Stress Analysis

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Summary

- Cauchy's formula
- Stress transformation
- Principal stresses
- Maximum shear stress
- Octahedral stress
- Stress decomposition
- Equilibrium equations

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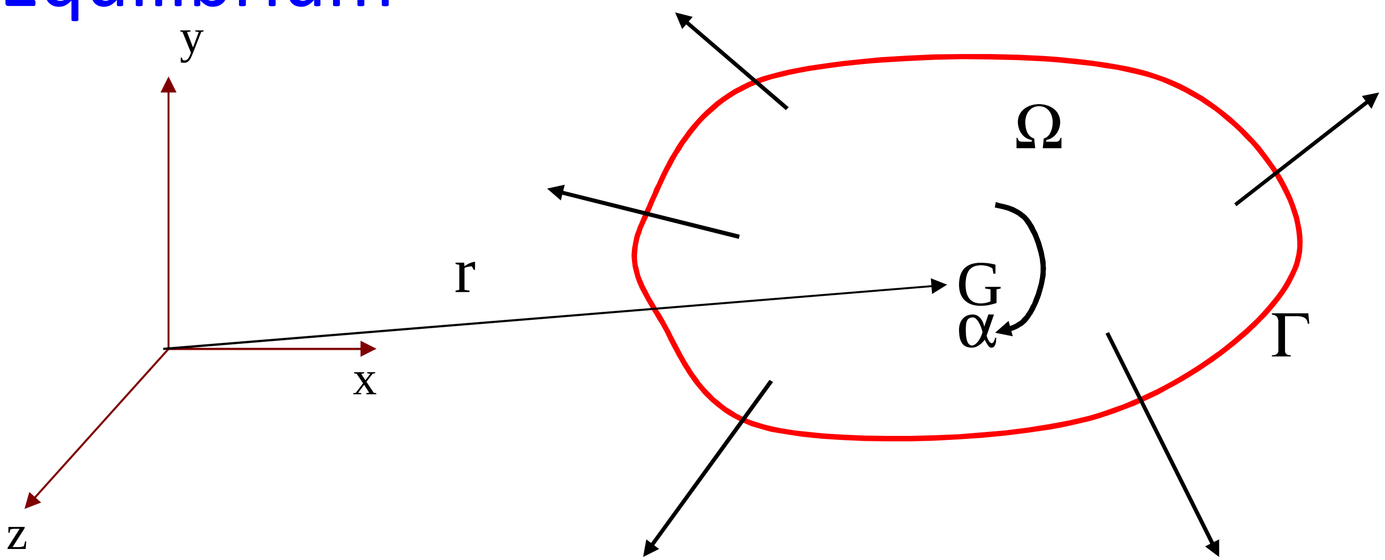
Introduction

- Body subjected to loading – internal resistance offered by body
- Internal forces – distributed continuously over the body
- Deformation – deformable body – finite elastic modulus
- Deformation – continuous – single valued
- Stress analysis – equilibrium of the body – every point is in equilibrium

Introduction

- Equilibrium



$$\sum \underset{\sim}{F} = m\,\underset{\sim}{a}$$

$$\sum \underset{\sim}{M} = \underset{\sim}{r} \times m\,\underset{\sim}{a} + I_G\,\underset{\sim}{\alpha}$$

Dynamic equilibrium

$$\sum \underset{\sim}{F} = 0 \quad \sum F_i = 0$$

$$\sum \underset{\sim}{M} = 0 \quad \sum M_i = 0$$

Static equilibrium

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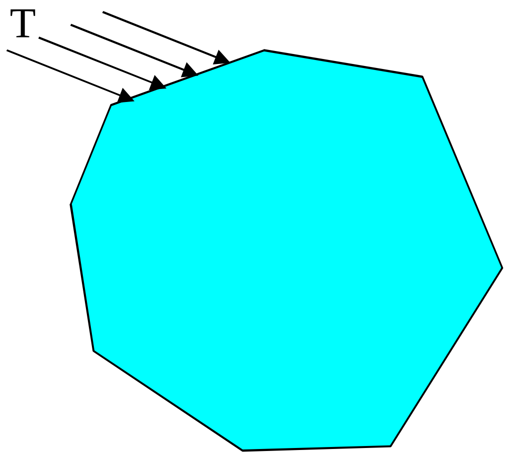
Forces

- Surface forces/tractions
  - Force per unit area
  - Acts on surfaces
  - Need not be perpendicular to surface area – reactions of other parts



$$d\,\underset{\sim}{F_s} = \underset{\sim}{T}\,dS$$

ds = infinitesimal area

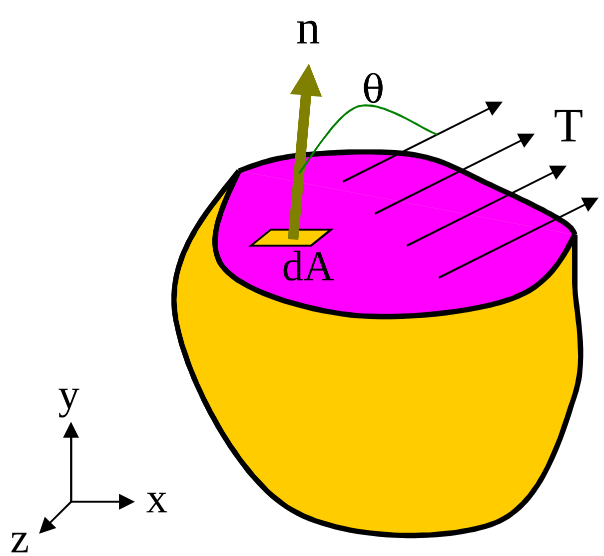
$$dF_{si} = T_i dS$$

$$\underset{\sim}{F_s} = \iint_S \underset{\sim}{T}\,dS$$

F_s = Force corresponds to traction acting over 'S'

Traction – Surface Normal

- Direction of traction and normal to area - different



dA – differential element

$\underset{\sim}{n}$ - normal to area

T – traction on area ‘A’

$$\underset{\sim}{T} = T_i e_i \qquad d\underset{\sim}{A} = \underset{\sim}{n}\, dA$$

Angle between traction and area vector => θ

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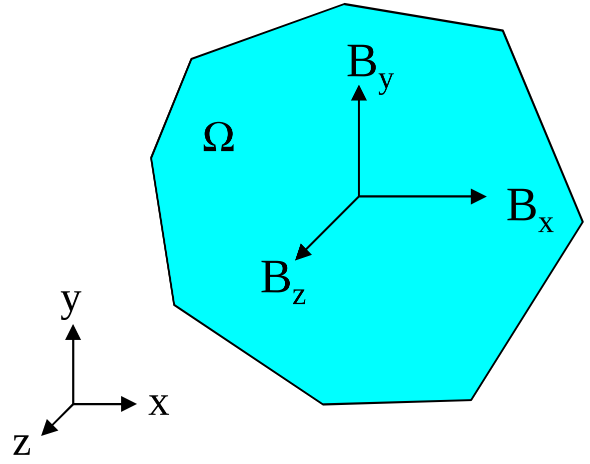
Forces

- Body load
  - Per unit volume or mass
  - Distributed over the body
  - Not a surface force
  - Acts at center of mass
  - Gravity force, centrifugal force



$$\underset{\sim}{B} = B_x \underset{\sim}{i} + B_y \underset{\sim}{j} + B_z \underset{\sim}{k}$$

Body force vector

$$\underset{\sim}{B} = B_i \underset{\sim}{e_i}$$

Indices notation

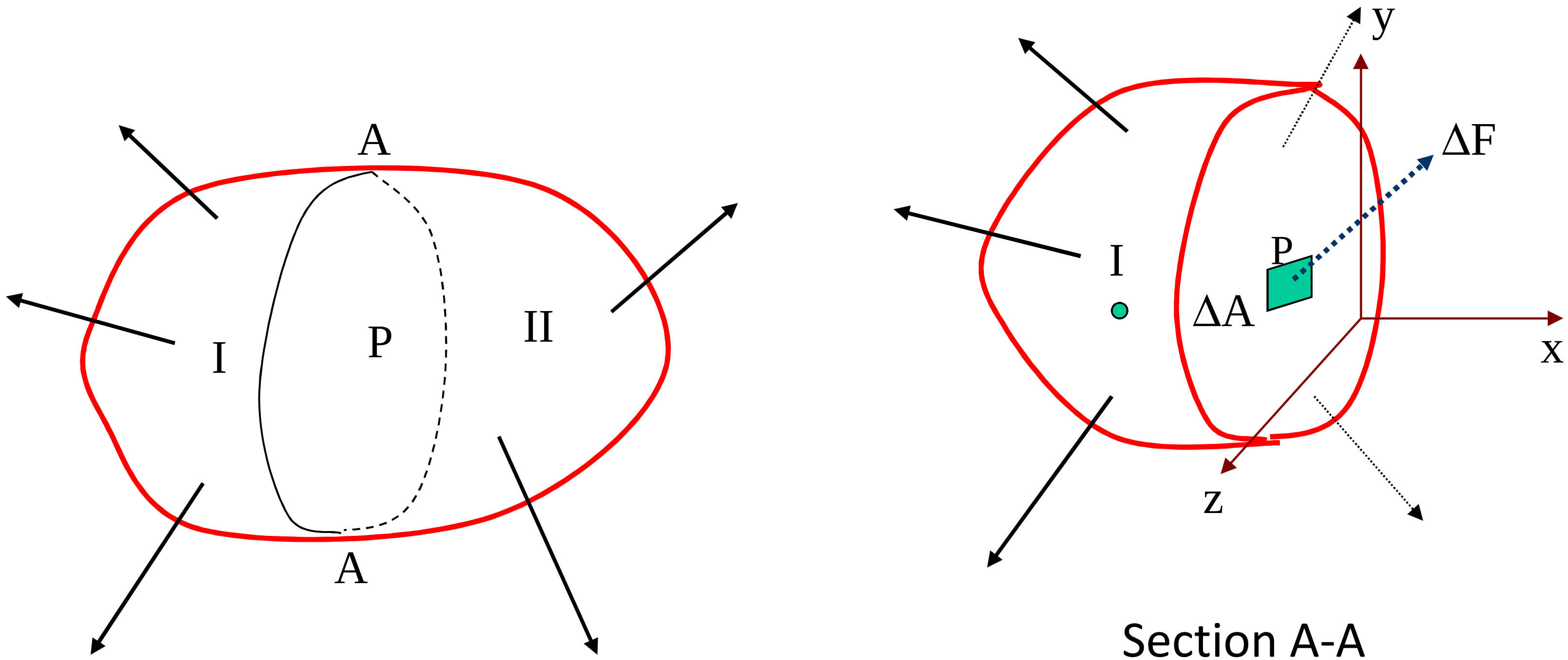
$$d\underset{\sim}{F}_B = \underset{\sim}{B}\, dV$$

Total force due to body load

$$\underset{\sim}{F}_B = \iiint_V \underset{\sim}{B}\, dV$$

Internal Forces

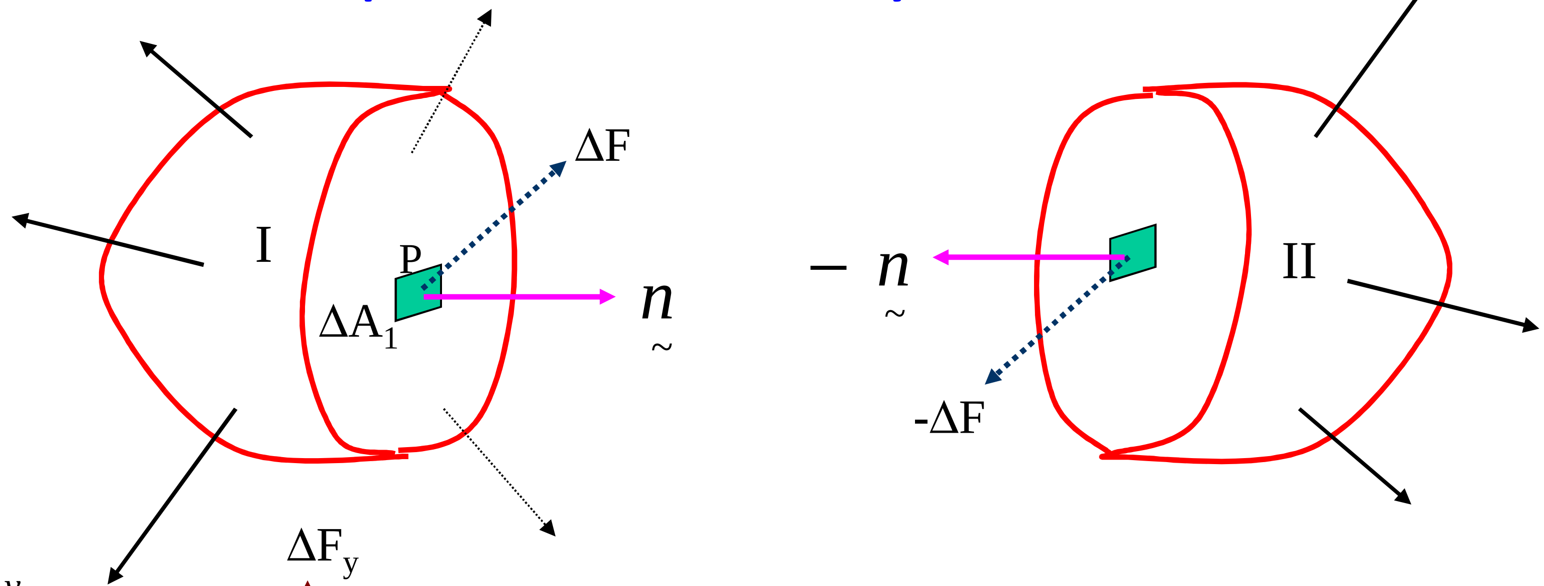
- Body under equilibrium – every particle under equilibrium

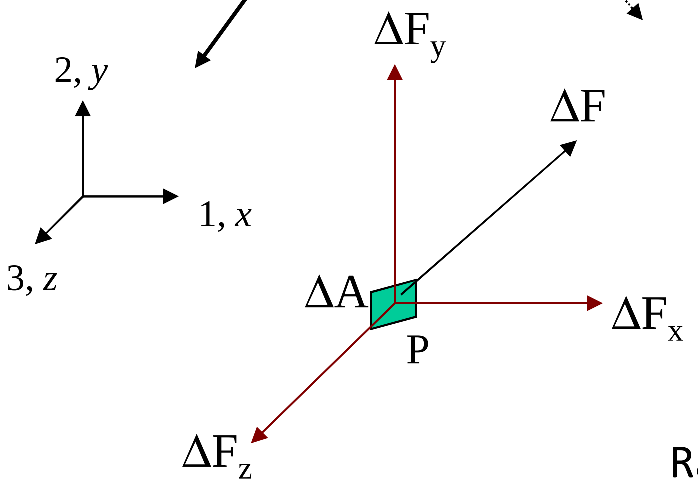


Section A-A
Perpendicular to X - axis

Internal Forces

- Second part of the body –





Decomposing forces

$$\underset{\sim}{\Delta F} = \Delta F_i \underset{\sim}{e_i} = \Delta F_1 \underset{\sim}{e_1} + \Delta F_2 \underset{\sim}{e_2} + \Delta F_3 \underset{\sim}{e_3}$$

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Internal Forces

- Normal stress –
  - Force perpendicular to area – $\sigma_{11} = \tau_{11} = \lim\limits_{\Delta A \to 0} \frac{\Delta F_1}{\Delta A_1} = \frac{dF_1}{dA_1}$
- Shear stress –
  - Force parallel to area – $\tau_{s1} = \lim\limits_{\Delta A \to 0} \frac{\Delta F_s}{\Delta A_1} = \frac{dF_s}{dA_1}$
  - ΔF_s acts in the plane – resolve into components

$$\tau_{21} = \lim_{\Delta A \to 0} \frac{\Delta F_2}{\Delta A_1} = \frac{dF_2}{dA_1} \qquad \tau_{31} = \lim_{\Delta A \to 0} \frac{\Delta F_3}{\Delta A_1} = \frac{dF_3}{dA_1}$$

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Internal Forces

- Stress components – depend on forces, orientation of the plane
- Notation – Two indices –

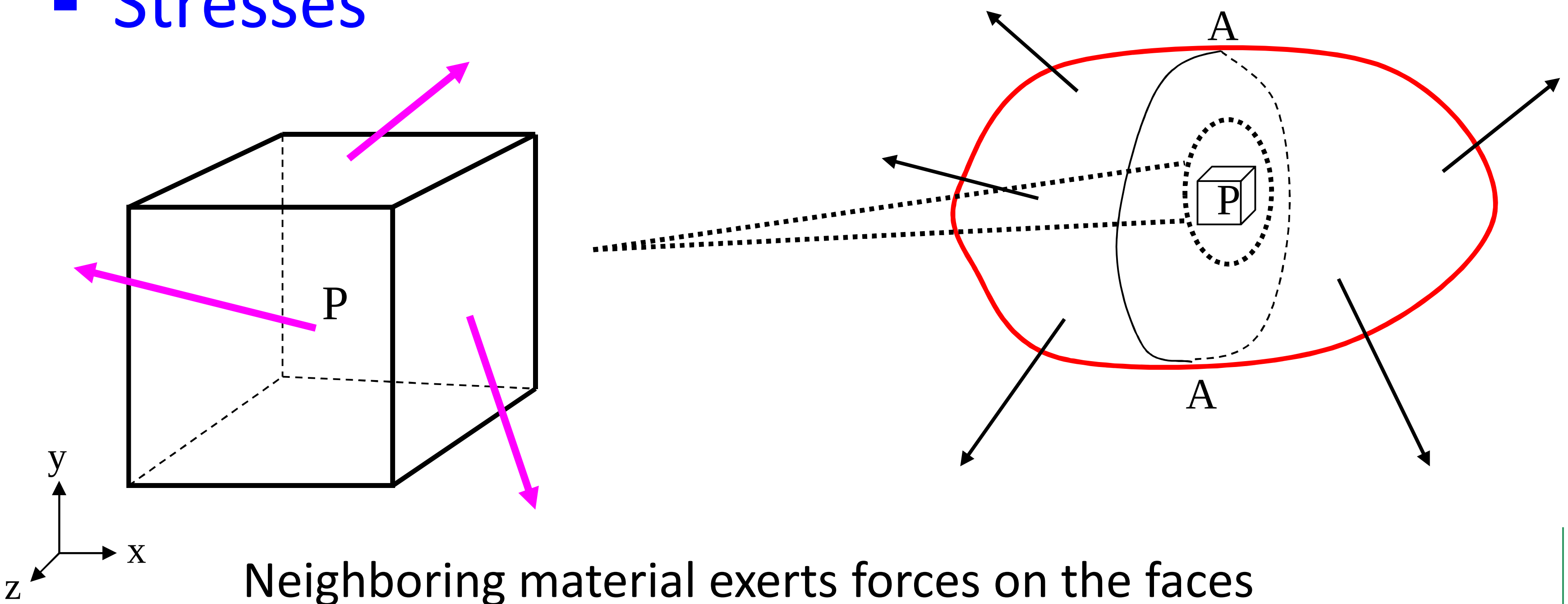
First index – plane & second index - direction

$$\tau_{yx} \qquad \tau_{ij} \qquad \tau_{xx}$$

Indices notation

Internal Forces

- Stresses



Neighboring material exerts forces on the faces of infinitesimal element

Decompose these forces – parallel & perpendicular to co-ordinate axes

Divide by area, dA => 0

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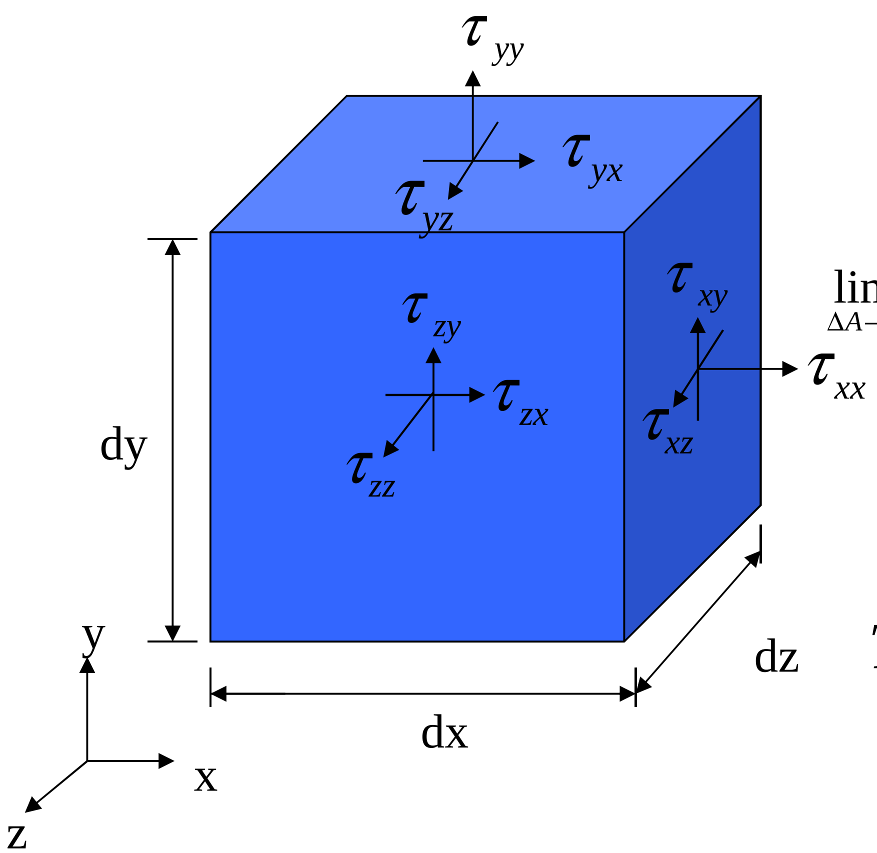
Stresses at a Point

- Stresses



Traction on each face of the element

$$\Delta \underset{\sim}{F} = \Delta F_i \underset{\sim}{e_i} = \Delta F_1 \underset{\sim}{e_1} + \Delta F_2 \underset{\sim}{e_2} + \Delta F_3 \underset{\sim}{e_3}$$

$$\lim_{\Delta A \to 0} \frac{\Delta \underset{\sim}{F}}{\Delta A} = \lim_{\Delta A \to 0} \frac{\Delta F_i}{\Delta A} \underset{\sim}{e_i} = \lim_{\Delta A \to 0} \frac{\Delta F_1}{\Delta A} \underset{\sim}{e_1} + \frac{\Delta F_2}{\Delta A} \underset{\sim}{e_2} + \frac{\Delta F_3}{\Delta A} \underset{\sim}{e_3}$$

$$\underset{\sim}{T}^1 = \sigma_{1i} \underset{\sim}{e_i} = \sigma_{11} \underset{\sim}{e_1} + \tau_{12} \underset{\sim}{e_2} + \tau_{13} \underset{\sim}{e_3}$$

$$\underset{\sim}{T}^2 = \sigma_{2i} \underset{\sim}{e_i} = \tau_{21} \underset{\sim}{e_1} + \sigma_{22} \underset{\sim}{e_2} + \tau_{23} \underset{\sim}{e_3}$$

$$\underset{\sim}{T}^3 = \sigma_{3i} \underset{\sim}{e_i} = \tau_{31} \underset{\sim}{e_1} + \tau_{32} \underset{\sim}{e_2} + \sigma_{33} \underset{\sim}{e_3}$$

Traction on each plane => Stresses acting on that plane

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Stresses at a Point

- Δx, Δy and Δz => 0 – it becomes a point

$$[\underset{\sim}{T}] = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \begin{bmatrix} \underset{\sim}{e_1} \\ \underset{\sim}{e_2} \\ \underset{\sim}{e_3} \end{bmatrix} \Rightarrow [\underset{\sim}{T}] = [\sigma][\underset{\sim}{e}]$$

Nine stress components

3 – Normal stresses, 6 – Shear stresses

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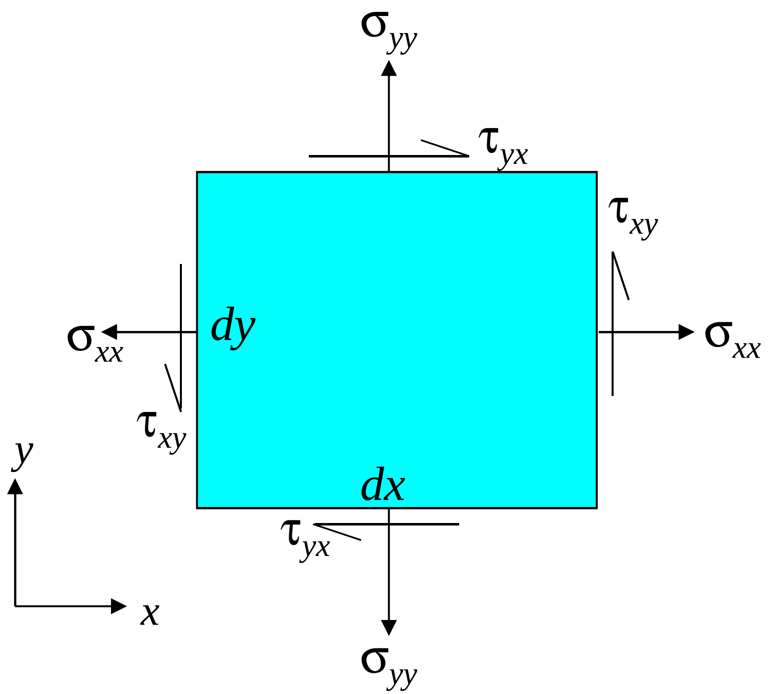
Stresses at a Point

- Shear stresses - Complementary



Static equilibrium of the element

$$\sum \underset{\sim}{F} = 0, \ \sum \underset{\sim}{M} = 0$$

$$\sum F_x = 0, \ \sum F_y = 0$$

$$\sum M_z = 0$$

This gives => $\tau_{xy} = \tau_{yx}$

Apply same method in other directions also, $\tau_{yz} = \tau_{zy}, \quad \tau_{zx} = \tau_{xz}$

Shear stresses are complementary in nature

Stresses at a Point

- Stresses at a point – 3D stress state
  - Nine stress components
  - Three normal stresses - τ_{xx}, τ_{yy}, τ_{zz}
  - Six shear stresses - τ_{yz}, τ_{zy}, τ_{xz}, τ_{zx}, τ_{xy}, τ_{yx}
  - Complementary => $\tau_{yz}= \tau_{zy}$; $\tau_{xz}= \tau_{zx}$; $\tau_{xy}= \tau_{yx}$
  - Three shear stress components
  - Total number of stress components at a point

    => three normal + three shear = six components

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Stresses at a Point

- Nine (six) stress – arrange in an array –

$$[\tau] = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

- Rows represent
  - First row – stresses on ‘x’ plane
  - Second row – stresses on ‘y’ plane
  - Third row – stresses on ‘z’ plane

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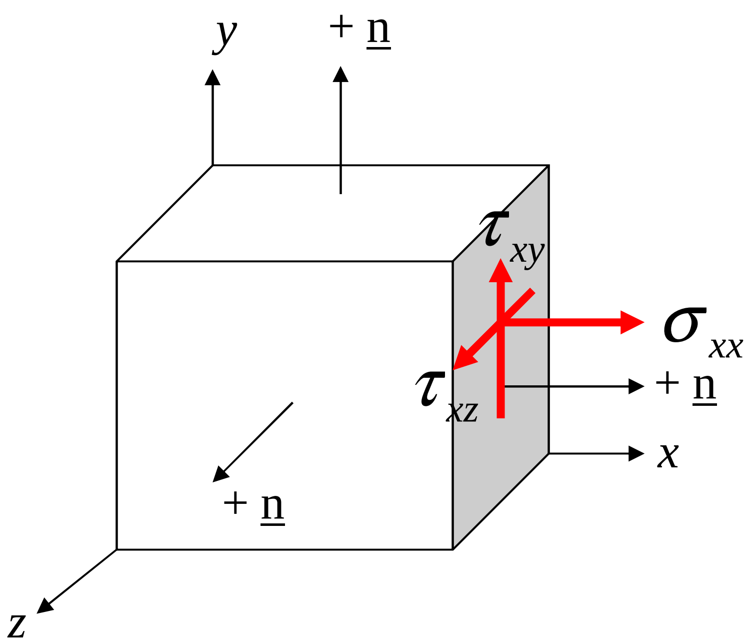
Sign Convention

- Positive or negative stress – depends on
  - Direction of area normal and
  - Force direction



σ_{xx} – Force in +ve x-direction

Area normal => +ve area
=> Stress is +ve - Tensile Stress

τ_{xy} – Force in +ve y – direction

Area => +ve area normal +ve
=> Stress is +ve

τ_{xz} – Force in +ve z – direction

Area => +ve area normal +ve
=> Stress is +ve

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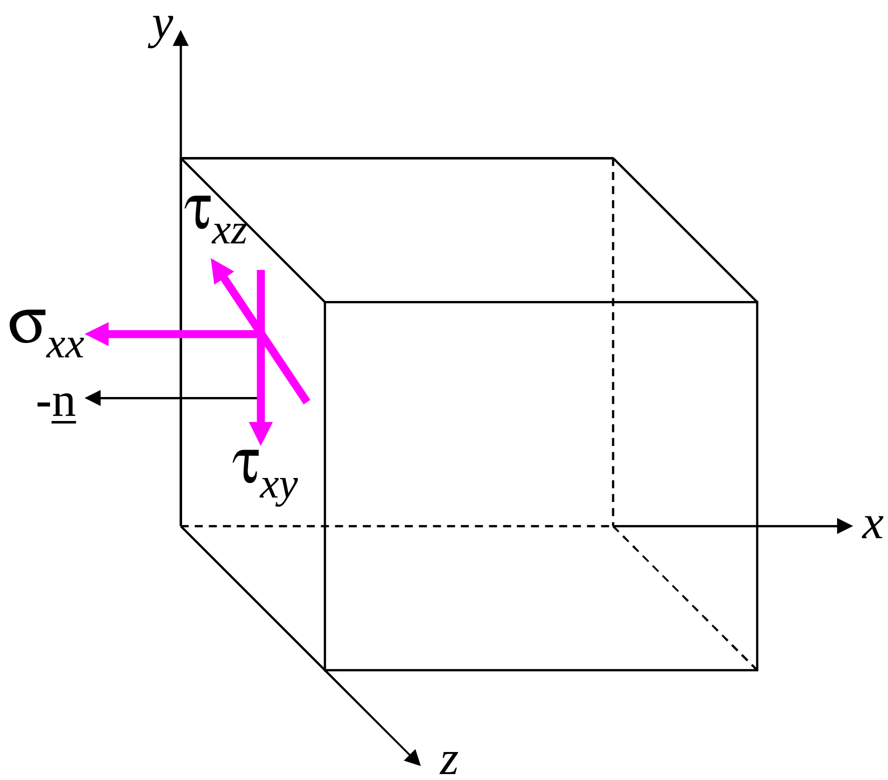
Sign Convention

- Positive/negative stresses



σ_{xx} – Force in -ve *x*-direction

Area normal => -ve area
=> Stress is +ve - Tensile Stress

τ_{xy} – Force in -ve *y* – direction

Area => -ve area normal
=> Stress is +ve

τ_{xz} – Force in -ve *z* – direction

Area => -ve area normal
=> Stress is +ve

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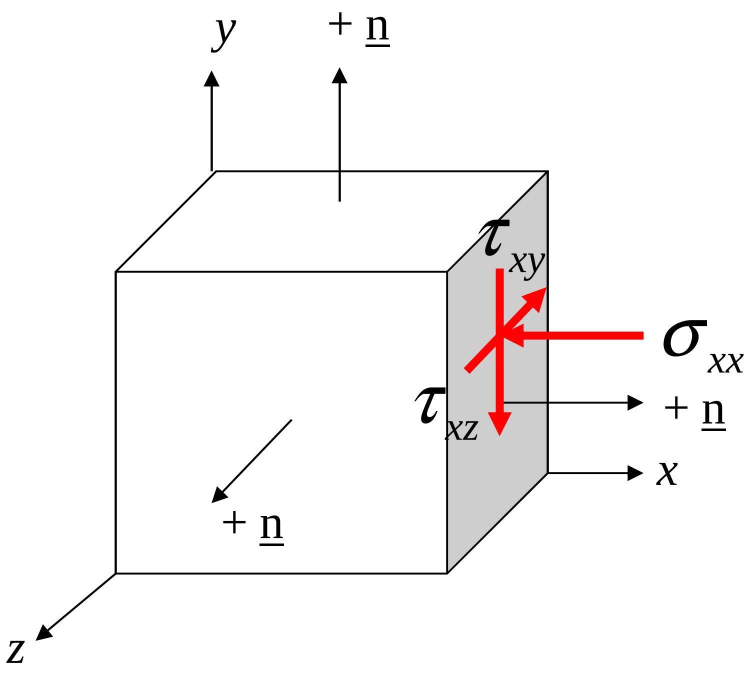
Sign Convention

- Positive/negative stresses



σ_{xx} – Force in -ve x-direction

Area normal => +ve area
=> Stress is -ve => Compressive Stress

τ_{xy} – Force in -ve y – direction

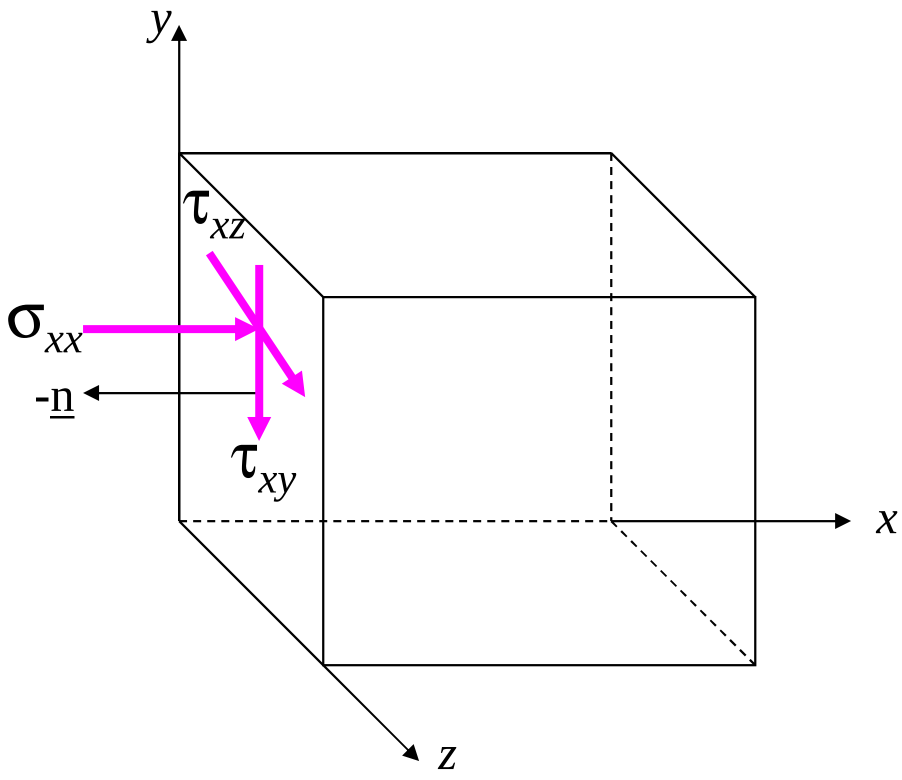
Area => +ve area normal
=> Stress is -ve

τ_{xz} – Force in -ve z – direction

Area => +ve area normal
=> Stress is -ve

Sign Convention

- Positive/negative stresses –



σ_{xx} – Force in +ve *x*-direction

Area normal => -ve area
=> Stress is -ve => Compressive Stress

τ_{xy} – Force in -ve *y* – direction

Area => -ve area normal
=> Stress is +ve

τ_{xz} – Force in +ve *z* – direction

Area => -ve area normal
=> Stress is -ve

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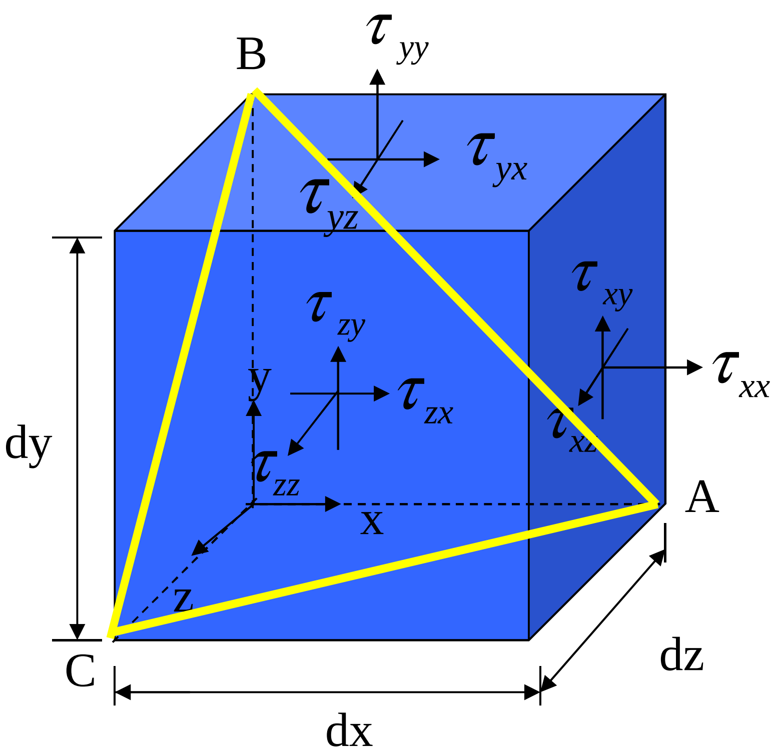
Cauchy's formula

- Stress components acting on an element



On each face three stress components are acting

Consider a plane ABC intersecting 'x', 'y' and 'z' axes

Normal has direction cosines *(l, m, n)*

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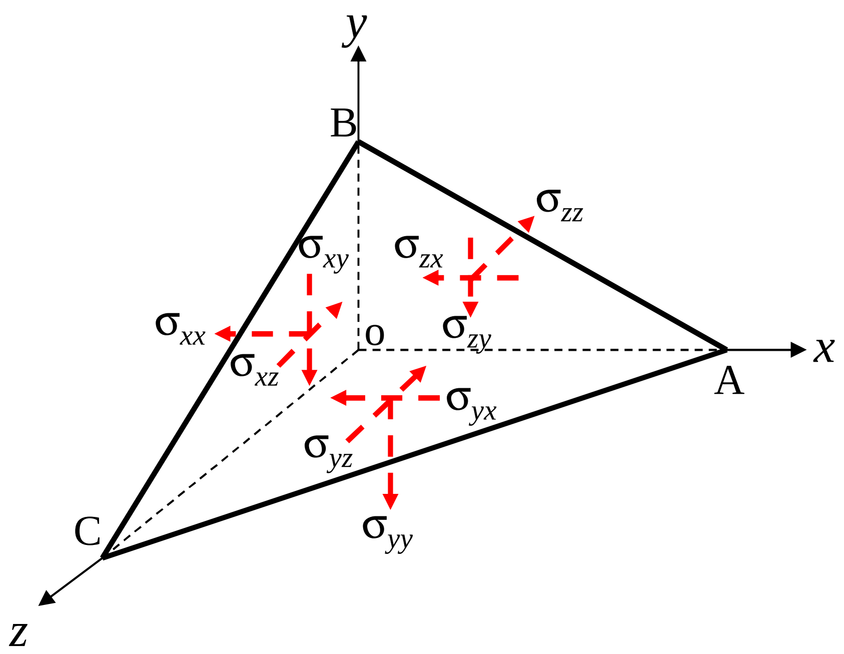
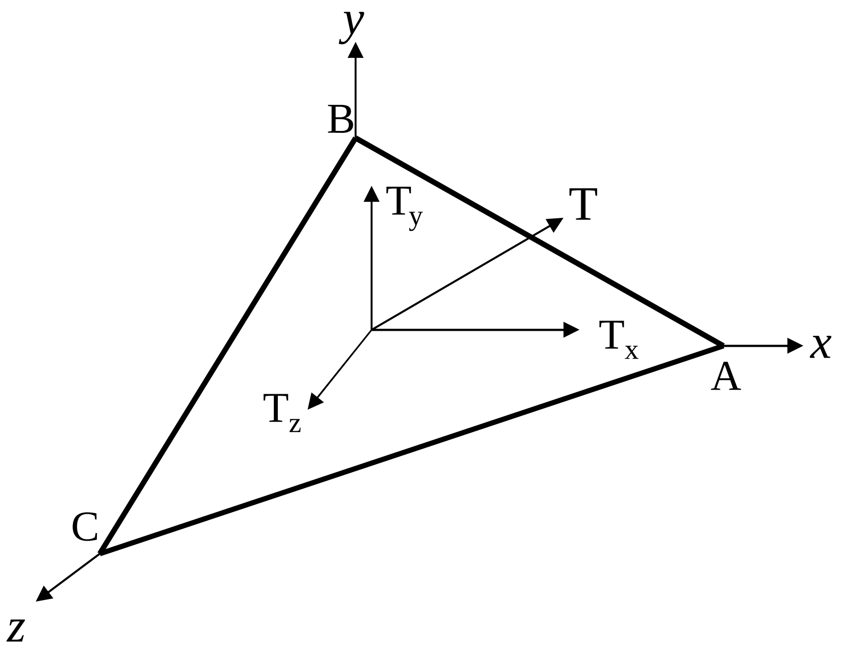
Cauchy's Formula

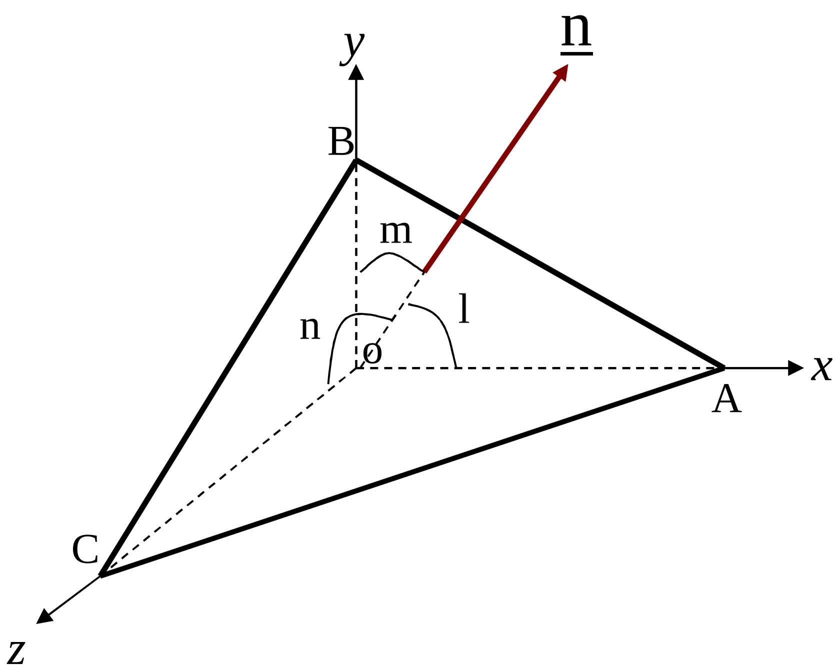
- Six stress components are given at point –
  Tractions on an inclined plane –





Cauchy’s Formula

- Direction cosines of ABC plane



$$\underset{\sim}{n} = l\,\underset{\sim}{e}_1 + m\,\underset{\sim}{e}_2 + n\,\underset{\sim}{e}_3 = l_i\,\underset{\sim}{e}_i$$

Equilibrium of tetrahedron

$$T_x A = \sigma_{xx} A_x + \tau_{yx} A_y + \tau_{zx} A_z$$

$$T_x = \sigma_{xx}\frac{A_x}{A} + \tau_{yx}\frac{A_y}{A} + \tau_{zx}\frac{A_z}{A}$$

$$T_x = \sigma_{xx} l + \tau_{yx} m + \tau_{zx} n$$

$$T_y A = \tau_{xy} A_x + \tau_{yy} A_y + \tau_{zy} A_z$$

$$T_y = \tau_{xy} l + \tau_{yy} m + \tau_{zy} n$$

$$T_z A = \tau_{xz} A_x + \tau_{yz} A_y + \sigma_{zz} A_z$$

$$T_z = \tau_{xz} l + \tau_{yz} m + \sigma_{zz} n$$

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Cauchy's Formula

- Stress state at a point and direction cosines of a plane => Tractions on that plane

$$T_x = \sigma_{xx} l + \tau_{yx} m + \tau_{zx} n$$

$$T_y = \tau_{xy} l + \sigma_{yy} m + \tau_{zy} n$$

$$T_z = \tau_{xz} l + \tau_{yz} m + \sigma_{zz} n$$

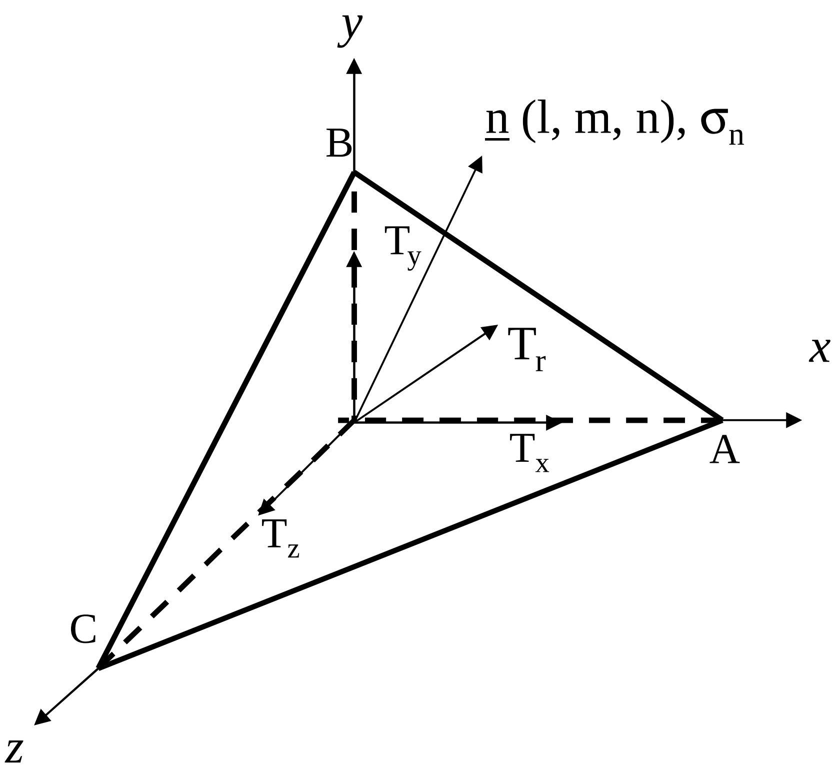
$$l^2 + m^2 + n^2 = 1$$

Indices notation

$$T_i = \sigma_{ji} l_j$$

$$\begin{Bmatrix} T_x \\ T_y \\ T_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} => \{T\} = [\sigma]\{\alpha\}$$

Normal Stress



- Resultant traction –

$$T_R^2 = T_x^2 + T_y^2 + T_z^2$$

- Direction cosines of T_R

$$l' = \frac{T_x}{T_R}, \quad m' = \frac{T_y}{T_R}, \quad n' = \frac{T_z}{T_R}$$

- Projections of T_i on $\underline{n}$

Normal stress on plane ABC

$$\sigma_n = lT_x + mT_y + nT_z$$

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Normal Stress

- Normal stress is given by,

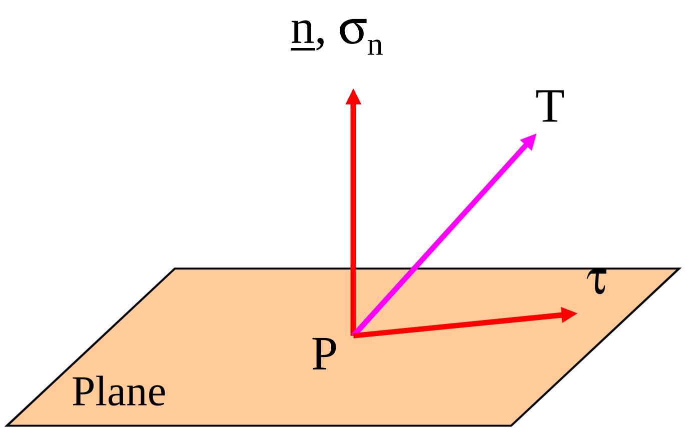
$$\sigma_n = lT_x + mT_y + nT_z$$

$$\sigma_n = l\left(l\sigma_{xx} + m\tau_{yx} + n\tau_{zx}\right) + m\left(l\tau_{xy} + m\sigma_{yy} + n\tau_{zy}\right) + n\left(l\tau_{xz} + m\tau_{yz} + n\sigma_{zz}\right)$$

$$\sigma_n = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2\left(lm\tau_{xy} + mn\tau_{yz} + nl\tau_{zx}\right)$$

Shear Stress

- Calculate shear stress from tractions and normal stress



$$T^2 = \sigma_n^2 + \tau^2$$

$$\tau^2 = T^2 - \sigma_n^2$$

$$\sigma_n = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2\left(lm\,\tau_{xy} + mn\,\tau_{yz} + nl\,\tau_{zx}\right)$$

$$\tau^2 = \left(T_x^2 + T_y^2 + T_z^2\right) - \sigma_n^2$$

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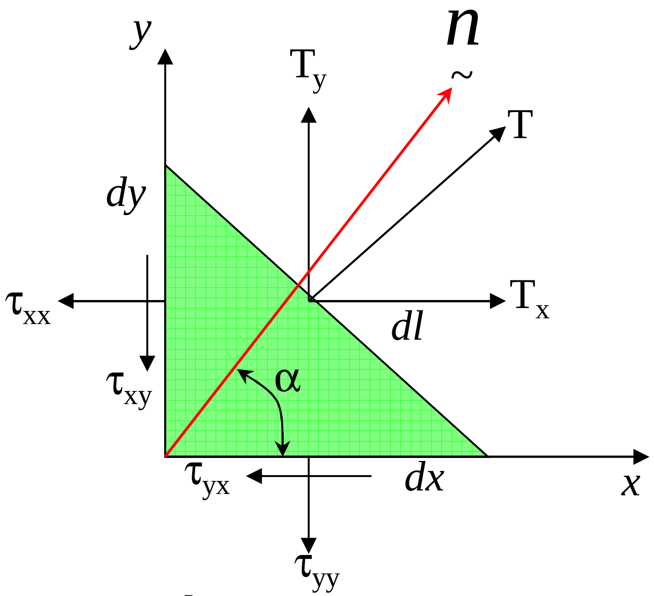
Stresses in 2D

- 2D stress state - σ_{xx}, σ_{yy} and τ_{xy}



$$T_x = l\tau_{xx} + m\tau_{yx}$$

$$T_y = l\tau_{xy} + m\tau_{yy}$$

$$\sigma_n = lT_x + mT_y$$

$$\sigma_n = \tau_{xx} l^2 + \tau_{yy} m^2 + 2lm\tau_{xy}$$

$$\cos\alpha = \frac{dy}{dl} = l$$

$$\cos(90-\alpha) = \sin\alpha = \frac{dx}{dl} = m$$

$$\underset{\sim}{n} = l\,\underset{\sim}{i} + m\,\underset{\sim}{j}$$

$$\sigma_n = \tau_{xx}\cos^2\alpha + \tau_{yy}\sin^2\alpha + \tau_{xy}\sin 2\alpha$$

$$\sigma_n = \left(\frac{\tau_{xx}+\tau_{yy}}{2}\right) + \left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\cos 2\alpha + \tau_{xy}\sin 2\alpha$$

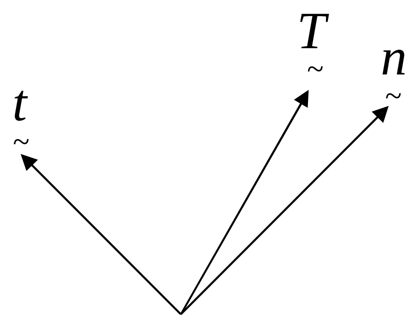
Stresses in 2D

- Normal stress in 't' direction

Angle => 90+α

$$\sigma_n = \left(\frac{\tau_{xx}+\tau_{yy}}{2}\right)+\left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\cos 2\alpha+\tau_{xy}\sin 2\alpha$$

$$\sigma_t = \left(\frac{\tau_{xx}+\tau_{yy}}{2}\right)-\left(\frac{\tau_{xx}-\tau_{yy}}{2}\right)\cos 2\alpha-\tau_{xy}\sin 2\alpha$$

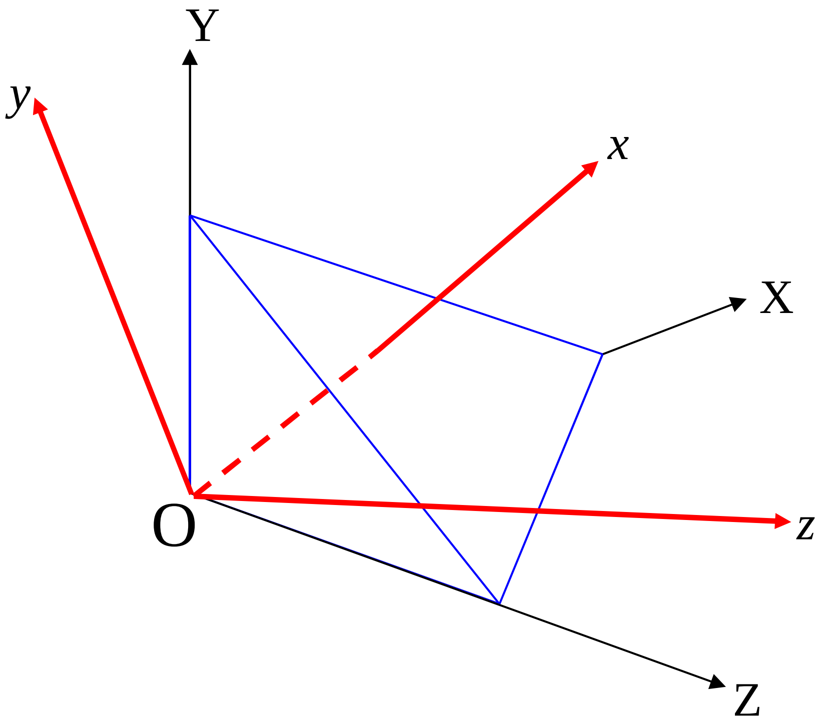


- Shear stress $\tau^2 = T^2 - \sigma_n^2$

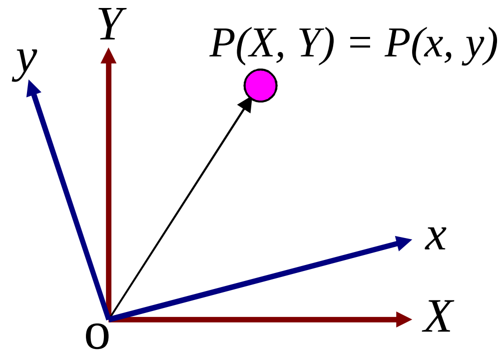
$$\tau = -\frac{1}{2}\left(\tau_{xx}-\tau_{yy}\right)\sin 2\alpha+\tau_{xy}\cos 2\alpha$$

Stress Transformation

- Transformation of stresses from one co-ordinate system (XYZ) to another co-ordinate system (*xyz*)



2D case



$$l_{11} = \cos\angle xoX \quad l_{12} = \cos\angle xoY$$

$$l_{21} = \cos\angle yoX \quad l_{22} = \cos\angle yoY$$

$P(x, y)$ => Vector

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Stress Transformation

- 2D Transformation –

$$x = Xl_{11} + Yl_{12} \quad \text{or}$$

$$x_1 = X_1 l_{11} + X_2 l_{12}$$

$$x_2 = X_1 l_{21} + X_2 l_{22}$$

Writing in indices form

$$x_i = l_{ij} X_j$$

Writing in matrix form

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} => \{x\} = [T]\{X\}$$

Where [T] is transformation matrix

$$[T] = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$$

Stress Transformation

- Stress is a second-order tensor – two indices
- Second order tensor => Outer product of two first order tensors

$$c_{ij} = a_i \otimes b_j \Rightarrow \underset{\approx}{c} = \underset{\sim}{a}\,\underset{\sim}{b}$$ Dyadic product in XYZ csys

- Transforming this product to *xyz* csys

$$c'_{ij} = a'_i b'_j = l_{im} a_m l_{jn} b_n = l_{im} l_{jn} a_m b_n = l_{im} l_{jn} c_{mn}$$

$$\sigma'_{ij} = l_{im} l_{jn} \sigma_{mn}$$

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Stress Transformation

- Writing in terms of transformation matrix

$$\sigma'_{ij} = l_{im} l_{jn} \sigma_{mn}$$

$$[\sigma'] = [T][\sigma][T]^T$$

Where,

$$[\sigma] = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$ => Stress tensor

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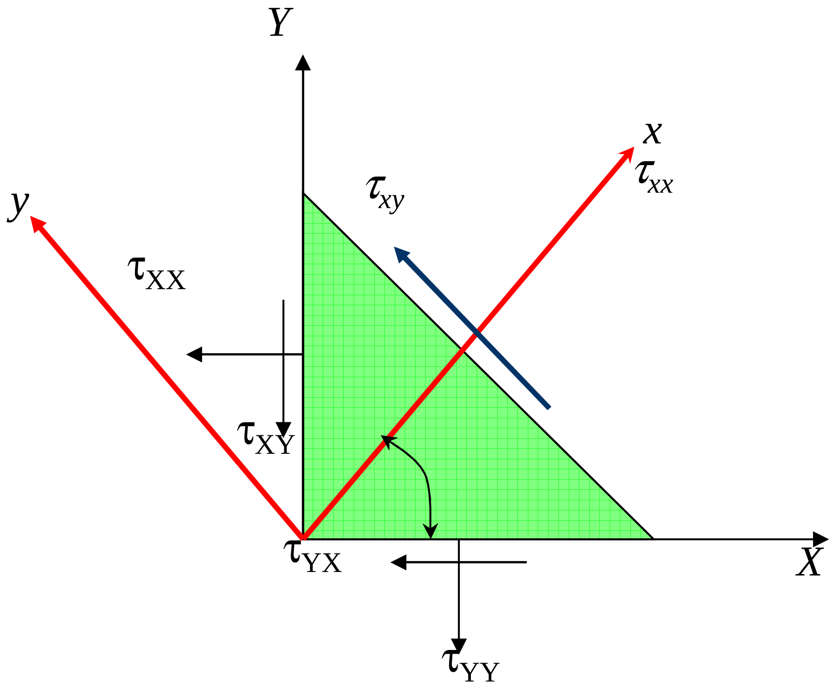
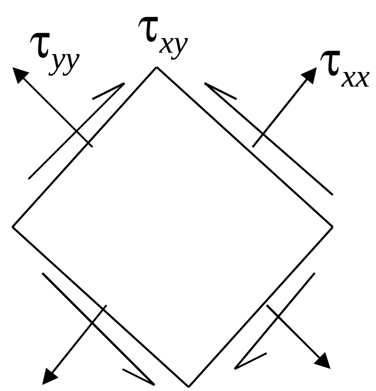
Stress Transformation

- Applying in 2D stress transformation –





Transformation matrix

$$[T] = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$$

Stress tensor in XYZ

$$[\sigma_{XYZ}] = \begin{bmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{bmatrix}$$

Stress tensor in *xyz*

$$[\sigma_{xyz}] = [T][\sigma_{XYZ}][T]^T$$

Stress Transformation

- 2D Stress transformation –

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} \sigma_{XX} & \tau_{XY} \\ \tau_{XY} & \sigma_{YY} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} \\ l_{12} & l_{22} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} l_{11}\sigma_{XX} + l_{12}\tau_{XY} & l_{11}\tau_{XY} + l_{12}\sigma_{YY} \\ l_{21}\sigma_{XX} + l_{22}\tau_{XY} & l_{21}\tau_{XY} + l_{22}\sigma_{YY} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} \\ l_{12} & l_{22} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} l_{11}^2\sigma_{XX} + 2l_{11}l_{12}\tau_{XY} + l_{12}^2\sigma_{YY} & l_{11}l_{21}\sigma_{XX} + l_{12}l_{21}\tau_{XY} + l_{11}l_{22}\tau_{XY} + l_{12}l_{22}\sigma_{YY} \\ l_{11}l_{21}\sigma_{XX} + l_{12}l_{21}\tau_{XY} + l_{11}l_{22}\tau_{XY} + l_{12}l_{22}\sigma_{YY} & l_{21}^2\sigma_{XX} + 2l_{22}l_{21}\tau_{XY} + l_{22}^2\sigma_{YY} \end{bmatrix}$$

$$\sigma_{xx} = l_{11}^2\sigma_{XX} + 2l_{11}l_{12}\tau_{XY} + l_{12}^2\sigma_{XX} \qquad \sigma_{yy} = l_{21}^2\sigma_{XX} + 2l_{22}l_{21}\tau_{XY} + l_{22}^2\sigma_{XX}$$

$$\tau_{xy} = l_{11}l_{21}\sigma_{XX} + \left(l_{12}l_{21} + l_{11}l_{22}\right)\tau_{XY} + l_{12}l_{22}\sigma_{YY}$$

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Principal Stresses

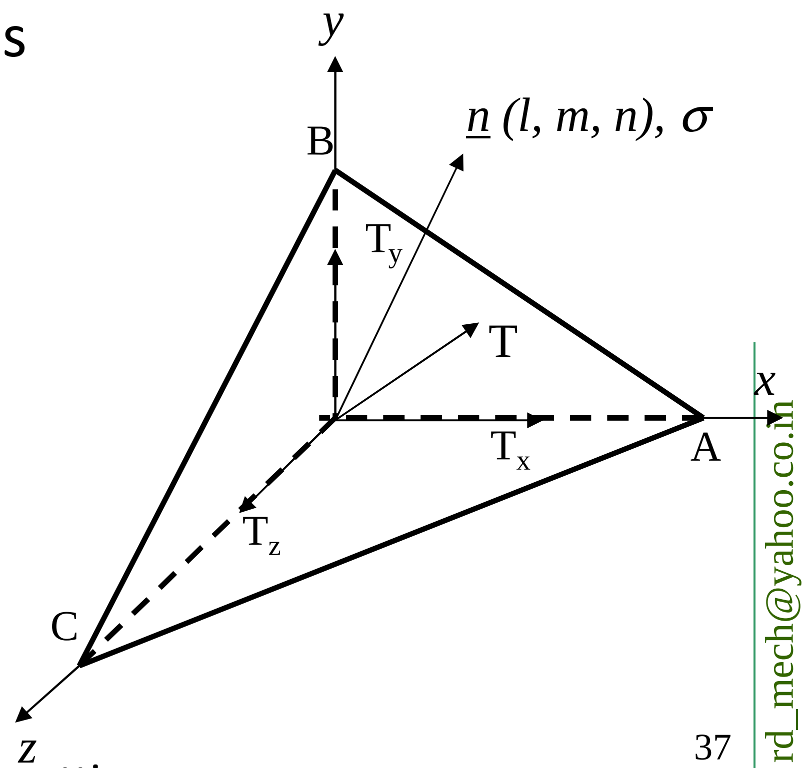
- Maximum normal stresses which act on the planes, where there is no shear stress

Tractions on ABC in x, y & z directions

$$T_x = l\sigma_{xx} + m\tau_{xy} + n\tau_{xz} = l\sigma$$

$$T_y = l\tau_{xy} + m\sigma_{yy} + n\tau_{yz} = m\sigma$$

$$T_z = l\tau_{xz} + m\tau_{yz} + n\sigma_{zz} = n\sigma$$



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Principal Stresses

- Writing in matrix form –

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix} = \sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} l \\ m \\ n \end{Bmatrix}$$

$$[\sigma]\{\alpha\} = \sigma [I]\{\alpha\}$$

$$[[\sigma] - \sigma [I]]\{\alpha\} = 0$$

Homogeneous algebraic equations – Eigenvalue problem

σ - Eigenvector, {α} - Eigenvector

Expanding – cubic equation – three roots

Principal Stresses

- For non-trivial solution – {α}≠ 0 – Determinant = 0

$$\left| [\sigma] - \sigma [I] \right| = 0$$

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$$

Principal Stresses

- Eigenvalue problem –
  - Homogeneous algebraic equations
  - Eigenvalues => Three principal stresses
  - Eigenvalues => Real values – definite matrix
  - Eigenvectors => Orientation of principal planes
  - Number of Eigenvalues => order of matrix

Principal Stresses

- Expanding the determinant –

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ symm & & \sigma_z \end{vmatrix}$$

Principal Stresses

- Principal stresses can be obtained from the following

$$\sigma_1 = \frac{I_1}{3} + 2\sqrt{-\frac{a}{3}}\cos\left(\frac{\phi}{3}\right) \qquad a = -\left(I_2 + \frac{I_1^2}{3}\right)$$

$$\sigma_2 = \frac{I_1}{3} + 2\sqrt{-\frac{a}{3}}\cos\left(120 + \frac{\phi}{3}\right) \qquad b = -\left(I_3 + \frac{2}{27}I_1^3 + \frac{I_1 I_2}{3}\right)$$

$$\sigma_3 = -\frac{I_1}{3} + 2\sqrt{-\frac{a}{3}}\cos\left(240 + \frac{\phi}{3}\right) \qquad \phi = \cos^{-1}\left(\frac{-b}{2\sqrt{-a^3/27}}\right)$$

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Principal Stresses

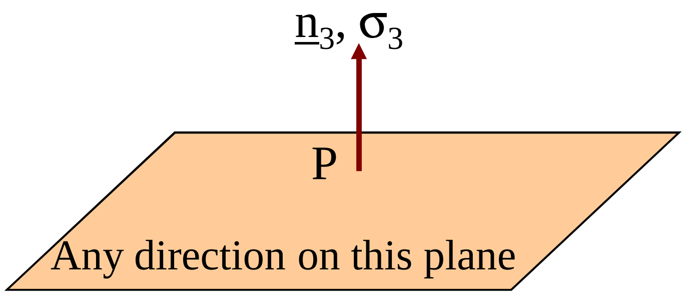
- Stress invariants – I_1, I_2 & I_3
  - Invariant – value doesn't change when the coordinate system is rotated – referring stresses wrt a different coordinate system
  - Same coefficients of the characteristic equation for any orientation
  - Independent of orientation of the planes
  - The principal stresses depend only on loading
  - For a given stress state – three mutually perpendicular planes - maximum stresses
  - Calculation of principal plane orientations – use direction cosines relation also

Principal Stresses

- Distinct values of σ_1, σ_2 and σ_3 – principal planes unique – mutually perpendicular

$$\underline{n}_1 . \underline{n}_2 = \underline{n}_2 . \underline{n}_3 = \underline{n}_3 . \underline{n}_1 = 0$$

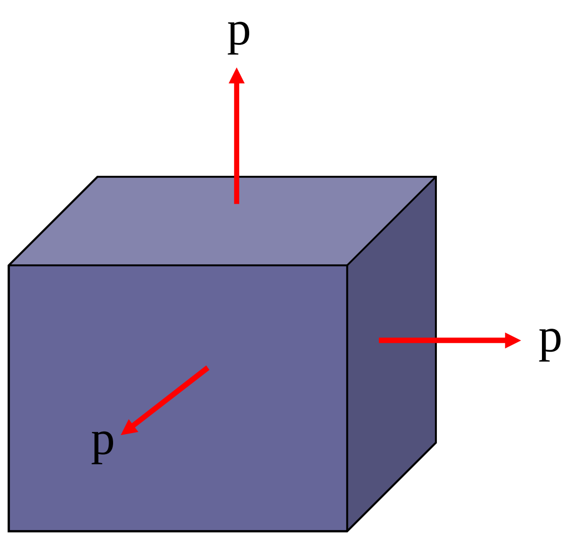
- If $\sigma_1 = \sigma_2$ & σ_3 different – any direction perpendicular to $\underline{n}_3$ => perpendicular direction



Principal Stresses

- If $\sigma_1 = \sigma_2 = \sigma_3 = p$

  every direction is principal direction – hydrostatic state of stress – any three orthogonal directions



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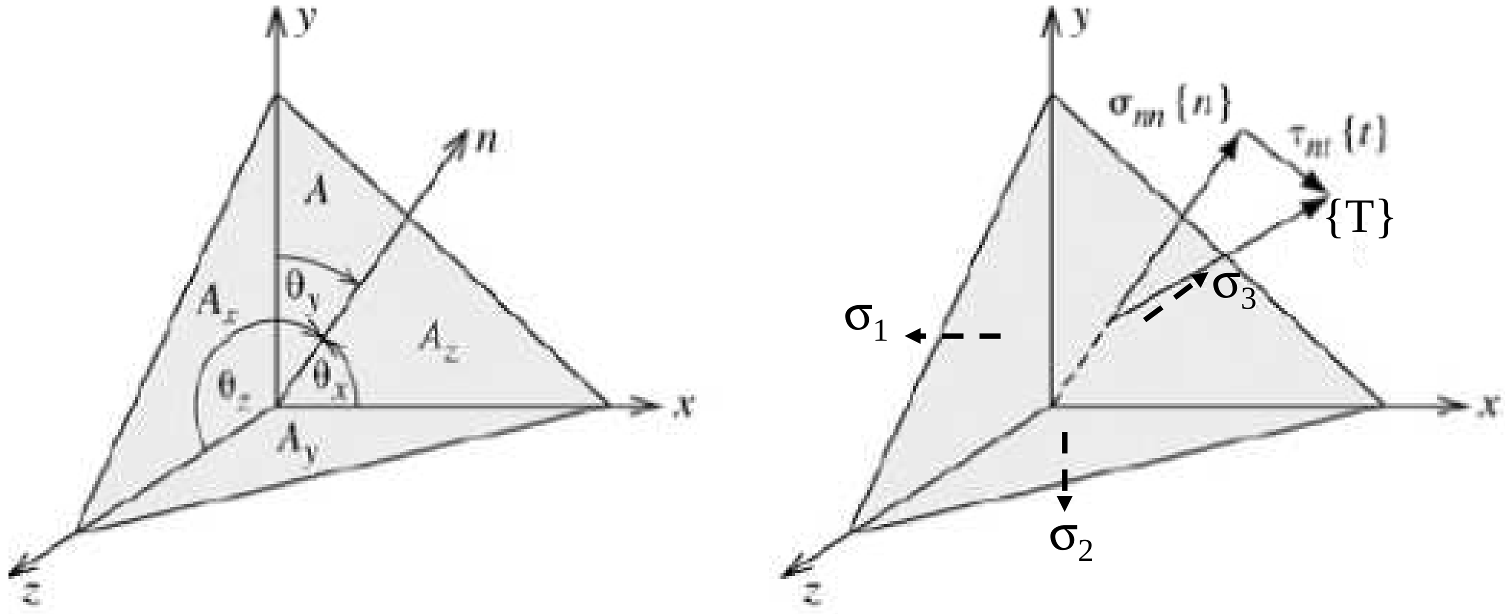
Maximum Shear Stress

- Given a stress state at a point



xyz csys coincides with principal stress directions

shear forces on A_x, A_y and A_z = 0

Maximum Shear Stress

- Tractions on plane 'A' => $T_i = \sigma_{ij}\, l_j$ Here, $\sigma_{ij} = 0$ for i ≠ j

$$T_1 = l\sigma_1, \quad T_2 = m\sigma_2, \quad T_3 = n\sigma_3$$

$$\underset{\sim}{T} = T_i\, \underset{\sim}{e_i}$$

$$T^2 = T_1^2 + T_2^2 + T_3^2 = (l\sigma_1)^2 + (m\sigma_2)^2 + (n\sigma_3)^2$$

- Normal Stress on plane 'A'

$$\sigma_{ij} = l_i T_j \qquad \text{for i = j => Normal stress}$$

$$\sigma_{ii} = l_i T_i$$

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Maximum Shear Stress

- Normal stress –

$$\sigma_n = lT_1 + mT_2 + nT_3 = l(l\sigma_1) + m(m\sigma_2) + n(n\sigma_3)$$

$$\sigma_n = l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3$$

- 'τ' => Shear stress – Normal and shear stresses are orthogonal

$$T^2 = \sigma_n^2 + \tau^2$$

$$\tau^2 = T^2 - \sigma_n^2 = \left[(l\sigma_1)^2 + (m\sigma_2)^2 + (n\sigma_3)^2\right] - \left[(l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3)^2\right]$$

$$l^2 + m^2 + n^2 = 1 \Rightarrow n^2 = 1 - l^2 - m^2$$

- For maximum shear stress

$$\frac{\partial \tau}{\partial l} = 0, \quad \frac{\partial \tau}{\partial m} = 0$$

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Maximum Shear Stress

- Differentiation gives the following expressions

$$l\left[l^2(\sigma_1-\sigma_3)+m^2(\sigma_2-\sigma_3)-\frac{\sigma_1-\sigma_3}{2}\right]=0$$

$$m\left[l^2(\sigma_1-\sigma_3)+m^2(\sigma_2-\sigma_3)-\frac{\sigma_2-\sigma_3}{2}\right]=0$$

- Solving & use $l^2+m^2+n^2 = 1$ – three sets of solutions

$$l=0,\ m=0,\ n=\pm 1$$

$$l=0,\ m=\pm\frac{1}{\sqrt{2}},\ n=\pm\frac{1}{\sqrt{2}}$$

$$l=\pm\frac{1}{\sqrt{2}},\ m=0,\ n=\pm\frac{1}{\sqrt{2}}$$

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Maximum Shear Stress

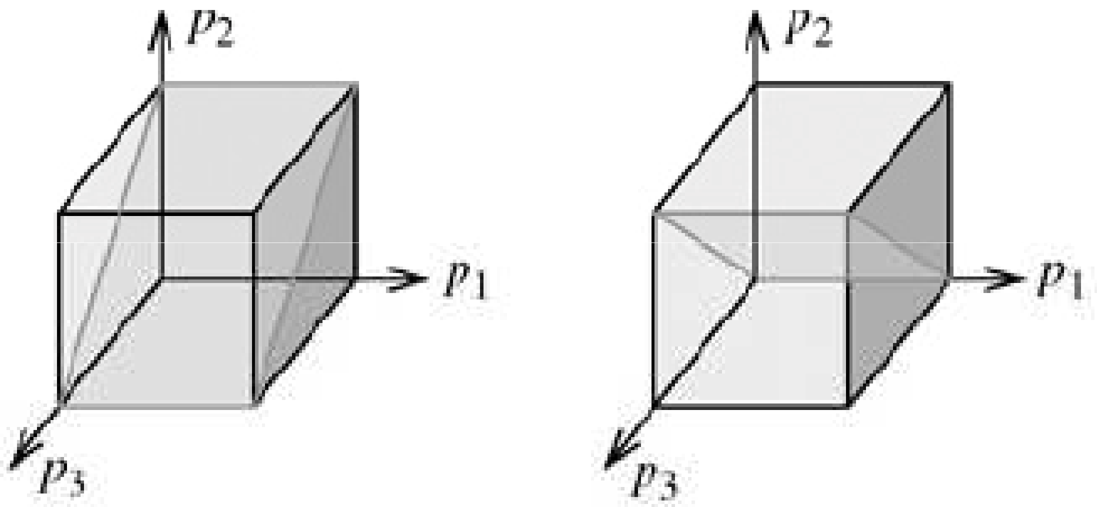
- Solutions – planes on which shear stresses are minimum and maximum

	1	2	3	4	5	6
ℓ	± 1	0	0	0	$\pm(1/2)^{0.5}$	$\pm(1/2)^{0.5}$
m	0	± 1	0	$\pm(1/2)^{0.5}$	0	$\pm(1/2)^{0.5}$
n	0	0	± 1	$\pm(1/2)^{0.5}$	$\pm(1/2)^{0.5}$	0

Min. shear planes Maximum shear planes

Maximum Shear Stress

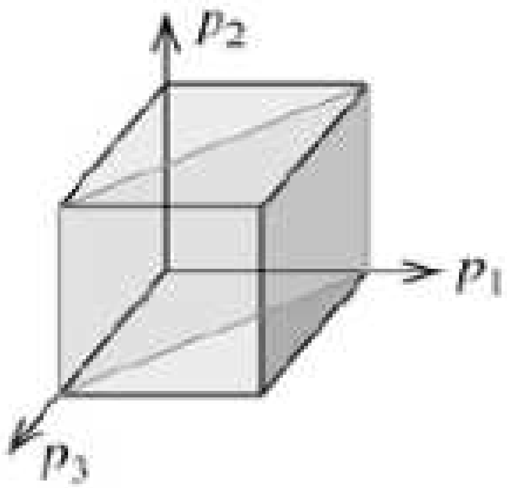
- Values of shear stresses and planes –

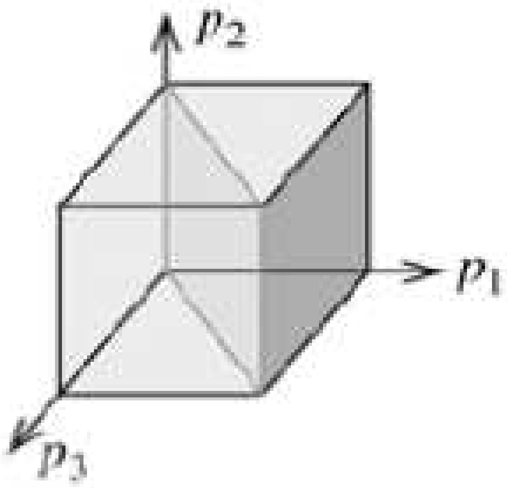


$$l = 0,\ m = \pm\sqrt{\frac{1}{2}},\ n = \pm\sqrt{\frac{1}{2}} \qquad \tau_{max} = \frac{1}{2}\left|\sigma_2 - \sigma_3\right|$$

Maximum Shear Stress

- Values of shear stresses and planes –

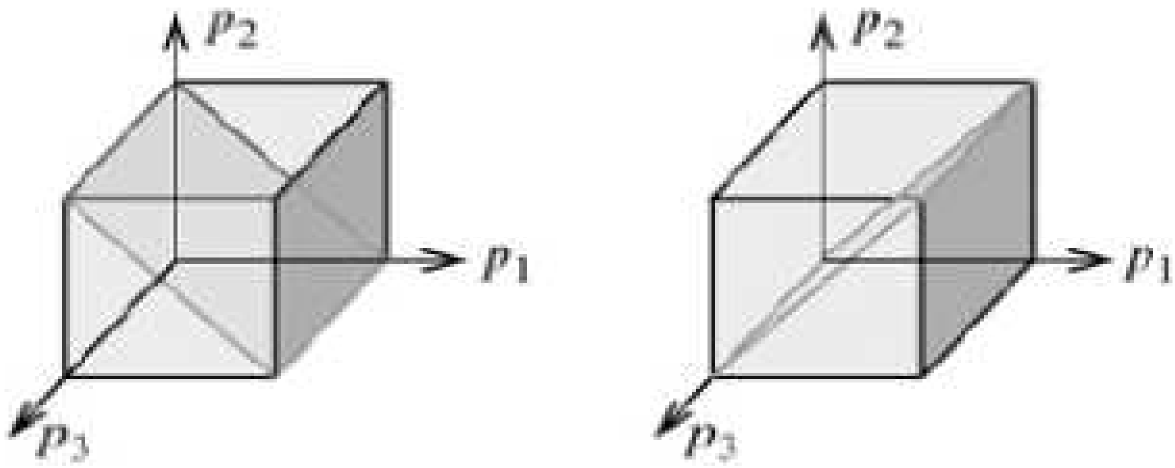




$$l = \pm\sqrt{\frac{1}{2}},\ m = 0,\ n = \pm\sqrt{\frac{1}{2}} \qquad \tau_{\max} = \frac{1}{2}\left|\sigma_3 - \sigma_1\right|$$

Maximum Shear Stress

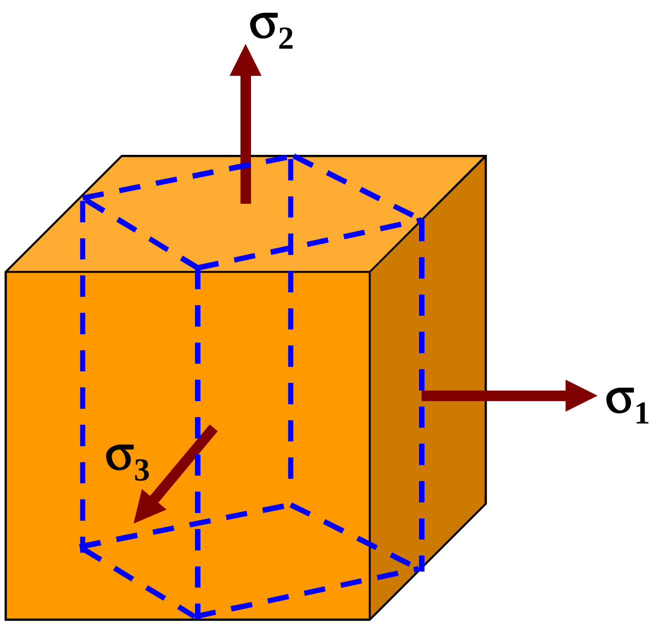
- Values of shear stresses and planes –

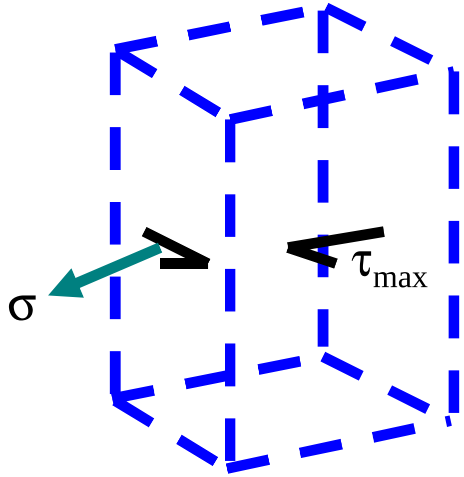


$$l = \pm\sqrt{\frac{1}{2}},\ m = \pm\sqrt{\frac{1}{2}},\ n = 0 \qquad \tau_{\max} = \frac{1}{2}\left|\sigma_1 - \sigma_3\right|$$

Maximum Shear Stress

- Maximum shear stress act at $\pm 45^0$ planes with reference to principal planes

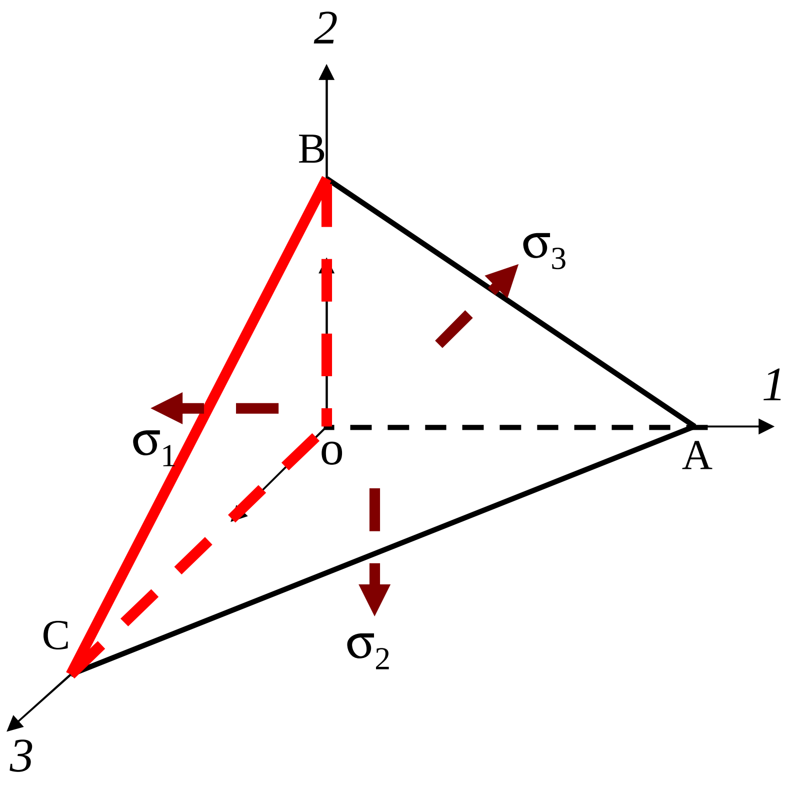




$$\tau_{max} = \frac{1}{2}\left|\sigma_3 - \sigma_1\right| \qquad \sigma = \frac{1}{2}\left(\sigma_3 + \sigma_1\right)$$

Minimum Shear Stress

- Plane of minimum shear stress



Plane OBC

Direction cosines

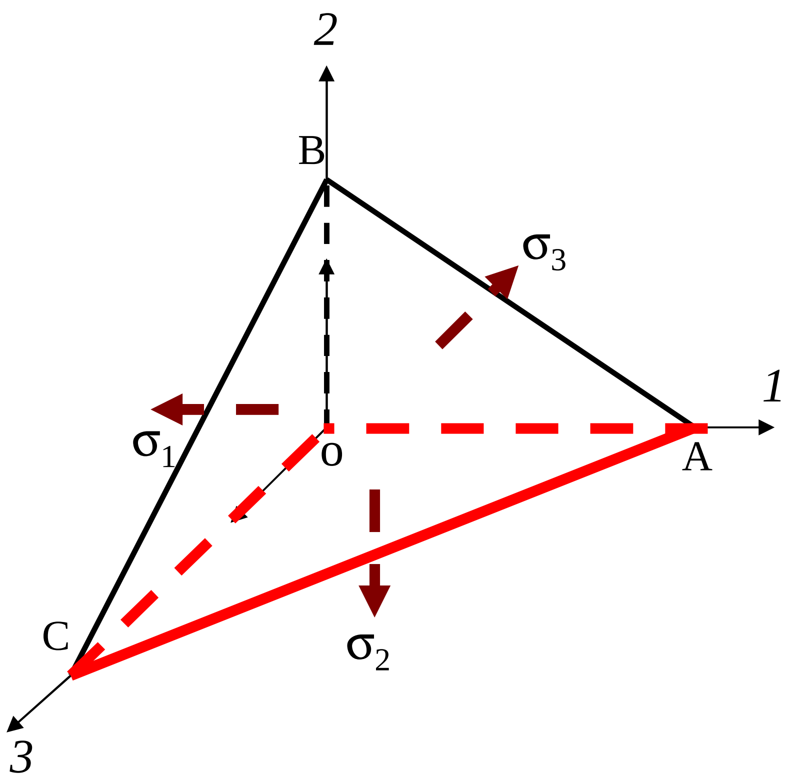
$l = \pm 1$, m = 0, n = 0

Maximum principal stress plane

Shear stress = 0

Minimum Shear Stress

- Plane of minimum shear stress



Plane OAC

Direction cosines

$\ell = 0$, $m = \pm 1$, $n = 0$

Principal stress plane - σ_2

Shear stress = 0

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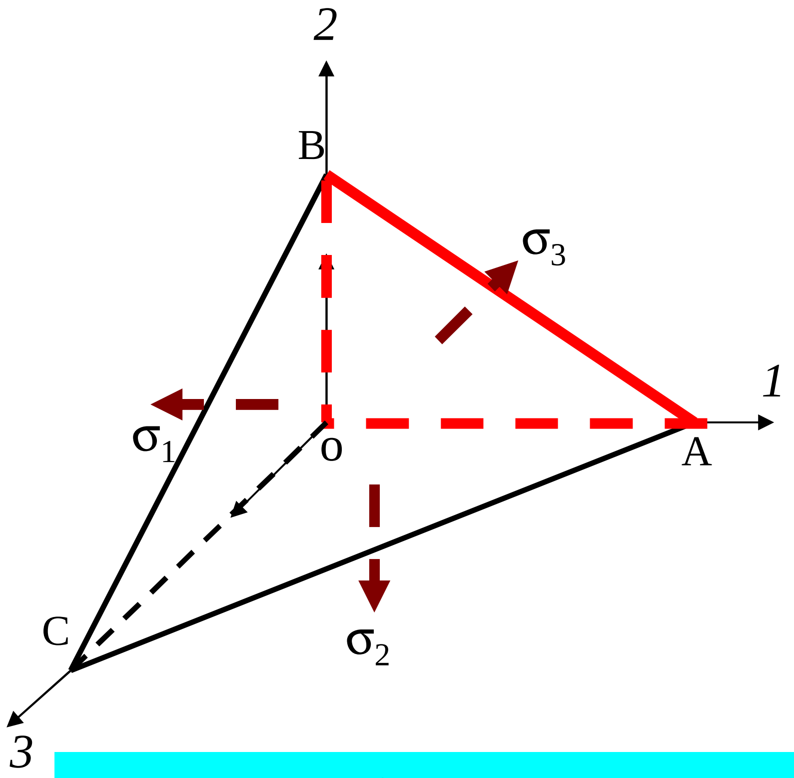
Minimum Shear Stress

- Plane of minimum shear stress



Plane OBC

Direction cosines

$\ell = 0$, m = 0, n = ±1

Principal stress plane - σ_3

Shear stress = 0

Maximum shear stresses are constant for a given stress state at a point

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Octahedral Stresses

- A plane that makes equal angles with principal planes – Octahedral plane – eight planes

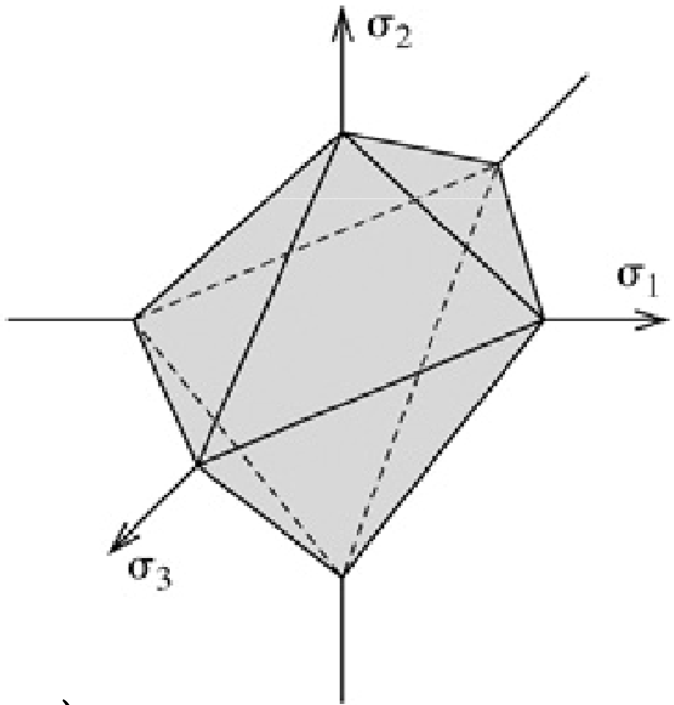
$$l^2 + m^2 + n^2 = 1$$

$$l = m = n$$

$$l = \pm\frac{1}{\sqrt{3}} = m = n$$

$$T_1 = l\sigma_1,\ T_2 = l\sigma_2,\ T_3 = l\sigma_3$$

$$\sigma_n = \sigma_{oct} = lT_1 + lT_2 + lT_3 = l^2\left(\sigma_1 + \sigma_2 + \sigma_3\right)$$



Equal angle with principal planes

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Octahedral Stress

- Equal angles with principal axes

$$\sigma_{oct} = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) = \frac{I_1}{3}$$

- Normal octahedral stress => Average of principal stresses
- It depends on first invariant
- For a given stress state => Octahedral normal stress - constant

Octahedral Stress

- Octahedral shear stress – shear stress from tractions is given by,

$$\tau^2 = T^2 - \sigma_n^2 \Rightarrow \tau_{oct}^2 = T^2 - \sigma_{oct}^2$$

$$\Rightarrow \tau_{oct}^2 = \left[\frac{1}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)\right] - \left[\frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right)\right]^2$$

Simplifying this expression,

$$\Rightarrow \tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sqrt{\left(I_1^2 - 3I_2\right)}$$

Octahedral shear stress => $\tau_{oct} = \tau(I_1, I_2)$ => Invariant

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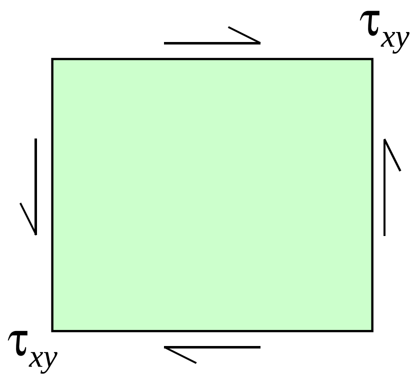
State of Pure Shear



- Only shear stress exist
- The stress state at a point

  six stress components, σ_{ij}
- Infinite sets of planes through this point
- For pure shear - Sum of main diagonal terms in stress tensor vanish

  $\Rightarrow Tr(\sigma) = 0$

State of Pure Shear

- Necessary and sufficient condition for existence of a state of pure shear

  => First invariant of stress tensor = 0

$$[\sigma] = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 0$$

Individual normal stress components need not be zero

Decomposition of Stress

- The stress tensor
- In indices notation, σ_{ij}
- Decompose [σ]

$$[\sigma]=\begin{bmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}\end{bmatrix}$$

Stress state = Hydrostatic state + Deviatoric state (shear)

$$\begin{bmatrix}\tau_{xx} & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy} & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}\end{bmatrix}=\begin{bmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{bmatrix}+\begin{bmatrix}\tau_{xx}-p & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \tau_{yy}-p & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \tau_{zz}-p\end{bmatrix}$$

In indices notation $\sigma_{ij} = p\delta_{ij} + \left(\sigma_{ij} - p\delta_{ij}\right) = p\delta_{ij} + S_{ij}$

Decomposition of Stress

- For pure shear => Tr(S_{ij})

$$\left(\sigma_{xx}-p\right)+\left(\sigma_{yy}-p\right)+\left(\sigma_{zz}-p\right)=0$$

$$p=\frac{1}{3}\left(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}\right)=\sigma_{mm}=\text{mean stress}$$

- Deviatoric stress tensor –

$$[S]=\begin{bmatrix} \frac{2}{3}\sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \frac{2}{3}\sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \frac{2}{3}\sigma_{zz} \end{bmatrix}=\begin{bmatrix} \sigma'_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma'_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma'_{zz} \end{bmatrix}$$

Decomposition of Stress

- Deviatoric stress – the first invariant is given by, => J_1 = Tr(S_{ij})

$$J_1 = (\sigma_{xx} - p) + (\sigma_{yy} - p) + (\sigma_{zz} - p)$$
$$=> J_1 = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) - 3p$$

But,

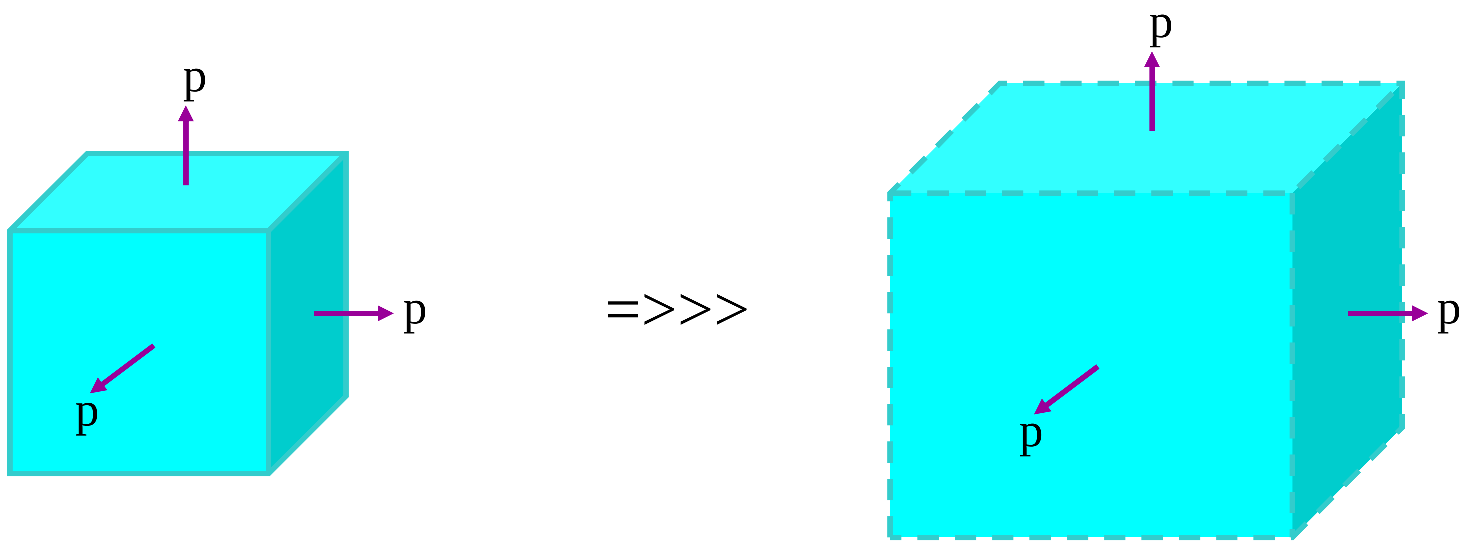
$$p = \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$ Use this relation

$$=> J_1 = 0$$

The first invariant of Deviatoric stress = 0

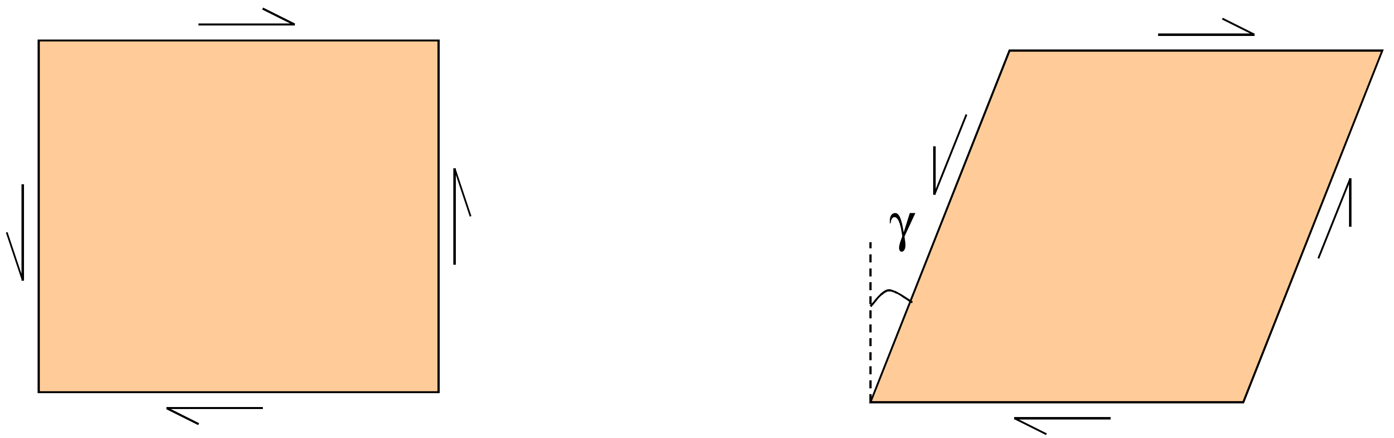
Decomposition of Stress

- Hydrostatic stress tensor – volumetric part of stress tensor – spherical stress tensor
- Hydrostatic part – responsible for volume change – size change



Decomposition of Stress

- Deviatoric part – responsible for deformation or shape change
- Change in angle between two adjacent sides => shape change – distortion - shear stress



Change in shape due to applied shear

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Decomposition of Stress

- Invariants of deviatoric stress tensor

$$[S]=\begin{bmatrix} \sigma'_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma'_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma'_{zz} \end{bmatrix}$$

$$J_1 = \sigma'_{xx} + \sigma'_{yy} + \sigma'_{zz} = 0$$

$$J_2 = \begin{vmatrix} \sigma'_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma'_{yy} \end{vmatrix} + \begin{vmatrix} \sigma'_{xx} & \tau_{xz} \\ \tau_{xz} & \sigma'_{zz} \end{vmatrix} + \begin{vmatrix} \sigma'_{yy} & \tau_{yz} \\ \tau_{yz} & \sigma'_{zz} \end{vmatrix}$$

$$J_3 = \begin{vmatrix} \sigma'_{xx} & \tau_{xy} & \tau_{xz} \\ & \sigma'_{yy} & \tau_{yz} \\ symm & & \sigma'_{zz} \end{vmatrix}$$

Mohr's Circle

- Graphical technique - Normal and shear stresses on a given plane – stress state at a point is known
- Stress state => σ_{ij} – Find principal stresses corresponding to this stress state

$$\text{Principal stresses} => \sigma_1, \sigma_2, \sigma_3$$

$$\sigma_1 > \sigma_2 > \sigma_3$$

Take principal directions as reference

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Mohr's Circle

- Tractions on plane *(l, m, n)* wrt principal directions

$$T_1 = l\sigma_1, \;\; T_2 = l\sigma_2, \;\; T_3 = l\sigma_3$$

$$T^2 = T_1^2 + T_2^2 + T_3^2 = l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2$$

$$T^2 = \sigma^2 + \tau^2 = l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2$$

But, $\quad l^2 + m^2 + n^2 = 1 \Rightarrow \; n^2 = 1 - l^2 - m^2$

$$\sigma^2 + \tau^2 = l^2\sigma_1^2 + m^2\sigma_2^2 + \left(1 - l^2 - m^2\right)\sigma_3^2$$

$$\sigma^2 + \tau^2 = l^2\left(\sigma_1^2 - \sigma_3^2\right) + m^2\left(\sigma_2^2 - \sigma_3^2\right) + \sigma_3^2$$

$$\left(\sigma^2 - \sigma_3^2\right) + \tau^2 = l^2\left(\sigma_1^2 - \sigma_3^2\right) + m^2\left(\sigma_2^2 - \sigma_3^2\right) - - (1)$$

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Mohr’s Circle

- Normal stress on this plane –

$$\sigma = lT_1 + mT_2 + nT_3 = l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3$$

$$\sigma = l^2\sigma_1 + m^2\sigma_2 + \left(1 - l^2 - m^2\right)\sigma_3$$

Eliminate ‘m’ from equations (1) and (2)

$$l^2 = \frac{\tau^2 + (\sigma - \sigma_2)(\sigma - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$

$$\tau^2 + \left(\sigma - \frac{\sigma_2 + \sigma_3}{2}\right)^2 = l^2(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

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Mohr's Circle

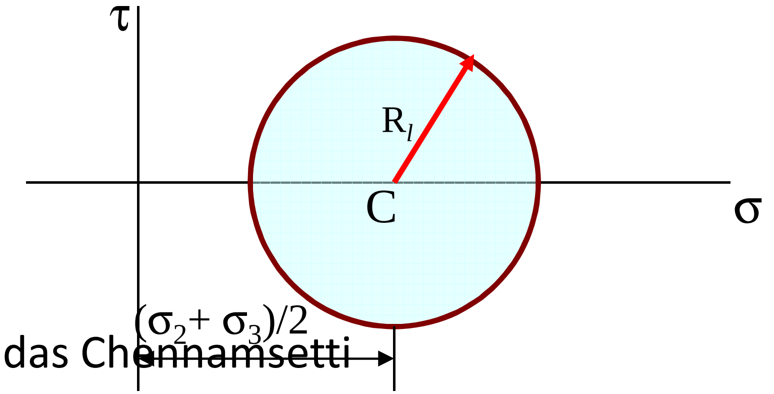
- The expression

$$\tau^2 + \left(\sigma - \frac{\sigma_2 + \sigma_3}{2}\right)^2 = l^2(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2 - (1)$$

represents a circle of center & radius

$$\left(\frac{\sigma_2 + \sigma_3}{2}, 0\right) \qquad R_l^2 = l^2(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2$$

on (σ, τ) plane



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Mohr's Circle

- Similar expressions as (1) can be obtained for direction cosines '*m*' and '*n*'

$$\tau^2 + \left(\sigma - \frac{\sigma_2 + \sigma_3}{2}\right)^2 = l^2(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3) + \left(\frac{\sigma_2 - \sigma_3}{2}\right)^2 \quad -(1)$$

$$\tau^2 + \left(\sigma - \frac{\sigma_1 + \sigma_3}{2}\right)^2 = m^2(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_3) + \left(\frac{\sigma_1 - \sigma_3}{2}\right)^2 \quad -(2)$$

$$\tau^2 + \left(\sigma - \frac{\sigma_1 + \sigma_2}{2}\right)^2 = n^2(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2) + \left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 \quad -(3)$$

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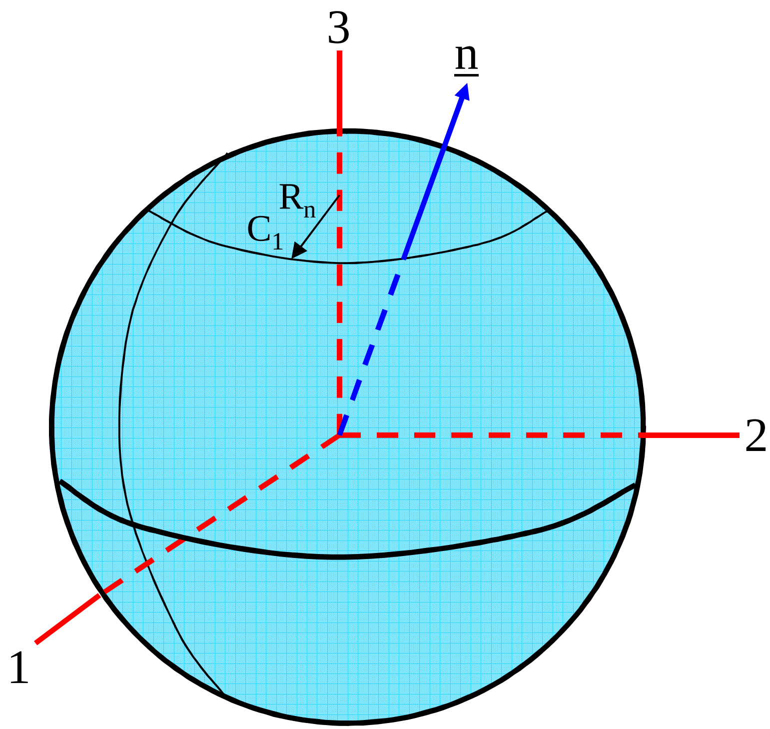
Mohr’s Circle

- Stress state [σ]

$$\tau^2+\left(\sigma-\frac{\sigma_1+\sigma_2}{2}\right)^2=n^2(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)+\left(\frac{\sigma_1-\sigma_2}{2}\right)^2 \quad -(3)$$



Direction cosines *(l, m, n)*

Say, ‘n = 0’

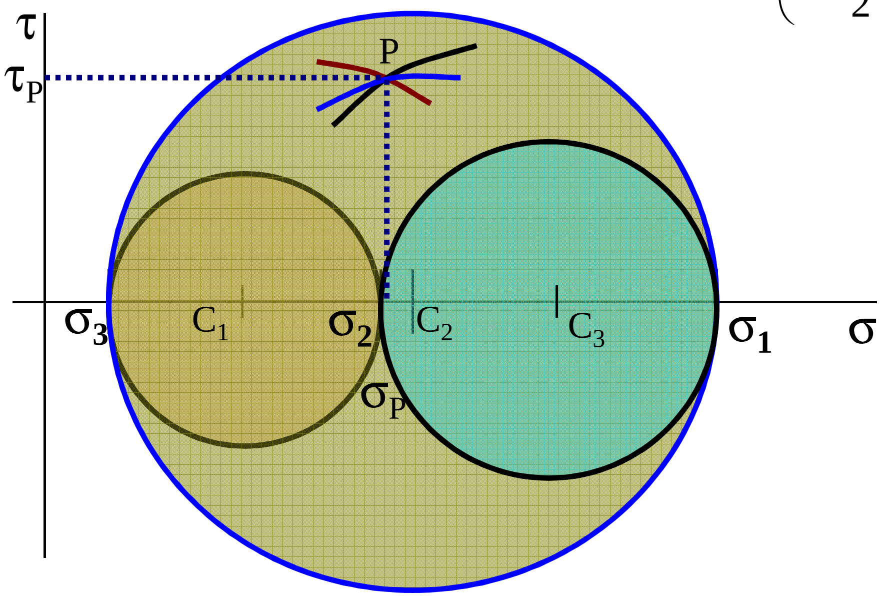
Circle of radius => $(\sigma_1 - \sigma_2)/2$

Center at $((\sigma_1 + \sigma_2)/2, 0)$

One of the Mohr’s circles

Mohr's Circle

- Similar circles can be drawn for equations (1) and (2) – plug *l* = 0 and m = 0



$$C_1\left(\frac{\sigma_2+\sigma_3}{2},0\right),\ R_1=\frac{\sigma_2-\sigma_3}{2} \qquad C_2\left(\frac{\sigma_1+\sigma_3}{2},0\right),\ R_2=\frac{\sigma_1-\sigma_3}{2}$$

$$C_3\left(\frac{\sigma_1+\sigma_2}{2},0\right),\ R_3=\frac{\sigma_1-\sigma_2}{2}$$

Draw arcs with following radii

$$R_l^2=l^2(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)+\left(\frac{\sigma_2-\sigma_3}{2}\right)^2$$

$$R_m^2=m^2(\sigma_2-\sigma_3)(\sigma_2-\sigma_1)+\left(\frac{\sigma_1-\sigma_3}{2}\right)^2$$

$$R_n^2=n^2(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)+\left(\frac{\sigma_1-\sigma_2}{2}\right)^2$$

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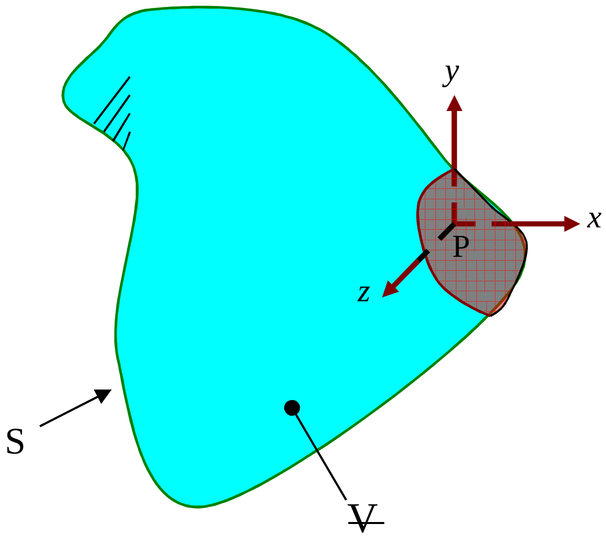
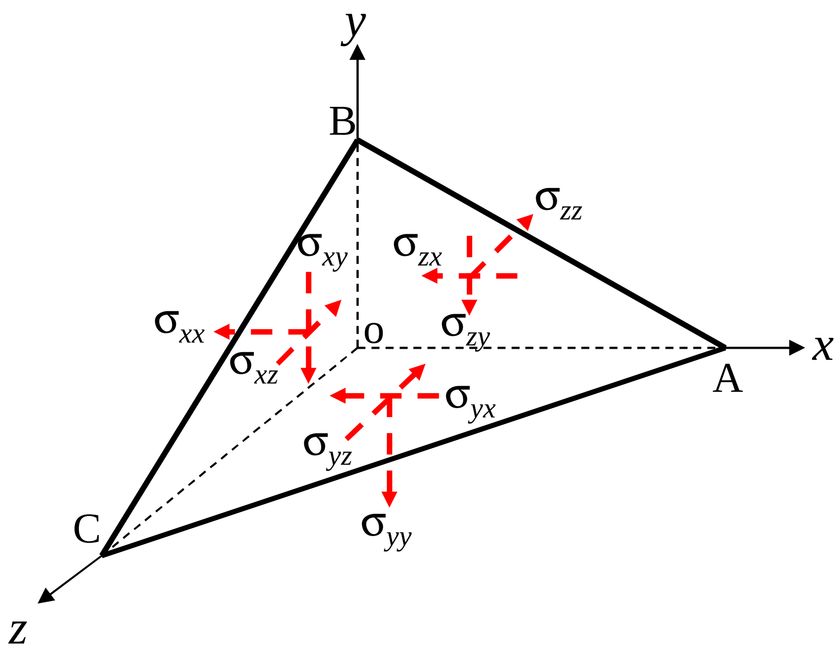
Equilibrium Equations

- Body under static equilibrium – every material point under equilibrium





Infinitesimal element, $dm = \rho \, dV$

Equilibrium Equations

- Body and Traction forces => F_b and F_t
- Newton's second law –

$$d\underset{\sim}{F}_t + d\underset{\sim}{F}_b = dm\,\frac{d\underset{\sim}{V}}{dt}$$

$$\underset{\sim}{V} = \underset{\sim}{V}(x, y, z, t)$$

$$\frac{d\underset{\sim}{V}}{dt} = \frac{\partial \underset{\sim}{V}}{\partial x}\frac{dx}{dt} + \frac{\partial \underset{\sim}{V}}{\partial y}\frac{dy}{dt} + \frac{\partial \underset{\sim}{V}}{\partial z}\frac{dz}{dt} + \frac{\partial \underset{\sim}{V}}{\partial t}\frac{dt}{dt}$$

$$\underset{\sim}{a} = \frac{\partial \underset{\sim}{V}}{\partial x}V_x + \frac{\partial \underset{\sim}{V}}{\partial y}V_y + \frac{\partial \underset{\sim}{V}}{\partial z}V_z + \frac{\partial \underset{\sim}{V}}{\partial t}$$

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Equilibrium Equations

- Rewrite,

$$\underset{\sim}{a} = \frac{\partial \underset{\sim}{V}}{\partial x} V_x + \frac{\partial \underset{\sim}{V}}{\partial y} V_y + \frac{\partial \underset{\sim}{V}}{\partial z} V_z + \frac{\partial \underset{\sim}{V}}{\partial t} = \overset{1}{\frac{\partial \underset{\sim}{V}}{\partial t}} + \overset{2}{\underset{\sim}{V} . \nabla \underset{\sim}{V}}$$

- Term – 1 => Local acceleration
- Term – 2 => Convective acceleration
- For small deformations, the particle of mass *'dm'* doesn't move substantially – so this is term – 2, which is very small compared to local acceleration term. Neglect term – 2

Equilibrium Equations

- This gives =>

$$\underset{\sim}{a} = \frac{d\underset{\sim}{V}}{dt} \approx \frac{\partial \underset{\sim}{V}}{\partial t}$$

$$d\underset{\sim}{F_t} + d\underset{\sim}{F_b} = \rho dxdydz\ \frac{\partial \underset{\sim}{V}}{\partial t}, \qquad d\underset{\sim}{F_t} = \underset{\sim}{T}\ dS\ , \qquad d\underset{\sim}{F_b} = \underset{\sim}{B}\ d\mathcal{V}$$

$$\underset{\sim}{T}\ dS + \underset{\sim}{B}\ d\mathcal{V} = \rho dxdydz\ \frac{\partial \underset{\sim}{V}}{\partial t},$$

Integrating this expression,

$$\oiint_S \underset{\sim}{T}\ dS + \iiint_{\mathcal{V}} \underset{\sim}{B}\ d\mathcal{V} = \iiint_{\mathcal{V}} \frac{\partial \underset{\sim}{V}}{\partial t} \rho d\mathcal{V}$$

In indices notation,

$$\oiint_S T_i dS + \iiint_{\mathcal{V}} B_i\ d\mathcal{V} = \iiint_{\mathcal{V}} \frac{\partial V_i}{\partial t} \rho d\mathcal{V}$$

Equilibrium Equations

- But tractions on surfaces related to stresses
  – Cauchy's formula $T_i = \sigma_{ij} l_j$ Use this relation

$$\oiint_S T_i \, dS + \iiint_{\forall} B_i \, d\forall = \iiint_{\forall} \frac{\partial V_i}{\partial t} \rho \, d\forall$$

$$\oiint_S \sigma_{ij} l_j \, dS + \iiint_{\forall} B_i \, d\forall = \iiint_{\forall} \frac{\partial V_i}{\partial t} \rho \, d\forall$$

Use Gauss divergence theorem – Volume integral to Surface integral

$$\iiint_{\forall} \nabla . \underset{\sim}{A} d\forall = \oiint_S \underset{\sim}{A} . \underset{\sim}{n} \, dS = \oiint_S \underset{\sim}{A} . d\underset{\sim}{S}$$

Indices notation $$\iiint_{\forall} \frac{\partial A_i}{\partial x_i} d\forall = \oiint_S A_i l_i dS$$

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Equilibrium Equations

- Use Gauss Divergence theorem –

$$\oiint_S \sigma_{ij} l_j \, dS + \iiint_{\forall} B_i \, d\forall = \iiint_{\forall} \frac{\partial V_i}{\partial t} \rho \, d\forall$$

$$\iiint_{\forall} \frac{\partial \sigma_{ij}}{\partial x_j} d\forall + \iiint_{\forall} B_i \, d\forall = \iiint_{\forall} \frac{\partial V_i}{\partial t} \rho \, d\forall$$

$$\iiint_{\forall} \left(\frac{\partial \sigma_{ij}}{\partial x_j} + B_i - \rho \frac{\partial V_i}{\partial t} \right) d\forall = 0$$

This gives -

$$\frac{\partial \sigma_{ij}}{\partial x_j} + B_i - \rho \frac{\partial V_i}{\partial t} = 0$$

Equilibrium Equations

- All three equilibrium equations –

$$\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+B_x=\rho\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+B_y=\rho\frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}+B_z=\rho\frac{\partial^2 w}{\partial t^2}$$

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Equilibrium Equations

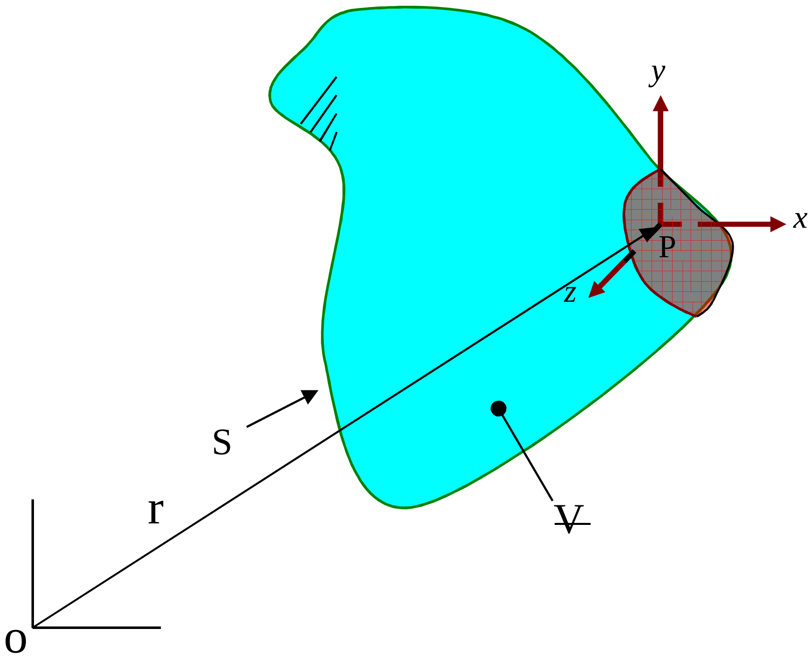
- Moment Equilibrium

$$\sum \underset{\sim}{M} = \underset{\sim}{r} \times m \underset{\sim}{a} + I_G \underset{\sim}{\alpha}$$

No rotations

$$\sum \underset{\sim}{M} = \underset{\sim}{r} \times m \underset{\sim}{a}$$

In differential form



$$d \underset{\sim}{M}_t + d \underset{\sim}{M}_b = \underset{\sim}{r} \times dm \frac{d \underset{\sim}{V}}{dt}$$

dM_t and dM_b moments due to traction and body loads

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Equilibrium Equations

- Plug dM_t and dM_b

$$d\,\underset{\sim}{M}_t + d\,\underset{\sim}{M}_b = \underset{\sim}{r}\times dm\frac{d\underset{\sim}{V}}{dt}$$

$$\underset{\sim}{r}\times\underset{\sim}{T}\,dA + \underset{\sim}{r}\times\underset{\sim}{B}\,d\forall = \underset{\sim}{r}\times dm\frac{d\underset{\sim}{V}}{dt}$$

Integrating -

$$\oiint_S \underset{\sim}{r}\times\underset{\sim}{T}\,dA + \iiint_{\forall}\underset{\sim}{r}\times\underset{\sim}{B}\,d\forall = \iiint_{\forall}\underset{\sim}{r}\times\rho\frac{d\underset{\sim}{V}}{dt}d\forall$$

Equilibrium Equations

- Writing in indices form –

$$\oint\!\!\oint_S r_i T_j \varepsilon_{ijk}\, \underset{\sim}{e_k}\, dA + \iiint_{V\!\!\!-} r_i B_j \varepsilon_{ijk}\, \underset{\sim}{e_k}\, dV\!\!\!- = \iiint_{V\!\!\!-} r_i \frac{\partial V_j}{\partial t} \rho \varepsilon_{ijk}\, \underset{\sim}{e_k}\, dV\!\!\!-$$

Use Cauchy's formula $\quad T_j = \sigma_{jm} l_m$

$$\varepsilon_{ijk}\left[\oint\!\!\oint_S r_i T_j dA + \iiint_{V\!\!\!-} r_i B_j dV\!\!\!- - \iiint_{V\!\!\!-} r_i \frac{\partial V_j}{\partial t} \rho dV\!\!\!-\right] = 0$$

$$\varepsilon_{ijk}\left[\oint\!\!\oint_S r_i \sigma_{jm} l_m dA + \iiint_{V\!\!\!-} r_i B_j dV\!\!\!- - \iiint_{V\!\!\!-} r_i \frac{\partial V_j}{\partial t} \rho dV\!\!\!-\right] = 0$$

Use Gauss Divergence theorem

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Equilibrium Equations

- Surface integral to volume integral

$$\varepsilon_{ijk}\left[\iiint_{V}\left(r_i\sigma_{jm}\right)_{,m}dV+\iiint_{V}r_iB_j\,dV-\iiint_{V}r_i\frac{\partial V_j}{\partial t}\rho\,dV\right]=0$$

Use integration by parts - first term

$$r_i\varepsilon_{ijk}\iiint_{V}\left(\sigma_{jm,m}+B_j-\frac{\partial V_j}{\partial t}\right)dV+\varepsilon_{ijk}r_{i,m}\sigma_{jm}=0$$

$$\sigma_{jm,m}+B_j-\frac{\partial V_j}{\partial t}=0$$

So,

$$\varepsilon_{ijk}\,r_{i,m}\,\sigma_{jm}=0$$

$$\varepsilon_{ijk}\,\delta_{im}\,\sigma_{jm}=0$$

$$\varepsilon_{ijk}\,\sigma_{ij}=0$$

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Equilibrium Equations

- Cyclic symbol is skew-symmetric in *i, j*

$$\varepsilon_{ijk} = -\varepsilon_{jik}$$

$$\varepsilon_{ijk}\sigma_{ij} = 0, \quad \varepsilon_{jik}\sigma_{ji} = 0$$

$$\varepsilon_{ijk}\sigma_{ij} + \varepsilon_{jik}\sigma_{ji} = 0$$

$$\varepsilon_{ijk}\left(\sigma_{ij} - \sigma_{ji}\right) = 0$$

$$\Rightarrow \sigma_{ij} - \sigma_{ji} = 0$$

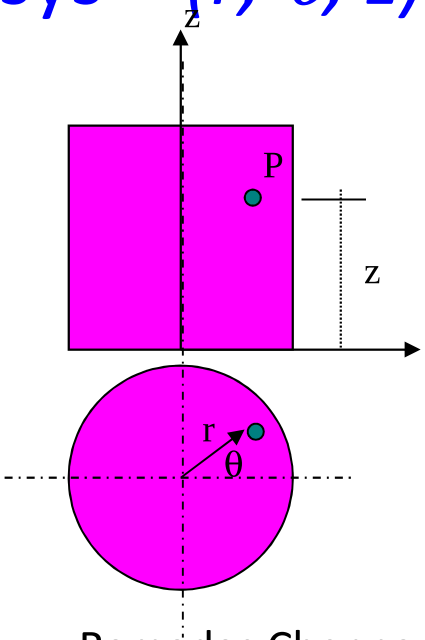
Shear stresses – complementary

$$\Rightarrow \sigma_{ij} = \sigma_{ji}$$

Stress tensor is symmetric -

Cylindrical Co-ordinate System

- Cartesian frame of reference – body processes straight boundaries
- Curved boundaries – Cylindrical, Spherical
- Cylindrical csys – $(r,\ \theta,\ z)$ – Spherical csys – $(r,\ \theta,\ \phi)$



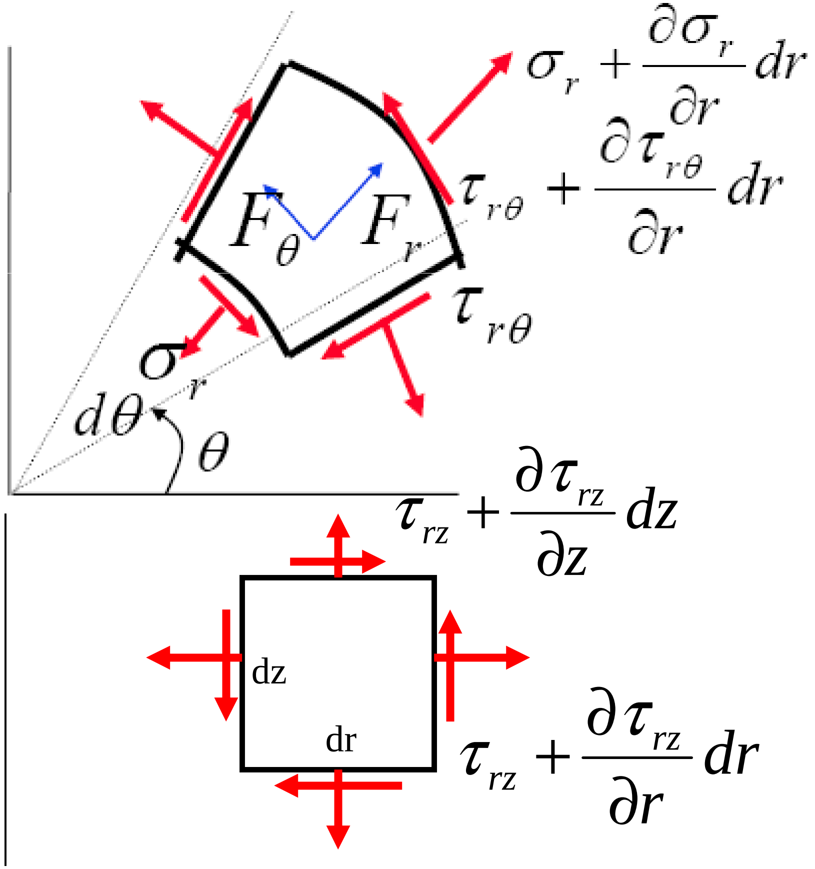
To completely specify the location of point 'P'

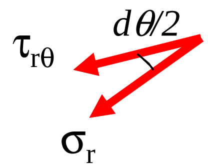
$(r,\ \theta,\ z)$ required

Cylindrical Co-ordinate System

R&DE (Engineers), DRDO

- Equilibrium equations





Equilibrium equations

$$\sum \underset{\sim}{F} = 0$$

$$\sum F_r = 0,$$

$$\sum F_\theta = 0,$$

$$\sum F_z = 0$$

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Cylindrical Co-ordinate System

- Equilibrium in radial direction –

$$\left[\left(\sigma_{rr}+\frac{\partial\sigma_{rr}}{\partial r}dr\right)(r+dr)d\theta dz-\sigma_{rr}rd\theta dz\right]+$$

$$\left[\left(\tau_{r\theta}+\frac{\partial\tau_{r\theta}}{\partial r}dr\right)drdz-\tau_{r\theta}drdz\right]\cos\left(\frac{d\theta}{2}\right)+$$

$$\left[\left(\tau_{rz}+\frac{\partial\tau_{rz}}{\partial z}dr\right)\left(r+\frac{dr}{2}\right)d\theta dr-\tau_{rz}\left(r+\frac{dr}{2}\right)d\theta dr\right]-$$

$$\left[\left(\sigma_{\theta\theta}+\frac{\partial\sigma_{\theta\theta}}{\partial\theta}\right)drdz+\sigma_{\theta\theta}drdz\right]\sin\left(\frac{d\theta}{2}\right)+$$

$$B_r\left(r+\frac{dr}{2}\right)d\theta dz=\rho\ddot{u}\left(r+\frac{dr}{2}\right)d\theta dz$$

Cylindrical Co-ordinate System

- Neglect higher order terms - simplifying this equation –

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\partial \tau_{zr}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+B_r=\rho\ddot{u}$$

- Similarly two more expression in θ and z directions

$$\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{z\theta}}{\partial z}+\frac{2\tau_{r\theta}}{r}+B_\theta=\rho\ddot{v}$$

$$\frac{\partial \tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}+\frac{\tau_{rz}}{r}+B_z=\rho\ddot{w}$$

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