium by requiring that the shear stress should vanish, $\tau = 0$. Thus, the fluid is in a state of thermodynamic equilibrium, specified by a temperature T and a pressure p.

Out of thermodynamic equilibrium. When the inequality in the above thermodynamic condition holds, $\tau\dot{\gamma} \ge 0$, the composite system is not in thermodynamic equilibrium. Newton's law of viscosity, $\tau = \eta\dot{\gamma}$, satisfies the thermodynamic condition $\tau\dot{\gamma} \ge 0$ if and only if the viscosity is non-negative:

$$\eta \ge 0 .$$

DILATION

Hydrostatic stress and dilation. Next consider the fluid under another special state of stress: a hydrostatic stress. Imagine a plane in the fluid. Let σ be the stress, namely, the force per unit area acting on the plane. In the state of hydrostatic stress, the force acting on each plane in the fluid is normal to the plane, and the magnitude of the stress is the same for planes in all orientations. We adopt the sign convention that $\sigma > 0$ means a state of hydrostatic tension.

The fluid has a fixed number of molecules, but the volume of the fluid can change with time, $V(t)$. Define the rate of dilation by

$$\Delta = \frac{dV}{Vdt}.$$

Potential energy of a force. We can arrange a setup to apply the hydrostatic stress using a hanging weight. As the fluid dilates, the hanging weight lowers its height, and the potential energy of the hanging weight changes at the rate $-\sigma dV/dt$. Thus, $-\sigma\Delta$ is the change in the potential energy of the applied force, per unit volume of the fluid, per unit time.

Thermodynamic condition. The fluid and the hanging weight together constitute a composite thermodynamic system. The composite system exchanges energy, but not molecules, with a heat bath of a fixed temperature. We restrict our analysis to isothermal processes, and do not vary the temperature *T*. As an idealization, we assume that the Helmholtz free energy of the fluid, $F(T,V)$, remains valid even when the fluid undergoes viscous flow. The Helmholtz free energy of the composite system is the sum over the parts (i.e., the fluid and the hanging weight). Thermodynamics requires that the Helmholtz free energy of the composite system should never increase:

$$\frac{dF}{dt} - \sigma\frac{dV}{dt} \le 0.$$

The equality holds when the fluid and the force are in thermodynamic equilibrium, whereas the inequality holds when the fluid and the force are not in thermodynamic equilibrium.

We have set the temperature to be a constant, so that $dF = -pdV$. Here *p* is the thermodynamic pressure defined by

$$p = -\frac{\partial F(T,V)}{\partial V}.$$

The thermodynamic pressure is a function of temperature and volume, $p(T,V)$. The above thermodynamic condition becomes that

$$(\sigma + p)\Delta \ge 0.$$

Thermodynamic equilibrium. When the equality in the above thermodynamic condition holds, $(\sigma + p)\Delta = 0$, the composite system is in thermodynamic equilibrium. We satisfy this condition of equilibrium by requiring

$$\sigma + p = 0.$$

Thus, the fluid is in a state of thermodynamic equilibrium, specified by a temperature *T* and a pressure *p*. To maintain this state of thermodynamic equilibrium, the hydrostatic stress due to the hanging weight must match with the thermodynamic pressure of the fluid, $\sigma = -p(T,V)$.

Out of thermodynamic equilibrium. When the inequality in the above thermodynamic condition holds, $(\sigma + p)\Delta > 0$, the composite system is not in thermodynamic equilibrium. We satisfy this inequality by prescribing a model of dilatation viscosity:

$$\sigma + p = \zeta\Delta ,$$

where ζ is the dilation viscosity. This model satisfies the thermodynamic inequality $(\sigma + p)\Delta > 0$ if and only if the dilation viscosity is non-negative:

$$\zeta \geq 0 .$$

The dilation of a fluid is viscoelastic. The thermodynamic pressure related to specific volume and temperature through a thermodynamic equation of state, $p = p(V,T)$. This expression describes the elasticity of the fluid in dilation. Recall the expression for the rate of dilation $\Delta = (dV/dt)/V$, and we rewrite $\sigma + p = \zeta\Delta$ as

$$\sigma = -p(V,T) + \zeta \frac{dV}{Vdt} .$$

This expression represents a spring and a dashpot in parallel, subject to an applied stress σ.

Note that in general $\sigma \neq -p$. When a stress σ is suddenly applied and held at a constant, the fluid creeps and the thermodynamic pressure of the fluid approaches this applied stress. The time needed is estimated by ζ / B, where B is bulk elastic modulus of the fluid. Take representative values for water at the room temperature, $\zeta = 10^{-3}\mathrm{Pa \cdot s}$ and $B = 10^{9}\mathrm{Pa}$, the relaxation time is $\zeta / B = 10^{-12}\mathrm{s}$.

For many substances, as the temperature drops, viscosity increases steeply, but the elastic modulus does not. It is conceivable that viscoelastic dilation can be important. It is also conceivable that when viscoelastic dilation is important, viscoelastic shear is also important. To describe shear, viscoelasticity of the Kelvin type is clearly wrong, and we have to invoke viscoelasticity of the Maxwell type, or some form of hybrid. The matter is complex. We should deal with viscoelastic fluids in a separate formulation of the theory.

Compressible, viscous fluid. It is perhaps wise to eliminate the dilation viscosity for fluids like water and air. Set $\eta \neq 0$, but $\zeta = 0$. That is, we assume that the fluid and the external forces are not in thermodynamic equilibrium with respect to shear, but are in thermodynamic equilibrium with respect to dilation. Thus, the hydrostatic stress matches with the thermodynamic pressure:

$$\sigma = -p(T,V).$$

Incompressible, viscous fluid. Molecules in a fluid are often nearly incompressible. As an idealization, we assume that the volume per molecule in the fluid remains constant, independent of the stress. The condition of incompressibility is $\Delta = 0$.

Consistent with this idealization, the Helmholtz free energy of the fluid is also a constant, so that we cannot determine the pressure p from the thermodynamic equation of state, $p(T,V)$; rather, we determine the pressure p from the applied stress, $p = -\sigma$.

HOMOGENEOUS STATE

We next generalize Newton's law of viscosity by generalizing the stress, the velocity gradient, and the linear relation between them.

Stress

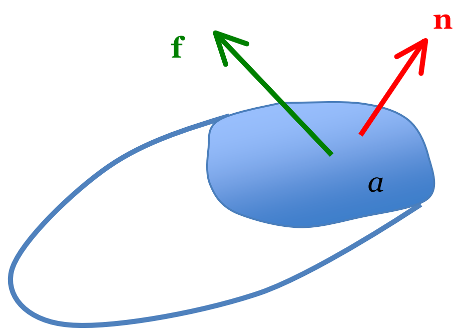
Tensile stress. We have just seen a fluid in a state of shear stress and in a state of hydrostatic stress. Here is one more example. When we pull a chewing gum, the gum elongates and its cross-sectional area reduces. The gum is in a state of uniaxial stress. In the current state, let f be the force pulling the gum, and a be the cross-sectional area of the gum. Define the tensile stress by the force per unit area:

$$\sigma = \frac{f}{a}.$$

Force per unit area. **Traction**. We now generalize these familiar examples to a fluid in a general homogeneous state. Inside the fluid, consider a planar region of area a, normal to a unit vector $\mathbf{n}$. Acting on the planar region is a force $\mathbf{f}$. In general, the force has components normal and tangential to the plane. Define the traction $\mathbf{t}$ by the force acting on the planar region divided by the area of the region:

$$\mathbf{t} = \frac{\mathbf{f}}{a}.$$

The fluid is in a homogeneous state. The traction is independent of the shape and the area of the region, but depends on the direction $\mathbf{n}$ of the region. For instance, for the chewing gum in tension, a plane not normal to the axial direction will have both normal and shear traction.



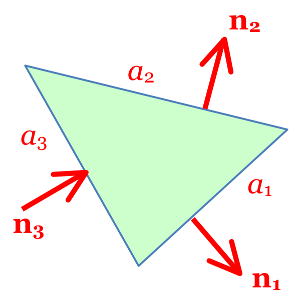
Stress. For a fluid in a homogeneous state, how many independent quantities do we need to specify its state of stress? The answer is six. To see this answer requires a deeper generalization than the definition of traction.

The product $a\mathbf{n}$ represents the planar region as a vector, written as $\mathbf{a} = a\mathbf{n}$. The force acting on the region, $\mathbf{f}$, depends on both the area and direction of the region. Write this relation as a function:

$$\mathbf{f} = \sigma(\mathbf{a}).$$

The set of planar regions form a vector space. We now confirm that the set of planar regions do form a vector space. The object $\mathbf{a} = a\mathbf{n}$ represents a planar region of area a, normal to the unit vector $\mathbf{n}$. The object $-\mathbf{a}$ represents a planar region of area a, also normal to the unit vector $\mathbf{n}$. Let β be a positive number. Thus, the object $\beta\mathbf{a}$ represents a planar region of area βa normal to the unit vector $\mathbf{n}$. Taken together, we have confirmed that, for every planar region $\mathbf{a}$ and every number β, the product $\beta\mathbf{a}$ is also a planar region.

Consider two planar regions represented by $\mathbf{a}_1$ and $\mathbf{a}_2$. Because the shapes of the planar regions do not affect the definition of the stress, we may choose the two regions as rectangular regions. The sum $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$ represents another planar region. The three planar regions form the surfaces of a prism. The cross section of the prism is shown in the figure. If the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ point toward the exterior of the prism, $\mathbf{n}_3$ points toward the interior of the prism.



Stress is a tensor. We next show that function $\mathbf{f}=\sigma(\mathbf{a})$ is a linear map. In linear algebra, a function that maps one vector to another vector is a linear map if

1. $\sigma(\beta\mathbf{a})=\beta\sigma(\mathbf{a})$ for every number β and every vector **a**, and
2. $\sigma(\mathbf{a}_1+\mathbf{a}_2)=\sigma(\mathbf{a}_1)+\sigma(\mathbf{a}_2)$ for any vectors.

Here one vector is the area vector representing a planar region in fluid in a homogeneous state, and the other vector is the force acting on the planar region.

Let β be a positive number. Because the fluid is in a homogeneous state, the force acting on the planer region $\beta\mathbf{a}$ is linear in β:

$$\sigma(\beta\mathbf{a})=\beta\sigma(\mathbf{a}).$$

Consider a thin slice of the fluid. Let **a** be one face of the slice, and $-\mathbf{a}$ be the other face of the slice. In each case, the unit vector normal to the face points outside the slice. The forces acting on the two faces are $\sigma(\mathbf{a})$ and $\sigma(-\mathbf{a})$. The balance of forces acting on the slice requires that

$$\sigma(-\mathbf{a})=-\sigma(\mathbf{a}).$$

The combination of the above statements shows that the function obeys

$$\sigma(\beta\mathbf{a})=\beta\sigma(\mathbf{a})$$

for every vector **a** and every number β.

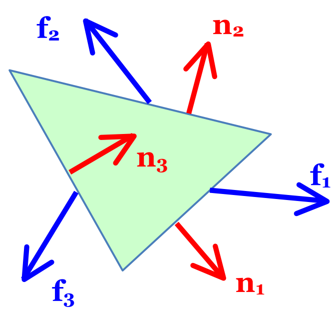
Consider two planar regions $\mathbf{a}_1$ and $\mathbf{a}_2$. Once again, because the shapes of the two regions do not affect the definition of the stress, we choose the two regions as rectangular regions. The sum $\mathbf{a}_3=\mathbf{a}_1+\mathbf{a}_2$ is another planar region. The three planar regions form the surfaces of a prism. The cross section of the prism is shown in the figure. If the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ point toward the exterior of the prism, $\mathbf{n}_3$ points toward the interior of the prism. The forces acting on the three faces of the prism are $\mathbf{f}_1=\sigma(\mathbf{a}_1)$, $\mathbf{f}_2=\sigma(\mathbf{a}_2)$ and $\mathbf{f}_3=\sigma(-\mathbf{a}_3)$.

The prism is a free-body diagram. The forces acting on the three faces are balanced, $\mathbf{f}_3 + \mathbf{f}_1 + \mathbf{f}_2 = \mathbf{o}$, so that

$$\sigma(\mathbf{a}_1 + \mathbf{a}_2) = \sigma(\mathbf{a}_1) + \sigma(\mathbf{a}_2).$$

This equation holds for any planar regions.

We have confirmed that the function $\mathbf{f} = \sigma(\mathbf{a})$ is a linear map that maps one vector (the area vector) to another vector (the force). In linear algebra, such a linear map is called a tensor. In mechanics, we call this linear map the stress.



Components of stress. The preceding definition is independent the choice of the basis of the vector space. We next choose an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Consider a unit cube of the fluid. Acting on the face $\mathbf{e}_1$ of the unit cube is the force $\sigma(\mathbf{e}_1)$. This force is a vector, which is also a linear combination of the three base vectors:

$$\sigma(\mathbf{e}_1) = \sigma_{11}\mathbf{e}_1 + \sigma_{21}\mathbf{e}_2 + \sigma_{31}\mathbf{e}_3,$$

where σ_{i1} are the three components of the force relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Similarly, we write

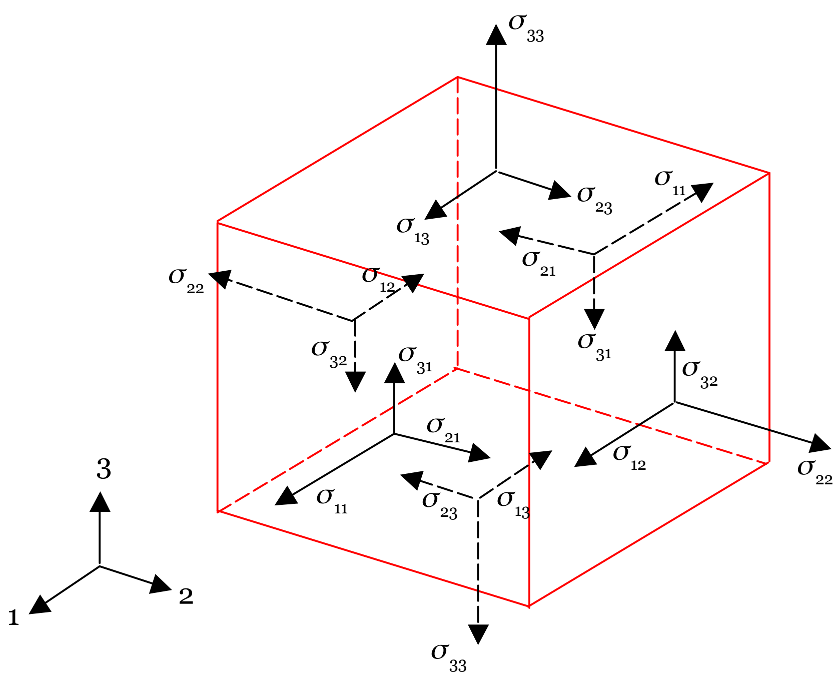
$$\sigma(\mathbf{e}_2) = \sigma_{12}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{32}\mathbf{e}_3,$$

$$\sigma(\mathbf{e}_3) = \sigma_{13}\mathbf{e}_1 + \sigma_{23}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3.$$

The force acting on the unit cube on the face whose normal is $-\mathbf{e}_1$ is given by $\sigma(-\mathbf{e}_1) = -\sigma(\mathbf{e}_1)$. This algebra is consistent with a physical requirement: the balance of the forces acting on the unit requires that the two forces acting on each pair of parallel faces of the unit cube be equal in magnitude and opposite in direction.

The nine quantities σ_{ij} are the components of stress. The first index indicates the direction of the force, and the second index indicates the direction of the vector normal to the face. Consider a piece of fluid of a fixed number of molecules. In the current state, the piece is in the shape of a unit cube, with faces parallel to the coordinate planes. Acting on each face is a force. Denote by σ_{ij} the component i of the force acting on the face of the cube normal to the axis j.

We adopt the following sign convention. When the outward normal vector of the face points in the positive direction of axis j, we take σ_{ij} to be positive if the component i of the force points in the positive direction of axis i. When the outward normal vector of the face points in the negative direction of the axis j, we take σ_{ij} to be positive if the component i of the force points in the negative direction of axis i.



Using the summation convention, we write the above three expressions as

$$\sigma\left(\mathbf{e}_j\right)=\sigma_{ij}\mathbf{e}_i .$$

The nine components of the stress can be listed as a matrix:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

The first index indicates the row, and the second the column.

Balance of moment. The matrix is symmetric, $\sigma_{ij} = \sigma_{ji}$, as required by the balance of moment acting on the cube. Consequently, a total of six independent components fully specify the state of stress for a fluid in a homogeneous state.

Traction-stress relation. Compare the definitions of traction $\mathbf{t} = \mathbf{f} / a$ and stress $\mathbf{f} = \sigma(\mathbf{a})$. The stress is a linear map, and the area a is a scalar, so that $\sigma(a\mathbf{n}) = a\sigma(\mathbf{n})$. We obtain that

$$\mathbf{t} = \sigma(\mathbf{n}).$$

The stress maps the unit vector normal to the plane to the traction acting on the plane.

The traction-stress relation can be expressed in terms of the components relative to a basis of the vector space, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The normal vector is a linear combination of the base vectors, $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$, where n_1, n_2, n_3 are the components of the unit vector. Recall the definition of the components of stress, $\sigma(\mathbf{e}_j) = \sigma_{ij}\mathbf{e}_i$. Consequently, the linear map of the vector is

$$\mathbf{t} = \sigma(\mathbf{n}) = \sigma(n_j\mathbf{e}_j) = \sigma(\mathbf{e}_j)n_j = n_j\sigma_{ij}\mathbf{e}_i n_j$$

The traction is also a linear combination of the base vectors, $\mathbf{t} = t_i\mathbf{e}_i$, where t_1, t_2, t_3 are the components of traction. A comparison of these expressions gives that

$$t_i = \sigma_{ij}n_j.$$

For a fluid in a homogeneous state, the stress tensor specifies the state of stress, and gives the traction vector on a plane of any direction.

Velocity Gradient

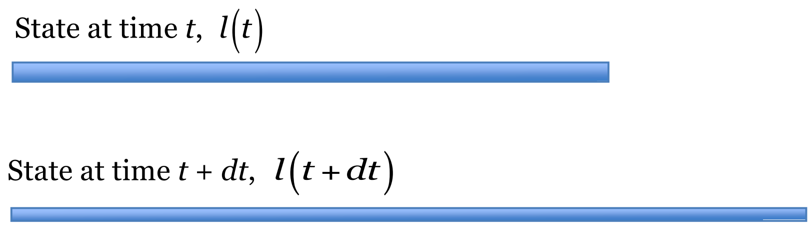
Rate of extension. Consider two material particles in a fluid. The two material particles are moving apart. The distance between the two material particles is a function of time, $l(t)$. Define the rate of extension along the line through the two material particles by

$$D = \frac{dl(t)}{ldt}.$$

That is, the rate of extension is the relative velocity between the two material particles divided by the distance between them.

At a given time, the rate of extension depends on the direction of the line.

Denote the rates of extension along the three coordinates by D_{11}, D_{22} and D_{33}.



Rate of dilation. A piece of fluid, consisting of a fixed number of molecules, changes its volume as a function of time, $V(t)$. Define the rate of dilation by

$$\frac{dV}{Vdt}.$$

We now relate the rate of dilation to the rates of extension. Consider a simple case that the fluid is of a rectangular shape, sides l_1, l_2, l_3 in the current state. The rates of extension of the sides are

$$D_{11} = \frac{dl_1}{l_1 dt}, \quad D_{22} = \frac{dl_2}{l_2 dt}, \quad D_{33} = \frac{dl_3}{l_3 dt}.$$

The volume of the fluid is $V = l_1 l_2 l_3$. The volume changes at the rate

$$\frac{dV}{dt} = l_2 l_3 \frac{dl_1}{dt} + l_3 l_1 \frac{dl_2}{dt} + l_1 l_2 \frac{dl_3}{dt}.$$

The rate of dilation is

$$\frac{dV}{Vdt} = D_{11} + D_{22} + D_{33}.$$

Even though we have derived this result using a fluid of rectangular shape, the conclusion is correct for a fluid of any shape, so long as the deformation is homogeneous. We will show that the three rates of extension are components of a tensor. The sum $D_{11} + D_{22} + D_{33}$ is the trace of the tensor, and is an invariant of the tensor. We write the trace by using the summation convention, D_{kk}.

Rate of shear. A piece of fluid undergoes homogeneous deformation. Consider the movements of three material particles P, A and B. At time *t*, line PA lies on axis 1, and line PB lies on axis 2. At time *t* + *dt*, both lines stretch and rotate. Let u be the horizontal velocity of particle B relative to that of particle P, and y be the distance between the two particles. Let v be the vertical velocity of particle A relative to that of particle P, and x be the distance between the two particles. Line PA rotates counterclockwise at rate v/x, and line PB rotates clockwise at rate u/y. The rate of change in the angle between the two lines defines the rate of shear:

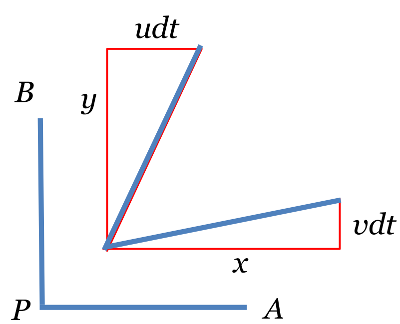
$$\dot{\gamma}_{12} = \frac{u}{y} + \frac{v}{x}.$$

For later convenience, we write

$$D_{12} = \frac{1}{2}\left(\frac{u}{y} + \frac{v}{x}\right).$$

Similarly define rates of shear D_{23} and D_{31}.

In defining the rates of shear, D_{23}, D_{31} and D_{12}, we have introduced a factor of 2. We do so to make the three rates of shear, along with the three rates of extension, D_{11}, D_{22} and D_{33}, constitute the components of a symmetric, second-rank tensor D_{ij}. This statement will be confirmed shortly.

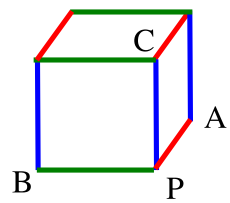


Velocity gradient. Consider two points P and A, of coordinates $\mathbf{x}^P$ and $\mathbf{x}^A$, and velocities $\mathbf{v}^P$ and $\mathbf{v}^A$. When a fluid undergoes homogeneous deformation, the velocity of material particle is linear in the position of the material particle:

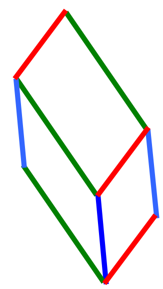
$$\mathbf{v}^A - \mathbf{v}^P = \mathbf{L}\left(\mathbf{x}^A - \mathbf{x}^P\right).$$

The linear map **L**, called the velocity gradient, maps one vector $\mathbf{x}^A - \mathbf{x}^P$ to another vector $\mathbf{v}^A - \mathbf{v}^P$. In linear algebra, a linear map is also known as a tensor. The deformation is homogeneous when the velocity gradient **L** is independent of position in space.

We can picture the tensor **L** as follows. Consider a piece of fluid of a fixed number of molecules, undergoing homogeneous deformation. At time *t*, the piece of fluid is a unit cube. At time *t* + *dt*, the piece deforms into a parallelepiped. This deformation stretches and rotates the three edges of the piece. Let the three edges be PA, PB, and PC. Thus, L_{i1} is the relative velocity between particles A and P, L_{i2} is the relative velocity between particles B and P, and L_{i3} is the relative velocity between particles C and P.



State at time *t*



State at time *t* + *dt*

Rate of deformation is a tensor. Using a picture, we can readily confirm that the rate of deformation relates to the velocity gradient as

$$D_{ij} = \frac{1}{2}\left(L_{ij} + L_{ji}\right).$$

Because the velocity gradient is a tensor, the rate of deformation is also a tensor.

Rate of extension of a line of material particles. A fluid is undergoing homogeneous deformation specified by the velocity gradient **L**. In the current state, a line of material particles in the fluid is in the direction of unit vector **n**, and is of length *l*. The line corresponds to vector $l\mathbf{n}$, and the two ends of the line move relative to each other at a velocity $\mathbf{L}\left(l\mathbf{n}\right)$. The line of material particles elongates at a rate $dl/dt = \mathbf{n}\cdot\left(\mathbf{L}\left(l\mathbf{n}\right)\right)$. By definition, the rate of extension of the line of material particles is $D_n = \left(dl/dt\right)/l$. A combination of the two expressions gives that $D_n = \mathbf{n}\cdot\left(\mathbf{L}\mathbf{n}\right)$. This expression is known as a quadratic form in linear algebra. We can confirm that the result only depends on the symmetric part of the tensor **L**. Thus, the rate of extension of the line is

$$D_n = D_{ij}n_i n_j .$$

For a fluid in a homogeneous state, the six independent components of **D** fully specify the rate of deformation, and give the rate of extension in any direction.

Rate of shear between two lines of material particles. The tensor D_{ij} also describes the rate of shear between two lines of material particles. In the current state, the two lines are two unit vectors normal to each other, **m** and **n**. By definition, the rate of shear with respect to the two vectors is $D_{mn} = \left(\mathbf{m}\cdot\left(\mathbf{Ln}\right)+\mathbf{n}\cdot\left(\mathbf{Lm}\right)\right)/2$. This expression is the same as

$$D_{mn} = D_{ij} m_i n_j .$$

Rate of rotation. Define another tensor by

$$W_{ij} = \frac{1}{2}\left(L_{ij} - L_{ji}\right).$$

This tensor is anti-symmetric, and has three independent components:

$$\begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix}$$

The velocity gradient, the tensor **L**, is the sum of its symmetric and anti-symmetric parts:

$$\mathbf{L} = \mathbf{D} + \mathbf{W} .$$

The symmetric part, **D**, fully characterizes the rate of deformation. What does the anti-symmetric part **W** do?

Once again consider the two material particles A and B, of coordinates $\mathbf{x}^A$ and $\mathbf{x}^B$ when the fluid is in the current state. Write the vector between the two particles as $\mathbf{r} = \mathbf{x}^B - \mathbf{x}^A$, and let the components of the vector be r_1, r_2, r_3. Thus

$$\mathbf{W}\left(\mathbf{x}^B - \mathbf{x}^A\right) = \begin{bmatrix} 0 & -W_{21} & W_{13} \\ W_{21} & 0 & -W_{32} \\ -W_{13} & W_{32} & 0 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} -W_{21}r_2 + W_{13}r_3 \\ W_{21}r_1 - W_{32}r_3 \\ -W_{13}r_1 + W_{32}r_2 \end{bmatrix}.$$

The tensor **W** rotates the vector $\mathbf{r}$, and is known as the rate of rotation, spin, and vorticity.

The above expression can be written in terms of vector algebra. Use the three independent components of **W** to define a vector:

$$\mathbf{w} = \begin{bmatrix} W_{32} \\ W_{13} \\ W_{21} \end{bmatrix}$$

The above expression is the same as

$$\mathbf{W}\left(\mathbf{x}^B - \mathbf{x}^A\right) = \mathbf{w} \times \mathbf{r} = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ w_1 & w_2 & w_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

Thus, the vector **w** is the angular velocity. The direction of **w** is the axis of the rotation, and the magnitude of **w** is the rate of rotation per unit time.

We now write $\mathbf{v}^B - \mathbf{v}^A = \mathbf{L}\left(\mathbf{x}^B - \mathbf{x}^A\right)$ as

$$\mathbf{v}^B = \mathbf{v}^A + \mathbf{w} \times \left(\mathbf{x}^B - \mathbf{x}^A\right) + \mathbf{D}\left(\mathbf{x}^B - \mathbf{x}^A\right).$$

The three terms correspond to translation, rotation, and deformation.

Thermodynamics

Potential energy of the external forces. A piece of a fluid is subject to external forces, represented by a set of hanging weights. The external forces apply to the fluid a state of stress σ_{ij}, and the fluid undergoes homogeneous deformation at the rate of deformation D_{ij}. Let V be the volume of the fluid in the current state.

By definition, the potential energy of a hanging weight is the weight times its height. You can confirm that the hanging weights change the potential energy at the rate $-V\sigma_{ij}D_{ij}$.

Thermodynamic condition. For a piece of fluid of a fixed number of molecules, the Helmholtz free energy is a function of temperature and volume, $F(T,V)$. The fluid and the hanging weights together constitute a composite thermodynamic system. The composite system exchanges energy, but not molecules, with a heat bath of a fixed temperature. We restrict our analysis to isothermal processes, and do not vary the temperature *T*. The Helmholtz free energy of the composite system is the sum over the parts (i.e., the fluid and the hanging weights). Thermodynamics requires that the Helmholtz free energy of the composite system should never increase:

$$\frac{dF}{dt} - V\sigma_{ij}D_{ij} \le 0.$$

The equality holds when the fluid and the forces are in thermodynamic equilibrium, whereas the inequality holds when the fluid and the forces are not in thermodynamic equilibrium.

The above thermodynamic condition is also commonly phrased in other words. The quantity $V\sigma_{ij}D_{ij}$ is the work done by the external forces per unit time. The thermodynamics says that the work done by the external forces can be no less than the change in the free energy of the fluid. The excess is the energy dissipated into the reservoir of the fixed temperature.

We have set the temperature to be a constant, so that $dF = -pdV$. Here p is the thermodynamic pressure defined by

$$p = -\frac{\partial F(T,V)}{\partial V}.$$

The thermodynamic pressure is a function of temperature and volume, $p(T,V)$. Recall a geometric relation $dV / dt = VD_{kk}$. The above thermodynamic condition becomes that

$$(\sigma_{ij} + p\delta_{ij})D_{ij} \geq 0.$$

Condition of thermodynamic equilibrium. When the equality in the above thermodynamic condition holds for arbitrary rate of deformation D_{ij}, the composite system is in thermodynamic equilibrium. This condition of equilibrium leads to

$$\sigma_{ij} + p\delta_{ij} = 0.$$

Thus, the fluid is in a state of thermodynamic equilibrium, specified by a temperature T and a pressure p. To maintain this state of thermodynamic equilibrium, all the shear stresses must vanish,

$$\sigma_{12} = \sigma_{23} = \sigma_{31} = 0,$$

and all the normal stresses must be the same as the thermodynamic pressure,

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p(T,V).$$

Linear, Isotropic, Viscous Fluids

When $\sigma_{ij} + p\delta_{ij} \neq 0$, the fluid is not in thermodynamic equilibrium with the stress, and the rate of deformation does not vanish, $D_{ij} \neq 0$. The fluid flows. We construct a rheological model to relate the two tensors, $\sigma_{ij} + p\delta_{ij}$ and D_{ij}. The model generalizes Newton's law of viscosity.

Linearity. As an idealization, we require that $\sigma_{ij} + p\delta_{ij}$ be linear in D_{ij}. The general form of linear relation between two tensors is

$$\sigma_{ij} + p\delta_{ij} = H_{ijkl}D_{kl}.$$

The coefficients H_{ijkl} constitute a fourth-rank tensor.

Isotropy. We further assume that the fluid is isotropic. For an isotropic fluid, the linear relation between two symmetric tensors takes the general form:

$$\sigma_{ij} + p\delta_{ij} = 2\eta\left(D_{ij} - \frac{1}{3}D_{kk}\delta_{ij}\right) + \zeta D_{kk}\delta_{ij},$$

where η and ζ are constants independent of the rate of deformation. This rheological model provides six equations that connect the two tensors, the stress and the rate of deformation.

This rheological model generalizes Newton's law of viscosity. The model describes linear, isotropic, viscous behavior in full generality. To see this generalization clearly, we should go through the same process as we obtain the generalized Hooke's law of elasticity. We should consider a fluid under a tensile stress, and look at the extension in the axial direction and contraction in the two transverse directions. We then relate them to the fluid under shear. We show that only two independent material constants are needed. This process is less elegant on paper, but more intuitive to the brain, than what we have put on paper.

Shear viscosity. For a fluid under a pure shear state D_{12}, the shear stress relates to the rate of shear as

$$\sigma_{12} = 2\eta D_{12},$$

where η is the shear viscosity. The factor 2 comes from the definition $\dot{\gamma}_{12} = 2D_{12}$. Because the fluid is isotropic, the viscosity η is the same in every shearing direction, so that

$$\sigma_{23} = 2\eta D_{23},$$

$$\sigma_{31} = 2\eta D_{31}.$$

Dilation viscosity. Next consider the fluid under another special state of stress: a hydrostatic stress:

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma, \quad \sigma_{12} = \sigma_{23} = \sigma_{31} = 0.$$

The rate of deformation is in a state of pure dilatation:

$$D_{11} = D_{22} = D_{33} = \frac{\Delta}{3}, \quad D_{12} = D_{23} = D_{31} = 0,$$

where $\Delta = \left(dV/dt\right)/V$ is the rate of dilatation.

When the external force equilibrates with the fluid, $\sigma + p = 0$. We are interested in the non-equilibrium condition $\sigma + p \neq 0$, and require that the $\sigma + p$ be linear in the rate of dilatation:

$$\sigma + p = \zeta\Delta,$$

where ζ is the dilation viscosity.

Trace and deviator. We can also motivate the model in yet another way. The trace of the rate of deformation, D_{kk}, describes the rate at which the piece of fluid changes its volume. The deviatoric part of the rate of deformation, $e_{ij} = D_{ij} - D_{kk}\delta_{ij}/3$, describes the rate at which the piece of fluid changes its shape. Define the mean stress by $\sigma_m = \sigma_{kk}/3$, and the deviatoric stress by $s_{ij} = \sigma_{ij} - \sigma_m\delta_{ij}$. Note an identity

$$\sigma_{ij}D_{ij} = s_{ij}e_{ij} + \sigma_m D_{kk}.$$

Thermodynamic condition in terms of trace and deviator. Rewrite the thermodynamic condition $\left(\sigma_{ij} + p\delta_{ij}\right)D_{ij} \ge 0$ as

$$s_{ij}e_{ij} + \left(\sigma_m + p\right)D_{kk} \ge 0.$$

The rheological model takes the form

$$s_{ij} = 2\eta e_{ij},$$

$$\sigma_m + p = \zeta D_{kk}.$$

The constant η represents the shear viscosity that resists the change in shape, and the constant ζ represents the dilation viscosity that resists the change in volume. The model satisfies the thermodynamic condition when the two viscosities are nonnegative.

Compressible, viscous fluid. It is perhaps wise to eliminate the dilation viscosity for fluids like water and air. Set $\eta \ne 0$, but $\zeta = 0$. That is, we assume that the fluid and the external forces are not in thermodynamic equilibrium with respect to shear, but are in thermodynamic equilibrium with respect to dilation:

$$s_{ij} = 2\eta e_{ij},$$

$$\sigma_m + p = 0,$$

where $p\left(V,T\right)$ is the thermodynamic pressure.

Incompressible, viscous fluid. Molecules in a fluid are often nearly incompressible. As an idealization, we assume that the volume per molecule in the fluid remains constant, independent of the stress. For a piece of fluid of a fixed number of molecules, the volume of the piece V is constant, independent of time. Recall that $dV/dt = VD_{kk}$. The condition of incompressibility is

$$D_{kk} = 0.$$

Consistent with this idealization, the Helmholtz free energy of the fluid is also a constant, so that we cannot determine the pressure p from the thermodynamic equation of state, $p = -\partial F(T,V)/\partial V$; rather, we determine the pressure p from the applied stress, $p = -\sigma_m$. This model is specified by the following equations:

$$D_{kk} = 0 .$$

$$\sigma_{ij} = -p\delta_{ij} + 2\eta D_{ij} .$$

INHOMOGENEOUS STATE

A body of fluid is a sum of many small pieces. As the body evolves by inhomogeneous deformation, each piece evolves through a sequence of homogeneous states, as described in the previous section. Different pieces communicate through the compatibility of geometry and the balance of forces.

Time derivative of a function of material particle. At time *t*, a material particle $\mathbf{X}$ moves to position $\mathbf{x}(\mathbf{X},t)$. The velocity of the material particle is

$$\mathbf{V} = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} .$$

Here, the independent variables are the time and the coordinates of the material particles when the body is in the reference state.

We can also use $\mathbf{x}$ as the independent variable. To do so we change the variable from $\mathbf{X}$ to $\mathbf{x}$ by using the function $\mathbf{X}(\mathbf{x},t)$, and then write

$$\mathbf{v}(\mathbf{x},t) = \mathbf{V}(\mathbf{X},t).$$

Both $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{V}(\mathbf{X},t)$ represent the same physical object: the velocity of material particle $\mathbf{X}$ at time *t*. Because of the change of variables, from $\mathbf{X}$ to $\mathbf{x}$, the two functions $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{V}(\mathbf{X},t)$ are different. We indicate this difference by using different symbols, $\mathbf{v}$ and $\mathbf{V}$.

This practice, however, is not always convenient. Often we simply use the same symbol for the velocity, and then indicate the independent variables: $\mathbf{v}(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{X},t)$. They represent the same physical object: the velocity of material particle $\mathbf{X}$ at time *t*.

Let $G(\mathbf{X},t)$ be a function of material particle and time. For example, *G* can be the temperature of material particle $\mathbf{X}$ at time *t*. The rate of change in temperature of the material particle is

$$\frac{\partial G(\mathbf{X},t)}{\partial t} .$$

This rate is known as the *material time derivative*.

We can calculate the material time derivative by an alternative approach. Change the variable from $\mathbf{X}$ to $\mathbf{x}$ by using the function $\mathbf{X}(\mathbf{x},t)$, and write

$$g(\mathbf{x},t) = G(\mathbf{X},t)$$

Using chain rule, we obtain that

$$\frac{\partial G(\mathbf{X},t)}{\partial t} = \frac{\partial g(\mathbf{x},t)}{\partial t} + \frac{\partial g(\mathbf{x},t)}{\partial x_i}\frac{\partial x_i(\mathbf{X},t)}{\partial t}.$$

Thus, we can calculate the material time derivative from

$$\frac{\partial G(\mathbf{X},t)}{\partial t} = \frac{\partial g(\mathbf{x},t)}{\partial t} + \frac{\partial g(\mathbf{x},t)}{\partial x_i} v_i(\mathbf{x},t).$$

A shorthand notation for the time derivative of a function of material particles is

$$\frac{Dg}{Dt} = \frac{\partial G(\mathbf{X},t)}{\partial t} = \frac{\partial g(\mathbf{x},t)}{\partial t} + \frac{\partial g(\mathbf{x},t)}{\partial x_i} v_i(\mathbf{x},t).$$

In particular, the acceleration of a material particle is

$$a_i(\mathbf{x},t) = \frac{\partial v_i(\mathbf{x},t)}{\partial t} + \frac{\partial v_i(\mathbf{x},t)}{\partial \mathbf{x}_j} v_j(\mathbf{x},t).$$

Compatibility of geometry. The rate of deformation relates to the gradient of velocity as

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

This relation expresses the compatibility of geometry.

Balance of forces. Consider a small volume in the body. The balance of forces now also include the inertial force:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \frac{Dv_i}{Dt}.$$

Material model. We adopt the generalized Newton's model of viscosity for an incompressible fluid, $D_{kk} = 0$ and $\sigma_{ij} = -p\delta_{ij} + 2\eta D_{ij}$. Because the material is incompressible, the density ρ is constant independent of time.

Navier-Stokes equations. A combination of the above expressions gives that

$$v_{k,k} = 0,$$

$$\eta \frac{\partial^2 v_i}{\partial x_j \partial x_j} - \frac{\partial p}{\partial x_i} + b_i = \rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right).$$

These four partial differential equations govern the four fields p and v_i.

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