Study of the simple extension tear test sample for rubber with Configurational Mechanics

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May 30, 2007

Abstract

The simple extension tear test-piece also referred to as the trousers sample is widely used to study crack propagation in rubber. The corresponding energy release rate, called tearing energy for rubber materials, was first established by Rivlin and Thomas (Rupture of rubber. I. Characteristic energy for tearing. *J. Polym. Sci.*, 10:291318, 1953); a second derivation was proposed later by Eshelby (The calculation of energy release rates. In G. C. Sih, H. C. van Elst, and D. Broek, editors, *Prospects of Fracture Mechanics*, 6984, Leyden, Noordhoff, 1975). We show here that the derivation of this result can be advantageously revisited through the scope of Configurational Mechanics. Our approach is based on the rigorous definition of the configurations of the body and on the physical significance of the configurational stress tensor. More precisely, it is demonstrated that the change in energy due to crack growth, and then the tearing energy, is directly related to the components of the configurational stress tensor in the body.

1   Introduction

In his seminal paper, Griffith [1920] proposed a criterion to determine the amount of energy involved during crack propagation in brittle materials. Denoting dU (< 0) the change of total energy (change of strain energy and work of external forces) and dA the increase in crack surface during crack growth in the body, the energy release rate G is defined by

$$G = -\frac{dU}{dA},\tag{1}$$

and the crack growth criterion can be simply written as

$$G < G_c,\tag{2}$$

where G_c is a critical value of the energy release rate which depends on the material and should be measured in experiments.

More than thirty years later, Rivlin and Thomas [1953] extended the Griffith theory to rubber-like materials considering the following assumptions:

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- the approach of Griffith is valid for large strain (in fact, no restriction was formulated in the original paper of Griffith),

- irreversible changes in energy due to crack growth take place only in the neighbourhood of the crack tip,

- the change in energy is independent of the shape and dimensions of the body.

Then, authors defined the tearing energy T by

$$T = -\left.\frac{\partial w}{\partial A}\right|_l, \tag{3}$$

where w is the strain energy, A is the crack surface and the suffix $\cdot_l$ denotes differentiation with constant displacement of the boundaries over which forces are applied. Considering thin samples (uniform thickness h_0) and denoting c the length of the crack, the tearing energy reduces to

$$T = -\frac{1}{h_0}\left.\frac{\partial w}{\partial c}\right|_l, \tag{4}$$

and the corresponding crack growth criterion is simply $T > T_c$, T_c being the critical value of the tearing energy that depends on the material.

In the same article, Rivlin and Thomas proposed several experimental samples to perform crack propagation experiments in rubber. One of these samples is the simple extension tear test-piece also referred to as the trousers sample. Authors calculated the corresponding expression for the tearing energy and used this sample to measure the critical tearing energy. Later, Eshelby [1975b] considered the same test-piece to illustrate the relevance of path-independent integrals for the calculation of configurational forces that drive the evolution of singularities. In his paper, Eshelby proposed a second derivation for the tearing energy of the trousers sample. The present paper discusses a third derivation of this result. It is demonstrated that the tearing energy of the simple extension tear test sample can be easily recovered using the general theory of Configurational Mechanics and more precisely the definition of the configurational stress tensor. Previous derivations are first recalled. Then, our proposal is detailed.

2 Description of the problem

The sample is a rectangular thin sheet of length b and width $2a$; the thickness h_0 is uniform. The test piece contains a cut of length c (with $c >> a$) parallel to the edges of length b. The sample geometry is presented in Figure 1.

The description of the problem is based on the Figure 2. Both the notations proposed by Rivlin and Thomas [1953, p. 302] and the three-dimensional sketch of Greensmith and Thomas [1956, p. 373] (considered later by Eshelby [1975b, p. 76]) are adopted. In the following, the emphasize is laid on the rigorous definition of body configurations. The initial position of the sample shown in Fig. 1 is adopted as the reference configuration $(\mathcal{C}_0)$ (see Fig. 2(a)). During the experiments, the 'legs of the trousers' are first spread; then, the force F is applied to the legs to produce tearing. It leads to the definition of the deformed configuration $(\mathcal{C})$ as shown in Fig. 2(b). In this configuration, the separation between legs
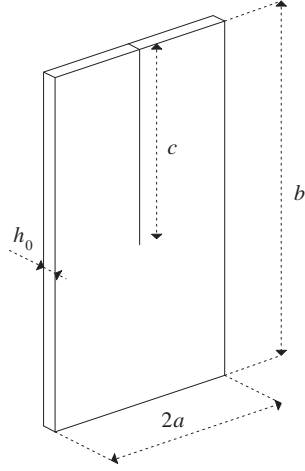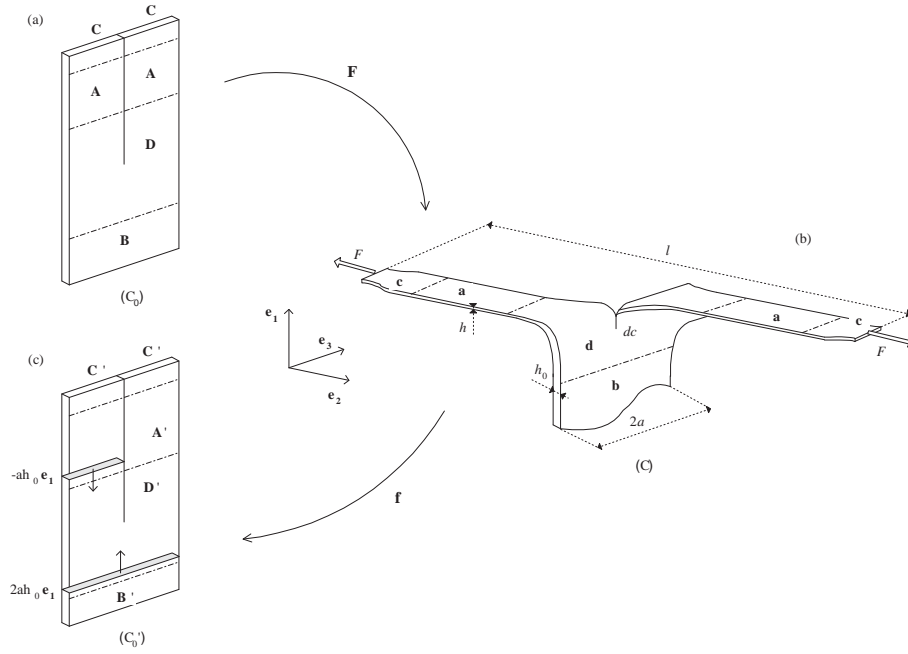
Figure 1: Trousers sample for rubber tearing

Figure 2: Definition of body configurations

3

extremities is l and the thickness h of the piece is no longer uniform. We consider now that the crack length increases by dc in the deformed configuration, the force F being kept constant. Thus, after unloading, i.e. $F = 0$, the sample occupies a new reference configuration $(\mathcal{C}'_0)$ depicted in Fig. 2(c). The motion between $(\mathcal{C})$ and $(\mathcal{C}'_0)$ is defined by its deformation gradient $\mathbf{f}$. Obviously, $\mathbf{f}$ is not exactly equal to $\mathbf{F}^{-1}$ because the crack has grown and $(\mathcal{C}'_0)$ differs from $(\mathcal{C}_0)$.

3 Previous derivations

3.1 Derivation of Rivlin and Thomas [1953]

The determination of the tearing energy as proposed by Rivlin and Thomas [1953] is based on an energetic analysis. For this purpose, authors considered Fig. 2(a) and (b).

The sample can be separated into four different regions in the deformed configuration (Fig. 2(b)):

- the region **a** of each of the legs which is substantially in uniaxial extension, the corresponding stretch ratio being denoted λ,

- the region **b** which is substantially undeformed if the uncut part of the sample is sufficiently long,

- the regions **c** and the region **d** in which strain and stress distributions are complicated, the former correspond to the neighbourhood of the points in which forces applies and the latter corresponds to the neighbourhood of the tip.

When the crack length increases from c to $c + dc$, authors consider that:

- the size of regions **a** increases and the size of the region **b** decreases of the same expense. In the reference configuration (Fig. 2(a)), the increase in volume of each region **A** (which are transformed into regions **a** by the motion) is $a\,h_0\,dc$; thus, the decrease in volume of the region **B** (which is transformed into region **b** by the motion) is equal to $2a\,h_0\,dc$,

- the regions **c** and the region **d** are only moved and their respective sizes (or the sizes of their undeformed counterparts **C** and **D**) do not change,

- moreover, after crack growth the region **a** is still in uniaxial extension with the same stretch ratio λ because F does not change, the region **b** is still undeformed, and the strain and stress distributions in regions **c** and **d** remain unchanged.

So, the length between legs extremities increases by $dl = 2\lambda\,dc$, and then,

$$\left.\frac{\partial l}{\partial c}\right|_F = 2\lambda. \tag{5}$$

Considering that the strain energy of the sample is a function of the two quantities l and c, its change dw is related to the changes of crack length dc and overall length dl by

$$dw = \left.\frac{\partial w}{\partial l}\right|_c dl + \left.\frac{\partial w}{\partial c}\right|_l dc. \tag{6}$$

Recalling that crack growth is due to the applied force F which remains constant, the previous equation can be written as

$$\left.\frac{\partial w}{\partial c}\right|_F = \left.\frac{\partial w}{\partial l}\right|_c \left.\frac{\partial l}{\partial c}\right|_F + \left.\frac{\partial w}{\partial c}\right|_l. \tag{7}$$

With respect to the previous statement about the increase in volume of the regions **a** and the decrease in volume of the region **b**, the change in energy can be determined in the reference configuration $(\mathcal{C}_0)$,

$$\left.\frac{\partial w}{\partial c}\right|_F = W\, 2a\, h_0, \tag{8}$$

where W stands for the strain energy density per unit of undeformed volume and $2a\, h_0$ is the cross-section of the undeformed sample. Moreover, the force F is related to the change in energy by

$$\left.\frac{\partial w}{\partial l}\right|_c = F. \tag{9}$$

Finally, recalling Eqs (5), (8) and (9), Eq. (7) becomes

$$\left.\frac{\partial w}{\partial c}\right|_l = W\, 2a\, h_0 - F\, 2\lambda, \tag{10}$$

the tearing energy being obviously obtained by Eq. (4).

3.2   Derivation of Eshelby [1975b]

In order to highlight the connection between the configurational stress tensor (also called the elastic energy momentum tensor by Eshelby) and the path-independent integral for energy release rate, Eshelby [1975b] proposed a new derivation of the tearing energy for the trousers sample.

In this way, he calculated the following surface integral defined in the reference configuration $(\mathcal{C}_0)$ shown in Fig. 2(a)

$$\gamma_1 = \int_S \left(W - P_{i1}\, u_{i,1}\right) dS \tag{11}$$

where S is a surface embracing the tip of the crack, $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor and $\mathbf{u}$ is the displacement vector. In fact, this scalar quantity corresponds to the first component of the configurational force

$$\boldsymbol{\gamma} = \int_S \overline{\boldsymbol{\Sigma}}\mathbf{n}\, dS \tag{12}$$

in which $\mathbf{n}$ is the unit vector normal to the surface dS and $\overline{\boldsymbol{\Sigma}}$ is the energy momentum (i.e. configurational stress) tensor proposed by Eshelby [1951, 1975a]

$$\overline{\boldsymbol{\Sigma}} = W\mathbf{I} - \nabla_{\mathbf{X}}^t \mathbf{u}\, \mathbf{P}, \tag{13}$$

where $\mathbf{I}$ is the 3×3 identity tensor and the superscript $\cdot^t$ denotes the transposition. γ_1 being path-independent, the author consider a surface S made up of

parts of the specimen surface with normal $\mathbf{e_2}$ and $\mathbf{e_3}$ in regions $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{B}$; a cross-section of the region $\mathbf{B}$ (which is not deformed); and cross-sections of both legs (in regions $\mathbf{A}$). The two first contributions are equal to zero; then γ_1 reduces to

$$\gamma_1 = 2 \left(W - P_{21} \frac{\partial u_2}{\partial X_1} \right) a \, h_0. \tag{14}$$

A given point with initial coordinates $(X_1, X_2 = 0)$ being first swung round then stretched, the displacement in the $\mathbf{e_2}$-direction is

$$u_2 = \pm \lambda \, X_1 - X_2, \tag{15}$$

where the sign depends on the leg: plus sign for the right-hand side leg and minus sign for the left-hand side leg (see Fig. 2). The stress is given by $P_{21} = \pm F/a \, h_0$, with the same remark for the signs. Thus, γ_1 reduces to

$$\gamma_1 = 2 \left(W \, a \, h_0 - F \, \lambda \right). \tag{16}$$

Recalling that the total energy release rate (tearing energy) is related to γ_1 by

$$T = -\frac{1}{h_0} \gamma_1, \tag{17}$$

in which the minus sign merely indicates that the tear will run to the bottom. So, the result of Rivlin and Thomas given by Eqs (4) and (10) is recovered.

4 A new derivation

Our method to determine the tearing energy of the trousers test sample is based on the general theory of Configurational Mechanics. Before presenting our approach, some basic results concerning the configurational stress tensor are recalled. Only results necessary for our derivation are given, for more details the reader can refer to Maugin [1993, 1995].

4.1 Basic definitions

The configurational stress tensor permits to quantify the evolution of the reference configuration of a given body subject to deformation. This tensor, denoted $\boldsymbol{\Sigma}$ through the rest of the paper, is defined as

$$\boldsymbol{\Sigma} = W\mathbf{I} - \mathbf{F}^t \, \mathbf{P}, \tag{18}$$

where $\mathbf{F}$ is the deformation gradient (equal to $\mathbf{I} + \nabla_{\mathbf{X}} \mathbf{u}$). Note that the expression of the configurational stress tensor is different than the one considered by Eshelby [1975b] and given in Eq. (13), the displacement gradient being replaced by the deformation gradient (see Section 5 for the discussion).

The derivation will be essentially based on the physical significance of the components of $\boldsymbol{\Sigma}$. Considering the material space $\mathcal{M}^3$ in which the body is defined as a set of particles (a reference configuration), the scalar $d\mathbf{U} \cdot \boldsymbol{\Sigma} \, dS_0 \mathbf{N}_0$ is the change in energy due to a material displacement, i.e. a displacement in $\mathcal{M}^3$, $d\mathbf{U}$ of the material surface $dS_0 \mathbf{N}_0$. So, as previously established by Kienzler and Herrmann [1997] for small strain, Σ_{ij} is the change in energy due to a unit material translation in the direction of the vector $\mathbf{e_i}$ of a unit material surface which normal vector is $\mathbf{e_j}$.

4.2 Application to the trousers test sample

As recalled above, the tearing energy is related to the change in energy between a body with a crack of length c and the same body with a crack of length $c + dc$ for a given motion. In order to determine the tearing energy for the trousers sample, let us examine Fig. 2. In this figure, the configurations $(\mathcal{C}_0)$ (Fig. 2(a)) and $(\mathcal{C}'_0)$ (Fig. 2(c)) are two reference configurations; then, with the help of the configurational stress tensor, it is possible to calculate the change in energy between them. In this way, only the material transformation, i.e. defined in $\mathcal{M}^3$, between $(\mathcal{C}_0)$ and $(\mathcal{C}'_0)$ has to be considered, the forces applied to the sample being known (Fig. 2(b)).

The analysis of the problem proposed by Rivlin and Thomas [1953] and recalled in Section 3.1 is adopted; regions $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ of $(\mathcal{C}_0)$ are transformed into $\mathbf{A}$', $\mathbf{B}$', $\mathbf{C}$' and $\mathbf{D}$' of $(\mathcal{C}'_0)$ in the following manner:

- the two regions $\mathbf{A}$ are transformed into the two regions $\mathbf{A}$' by the material translation $-dc\,\mathbf{e_1}$ of the material surface $a\,h_0\,(-\mathbf{e_1})$ (grey cross-section in Fig. 2(c)),

- the region $\mathbf{B}$ is transformed into the region $\mathbf{B}$' by the material translation $-dc\,\mathbf{e_1}$ of the material surface $2a\,h_0\,\mathbf{e_1}$ (grey cross-section in Fig. 2(c)),

- the regions $\mathbf{C}$ and $\mathbf{D}$ are transformed into regions $\mathbf{C}$' et $\mathbf{D}$' by a rigid body motion.

Then, the change in energy between configurations $(\mathcal{C}_0)$ and $(\mathcal{C}'_0)$ is

$$dw = dw_{\mathbf{A}\rightarrow\mathbf{A}'} + dw_{\mathbf{B}\rightarrow\mathbf{B}'} + dw_{\mathbf{C}\rightarrow\mathbf{C}'} + dw_{\mathbf{D}\rightarrow\mathbf{D}'} \qquad (19)$$

with

$$\begin{aligned} dw_{\mathbf{A}\rightarrow\mathbf{A}'} &= 2\left[-dc\,\mathbf{e_1}\cdot\boldsymbol{\Sigma}^{\mathbf{A}}\,a\,h_0\,(-\mathbf{e_1})\right] \\ &= 2\,dc\,a\,h_0\,\Sigma^{\mathbf{A}}_{11}, \end{aligned} \qquad (20)$$

$$\begin{aligned} dw_{\mathbf{B}\rightarrow\mathbf{B}'} &= -dc\,\mathbf{e_1}\cdot\boldsymbol{\Sigma}^{\mathbf{B}}\,2a\,h_0\,\mathbf{e_1} \\ &= -2\,dc\,a\,h_0\,\Sigma^{\mathbf{B}}_{11}, \end{aligned} \qquad (21)$$

and

$$dw_{\mathbf{C}\rightarrow\mathbf{C}'} = 0 \quad \text{and} \quad dw_{\mathbf{D}\rightarrow\mathbf{D}'} = 0. \qquad (22)$$

In these equations $\boldsymbol{\Sigma}^{\mathbf{A}}$ and $\boldsymbol{\Sigma}^{\mathbf{B}}$ stand for the values of the configurational stress tensor in regions $\mathbf{A}$ and $\mathbf{B}$, respectively. Eq. (22) is obvious because regions $\mathbf{C}$ and $\mathbf{D}$ are statically moved in $\mathcal{M}^3$ and consequently their energies are unchanged. Moreover, the region $\mathbf{B}$ being undeformed and stress-free, $\boldsymbol{\Sigma}^{\mathbf{B}} = \mathbf{0}$, and

$$dw_{\mathbf{B}\rightarrow\mathbf{B}'} = 0. \qquad (23)$$

Finally, the change in energy of the body is only due to the transformation of the regions $\mathbf{A}$ into the regions $\mathbf{A}$'. These regions being in simple extension and assuming that the material is incompressible, the deformation gradient and the first Piola-Kirchhoff stress tensor are:

- for the right-hand side leg of the trousers

$$\mathbf{F} = \lambda\,\mathbf{e_2} \otimes \mathbf{e_1} - \lambda^{-\frac{1}{2}}\mathbf{e_1} \otimes \mathbf{e_2} + \lambda^{-\frac{1}{2}}\mathbf{e_3} \otimes \mathbf{e_3} \qquad (24)$$

and

$$\mathbf{P} = \frac{F}{a\,h_0}\,\mathbf{e_2} \otimes \mathbf{e_1}, \qquad (25)$$

- for the left-hand side leg of the trousers

$$\mathbf{F} = -\lambda\,\mathbf{e_2} \otimes \mathbf{e_1} + \lambda^{-\frac{1}{2}}\mathbf{e_1} \otimes \mathbf{e_2} + \lambda^{-\frac{1}{2}}\mathbf{e_3} \otimes \mathbf{e_3} \qquad (26)$$

and

$$\mathbf{P} = -\frac{F}{a\,h_0}\,\mathbf{e_2} \otimes \mathbf{e_1}. \qquad (27)$$

For both legs, the first component of $\mathbf{\Sigma^A}$ reduces to

$$\Sigma_{11}^{\mathbf{A}} = W - F_{j1}\,P_{j1} = W - \lambda\,\frac{F}{a\,h_0}. \qquad (28)$$

So, the change in energy of the regions $\mathbf{A}$ is

$$\begin{aligned} dw_{\mathbf{A}\to\mathbf{A'}} &= 2dc\left(W - \lambda\,\frac{F}{a\,h_0}\right)a\,h_0 \\ &= (W\,2a\,h_0 - F\,2\lambda)\,dc. \end{aligned} \qquad (29)$$

So considering Eqs (19), (22) and (23), it is also equal to the total change in energy between configurations $(\mathcal{C}_0)$ and $(\mathcal{C}_0')$. It can also be written as

$$\frac{\partial w}{\partial c} = W\,2a\,h_0 - F\,2\lambda. \qquad (30)$$

So the result due to Rivlin and Thomas, and Eshelby is recovered.

5 Final remark

It should be noted that the derivation proposed above does not depend on the choice of the reference configuration. As an example, consider the configuration depicted in Figure 3. During experiments, the legs of the trousers are first spread then extended. Thus this configuration is undeformed and stress-free because the motion between the configuration $(\mathcal{C}_0)$ shown in Fig. 2(a) and this one reduces to a rigid body motion of a part of the sample. So, it can be adopted as a reference configuration. In this case, the change in energy between regions $\mathbf{A}$ and $\mathbf{A'}$ is simply due to a material unit translation $-dc\,\mathbf{e_2}$ of the material surface $a\,h_0\,(-\mathbf{e_2})$ for the right-hand side leg and to a material unit translation $dc\,\mathbf{e_2}$ of the material surface $a\,h_0\,\mathbf{e_2}$ for the right-hand side leg (see the grey cross-sections in Fig. 3)

$$\begin{aligned} dw_{\mathbf{A}\to\mathbf{A'}} &= dc\,\mathbf{e_2} \cdot \mathbf{\Sigma^A}\,a\,h_0\,\mathbf{e_2} \\ &\quad +dc\,(-\mathbf{e_2}) \cdot \mathbf{\Sigma^A}\,a\,h_0\,(-\mathbf{e_2}) \\ &= 2\,dc\,a\,h_0\,\Sigma_{22}^{\mathbf{A}}. \end{aligned} \qquad (31)$$
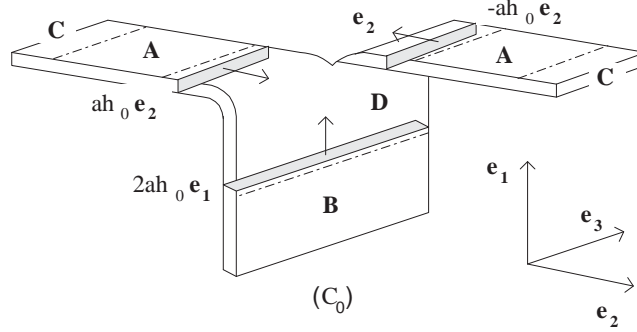
Figure 3: A new reference configuration for the trousers sample

The changes in energy due to the other regions remain equal to zero. For both legs of the sample, the deformation gradient and the first Piola-Kirchhoff stress tensor are identical and respectively given by

$$\mathbf{F} = \lambda \, \mathbf{e_2} \otimes \mathbf{e_2} + \lambda^{-\frac{1}{2}} \, \mathbf{e_1} \otimes \mathbf{e_1} + \lambda^{-\frac{1}{2}} \, \mathbf{e_3} \otimes \mathbf{e_3} \tag{32}$$

and

$$\mathbf{P} = \frac{F}{a \, h_0} \, \mathbf{e_2} \otimes \mathbf{e_2}. \tag{33}$$

With these expressions, the result Eq. (30) is easily recovered.

As noted above, Eshelby [1975b] considered the configurational stress tensor $\overline{\boldsymbol{\Sigma}}$ defined in terms of the displacement gradient (Eq. (13)) rather than $\boldsymbol{\Sigma}$ defined in terms of the deformation gradient (Eq. (18)). The former definition of the configurational stress tensor was first proposed by Eshelby [1951] for small strain, but its use for large strain problems is revealed inappropriate [Maugin, 1993]. This can be shown by the present study. Indeed, adopting the tensor $\overline{\boldsymbol{\Sigma}}$ and making the derivation with the reference configuration shown in Fig. 2(a) leads to the right expression of the tearing energy. Nevertheless, making it with the reference configuration depicted in Fig. 3 fails.

References

J. D. Eshelby. The force on an elastic singularity. *Phil. Trans. R. Soc. Lond.*, A. 244:87–112, 1951.

J. D. Eshelby. The elastic energy-momentum tensor. *J. Elast.*, 5(3-4):321–335, 1975a.

J. D. Eshelby. The calculation of energy release rates. In G. C. Sih, H. C. van Elst, and D. Broek, editors, *Prospects of Fracture Mechanics*, pages 69–84, Leyden, 1975b. Noordhoff.

H. W. Greensmith and A. G. Thomas. Rupture of rubber - III - Determination of tear properties. *Rubber Chem. Technol.*, 29:372–381, 1956.

A.A. Griffith. The phenomena of rupture and flow in solids. *Phil. Trans. R. Soc. Lond. A*, 221:163–198, 1920.

R. Kienzler and G. Herrmann. On the properties of the Eshelby tensor. *Acta Mech.*, 125:73–91, 1997.

G. A. Maugin. *Material Inhomogeneities in Elasticity*. Chapman and Hall, London, 1993.

G. A. Maugin. Material forces: concepts and applications. *Appl. Mech. Rev.*, 48:213–245, 1995.

R. S. Rivlin and A. G. Thomas. Rupture of rubber. I. Characteristic energy for tearing. *J. Polym. Sci.*, 10:291–318, 1953.