Strain Analysis

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Summary

- Deformation gradient
- Green-Lagrange strain
- Almansi strain
- Rotation vector
- Strain transformation
- Principal strains
- Strain decomposition
- Strain compatibility

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Introduction

- The effect of forces applied to a body – Newton's second law – Stress analysis
- Applied forces => deformations
- Concerned with study of deformations – geometric problem – no material properties
- Strains => kinematics of material deformation
- Change in relative position of any two points in a continuous body => deformed or strained
  - Distance remains constant – rigid body motion
- Rigid body => Translations & Rotations
- Continuum mechanics approach

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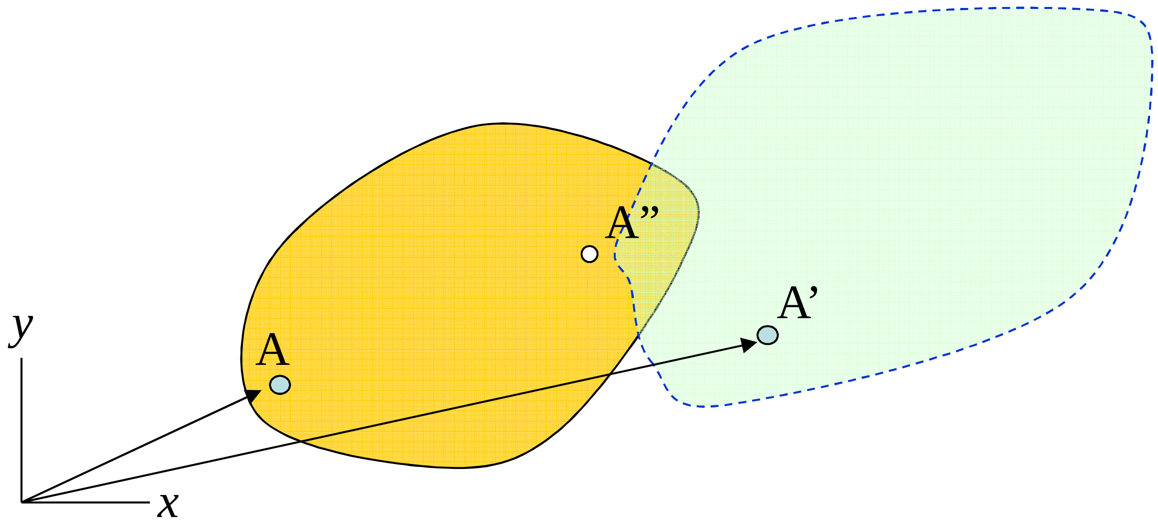
Introduction

- *Deformation* – consists of relative motion between particles and rigid body motion



Change in position

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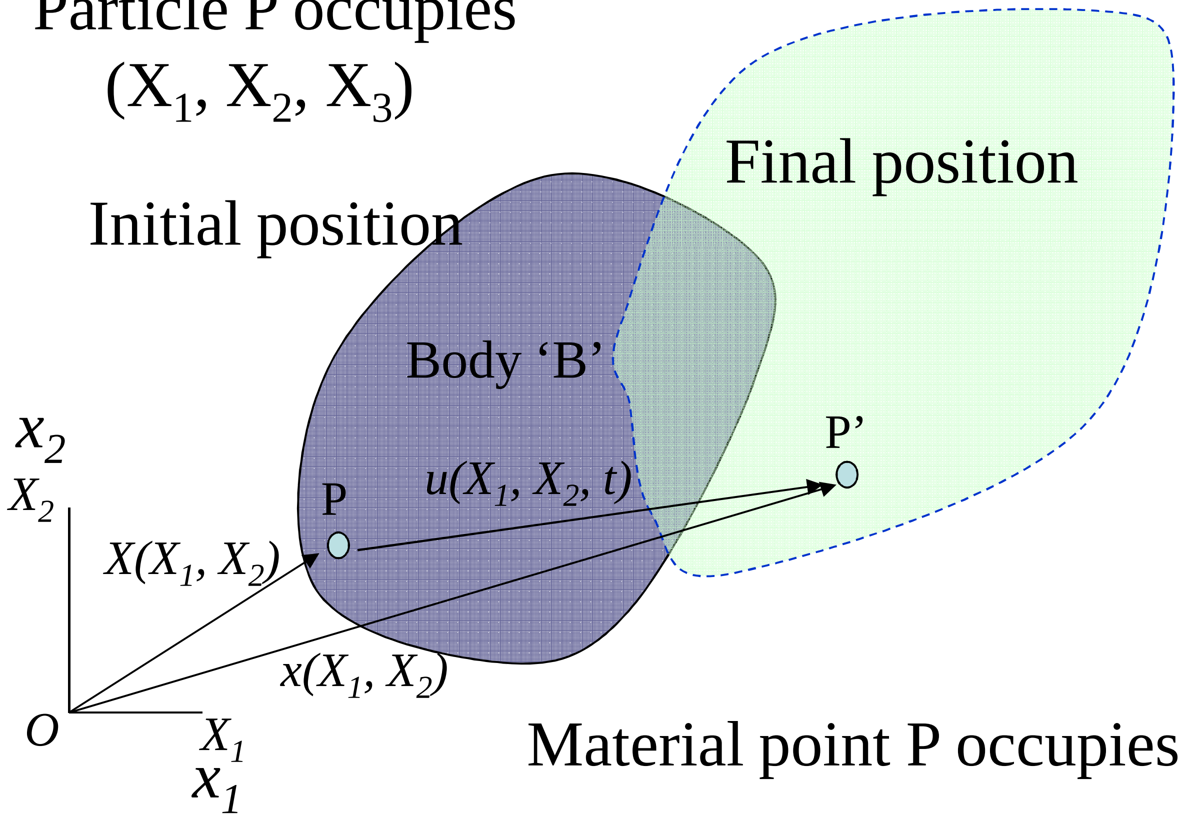
Introduction

- Description of motion

Particle P occupies (X_1, X_2, X_3)



Body undergoes macroscopic changes within the body 'Deformation'

Co-ordinate system refers to material particles in the body => Material co-ordinate system

Material point P occupies P' in final config

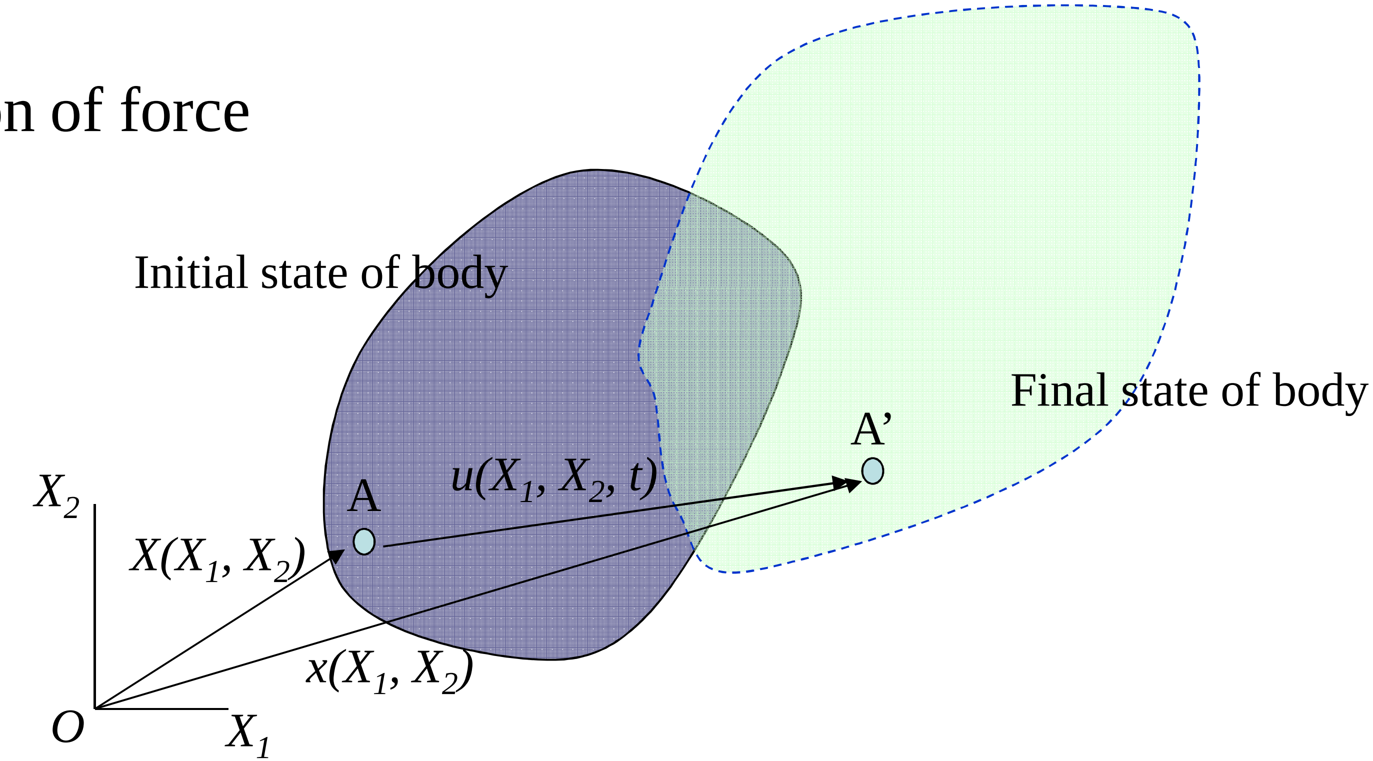
Deformation csys $x_1x_2x_3$ => Spatial co-ordinate system

Displacement

- Displacement in a body

Mapping from reference csys to deformed csys - function

Application of force



$$\underset{\sim}{OA} = \underset{\sim}{X}\left(X_1, X_2, X_3\right) \qquad \underset{\sim}{OA'} = \underset{\sim}{x}\left(X_1, X_2, X_3\right) \qquad \underset{\sim}{AA'} = \underset{\sim}{U}\left(X_1, X_2, X_3\right)$$

$$\underset{\sim}{OA'} = \underset{\sim}{OA} + \underset{\sim}{AA'} \Rightarrow \underset{\sim}{x} = \underset{\sim}{X} + \underset{\sim}{U}$$

Mapping

- Material co-ordinate system – spatial co-ordinate system

$$\chi : P \rightarrow P'$$

$$\Rightarrow \underset{\sim}{x} = \chi\left(\underset{\sim}{X}\right)$$

Substitute material co-ordinates of the particle in mapping function => Final position of the particle

Existence of inverse mapping

$$\underset{\sim}{X} = \chi^{-1}(\underset{\sim}{x})$$

Substitute spatial co-ordinates => Material co-ordinates – which material occupies the spatial position

Mapping

- Variable – velocity, density etc vector / scalar

$$\phi = \phi\left(\underset{\sim}{X}, t\right) \quad \text{Expressed in material csys}$$

$$\text{Mapping} \; : \; \underset{\sim}{x} = \chi\left(\underset{\sim}{X}\right)$$

$$\underset{\sim}{X} = \chi^{-1}\left(\underset{\sim}{x}\right)$$

$$=> \phi = \phi\left(\underset{\sim}{X}, t\right) = \phi\left(\chi^{-1}\left(\underset{\sim}{x}\right), t\right)$$

$$=> \phi = \phi\left(\underset{\sim}{x}, t\right)$$

Expressed in spatial co-ordinate system

Displacement

- In indices notation

$$\underset{\sim}{x} = \underset{\sim}{X} + \underset{\sim}{U} \Rightarrow x_i = X_i + u_i(X_1, X_2, X_3)$$

- This relates the current deformed geometry with initial undeformed geometry of a point
- Displacement field => movement of each point in initial configuration to final configuration
- U̲ => rigid body motion and deformation in the body
- Deformation – important

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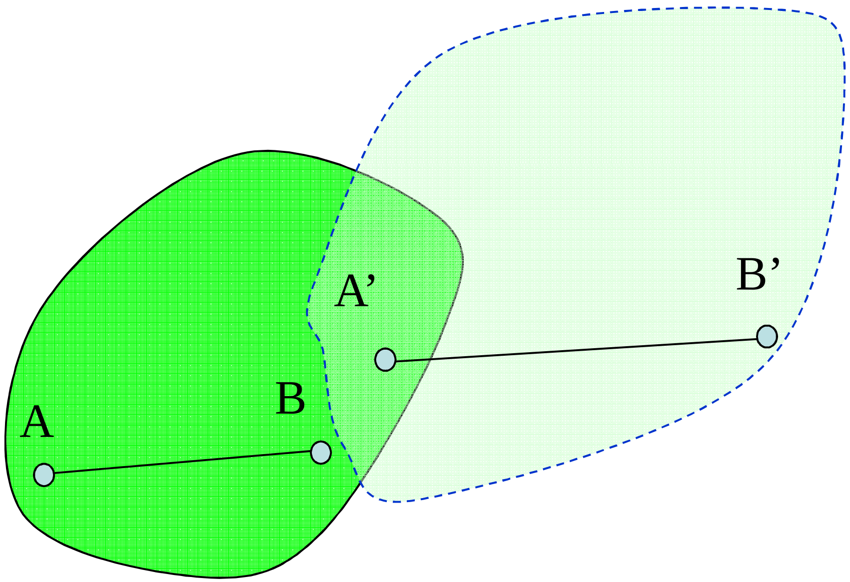
Normal strain

- Body subjected to loads –



AB => A'B'

AB // A'B' & AB = A'B'

⇒Rigid body translation

AB $\nparallel$ A'B' & AB = A'B'

⇒Rigid body rotation

AB // A'B' & AB ≠ A'B'

⇒Deformation / normal Strain

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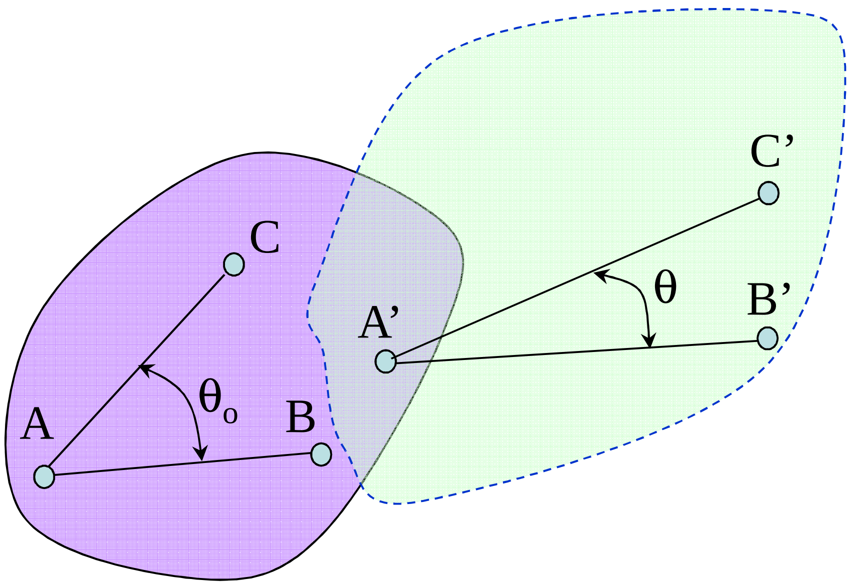
Shear strain

- Shear strain => distortion



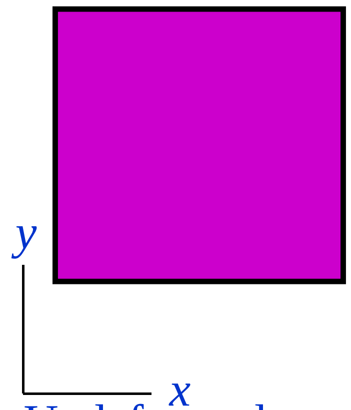
Change in angle between two lines => shear strain

No change in angle => shear strain = 0

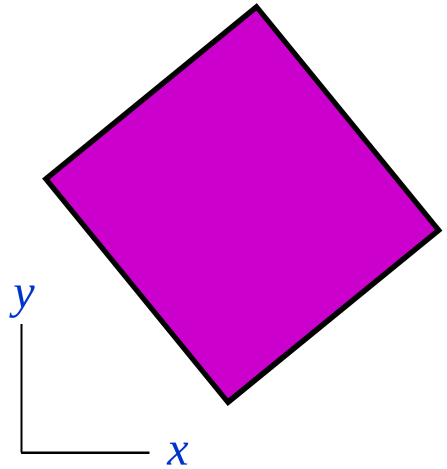
Change in angle – measure of shear strain / distortion

Displacement

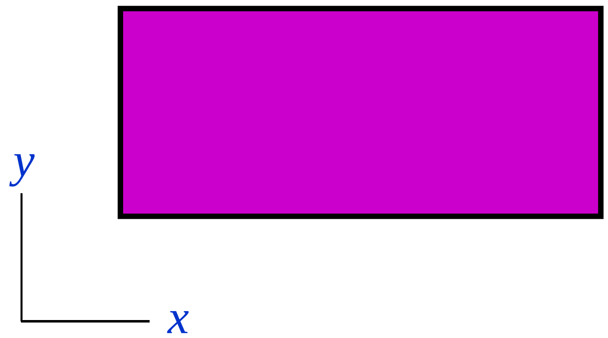
- General motion of a solid –



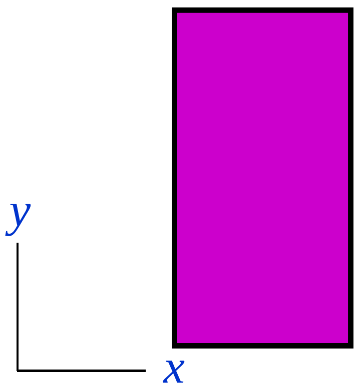
Undeformed state



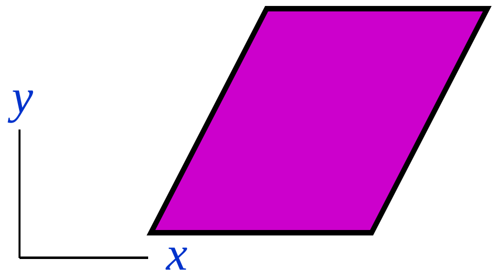
Rigid body rotation



Horizontal extension



Vertical extension



Shear deformation

- Rigid body motion doesn’t effect strain field – no effect on stresses

Strain

- Strain – measure of displacement of a body
- Various measures of strain - definition of strain is not unique – equally valid definition of strain
- Engineering normal strain – change in length/original length
- Green-Lagrange strain

$$\underset{\sim}{\underset{\sim}{E}} = \frac{(\text{final length})^2 - (\text{initial length})^2}{(\text{initial length})^2}$$

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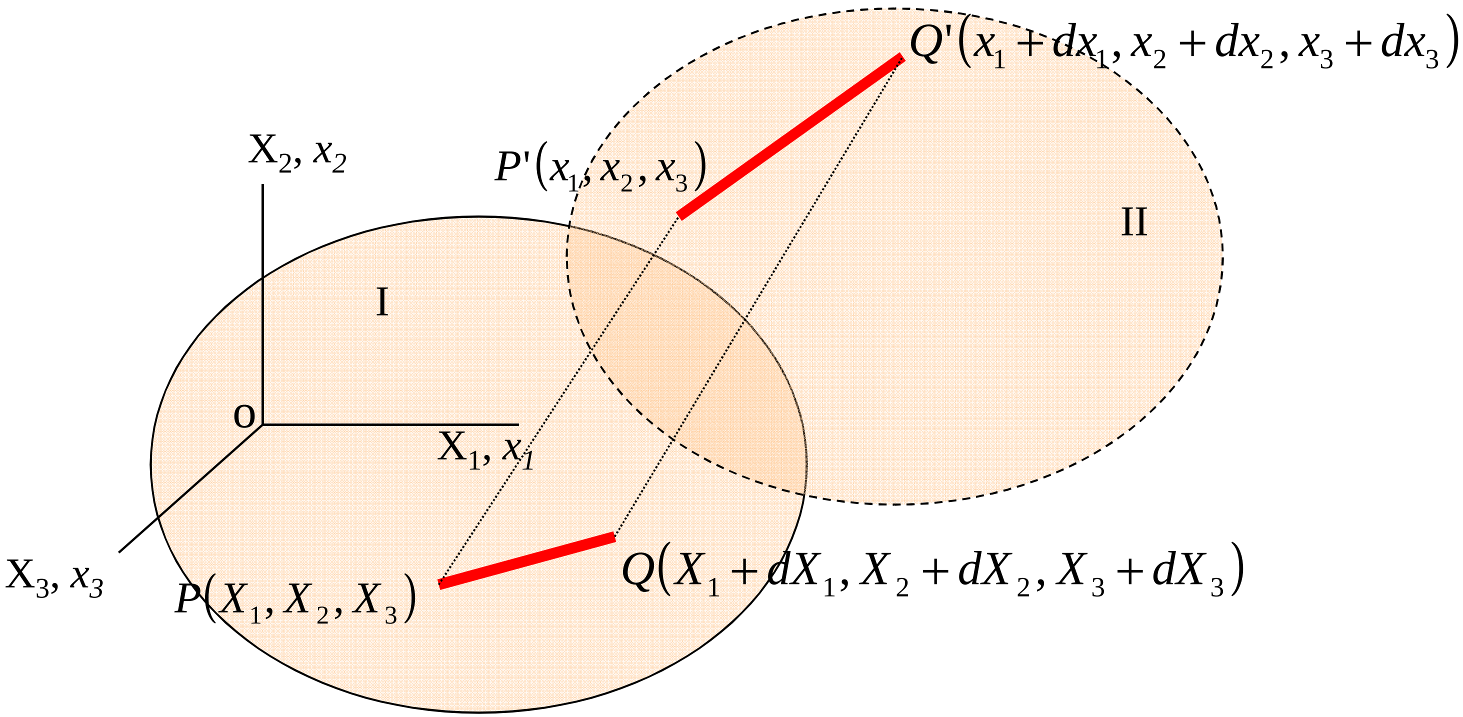
Green – Lagrange strain

- Green – Lagrange strain



X_i – undeformed configuration, x_i – deformed configuration

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Green – Lagrange strain

PQ = dS, P'Q' = ds

$$PQ^2 = d\underset{\sim}{S} . d\underset{\sim}{S} = dS_i \underset{\sim}{e_i} . dS_j \underset{\sim}{e_j} = dS_i dS_j \underset{\sim}{e_i} . \underset{\sim}{e_j}$$

$$=> PQ^2 = dS_i dS_j \delta_{ij} = dS_i dS_i = dS_i^2 = dS^2$$

Similarly, $$P'Q'^2 = ds_i^2 = ds^2$$

Green – Lagrange strain (E) is defined as,

$$2.d\underset{\sim}{X} . \underset{\sim}{\underset{\sim}{E}} . d\underset{\sim}{X} = ds^2 - dS^2 = d\underset{\sim}{x} . d\underset{\sim}{x}^T - d\underset{\sim}{X} . d\underset{\sim}{X}^T$$

$$d\underset{\sim}{x} = d\underset{\sim}{X} + d\underset{\sim}{U}$$

$$d\underset{\sim}{X} = dX_1 \underset{\sim}{e_1} + dX_2 \underset{\sim}{e_2} + dX_3 \underset{\sim}{e_3}$$

$$d\underset{\sim}{x} = dx_1 \underset{\sim}{e_1} + dx_2 \underset{\sim}{e_2} + dx_3 \underset{\sim}{e_3}$$

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Green – Lagrange strain

- dU – infinitesimal displacement
  function of (X_1, X_2, X_3)

$$\underset{\sim}{dU} = \underset{\sim}{dU}\left(X_1, X_2, X_3\right)$$

$$\text{it is a vector} \Rightarrow \underset{\sim}{dU} = du_1 \underset{\sim}{e_1} + du_2 \underset{\sim}{e_2} + du_3 \underset{\sim}{e_3}$$

$$du_i = du_i\left(X_1, X_2, X_3\right)$$

Use chain rule

$$du_i = \frac{\partial u_i}{\partial X_1} dX_1 + \frac{\partial u_i}{\partial X_2} dX_2 + \frac{\partial u_i}{\partial X_3} dX_3 = \nabla u_i \bullet \underset{\sim}{dX}$$

Green – Lagrange strain

- Writing in vector form

$$\underset{\sim}{dU} = du_i \underset{\sim}{e_i} = \left(\nabla u_i \bullet \underset{\sim}{dX}\right)\underset{\sim}{e_i}$$

$$\underset{\sim}{dU} = \left(\underset{\sim}{e_j}\frac{\partial u_i}{\partial x_j}\bullet dX_k \underset{\sim}{e_k}\right)\underset{\sim}{e_i} = \left(\frac{\partial u_i}{\partial x_j}dX_k \underset{\sim}{e_j}\bullet \underset{\sim}{e_k}\right)\underset{\sim}{e_i}$$

$$=> \underset{\sim}{dU} = \left(\frac{\partial u_i}{\partial X_j}dX_k \delta_{jk}\right)\underset{\sim}{e_i} = \frac{\partial u_i}{\partial X_j}dX_j \underset{\sim}{e_i} = du_i \underset{\sim}{e_i}$$

$$=> du_i = \frac{\partial u_i}{\partial X_j}dX_j$$

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Green – Lagrange strain

- The new position is, $dx_i = dX_i + du_i$

use $du_i = \frac{\partial u_i}{\partial X_j} dX_j$

$$dx_i = dX_i + \frac{\partial u_i}{\partial X_j} dX_j$$

$$\Rightarrow dx_i = \delta_{ij} dX_j + \frac{\partial u_i}{\partial X_j} dX_j = dX_j \left(\delta_{ij} + \frac{\partial u_i}{\partial X_j} \right)$$

$$\Rightarrow \underset{\sim}{dx} = \underset{\sim}{dX} \bullet \left(\underset{\sim}{\underset{\sim}{I}} + \nabla \underset{\sim}{U} \right)$$

Green – Lagrange strain

- Green-Lagrange strain is defined as,

$$2.\underset{\sim}{dX}.\underset{\sim}{\underset{\sim}{E}}.\underset{\sim}{dX}^T = \underset{\sim}{dx}.\underset{\sim}{dx}^T - \underset{\sim}{dX}.\underset{\sim}{dX}^T$$

$$2.\underset{\sim}{dX}.\underset{\sim}{\underset{\sim}{E}}.\underset{\sim}{dX}^T = \left(\underset{\sim}{dX}.\left(\underset{\sim}{\underset{\sim}{I}}+\nabla\underset{\sim}{U}\right)\right).\left(\underset{\sim}{dX}.\left(\underset{\sim}{\underset{\sim}{I}}+\nabla\underset{\sim}{U}\right)\right)^T - \underset{\sim}{dX}.\underset{\sim}{dX}^T$$

$$2.\underset{\sim}{dX}.\underset{\sim}{\underset{\sim}{E}}.\underset{\sim}{dX}^T = \left(\underset{\sim}{dX}.\left(\underset{\sim}{\underset{\sim}{I}}+\nabla\underset{\sim}{U}\right)\right).\left(\left(\underset{\sim}{\underset{\sim}{I}}+\nabla\underset{\sim}{U}^T\right).\underset{\sim}{dX}^T\right) - \underset{\sim}{dX}.\underset{\sim}{dX}^T$$

$$2.\underset{\sim}{dX}.\underset{\sim}{\underset{\sim}{E}}.\underset{\sim}{dX}^T = \underset{\sim}{dX}.\left(\underset{\sim}{\underset{\sim}{I}}+\nabla\underset{\sim}{U}+\nabla\underset{\sim}{U}^T+\nabla\underset{\sim}{U}.\nabla\underset{\sim}{U}^T\right).\underset{\sim}{dX}^T - \underset{\sim}{dX}.\underset{\sim}{dX}^T$$

$$\underset{\sim}{\underset{\sim}{E}} = \frac{1}{2}\left(\nabla\underset{\sim}{U}+\nabla\underset{\sim}{U}^T+\nabla\underset{\sim}{U}.\nabla\underset{\sim}{U}^T\right)$$

Green – Lagrange strain

Green-Lagrange Strain – Normal strains

$$\varepsilon_X = \frac{\partial u}{\partial X} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial X}\right)^2 + \left(\frac{\partial v}{\partial X}\right)^2 + \left(\frac{\partial w}{\partial X}\right)^2\right]$$

$$\varepsilon_Y = \frac{\partial v}{\partial Y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Y}\right)^2 + \left(\frac{\partial v}{\partial Y}\right)^2 + \left(\frac{\partial w}{\partial Y}\right)^2\right]$$

$$\varepsilon_Z = \frac{\partial w}{\partial Z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial Z}\right)^2 + \left(\frac{\partial v}{\partial Z}\right)^2 + \left(\frac{\partial w}{\partial Z}\right)^2\right]$$

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Green – Lagrange strain

Green-Lagrange Strain – Shear strains

$$\gamma_{XY} = \frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X} + \left[\frac{\partial u}{\partial X}\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\frac{\partial v}{\partial Y} + \frac{\partial w}{\partial X}\frac{\partial w}{\partial Y} \right]$$

$$\gamma_{YZ} = \frac{\partial w}{\partial Y} + \frac{\partial v}{\partial Z} + \left[\frac{\partial u}{\partial Z}\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial Z}\frac{\partial v}{\partial Y} + \frac{\partial w}{\partial Z}\frac{\partial w}{\partial Y} \right]$$

$$\gamma_{XZ} = \frac{\partial u}{\partial Z} + \frac{\partial w}{\partial X} + \left[\frac{\partial u}{\partial X}\frac{\partial u}{\partial Z} + \frac{\partial v}{\partial X}\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial X}\frac{\partial w}{\partial Z} \right]$$

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Green – Lagrange strain

- Green-Lagrange strain captures finite deflections, strains and rotations – non-linear terms
- Infinitesimal strains, rotations => linear terms – non-linear terms – much smaller than linear terms

$$\left| \frac{\partial u_i}{\partial x_j} \right| << 1$$

- Small strains known as engineering strains
- Engineering strains – denoted by => ε_{ij}

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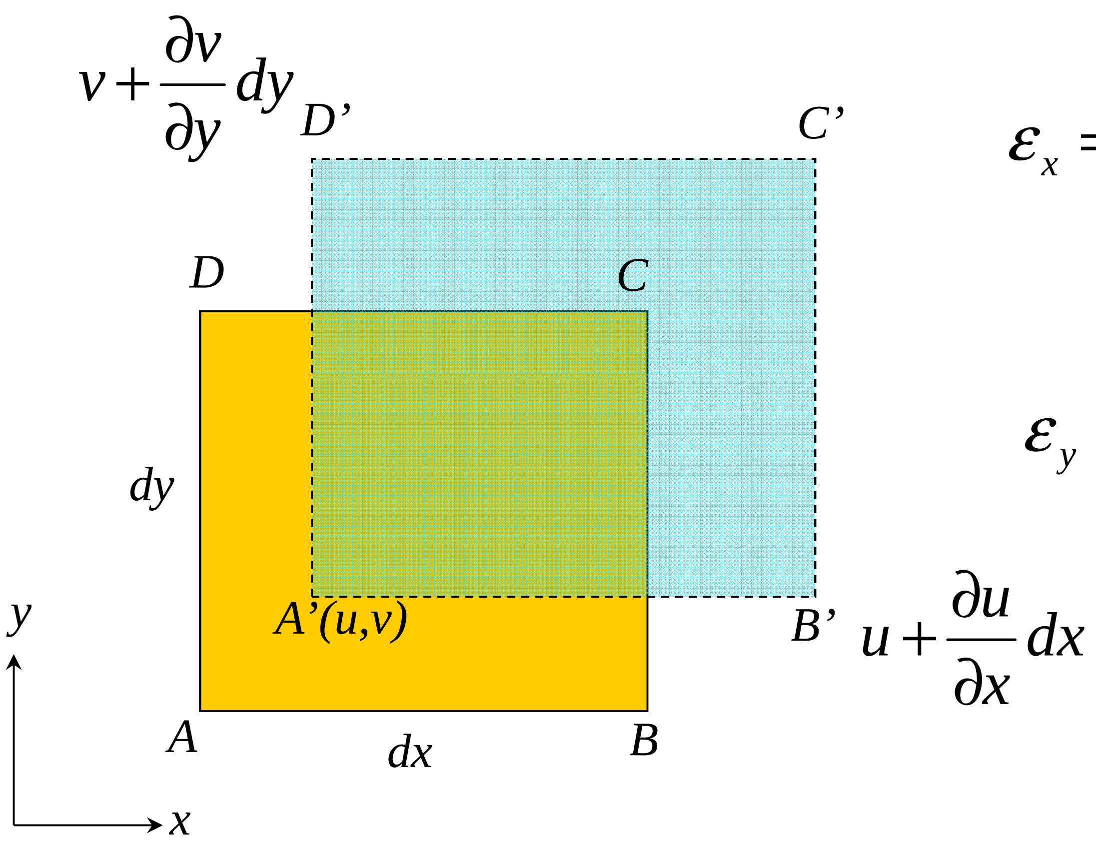
Engineering strain



$$\varepsilon_x = \frac{u+\frac{\partial u}{\partial x}dx-u}{dx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{v+\frac{\partial v}{\partial y}dy-v}{dy} = \frac{\partial v}{\partial y}$$

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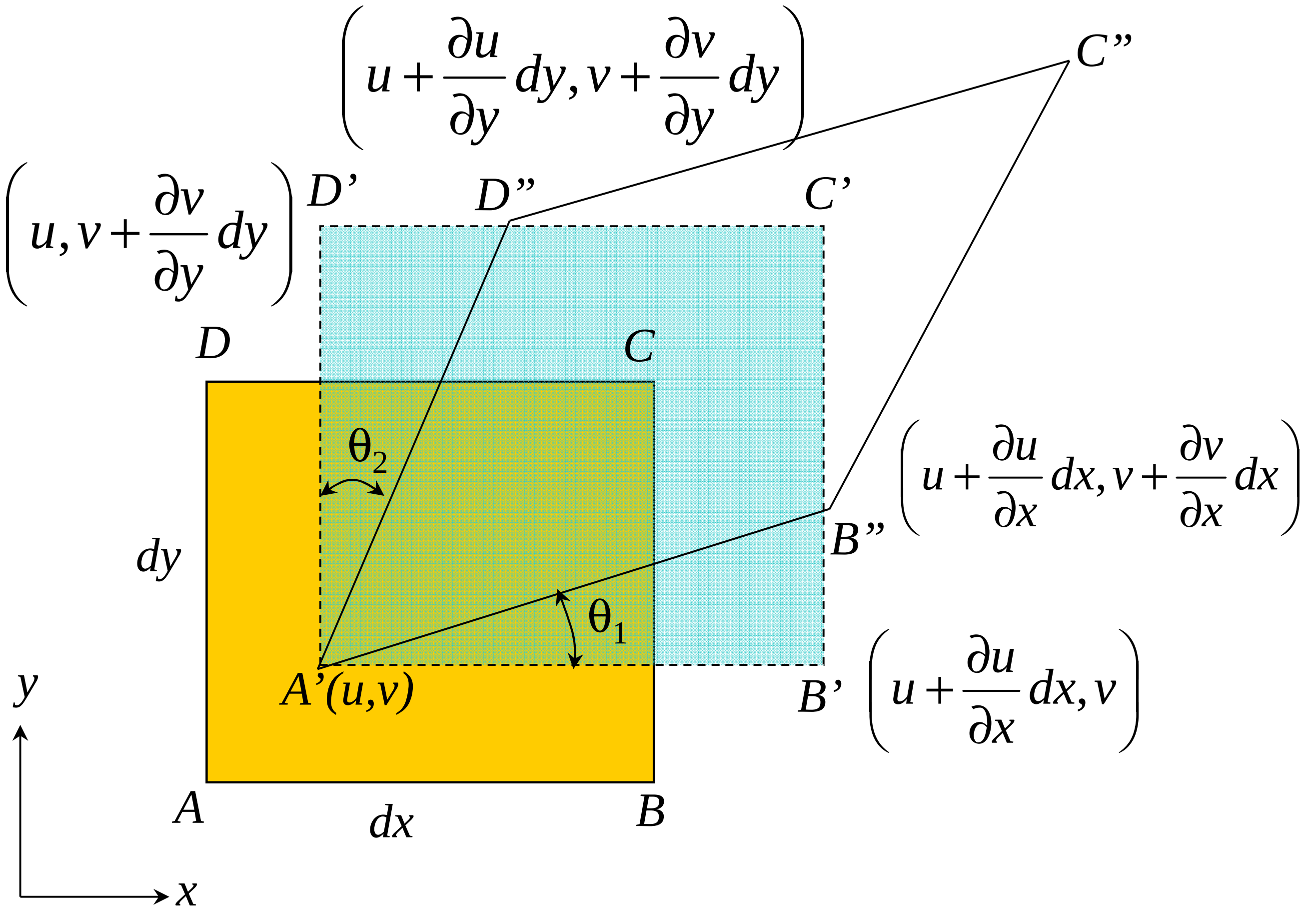
Engineering strain



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Engineering strain

- Shear strain – Change in the angle
  - Initial angle => 90^0
  - Final angle => $(90^0 - \theta_1 - \theta_2)$
  - Change in angle => $\theta = \theta_1 + \theta_2$

$$\tan\theta_1 = \frac{v + \frac{\partial v}{\partial x}dx - v}{dx} = \frac{\partial v}{\partial x} \qquad \tan\theta_2 = \frac{u + \frac{\partial u}{\partial y}dy - u}{dy} = \frac{\partial u}{\partial y}$$

$$\theta_1 \approx \frac{\partial v}{\partial x} \qquad \theta_2 \approx \frac{\partial u}{\partial y}$$

$$\gamma_{xy} = \theta_1 + \theta_2 \approx \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Engineering strain

- The six strain components –
  - Normal strains –

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

  - Shear strains –

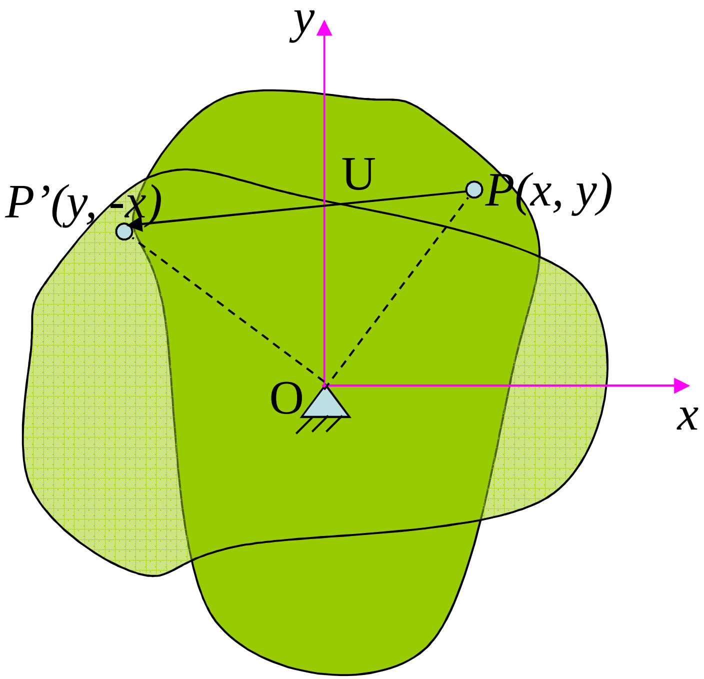
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Indices notation - $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

- Infinitesimal strains => Green-Lagrange strain = Engineering Strain

Engineering strain

- Rigid body rotations –



Rotated by 90^0

$$\underset{\sim}{OP}' = \underset{\sim}{OP} + \underset{\sim}{U}$$

$$\underset{\sim}{OP} = x\,\underset{\sim}{i} + y\,\underset{\sim}{j}$$

$$\underset{\sim}{OP}' = -y\,\underset{\sim}{i} + x\,\underset{\sim}{j}$$

Linear strains –

$$\varepsilon_{xx} = \varepsilon_{yy} = -1,\ \gamma_{xy} = 0$$

Non-linear strains –

$$\varepsilon_{xx} = \varepsilon_{yy} = 0,\ \gamma_{xy} = 0$$

- Linear strains can't capture large rotations use non-linear strains

Engineering strain

- Assuming engineering – linear strains

Strain tensor

$$[\varepsilon]=\begin{bmatrix}\varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz}\\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz}\\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_y\end{bmatrix}=\begin{bmatrix}\varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz}\\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz}\\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_y\end{bmatrix}$$

ε_{ij} – Tensorial strain

γ_{ij} – Engineering shear strain

$$\varepsilon_{ij}=\frac{1}{2}\gamma_{ij}$$

Displacement gradient

- Displacement gradient tensor – second order tensor

$$H_{ij} = \frac{\partial u_i}{\partial x_j} \Rightarrow \underset{\sim}{\underset{\sim}{H}} = \nabla \underset{\sim}{U}$$

In matrix form

$$H_{ij} = u_{i,j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

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Displacement gradient

- Square matrix – sum of symmetric and skew symmetric matrices

$$[M] = \frac{1}{2}\left(M + M^T\right) + \frac{1}{2}\left(M - M^T\right)$$

$$[M] = [S] + [A]$$

$$[S] = \frac{1}{2}\left(M + M^T\right), \quad [A] = \frac{1}{2}\left(M - M^T\right)$$

$$[S]^T = [S], \ [A]^T = -[A]$$

- [S] – Symmetric tensor, [A] – Skew-symmetric tensor – Main diagonal terms => 0

Displacement gradient

- Decomposing displacement gradient

$$u_{i,j}=\begin{bmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{bmatrix}=\begin{bmatrix}\frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)\\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)\\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3}\end{bmatrix}+$$

$$\begin{bmatrix}0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right)\\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right)\\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right) & 0\end{bmatrix}$$

Zeros in main diagonal

Displacement gradient

- Displacement gradient is sum of infinitesimal strain tensor and rotation tensor

$$\underset{\sim}{\underset{\sim}{H}} = \underset{\sim}{\underset{\sim}{\varepsilon}} + \underset{\sim}{\underset{\sim}{\Omega}}$$

$$u_{i,j} = \varepsilon_{ij} + \Omega_{ij}$$

$$u_{i,j} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}\left(u_{i,j} - u_{j,i}\right)$$

$$=> \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) => \text{Symmetric tensor}$$

$$\Omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) => \text{Skew - symmetric tensor}$$

Displacement gradient

- If displacement gradient is symmetric
  => $u_{i,j} = u_{j,i}$
- Rotation tensor becomes zero – only strain tensor exists
- Displacement gradient = strain tensor

$$u_{i,j} = u_{j,i}$$

$$u_{i,j} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

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Rotation vector

- Rotation vector is defined as

$$\underset{\sim}{\omega} = \frac{1}{2}\nabla \times \underset{\sim}{U}$$

$$\underset{\sim}{\omega} = \frac{1}{2}\underset{\sim}{e_i}\frac{\partial}{\partial x_i} \times u_j \underset{\sim}{e_j} = \frac{1}{2}\frac{\partial u_j}{\partial x_i}\underset{\sim}{e_i} \times \underset{\sim}{e_j}$$

$$\underset{\sim}{\omega} = \frac{1}{2}\frac{\partial u_j}{\partial x_i}\varepsilon_{ijk}\underset{\sim}{e_k} => \omega_k = \frac{1}{2}\frac{\partial u_j}{\partial x_i}\varepsilon_{ijk}$$

$$\text{Say, k} = 3, \text{i, j} = 1, 2 => \omega_3 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)$$

- ω represents components of rigid body rotations

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Rotation vector

- Relation between rotation vector and tensor

$$\omega_k = \frac{1}{2}\frac{\partial u_j}{\partial x_i} e_{ijk}$$

$$\text{Displacement gradient} \Rightarrow \frac{\partial u_j}{\partial x_i} = \varepsilon_{ji} + \Omega_{ji}$$

$$\omega_k = \frac{1}{2}\left(\varepsilon_{ji} + \Omega_{ji}\right)e_{ijk} = -\frac{1}{2}\left(\varepsilon_{ji}e_{jik} + \Omega_{ji}e_{jik}\right)$$

e_{ijk} – skew symmetric matrix in 'i', 'j'

$$\omega_k = -\frac{1}{2}\Omega_{ji}e_{jik} \Rightarrow e_{lmk}\omega_k = -\frac{1}{2}\Omega_{ji}e_{jik}e_{lmk}$$

Rotation vector

- Multiplication of two permutations

$$\omega_k = -\frac{1}{2}\Omega_{ji} e_{jik} \Rightarrow e_{lmk}\omega_k = -\frac{1}{2}\Omega_{ji} e_{jik} e_{lmk}$$

$$e_{jik} e_{lmk} = \delta_{jl}\delta_{im} - \delta_{jm}\delta_{il}$$

$$e_{lmk}\omega_k = -\frac{1}{2}\Omega_{ji}\left(\delta_{jl}\delta_{im} - \delta_{jm}\delta_{il}\right) = -\frac{1}{2}\left(\Omega_{lm} - \Omega_{ml}\right)$$

$$-e_{lmk}\omega_k = \Omega_{lm}$$

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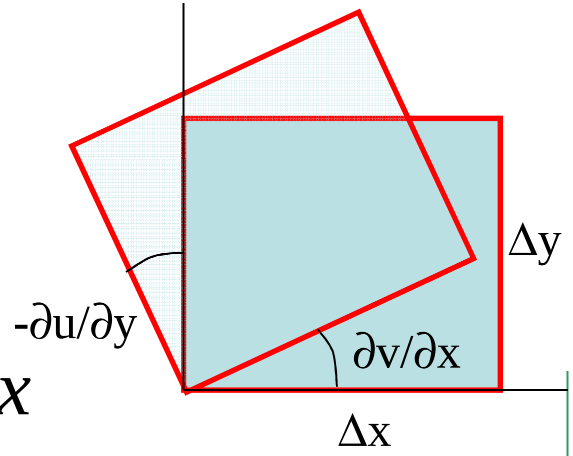
Rotation vector

- From shear strain

$$\gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \text{constant} = c$$

$$\Rightarrow \frac{\partial v}{\partial x} = c,\ \frac{\partial u}{\partial y} = -c$$



Integrating these two equations

$$u = u_o - cy,\ v = v_o + cx$$

For constant rotation

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \Rightarrow \frac{1}{2}(c+c) \Rightarrow c = \omega_z$$

Rotation vector

- The displacements are

$$u = u_o - \omega_z y, \;\; v = v_o + \omega_z x$$

- Extending the same procedure to 3D rigid body rotations – all shear strains => 0

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0$$

- Integrate these equations – integration constants

Rotation vector

- Integration constants – rigid body translations and rotations

$$u = u_o - \omega_z y + \omega_y z$$

$$v = v_o - \omega_x z + \omega_z x$$

$$w = w_o - \omega_y x + \omega_x y$$

- Rigid body translations and rotations don't contribute to strains and stresses – may be dropped
- Basic assumption – small deformation theory
- Small deformation theory – Principle of superposition

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Transformation of strain

- Strain – second order tensor
- Transformation rules of any second order tensor – applicable – ex. Stress tensor
- Transformation of stress tensor $\sigma'_{ij} = l_{im} l_{jn} \sigma_{mn}$
- Transformation of strain tensor $\varepsilon'_{ij} = l_{im} l_{jn} \varepsilon_{mn}$
- In matrix notation

$$[\varepsilon'] = [T][\varepsilon][T]^T$$

[T] – Transformation matrix => direction cosines

Principal strains

- Principal strains – similar to principal stresses – maximum strains acting on three mutually perpendicular planes, where shear stresses => 0

$$\varepsilon_x = l\varepsilon_{xx} + m\varepsilon_{xy} + n\varepsilon_{xz} = l\varepsilon$$

$$\varepsilon_y = l\varepsilon_{xy} + m\varepsilon_{yy} + n\varepsilon_{yz} = m\varepsilon$$

$$\varepsilon_z = l\varepsilon_{xz} + m\varepsilon_{yz} + n\varepsilon_{zz} = n\varepsilon$$

Writing in indices notation

$$\left(\varepsilon_{ij} l_j - \delta_{ij} \varepsilon l_j\right) = 0$$

For non-trivial solution

determinant = 0

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Principal strains

- Determinant $\left(\varepsilon_{ij} - \delta_{ij}\varepsilon\right) l_j = 0$

$$=> \left|\varepsilon_{ij} - \delta_{ij}\varepsilon\right| = 0$$

Expand this

$$\varepsilon^3 - I_1\varepsilon^2 + I_2\varepsilon - I_3 = 0$$

I_i - Invariant

$$I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$I_2 = \begin{vmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_y \end{vmatrix} + \begin{vmatrix} \varepsilon_x & \varepsilon_{xz} \\ \varepsilon_{xz} & \varepsilon_z \end{vmatrix} + \begin{vmatrix} \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{yz} & \varepsilon_z \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ symm & & \varepsilon_z \end{vmatrix}$$

First invariant I_1 known as cubical dilatation

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Strain decomposition

- Hydrostatic part represents volume deformation – isotropic tensor – same in all direction
- Volumetric strains – accounts for change in size or volume only
- Deviatoric strain tensor – accounts for change in shape – distortion of material
- Physically ε_d => deviation of strain from $1/3^{rd}$ of pure cubical dilatation
- $Tr[\varepsilon_d] = 0$ => Trace of deviatoric strain tensor is zero

Strain compatibility

- Two dimensional stress analysis – three strain components => ε_{xx}, ε_{yy} and γ_{xy}
- Writing in terms of displacements

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u}{\partial y^2 \partial x} = \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial u}{\partial y}\right) \qquad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y} = \frac{\partial^2}{\partial y \partial x}\left(\frac{\partial v}{\partial x}\right)$$

Add these two equations

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

One compatibility equation

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Strain compatibility

- In 3D => six strain compatibility equations

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$
$$(i = j = 1, k = l = 2)$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$
$$(i = j = 2, k = l = 3)$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}$$
$$(i = j = 3, \ k = l = 1)$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 \varepsilon_{xx}}{\partial z \partial y}$$
$$(i = j = 1, k = 2, l = 3)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 \varepsilon_{yy}}{\partial x \partial z}$$
$$(i = j = 2, k = 3, l = 1)$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y}$$
$$(i = j = 3, k = 1, l = 2)$$

Strain compatibility

- Six strain compatibility equations – no displacement term appears
- Displacements – integration of strain – displacement relations
- Six equations – three displacements => over constraining – can't get a single valued displacements – ill-conditioned
- For well conditioned system => Number of equations = number of unknowns
- Three more extra equations required

Strain compatibility

- There exist relations among six strain components => Strain compatibility equations – six equation
- Only three of them are independent
- Equation -

Displacements => 3

Strain-displacement relations => 6

Independent strain compatibility equations => 3

=> Number of unknowns = number of equations

Strain compatibility

- Satisfy strain compatibility equations => single valued displacement in the domain
- Conversely – all six strain compatibility equations satisfied if displacement field is single valued
- Arbitrary selection of strains – fails to satisfy strain compatibility equations

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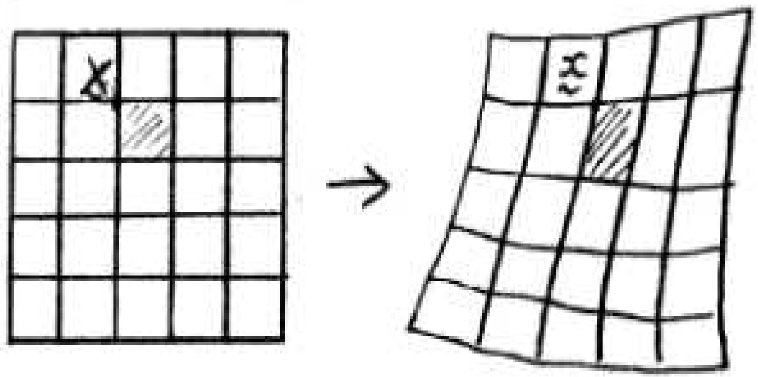
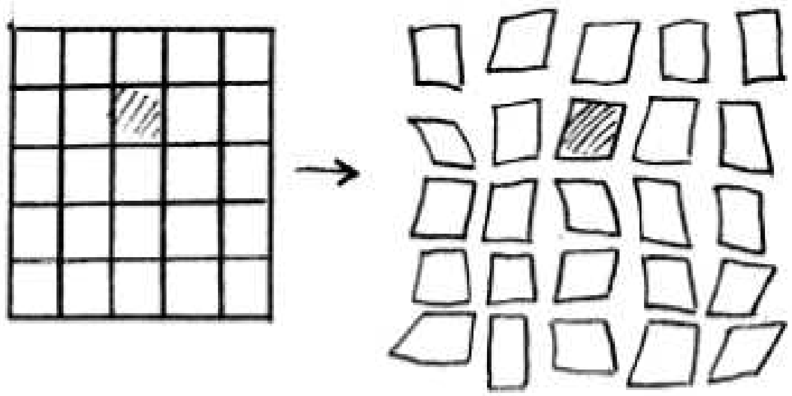
Strain compatibility



- Strains are calculated from displacements
- Compatible – no gaps and overlaps



- Arbitrary strain field not satisfying compatibility eqns.
- Integrated to get displacements
- Gaps & overlaps - incompatibility

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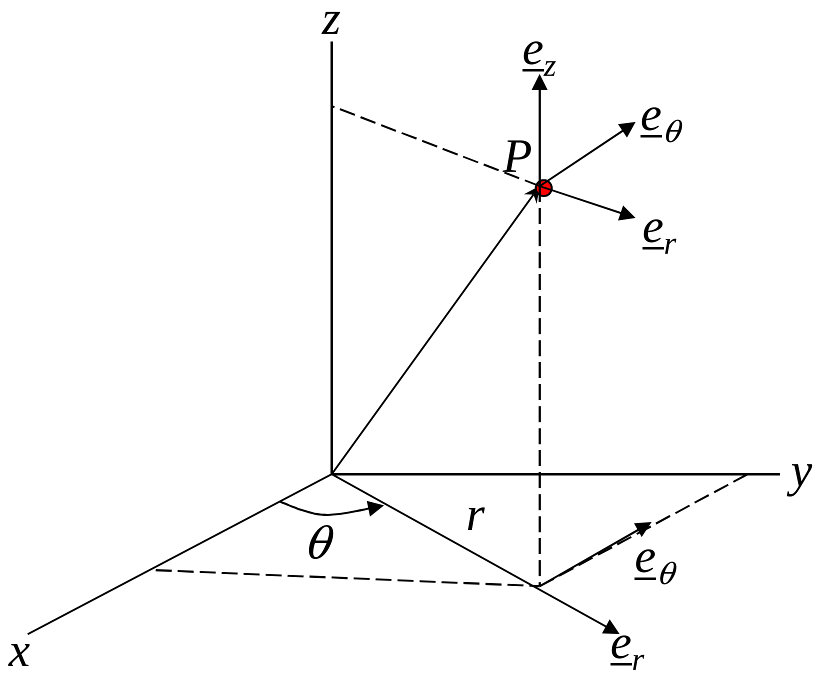
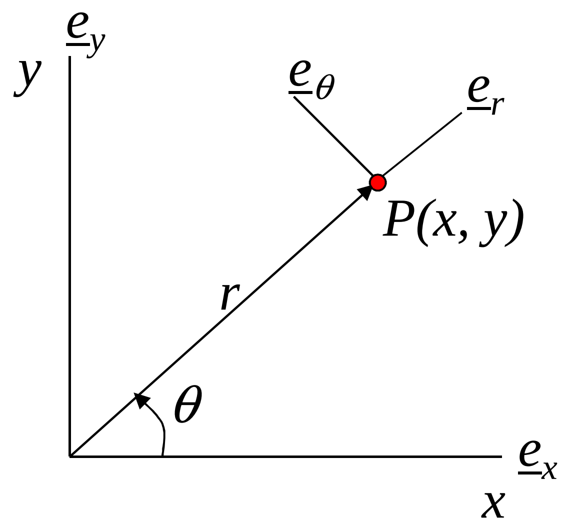
Polar co-ordinates

- Some geometries – cylindrical – easy to handle in cylindrical co-ordinates
- Evaluate stress, strains, displacements etc in that co-ordinate system – ex. Radial and Hoop stresses
- Two different ways –
  - Use strain-displacement relations in cartesian co-ordinates – convert to polar co-ordinates
  - Derive from polar co-ordinates

Polar co-ordinates

- From cartesian csys – e_x, e_y – unit vectors in cartesian csys - e_r, e_θ - unit vectors in polar csys





Cartesian strain => $\varepsilon_{ij} = \frac{1}{2}\left(\nabla \underset{\sim}{U} + \nabla \underset{\sim}{U}^T\right)$

Polar co-ordinates

- Relation between unit vectors –

$$\underset{\sim}{e_x} = \underset{\sim}{e_r}\cos\theta - \underset{\sim}{e_\theta}\sin\theta$$

$$\underset{\sim}{e_y} = \underset{\sim}{e_r}\sin\theta + \underset{\sim}{e_\theta}\cos\theta$$

$$\underset{\sim}{e_z} = \underset{\sim}{e_z}$$

$$\begin{Bmatrix} \underset{\sim}{e_x} \\ \underset{\sim}{e_y} \\ \underset{\sim}{e_z} \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \underset{\sim}{e_r} \\ \underset{\sim}{e_\theta} \\ \underset{\sim}{e_z} \end{Bmatrix}$$

$$\left\{\underset{\sim}{e_{xyz}}\right\} = [T]\left\{\underset{\sim}{e_{r\theta z}}\right\}$$

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$

$$y = r\sin\theta$$

$$Tan\,\theta = \frac{y}{x}$$

$$z = z$$

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Polar co-ordinates

- In cartesian csys - gradient

$$\nabla = \underset{\sim}{e}_x \frac{\partial}{\partial x} + \underset{\sim}{e}_y \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

Substitute all these

$$\nabla = \underset{\sim}{e}_r \frac{\partial}{\partial r} + \underset{\sim}{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \underset{\sim}{e}_z \frac{\partial}{\partial z}$$

Polar co-ordinates

- Displacement vector in polar csys –

$$\underset{\sim}{U} = u_r \underset{\sim}{e}_r + u_\theta \underset{\sim}{e}_\theta + u_z \underset{\sim}{e}_z$$

- Gradient of displacement vector –

$$\nabla \underset{\sim}{U} = \left[\frac{\partial u_r}{\partial r} \underset{\sim}{e}_r \underset{\sim}{e}_r + \frac{\partial u_\theta}{\partial r} \underset{\sim}{e}_r \underset{\sim}{e}_\theta + \frac{\partial u_z}{\partial r} \underset{\sim}{e}_r \underset{\sim}{e}_z \right] +$$

$$\left[\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta \right) \underset{\sim}{e}_\theta \underset{\sim}{e}_r + \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta} \right) \underset{\sim}{e}_\theta \underset{\sim}{e}_\theta + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \underset{\sim}{e}_\theta \underset{\sim}{e}_z \right] +$$

$$\left[\frac{\partial u_r}{\partial z} \underset{\sim}{e}_z \underset{\sim}{e}_r + \frac{\partial u_\theta}{\partial z} \underset{\sim}{e}_z \underset{\sim}{e}_\theta + \frac{\partial u_z}{\partial z} \underset{\sim}{e}_z \underset{\sim}{e}_z \right]$$

Polar co-ordinates

- Strain =>

$$\underset{\sim}{\underset{\sim}{\varepsilon}} = \frac{1}{2}\left[\nabla \underset{\sim}{U} + \left(\nabla \underset{\sim}{U}\right)^{T}\right]$$

- Stain components in polar co-ordinates

Normal strains

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r},\ \varepsilon_{\theta\theta} = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right),\quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

Shear strains

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right),\quad \varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right),\ \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

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