SCALAR

A collection of things is called a *scalar set* if these things are additive to one another, scalable by numbers, and proportional to one another. Each thing in the collection is called a *scalar*. We will define the words *additive*, *scalable* and *proportional* with care.

Scalars in two worlds. Scalars model many things in the real world. In physics, scalars model length, time, mass, charge, energy, and entropy. In economics, scalars model money, labor, and commodities. In politics, scalars model population, votes, and polls. In social networks, scalars model friends, posts, and likes.

Scalars are the fundamental building blocks of a virtual world, called linear algebra. Being linear is being scalars. Scalars build vectors. Scalars and vectors then build everything else in linear algebra: linear maps, linear forms, bilinear forms, quadratic forms, inner products, and tensors. These characters play and interplay in the virtual world of linear algebra.

Give us scalars. We build a virtual world that models many aspects of the real world.

Happiness does not scale. Scalars, however, do not model everything in the world. We have some notion of various states of happiness. But the states of happiness are not additive to one another, or scalable by numbers, or proportional to one another. States of happiness do not form a scalar set.

Indeed, many things in the real world do not scale. Love does not scale. Nor does intelligence, color, or temperature. The virtual world of linear algebra, which is built from scalars, can only represent part of the real world. Such is the limitation of linear algebra. To learn a subject is to learn its interior, as well as its boundary.

March toward scalars. The boundary of a subject moves. One subject morphs to another. The known morphs to the unknown.

For thousands of years, people have devised many ways to map temperatures to numbers. Experience tells us that temperatures form an ordered set, but not a scalar set. Temperatures are not additive to one another, or scalable by numbers, or proportional to one another. In the nineteenth century, a breakthrough took place. People discovered two new scalars, energy U and entropy S, and related temperature T to these scalars:

$$T = \frac{\partial U}{\partial S}.$$

This relation connected a non-scalar to two scalars. The new science of thermodynamics was born.

Much of science in general is a program to relate things in the real world to scalars. Quantities in science are mostly scalars, or things related to scalars. We do not know how far we can push this program. It is conceivable that someone in your generation might discover ways to relate more non-scalars, such as happiness, intelligence and love, to scalars.

Axioms of Scalar Set

A *scalar set over a number field* means five items $(S,+,0,F,*)$ that obey twelve axioms.

1. S is a set, each element of which is called a *scalar*. To every two elements x and y in S, there corresponds a unique element in S, written $x+y$. The binary map, $+:S\times S\to S$, $(x,y)\mapsto x+y$, is called the *scalar-scalar addition*. That is, elements in S are *additive*, and the set S is *closed* under the addition.
2. $(x+y)+z=x+(y+z)$ for every x, y and z in S. That is, the scalar-scalar addition is *associative*.
3. There exists an element in S, called *zero scalar*, written 0, such that $x+0=x$ for every x in S. That is, there exists an *identity element* for the scalar-scalar addition.
4. For every x in S there exists an element in S, called the *negative element*, written $-x$, such that $x+(-x)=0$. That is, scalars are *subtractive*.
5. $x+y=y+x$ for every x and y in S. That is, the scalar-scalar addition is *commutative*.
6. F is a number field, each element of which is called a *number*. To every element α in F and every element x in S, there corresponds a unique element in S, written $\alpha * x$, or simply αx. The binary map, $*:F\times S\to S$, $(\alpha,x)\mapsto \alpha x$, is called the *number-scalar multiplication*. That is, the elements in S are *scalable* by elements in F, and the set S is *closed* under the number-scalar multiplication.
7. $1x=x$ for every x in S, where 1 is the identity element in F for the number-number multiplication in F.
8. $\alpha(\beta x)=(\alpha\beta)x$ for every x in S and for every α and β in F. The equation contains multiplications of two types: the number-scalar multiplication and the number-number multiplication.
9. $(\alpha+\beta)x=\alpha x+\beta x$ for every x in S and for every α and β in F. That is, the number-scalar multiplication *distributes* over the number-number addition.
10. $\alpha(x+y)=\alpha x+\alpha y$ for every x and y in S and every α in F. That is, the number-scalar multiplication *distributes* over the scalar-scalar addition.
11. There exists at least one nonzero element in S.

12. For a nonzero scalar u in S and any scalar x in S, there exists a number α in F, such that $x = \alpha u$. The nonzero scalar u is called a *unit* of the scalar set S, and the number α is called the *magnitude* of the scalar x in the unit u. That is, elements in S are *proportional to* one another.

Apples

Let us watch a concrete scalar set specify the five items $(S, +, 0, F, *)$ and fulfill the twelve axioms.

Piles of apples. Piles and piles of apples form a set, denoted APPLE. Each element in the set APPLE is a pile containing a distinct quantity of apples. Here are some elements in the set APPLE:

1 apple, 2 apples, 100 apples, half of an apple.

In general, we denote a pile of apples by a symbol, such as x. Using symbols to represent real things is an ancient invention. For example, we name mountains, rivers, and people.

We next verify that the set APPLE is a scalar set over the field of real numbers. Each pile of apples is a scalar in the scalar set APPLE.

Piles of apples are additive to one another. Two piles of apples, x and y, can be put together to give another pile of apples, denoted $x + y$. This *pile-pile addition* lets us add apples to apples, but does not let us add apples to oranges. A pile containing no apple is called the *zero pile*, denoted 0. For any pile of apples, x, the negative element $-x$ means the deficit of a pile of apples of the same amount as x. We can confirm that the pile-pile addition fulfills Axioms 1-5.

Piles of apples are scalable by real numbers. A pile x of apples can be multiplied by a real number α to give another pile α times the quantity of applies, denoted αx. This *number-pile multiplication* lets us multiply apples by real numbers, but does not let us multiply apples by apples. We can confirm that the number-pile multiplication fulfills Axioms 6-10.

Piles of apples are proportional to one another. For a nonzero pile u in APPLE and any pile x in APPLE, there exists a unique real number α such that $x = \alpha u$. That is, the set APPLE fulfills Axioms 11 and 12.

Model and Reality*

Ignore inconvenient truth. When we model the set APPLE as a scalar set over the field of real numbers, the axioms of scalar set require that an element in APPLE multiplying any real number be an element in APPLE. If the real number is too large, we do not have that many apples. If the real number is too small, we reach subatomic dimension, in which case the "pile" no longer contains any apple. Also, the axioms of scalar set require that negative quantities of apples be in APPLE.

In representing the reality with a model, we ignore inconvenient truths. (All models are wrong, but some are useful.) But we do check what the model predicts against the reality. If the model predicts a negative quantity of apples, it means that we are in deficit. If the model predicts a non-integer quantity of apples, we cut apples in pieces, or just round up. If a piece is too small to preserve its appleness, we may approximate the piece as the zero scalar.

A model achieves the economics of abstraction. A model is not the reality. Why do we link the reality to a model then? A map of a city is not the city, but the map lets us plan a tour without walking through the city. The map would be useless if it were as large and as detailed as the city. The model abstracts: it subtracts most details, retains a few, and idealizes them. Abstraction is value. A painting of a shrimp by Qi Baishi is worth a lot more than a photo of a shrimp, or the shrimp itself.

The model of the piles of apples lets us reason something about apples without having apples. The reasoning is about the quantities of apple; it abstracts the ideas of being *additive*, *scalable*, and *proportional*. This model ignores all other aspects of apples—the smell, the taste, the color, etc.

This particular model—the scalar set—has been reasoned thoroughly. The model applies to piles of apples, and to piles of oranges. The model, generalized to vector space, applies to piles containing both apples and oranges. The model applies to durations of time. The model, with one more feature (the metric), applies to displacements in space. The model, with yet one more feature (the constancy of the speed of light), applies to spacetime. The model achieves the economics of abstraction, as well as the economics of scale.

Different amounts of the same thing. The axioms of scalar set formalize a primitive notion. Elements in a scalar set are different amounts of the same things.

In modeling the world, we have considerable flexibility to choose the level of aggregation. For example, we may regard all kinds of apples as a single scalar set, or regard each species of apples as a distinct scalar set, or even regard each species of apples of the same size as a single scalar set.

Remarks on the Axioms of Scalar Set

Zero scalar. The zero scalar by itself forms a set. This single-element set fulfills Axioms 1-10, but violates Axioms 11 and 12. Thus, this single-element set is not a scalar set. We will learn later that the single-element set is the zero-dimensional vector space.

One-dimensional vector space. The twelve axioms of scalar set fall into several categories:

- Axioms 1-5 say that scalars in *S* are *additive* to one another.
- Axioms 6-10 say that scalars in *S* are *scalable* by numbers in *F*.
- Axioms 11 and 12 say that scalars in *S* are *proportional* to one another.

As we will learn later, the scalar set is simply a one-dimensional vector space. Axioms 1-10 of scalar set are identical to those of vector space. Axioms 11 and 12 specify the dimension.

Two sets. The axioms of scalar set mention two sets, *S* and *F*. In most applications, *F* stands for either the set of real numbers *R*, or the set of complex numbers *C*. On rare occasions, *F* stands for the field of rational numbers *Q*. A scalar set over *R* is called a *real scalar set*, a scalar set over *C* is called a *complex scalar set*, and a scalar set over *Q* is called a *rational scalar set*.

The axioms of scalar set do not say what a scalar set *S* is, or what scalars are. Such is the nature of abstraction. The axioms call for actions: look at the world, find things that fulfill the axioms, and call these things scalars.

The axioms also call for another type of actions: use the axioms to deduce logical consequences, use scalars to build new concepts, and use the new concepts to create a virtual world. Examples of new concepts include vectors, linear maps, bilinear maps, and inner products. The virtual world built with scalars is called linear algebra. Once a collection of things in the real world is identified as a scalar set, linear algebra provides tools to relate this scalar set to other scalar sets.

Three identity elements. A scalar set *S* over a number field *F* requires three identity elements:

- The identity element in *F* for the number-number multiplication (i.e., the number 1).
- The identity element in *F* for the number-number addition (i.e., the number 0).
- The identity element in *S* for the scalar-scalar addition (i.e., the zero scalar 0).

The element 1 is the identity element in *F* for the number-number multiplication. The same number 1 appears in Axiom 7 as the identity element in *F* for the number-scalar multiplication. The axioms of scalar set do *not* introduce any identity element in *S* for the number-scalar multiplication.

The element zero in the set *F* is an object different from the element zero in the set *S*. The two objects have the same notation, 0. We tell them apart by seeing them in context.

Look at an equation

$$0x = 0 .$$

Here *x* is any element in *S*, 0 on the left side is the zero element in *F*, and 0 on the right side is the zero element in *S*. We can deduce this equation from the axioms of scalars. Axiom 9 says that $(\alpha + \beta)x = \alpha x + \beta x$ for every *x* in *S* and for every α

and β in *F*. Let $\alpha = \beta = 0$, the axiom says that $0x = 0x + 0x$. Axiom 4 lets us subtract the same scalar from both sides of the equation, giving $0 = 0x$.

Now look at another equation

$$\alpha 0 = 0 .$$

Here α is any element in *F*, and 0 on both sides is the zero element in *S*. We can also deduce this equation from the axioms. Axiom 10 says that $\alpha(x+y) = \alpha x + \alpha y$ for every *x* and *y* in *S* and for every α in *F*. Let $x = y = 0$, the axiom says that $\alpha 0 = \alpha 0 + \alpha 0$. Axiom 4 lets us subtract the same scalar from both sides of the equation, giving $0 = \alpha 0$.

The axioms also lead to another statement. For any α in *F* and *x* in *S*, the equation

$$\alpha x = 0$$

implies that at least one of the following is true:

$$\alpha = 0,$$

$$x = 0.$$

We prove this statement as follows. The statement is obviously true if $\alpha = 0$. Now suppose that $\alpha \neq 0$, and we need to prove that $\alpha x = 0$ implies that $x = 0$. The axioms of number field say that there exists a nonzero element α^{-1} in *F*, such that $\alpha^{-1}\alpha = 1$. Multiply both sides of $\alpha x = 0$ by α^{-1}, and we obtain that $\left(\alpha^{-1}\alpha\right)x = \alpha^{-1}0$. Axiom 7 says $1x = x$. In the above we have just proved $\alpha^{-1}0 = 0$. Consequently, $\left(\alpha^{-1}\alpha\right)x = \alpha^{-1}0$ implies that $x = 0$.

We can paraphrase the above statement as follows: $\alpha x \neq 0$ for every nonzero α in *F* and every nonzero *x* in *S*.

Four binary maps. The axioms of scalar set mention four binary maps. Two of them come with the number field *F*. Adding two elements in *F* gives a unique element in *F*:

$$F \times F \xrightarrow{\text{add}} F, \quad (\alpha,\beta) \mapsto \alpha + \beta .$$

The number field *F* is a commutative group under the number-number addition. Multiplying two elements in *F* gives a unique element in *F*:

$$F \times F \xrightarrow{\text{multiply}} F, \quad (\alpha,\beta) \mapsto \alpha\beta .$$

The number field *F*, with element zero removed, is a commutative group under the number-number multiplication. The number-number multiplication distributes over the number-number addition.

Axioms 1-5 of scalar set do not mention the set *F*, and are devoted entirely to the scalar-scalar addition on the set *S*. Adding two elements in *S* gives a unique element in *S*:

$$S \times S \xrightarrow{\text{add}} S, \quad (x,y) \mapsto x + y .$$

Axioms 1-5 define S as a commutative group, with the individual scalars as the elements, the scalar-scalar addition as the operation, and the zero scalar as the identity element.

Axioms 6-8 introduce another binary map. Multiplying an element in F and an element in S gives a unique element in S:

$$F \times S \xrightarrow{\text{multiply}} S, \quad (\alpha, x) \mapsto \alpha x .$$

The number-scalar multiplication distributes over both the number-number addition (Axiom 9), and distributes over the scalar-scalar addition (Axiom 10).

Given two sets S and F, we can think of many possible operations to combine elements in the two sets. Most operations, however, do not appear in the axioms of scalar set. In particular, we exclude from the axioms any operation that might represent the addition of an element in F and an element in S, or represent the multiplication of two elements in S.

Five items. We have modeled piles of apples as a scalar set over a number field, and identified the five items $(S,+,0,F,*)$ as follows.

- $S = \text{APPLE}$, each element of which is a pile of apples.
- + means putting two piles of apples together.
- 0 means a pile containing no apple.
- $F = R$, the field of real numbers.
- $*$ means scaling a pile of apples by a real number.

Listing five items $(S,+,0,F,*)$ all the time is tiresome. We often say that apples form a scalar set, assuming that the reader knows the five items.

Change of Unit

Infinitely many units. Axiom 3 says S has a zero element. Indeed, a set that contains only the zero element fulfills Axioms 1-10. Axiom 11 says that there exists at least one nonzero element u in S. For any nonzero α in F, we have just proved that αu is a nonzero scalar in S. A number field, such as Q, R and C, has infinitely many elements. Consequently, a scalar set S over such a number field has infinitely many nonzero scalars. Every nonzero scalar can serve as a unit of S. A unit characterizes a *type* of scalars. A number characterizes the *size* of an individual scalar in the set.

Change of unit. Let u and $\tilde{u}$ be two nonzero scalars in a scalar set S over a number field F. Axiom 12 ensures that the two scalars are proportional to each other:

$$\tilde{u} = pu ,$$

where p is a number in F, and is the magnitude of the scalar $\tilde{u}$ in the unit u.

A scalar x in S scales with either unit:

$$x = \alpha u,$$

$$x = \tilde{\alpha}\tilde{u}.$$

where the number α in F and is the magnitude of the scalar s in the unit u, and the number $\tilde{\alpha}$ is the magnitude of the scalar s in the unit $\tilde{u}$.

A combination of the above expressions gives that

$$\alpha = p\tilde{\alpha} .$$

This expression relates three numbers in F. The magnitude of a scalar converts in a way opposite to the way in which the unit of the scalar set converts. We say that the scalar set is *contravariant*.

For example, kilogram and pound are two units of mass. They convert to each other by

1 kilogram = 2.20462 pounds.

Thus, a 10-pound turkey is 4.5 kilograms.

For the scalar set APPLE, we use a kilogram of apples as a unit, or a pound of apples as a unit, or a dozen of apples as a unit.

Change of unit is busy work. When we change the unit of mass from a pound to a kilogram, we do not change the nature of things. Any change in unit gives a factor of conversion, often an unsightly number. Consider one more example:

1 year = 31536000 seconds.

Change of unit generates busy work for committees and students, but does not change the nature of things. We can use any nonzero element in a scalar set as a unit.

More Examples of Scalar Sets

Gold. Pieces of gold form a set, denoted GOLD. We can add two pieces, but not multiply two pieces. Thus, the set GOLD is *not* a number field.

The set GOLD, however, is a real scalar set. Adding two pieces of gold corresponds to a piece of gold. Multiplying a real number α and a piece of gold corresponds to a piece of gold α times the amount. Pieces of gold are proportional to one another. Any nonzero piece of gold serves as a unit of the scalar set. For example, we can use 1 gram of gold as a unit, or 120 gold atoms as a unit.

Commodities. Apples and gold are examples of commodities. Quantities of each commodity are additive to one another, scalable by real numbers, and proportional to one another. Thus, quantities of each commodity form a real scalar set. This fact enables the application of linear algebra to economics.

Money. Money also forms a real scalar set. Historically, money meant commodities like gold and silver. In modern time, money often takes the form of currencies. The dollars and the euros form different scalar sets. Within the scalar set of dollars, the addition of two amounts means putting the two amounts together. Multiplying a real number α and an amount of dollars corresponds to α times the amount of dollars. Amounts of dollars are proportional to one another.

Money in the United States comes in many units: penny, nickel, dime, quarter, and dollar. These units correspond to distinct physical objects. Note the factors of conversion:

1 penny = 0.01$,

1 nickel = 0.05$,

1 dime = 0.1$,

1 quarter = 0.25$.

Also in common use are banknotes of 1, 5, 10, 20, 50, and 100 dollars. Each of them can also serve as a unit of money.

Goldsilver. Consider a set, denoted GOLDSILVER, each element of which is a piece containing some gold and some silver. Here are some pieces of goldsilver:

$$(\text{2 grams of gold, 3 grams of silver}),$$

$$(\text{200 gold atoms, 500 silver atoms}).$$

Thus, the set GOLDSILVER is the Cartesian product of the two scalar sets APPLE and ORANGE:

$$\text{GOLDSILVER} = \text{GOLD} \times \text{SILVER} .$$

Two pieces of goldsilver can be put together to form another piece of goldsilver. Multiplying a piece of goldsilver by a real number α means finding another piece α times the amounts of gold and silver. Thus, the pieces of goldsilver are additive to one another, and scalable by real numbers. Indeed, the pieces of goldsilver fulfill Axioms 1-11.

But the pieces of goldsilver violate Axiom 12. The pieces containing different proportions of gold and silver are, of course, not proportional to one another. Thus, the pieces of goldsilver do not form a scalar set. We will see later that the pieces of goldsilver form a two-dimensional vector space. Indeed, the Cartesian product of any two scalar sets is a two-dimensional vector space.

Proportional goldsilver. Consider the collection of pieces of goldsilver of the form

$$\alpha(\text{2 gold atoms, 3 silver atoms}),$$

where α is a real number. Pieces in this collection are additive to one another, scalable by numbers, and proportional to one another. The collection satisfies Axioms 1-12, and is a real scalar set.

Ordered pair of real numbers. The example of GOLDSILVER is similar to the collection of ordered pair of real numbers, R^2. Here are some ordered pairs of real numbers

$$(0,0),(1,2),(1,3),(100,200).$$

Ordered pairs of numbers are additive to one another. For example,

$$(2,3)+(100,500)=(102,503).$$

The addition of two pairs requires two additions in parallel: adding the first items in the two pairs, and adding the second items in the two pairs. The addition of two pairs produces another pair. The pair-pair addition fulfills Axioms 1-5.

Ordered pairs of numbers are scalable by numbers. For example,

$$7(2,3)=(14,21),$$

$$\sqrt{2}(3,5)=\left(3\sqrt{2},5\sqrt{2}\right).$$

The multiplication of a number α and a pair requires two multiplications in parallel: multiplying α with the first item of the pair, and multiplying α with the second item of the pair. The multiplication a number and a pair produces another pair, and fulfills Axioms 6-10.

The set of ordered pairs R^2 also fulfills Axiom 11, but violates Axiom 12. Thus, R^2 is not a scalar set. We will learn later that R^2 is a two-dimensional vector space.

But proportional ordered pairs of real numbers form a scalar set. Consider the set

$$S=\left\{s\middle|s=\alpha(1,2),\alpha\in R\right\}.$$

This set, along with the pair-pair addition and the number-pair multiplication defined above, fulfills Axioms 1-12. The set S is a real scalar set.

The above considerations are readily generalized to n-tuples over a number field, F^n.

Intelligence. Like happiness and love, intelligence does not scale. We readily appreciate intelligence in people, animals, or even plants. We even devise tests that purport to measure intelligence quotients (IQ) of individuals. But we cannot model intelligence by a scalar set.

Rational multiples of an irrational number. The set of all numbers of form $p\sqrt{2}$, where p is a rational number, is a rational scalar set.

The set of all numbers of form $p\sqrt{2}+q\sqrt{3}$, where p are q are rational numbers, is not a scalar set. The set obeys Axioms 1-11, but violates Axiom 12.

Complex numbers. The set of all numbers of form bi, where b is a real number and $i=\sqrt{-1}$, is a real scalar set.

The set of all numbers of form $a + bi$, where a are b are real numbers and $i = \sqrt{-1}$, is not a real scalar set. The set obeys Axioms 1-11, but violates Axiom 12. The field of complex numbers, however, is a complex scalar set.

Scalars in Nature*

Like geometry and analysis, algebra has long been a part of the language that describes the world. We have shown that a scalar set models different quantities of the same thing, such as piles of apples, pieces of gold, and amounts of money. Here we look at fundamental scalar sets in Nature.

International System of Units (SI). The International System of Units is built on seven base units:

- second for time
- meter for length
- kilogram for mass
- ampere for electric current
- kelvin for temperature
- mole for amount of substance
- candela for luminous intensity

Of the seven types of quantities, only temperature is not a scalar.

Time. The set of all moments in time is not a scalar set, because it is unclear how we define the addition of two moments, or the multiplication of a moment and a number. However, we can form a scalar set by the following procedure. Mark a particular moment as the reference. The difference of any other moment relative to this reference defines a directed interval, called duration. All durations form a scalar set over the field of real numbers. The addition of two durations x and y is a durations, formed by placing the tail of x at the reference, and placing the tail of y at the head of x. The multiplication of a real number α and a duration x is a duration α times that of x.

Any nonzero duration can serve as a unit of time. The unit of time can be, for example, the duration for a full spin of the Earth, called a day. The modern unit of time is second, defined as 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

Length. Points on a straight line do not form a scalar set, because it is unclear how we define the addition of two points, or the multiplication of a point and a number. However, we can form a scalar set by the following procedure. Mark a particular point on the line as the origin. Any point on the line relative to the origin defines an arrow. Each arrow has a direction and a length.

As we discussed before, the arrows in the line form a scalar set over the field of real numbers. The addition of two arrows x and y is an arrow, formed by placing the tail of x at the origin, and placing the tail of y at the head of x. The multiplication of a real number α and an arrow x is an arrow of length α times that of x.

Any distance can serve as a unit of length. The rod preserved near Paris defines a unit of length, the *meter*. To change one unit of length to another, we need to know the factor of conversion. For example,

1 m = 39.375 inch.

We now know that the speed of light in vacuum is a constant, independent of the direction of propagation and the velocity of the emitter of the light. In a rational system of units, the speed of light in vacuum is set to be 1. Consequently, length and time have the same unit. In this system of units, second and meter are just two units of spacetime. They convert according to

1 s = 299,792,458 m.

The factor of conversion is exact by convention.

Thus, if we use second as a unit of time, and use meter a unit of length, the speed of light in vacuum is

c = 299,792,458 m/s.

This number is exact by convention. Nature creates a beauty. Human beings turn it into a beast.

Mass. Quantities of mass form a scalar set. You might as well regard this statement as a law of physics.

For this scalar set, the unit of mass, kilogram, is the mass of a block metal, called the International Prototype Kilogram (IPK), preserved in a vault located in Sevres, France. Any other mass equals this unit times a real number. For example, 1.7 kg means a mass, which is 1.7 times the mass of the IPK. The mass of the IPK is an element in the scalar set of masses, and by an international convention we agree to call it a unit of mass. The mass 1.7 kg is another element in the set.

Charge. Quantities of electric charge form a scalar set. You should regard this statement as a law of physics. Electric charges are quantized. Every electron carries the same charge. Every proton carries the same charge. The charge on an electron is negative to a charge on a proton.

We can of course use the charge on a proton as a unit for charge, a unit known as the elementary charge. The SI unit for charge is called coulomb, which is defined in a convoluted way. The two units convert as follows:

$$\text{change on one proton} = 1.60217662 \times 10^{-19} \text{coulombs}.$$

Amount. Amounts of a substance form a scalar set. Each element in the set is a certain amount of the substance. We have looked at an example: pieces of gold form a scalar set. If a substance is an aggregate of a single species of atoms, such as gold, we can use one atom as the unit for the scalar set. If a substance is an aggregate of a single species of molecules, such as water, we can use one molecule as the unit of the scalar set.

Of course, any nonzero amount of the substance can serve as a unit. A commonly used unit is mole, defined by the number of atoms in 12 grams of C-12, an isotope of carbon. Here is the factor of conversion between the two units:

$$1 \text{ mole} = 6.022 \times 10^{23} \text{number of atoms (or molecules)}.$$

This factor of conversion is known as the Avogadro constant.

Temperature is not a scalar. Since antiquity, people have been devising ways to map temperature to numbers. Many scientists call the temperature a scalar. This usage is inconsistent with our definition of scalar. Temperatures form an ordered set, but *not* a scalar set. The addition of two temperatures is meaningless. The multiplication of a temperature and a number is also meaningless. Temperatures are not proportional to one another.

By an unfortunate coincidence of English, a system of numerical labeling of temperatures is called a scale of temperature. Of course, temperatures do *not* scale in the sense of being scalars. A scale of temperatures is akin to a scale of earthquakes, or a scale of terrorist threat. Such a scale merely indicates the order of things, not the proportion of things.

The SI unit for temperature is kelvin, defined in a bizarre way, which you can find online. However, a rational definition of the unit for temperature is also available. It relies on the relation between temperature, energy, and entropy:

$$T = \frac{\partial U}{\partial S}.$$

Entropy is a dimensionless number. Consequently, a unit for temperature can be identical to the unit for energy, which is joule.

The two units for temperature convert according to

$$1 \text{ kelvin} \approx 1.38 \times 10^{-23} \text{ joule}.$$

The factor of conversion is called Boltzmann's constant. Why do we honor the three great scientists in this bizarre way?

Luminous intensity. Search online.

Properties of a Substance*

Substance, state, and property. A pure *substance* is an aggregate of atoms or molecules of a single species. Water is a pure substance; it aggregates a single species of molecules, H_2O. Wine is not a pure substance; it aggregates multiple species of molecules.

Let us focus on a pure substance. The substance reaches a *state of equilibrium* when the *properties* of the substance are constant in time. Commonly measured properties include temperature, pressure, amount, mass, volume, energy, and entropy. Of these seven properties, the first five were known in antiquity, but energy and entropy were discovered in the nineteenth century.

Extensive property and intensive property. Properties of a substance have different algebraic structures. Of the seven properties, temperature is not a scalar, but the other six are scalars.

Of the six scalars, pressure is peculiar: it does not share another attribute common to amount, mass, volume, energy, and entropy. The last five properties are proportional to one another when temperature and pressure are fixed. Given a substance at fixed temperature and pressure, when the amount of a substance doubles, the mass, volume, energy, and entropy also double. Any property proportional to the amount of a substance at fixed temperature and pressure is called an *extensive property*. Temperature and pressure are called *intensive properties*.

Every extensive property is a scalar. Some intensive properties are scalars, but others are not.

	scalar	*non-scalar*
extensive	amount, mass, volume, energy, entropy	impossible
intensive	pressure	temperature

An isolated system conserves mass, volume and energy, but maximizes entropy. Of all extensive properties, entropy differs from the rest in one aspect. When a substance is just isolated from the rest of the world, the substance is not in a state of equilibrium. For example, when a half bottle of liquid water is just isolated from the rest of the world, the liquid may move violently, some molecules will vaporize to fill the other half of the bottle, and some molecules might even freeze into ice. After being isolated for some time, the substance approaches a state equilibrium.

As the isolated substance evolves in time, its mass, volume and energy remains constant: these quantities are *conserved*. Entropy of the isolated substance, however, is *not* conserved. As the isolated substance evolves in time, its entropy increases, and attains maximum when the substance reaches a state of equilibrium.

Number Field as Scalar Set

A number field is a scalar set over itself. A number field and a scalar set have different algebraic structures. The axioms of number field and the axioms of scalar set are different, but have many similarities. Indeed, a number field F is a scalar set over itself.

Axioms 1-5 of scalar set are identical to those of number field. Axioms 6-10 of scalar set reduce to those of number field, provided the elements in the scalar set are identical to those in F. For any two elements α and x in F, the multiplication αx is an element in F. Any nonzero number u in F serves as a unit for the scalar set. For any number x in F, there exists a number α in F, such that $x = \alpha u$.

Numerical representation of a scalar set. We have just confirmed that a number field is a scalar set over itself. The converse is not true. A scalar set, in general, is not a number field. As we have noted, pieces of gold form a scalar set, but not a number field.

Even though a scalar set in general is not a number field, we can *represent* every scalar set by a number field. Let S be a scalar set over a number field F. For a fixed nonzero scalar u in S, Axiom 12 says that every element x in S is proportional to u—that is, for every x in S, there exists a number α in F such that $x = \alpha u$. We can make a stronger statement. Given a nonzero u of S, for every x in S, there exists a *unique* number α in F such that $x = \alpha u$. To see this uniqueness, write a scalar x in S as

$$x = \alpha u,$$

$$x = \beta u,$$

where α and β are numbers in F. The difference of the two expressions is

$$0 = (\alpha - \beta) u.$$

This statement implies that

$$\alpha = \beta.$$

Thus, we can specify a scalar set by a nonzero element u and a number field F. We generate all other elements in the scalar set by numerical multiples of u.

Once we fix a nonzero scalar u in S, we have just shown that, for every x in S, there exists a unique α in F such that

$$x = \alpha u.$$

This equation defines a *bijection* between the sets S and F—that is, a one-one correspondence between an element x in S and an element α in F:

$$x \leftrightarrow \alpha.$$

In this sense, the number α in F represents the scalar x in S.

Number-scalar bijection preserves proportion. This bijection, $x = \alpha u$, $x \leftrightarrow \alpha$, preserves proportion. For any number λ in F, λx is also a scalar in S, and the magnitude of this scalar in the unit u is the number $\lambda \alpha$. Write

$$\lambda x = \lambda \alpha u.$$

Consequently, the bijection also associates the scalar λx to its magnitude:

$$\lambda x \leftrightarrow \lambda \alpha.$$

The numerical representation of a scalar set is a proportion-preserving bijection between two scalar sets S and F.

Ordered set and scalar set. The field of real numbers, R, is an ordered set. There exists a bijection between R and real scalar set S. Thus, any real scalar set is an ordered set.

The converse is not true. Not every ordered set is a scalar set. For example, temperatures form an ordered set, but not a scalar set. Addresses of buildings in a street form an ordered set, but not a scalar set. Buildings are not additive to one another, or scalable by numbers, or proportional to one another.

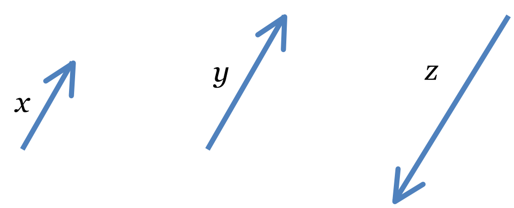
The field of complex numbers, C, is an unordered set. There exists a bijection between C and any complex scalar set. Thus, any complex scalar set is an unordered set.

Arrows in a Line

This scalar set is so important that we single it out for a detailed discussion. Watch how this example specifies the five items $(S,+,0,F,*)$ and fulfills the twelve axioms of scalars. Arrows in a line are important because we use them to represent any real scalar set on a piece of paper.

Arrows form a set. An arrow drawn in a straight line has a length and a direction, pointing from the tail to the head. The collection of all such arrows forms a set S. Two arrows are regarded as the same element in S if one of them can be translated onto the other. We designate each arrow by a symbol, such as x, y, and z.

Of course, arrows do not have to be confined to a straight line, so long as they are parallel, and two arrows are regarded as the same element in S if one of them can be translated onto the other.



We next confirm that the set of arrows S in a straight line is a real scalar set. To save space, we will draw arrows on a horizontal line.

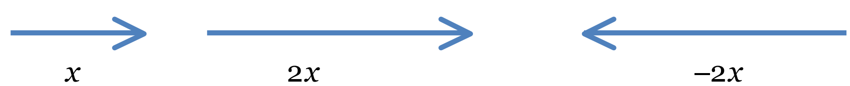
Arrows are additive to one another. To fulfill Axiom 1, define the arrow-arrow addition as follows. For two arrows x and y, translate the arrows to make the tail of y coincide with the head of x. Call the combination the arrow $x+y$, with its tail coinciding with the tail of x, and its head coinciding with the head of y.

We can confirm that the arrow-arrow addition is associative—that is the operation fulfills Axiom 2.

To fulfill Axiom 3, we include in the set S the zero arrow, which has zero length and unspecified direction. To fulfill Axiom 4, for every arrow x, we define $-x$ as an arrow of the same length but in the opposite direction.

The arrow-arrow addition clearly commutes. Indeed, the arrow-arrow addition so defined fulfills Axioms 1-5. That is, the arrows form a commutative group.

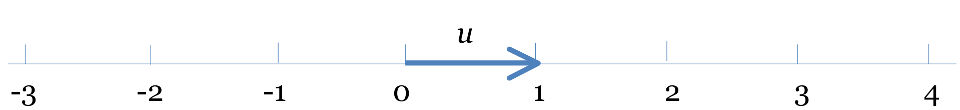
Arrows are scalable by real numbers. Define the number-arrow multiplication as follows. For a positive real number α and an arrow x, we define αx to be a arrow of the same direction as x, and of length α times that of x. For a negative number α, we define αx to be an arrow in the opposite direction of x, of length $|\alpha|$ times that of x. The number-arrow multiplication so defined fulfills Axioms 6-10.



Arrows in a line are proportional to one another. Any nonzero arrow u serves as a unit of the scalar set S. For any arrow x in the line, there exists a real number α, such that $x = \alpha u$.

Graphical representation of a real scalar set. We have just confirmed that arrows in a line form a real scalar set. We now use the arrows to represent any real scalar set.

For example, consider the scalar set GOLD. Choose any nonzero piece of gold as a unit u. For example, u can be one gram of gold. Represent the piece u as an arrow in a line. The line itself is called *coordinate*. The origin of the coordinate represents the zero scalar (i.e., a piece containing no gold). Each arrow from the origin to a point in the coordinate represents a piece of gold. To guide the eye, mark on the coordinate several scalars proportional to u.



Arrow-gold bijection preserves proportion. In representing pieces of gold by arrows in a line, we establish a bijection between the two sets as follows. We choose a piece of gold u as a unit of the scalar set GOLD. We choose an arrow v as a unit of the scalar set of arrows in a line. The bijection associates the unit of GOLD to the unit of ARROW:

$$u \leftrightarrow v.$$

For any real number α, the bijection associates the piece of gold αu to the arrow αv:

$$\alpha u \leftrightarrow \alpha v.$$

The bijection preserves the proportion of the two scalar sets.

Isomorphism

Piles of apples, pieces of gold, field of real numbers, and arrows in a line are things of different types. Yet one type of things can represent another. We now make this representation precise.

Isomorphism. Two scalar sets S and T over a number field F are called *isomorphic* if there exists a bijection between S and T such that, if the bijection associates a scalar s in S with a scalar t in T, then the bijection also associates αs and αt for every α in F. That is,

$$s \leftrightarrow t \text{ implies } \alpha s \leftrightarrow \alpha t.$$

Such a bijection between two scalar sets is called an *isomorphism*.

An isomorphism is not just a bijection between two unstructured sets, but is a bijection of a particular type, taking advantage of S and T being scalar sets. An isomorphism between two sets is a bijection that preserves proportion.

Isomorphism is an equivalence relation. Being isomorphic is an *equivalence relation* on the collection of all scalar sets. We can divide the collection of all scalar sets by a *partition*, each part of which is an *equivalent class*, consisting of scalar sets over the same number field F. Any scalar set over a number field F is isomorphic (equivalent) to F. We may designate each equivalent class by a reprehensive, F.

We have seen the equivalence of the real scalar sets:

$$\text{APPLE} \sim \text{GOLD} \sim R \sim (\text{arrows in a line}).$$

Such is the power of abstraction.

Two scalar sets are isomorphic if and only if they are over the same number field. GOLD is not isomorphic to GOLDSILVER. The former is a real scalar set (a one-dimensional vector space), and the latter is a two-dimensional real vector space. However, GOLD is isomorphic to a collection of proportional pieces of goldsilver. The latter is also a real scalar set.

Of course, all complex scalar sets are isomorphic. But a real scalar set is not isomorphic to a complex scalar set.

Being isomorphic does not mean being the same. We do not confuse a piece of gold with a pile of apples. We often say that we model a quantity, such as distance or duration, by a real variable. What we really mean is that we model distances by one real scalar set, and model durations by another real scalar set.

Commodity and money. Quantities of each commodity form a real scalar set. All commodities are isomorphic. We trade commodities. A piece of cloth can trade for a bushel of wheat. Money is a "universal scalar" to help us trade commodities. Money is isomorphic to every real scalar set. We even say "time is money".

Bean counters abuse isomorphism. Human beings have been obsessive bean counters since antiquity. We use a bag of beans to represent a piece of gold, a parcel of land, or a pile of apples. We count beans whenever we can. Bags of beans form a real scalar set, which is isomorphic to any other real scalar set.

We count beans even when we should not. Subconsciously we judge people by their heights, weights, or salaries. Making bags of beans isomorphic to things in the world is often impossible, because many things in the world do not form scalar sets. Love does not scale. Happiness does not scale.

An intelligence quotient (IQ) is a score obtained by a person on a test purported to map human intelligence to a set of numbers. Various kinds of intelligence do not form a scalar set. They do not even form an ordered set. Some people are good at playing basketballs, others are good at counting scores, and still others are good at making money out of the games. These skills are different kinds of intelligence. They are not additive to one another, or scalable by numbers, or proportional to one another. No isomorphism will ever exist between various kinds of intelligence and the set of real numbers.

We measure the accomplishments of researchers by the numbers of citations to their published papers. The numbers of citations form a scalar set, but the accomplishments of researchers do not. No isomorphism will ever exist between the two sets.

Linear Map between Two Scalar Sets

Two scalar sets S and T over a number field F are isomorphic because there exists a proportion-preserving bijection between the two scalar sets. If the bijection relates an element s in S to an element t in T,

$$s \leftrightarrow t,$$

then the bijection also relates αs to αt,

$$\alpha s \leftrightarrow \alpha t,$$

for every number α in F. Isomorphism is an equivalence relation:

$$S \sim T.$$

Often many proportion-preserving bijections exist between two scalar sets. The above notation does not differentiate different bijections.

Linear map. Let S and T be two scalar sets over a number field F. A map g associates every element s in S to a unique element t in T. Write

$$g: S \rightarrow T,$$

$$t = g(s).$$

The map is a *linear map* if

$$g(\alpha x) = \alpha g(x)$$

for any α in F and any x in S.

An isomorphism is a linear map, but not every linear map is an isomorphism. The definition of linear map allows zero map, a map that sends every element in S to the zero element in T. The zero map is not bijective, and is not an isomorphism. All nonzero linear maps between S and T are isomorphism.

Examples. *Rate of exchange between two currencies*. Amounts of dollars form one real scalar set D, and amounts of euros form another real scalar set E. A rate of exchange between the two currencies is a linear map, $r: D \rightarrow E$. For example, a rate of exchange,

$$r = 0.9 \text{ euros/dolar},$$

means that 0.9 euros exchange with 1 dollar. When the amount of dollars d exchanges with the amount of euros e, we write

$$e = rd.$$

Price of a commodity. Amounts of gold form one real scalar set G, and amounts of dollars form another real scalar set D. A price of gold is a linear map, $p: G \rightarrow D$. A common unit for price of gold is dollars per ounce of gold. The price of gold changes from time to time. Each price is a linear map between the two scalar sets.

Density of a substance. Let M be the set of different masses of gold, and V be the set of different volumes. The linear map $\rho: V \rightarrow M$ is the density of gold. The density of gold changes with temperature.

Atomic mass. Let M be the mass of a piece of gold, and N be the number of gold atom in the piece. The linear map $\mu: N \rightarrow M$ is the atomic mass of gold. The atomic mass of gold is fixed.

Speed. Let T be the set of directed intervals of time, and D be the set of directed segments on a geometric line. The linear map $c: T \rightarrow D$ is the speed.

Power. Let T be the set of directed intervals of time, and E be the set of differences in energy. The linear map $P: T \rightarrow E$ is the power.

Electric current. Let T be the set of directed intervals of time, and C be the set of various amounts of electric charge. The linear map $I: T \rightarrow C$ is the electric current.

Electric potential. Let C be the set of various amounts of electric charge, and E be the set of various amounts of energy. The linear map $\phi: C \rightarrow E$ is the electric potential.

Chemical potential. Let N be the set of various amounts of a species of molecules, and E be the set of various amounts of energy. The linear map $\mu : N \to E$ is the chemical potential.

Magnitude of a linear map. The magnitude of a linear map is relative to the units of the two scalar sets. Let S and T be two scalar sets over a number field F, and $g : S \to T$ be a linear map. The linear map associates an element s in S to an element t in T:

$$t = gs .$$

Let u be a unit of S. An element s in S scales with u, namely,

$$s = s_M u .$$

This expression defines the number s_M in F as the magnitude of the scalar s relative to the unit u. Similarly, let v be a unit of T. An element t in T scales with v, namely,

$$t = t_M v .$$

This expression defines the number t_M in F as the magnitude of the scalar t relative to the unit v.

Because gs is a linear map, we write

$$gs = g\left(s_M u\right) = s_M gu .$$

Note that u is an element in S, so that gu is an element in T, and must scale with v. We write

$$gu = g_M v .$$

This expression maps the unit u of S to the unit v of T. We call the number g_M in F as the magnitude of the linear map in the units u and v.

A combination of the above expressions gives that

$$t_M v = g_M s_M v .$$

The components must equal:

$$t_M = g_M s_M$$

This expression relates the three magnitudes. They are all numbers in F.

Change of units of the two scalar sets. Let u and $\tilde{u}$ be two non-zero scalars in S. The two scalars are proportional to each other:

$$\tilde{u} = pu ,$$

where p is a number in F, and is the magnitude of the scalar $\tilde{u}$ relative to the scalar u. Let v and $\tilde{v}$ be two non-zero scalars in S. The two scalars are proportional to each other:

$$\tilde{v} = rv ,$$

where r is a number in F, and is the magnitude of the scalar $\tilde{v}$ relative to the scalar v.

Recall that $gu = g_M v$ defines the number g_M in *F* as the magnitude of the linear map *g* relative to the two units *u* and *v*. Similarly, $g(\tilde{u}) = \tilde{g}_M \tilde{v}$ defines the number $\tilde{g}_M$ in *F* as the magnitude of the linear map *g* relative to the two units $\tilde{u}$ and $\tilde{v}$. The two magnitudes g_M and $\tilde{g}_M$ are related as

$$\tilde{g}_M = r g_M / p .$$

Thus the magnitude of the linear map is covariant with respect to the unit of *T*, but contravariant with respect to the unit of *S*.

Examples. *Rate of exchange between two currencies*. Say the rate of exchange between dollars and Euros is

$$1 \text{ dollar} = 0.9 \text{ euros} .$$

This rate of exchange between the two currencies is stated when the unit of one currency is a dollar, and the unit of the other currency is a euro.

If we use a dime as a unit for the currency in the US, then the same rate of exchange between the two currencies is written as

$$1 \text{ dime} = 0.09 \text{ euros} .$$

The magnitude of a rate of exchange depends on the units of the two currencies.

Linear map between two sets of numbers. Let *S* be the set of all numbers of form $q\sqrt{2}$, and *T* be the set of all numbers of form bi, where q and b are rational numbers, and $i = \sqrt{-1}$. A particular a linear map $g: S \to T$ associates an element $q\sqrt{2}$ in *S* to an element $1.6qi$ in *T*; that is, $g(q\sqrt{2}) = 1.6qi$. This description is independent of the choice of units of the two scalar sets.

We can also choose units for the two scalar sets, say $u = 2\sqrt{2}$ as a unit of *S*, and $v = 3i$ as a unit of *T*. Write an element *s* in *S* as $s = \hat{s}2\sqrt{2}$, and an element *t* in *T* as $t = \hat{t}3i$, where $\hat{s}$ and $\hat{t}$ are rational numbers, and are the magnitudes of the two scalars relative the two units.

Recall that $g(u) = \hat{g}v$ defines the rational number $\hat{g}$ as the magnitude of the linear map *g* relative to the two units *u* and *v*. Write

$$g(2\sqrt{2}) = (1.6)2i = (1.6)(2/3)(3i),$$

so that the magnitude of the linear map *g* is $\hat{g} = (1.6)(2/3)$.

The collection of all linear maps from one scalar set to another scalar set. Let *S* and *T* be scalar sets over a number field *F*. Let $L(S,T)$ be the collection of all linear maps from *S* to *T*. For every element *s* in *S*, every number

α in *F*, and every linear map $g:S \to T$, $\alpha(gs)$ is an element in *T*. The multiplication of α and *g*, written as αg, is defined by

$$(\alpha g)s = \alpha(gs).$$

That is, the collection $L(S,T)$ is also a scalar set over the number field *F*.

Examples. *Prices of a commodity*. A price of a commodity is the amount money for a unit amount of the commodity. The price need not be fixed. Any two prices are proportional to one another. The collection of all prices of a commodity forms a scalar set.

Maps between numbers. Let *S* be the set of all numbers of form $q\sqrt{2}$, and *T* be the set of all numbers of form bi, where *q* and *b* are rational numbers, and $i = \sqrt{-1}$. Each linear from *S* to *T* associates an element $q\sqrt{2}$ in *S* to an element rqi in *T*, where *r* is a particular rational number. That is, each rational number *r* corresponds to a distinct linear map from *S* to *T*. All these linear maps constitute $L(S,T)$.

Successive linear maps. Let *S*, *T* and *U* be three scalar sets over a number field *F*. Consider two successive linear maps, $g:S \to T$ and $h:T \to U$. We can use the successive maps to define a map $hg:S \to U$.

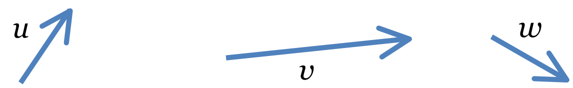
For example, we can sell one commodity for money, and then use the money to buy another commodity. The two successive linear maps give a map that from one commodity to the other commodity.

Complex Scalar Set

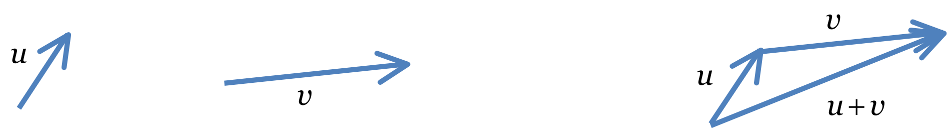
In the above development, we have mostly assumed that the number field is the field of real numbers. Much of the development is applicable to the field of complex numbers, with two significant exceptions. First, real scalar sets are ordered sets, but complex scalar sets are unordered sets. Second, the graphical representation of a real scalar set does not work for a complex scalar set. We have discussed the first difference before, and we now discuss the second difference.

Complex scalar set. Let *S* be a scalar set over the field of complex numbers *C*. We call such a scalar set a complex scalar. We have shown that every scalar set over a number field *F* is isomorphic to *F*. Consequently, every complex scalar set *S* is isomorphic to the field of complex numbers *C*. Since we have represented complex numbers by arrows in a plane, the complex scalar set *S* must be isomorphic to arrows in a plane.

We next create a bijection between the complex scalar set S and the arrows in the plane. We represent each scalar in S by an arrow in the plane, and label each scalar by a symbol, such as u, v, and w.



Complex scalars are additive to one another. The addition of two arrows u and v follows the same rule in geometry. Translate arrow v so that the tail of v coincides with the head of u. The arrow from the tail of u to the head of v gives the arrow $u+v$.

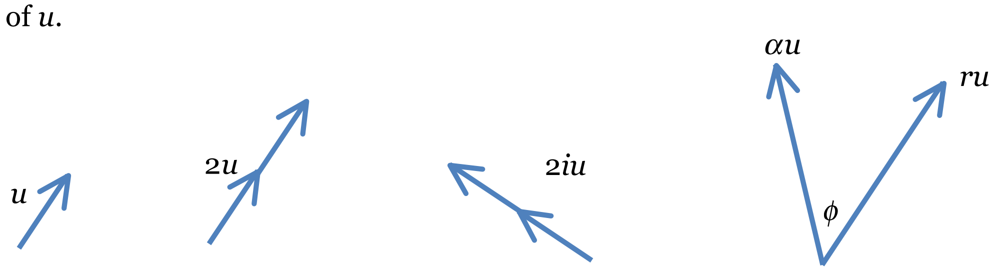


Complex scalars are scalable by complex numbers. Represent a scalar u by an arrow in a plane. The scalar $2u$ is an arrow in the same direction as u and twice the length of u. The scalar $2iu$ is an arrow in the direction rotated 90 degrees from that of u and twice the length of u.

In general write a complex number as

$$\alpha = r\exp(i\phi),$$

where r is modulus of α, and ϕ the phase of α. Thus, the scalar αu is an arrow in the direction rotated by an angle ϕ from that of u, and is of length r times that of u.



Complex scalars are proportional to one another. Once we represent a nonzero scalar u in S as an arrow in a plane, any other scalar v in S is also an arrow in the plane, obtained by scaling by a complex number α:

$$v = \alpha u.$$

By the number-scalar multiplication, the complex number α stretches and rotates the scalar u.

Do not confuse a complex scalar set with a two-dimensional real vector space. A complex scalar set is isomorphic to C, and a two-dimensional real vector space is isomorphic to R^2. The two sets, C and R^2, have different algebraic structures.

Examples of complex scalars. In analyzing oscillatory electrical circuits and mechanical structures, we often represent quantities as complex scalar sets.

Scalars Build Linear Algebra

We have listed the axioms of scalars, used scalars to model things in the real world, and deduced logical consequences of the axioms. We are ready to use scalars to build the virtual world of linear algebra. For example, we will build the following players in this virtual world.

Vector space. The Cartesian product of *n* scalar sets $S_1,...,S_n$ over a number field *F* is called an *n*-dimensional vector space *V* over *F*. Write

$$V = S_1 \times ... \times S_n .$$

Each element *x* in *V* is called a *vector*, and is an *n*-tuple of scalars s_1 in S_1,..., s_n in S_n:

$$x = \left(s_1,...,s_n\right).$$

Vectors are additive to one another. Let $x = \left(a_1,...,a_n\right)$ and $y = \left(b_1,...,b_n\right)$ be two vectors in *V*. Define the vector-vector addition by the scalar-scalar additions:

$$x + y = \left(a_1 + b_1,...,a_n + b_n\right).$$

Vectors are scalable by numbers in *F*. Let $x = \left(s_1,...,s_n\right)$ be a vector in *V* and α be a number in *F*. Define the number-vector multiplication by the number-scalar multiplications:

$$\alpha x = \left(\alpha s_1,...,\alpha s_n\right).$$

In general, vectors are *not* proportional to one another. But we can represent any vector *x* in *V* using units $u_1,...,u_n$ in the scalar sets $S_1,...,S_n$:

$$x = \left(\alpha_1 u_1,...,\alpha_2 u_n\right),$$

where $\alpha_1,...,\alpha_2$ are numbers in *F*.

Linear map. Let *V* and *W* be two vector spaces over a number field *F*. A linear map is a map

$$A : V \to S, \quad x \mapsto A\left(x\right)$$

such that

$$A\left(\alpha x + \beta y\right) = \alpha A\left(x\right) + \beta A\left(y\right)$$

for every *x* and *y* in *V* and every α and β in *F*.

Linear form. Let S be a scalar set and V be an n-dimensional vector space, both over a number field F. A linear form is a map $f:V \to S$ such that

$$f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$$

for every x and y in V and every α and β in F.

Given V, S, and F, we can define many linear forms. The collection of all the linear forms is an n-dimensional vector space over F, called the dual space of V with respect to S.

Bilinear form. Let V and W be two vector spaces and S be a scalar set, all over a number field F. A bilinear form is a map $B:V\times W \to S$ such that $B(v,w)$ is a linear form in v for every fixed w, and a linear form in w for every fixed v.

Tensors. Let V be an n-dimensional vector space over a number field F, and $S_1,...,S_k$ be several scalar sets also over F. From the list $(V,S_1,...,S_k,F)$ we create things—such as forms, dual spaces, and maps—that preserve the two properties: additivity and scalability. These things are collectively called *tensors* over $(V,S_1,...,S_k,F)$.

Tensors are of particular significance in physics. For example, classical physics is built upon a single three-dimensional physical space, along with a number of scalar sets like time, mass, energy, and charge. The vector space, as well as every scalar set, is over the field of real numbers. As we will learn later, other vectors, such as forces and electric fields, are derived from the physical space and scalar sets.

Scalars in Textbooks

Scalars in textbooks of linear algebra. Textbooks of linear algebra use the word "scalar" in two ways. They correspond to two different algebraic structures.

In the first usage, the word scalar is a synonym to the word number, an element in a number field F. As we have seen before, a number field is a set F closed under two operations: addition and multiplication. Adding two elements in F gives an element in F, and multiplying two elements in F gives an element in F.

In the second usage, the word scalar appears in the definitions of linear form, bilinear form, quadratic form, and inner product. In this second usage, the scalar is an element of a set S, which we call a scalar set over a number field F. Adding two elements in S gives an element in S, multiplying an element in S with an element in F gives in an element in S, but multiplying two elements in S does not give an element in S. In fact, the last operation is meaningless in the axioms of scalar set.

Scalars in textbooks of physics. In physics, the word scalar is used to indicate a property like mass, volume, charge, and energy. Quantities of such a property are additive to one another, scalable by numbers, and proportional to one another. The usage in physics is consistent with the second usage in linear algebra, but is inconsistent with the first one in several ways:

- A physical property like mass is more than just a number; it has a unit. A unit of a scalar set is a concrete thing, such as a pile of apples, a piece of gold, and a parcel of land.
- The multiplication defined on a number field makes no sense to a physical quantity like mass: the multiplication of two elements in a number field F gives yet another element in F, but the multiplication of two masses does not give another mass.
- If we regard both mass and volume as elements in a number field F, then we need to assign a meaning to the addition of mass and volume. What does that even mean? There is only one field of real numbers, but there are many real scalar sets. Elements in different scalar sets do not add.

Do not confuse number and scalar. For a scalar set S over a number field F, given a unit u, a scalar x in S scales with the unit, $x = \alpha u$, where the magnitude α is a number in F. For the set GOLD, the scalar is a physical thing, a piece of gold, and the magnitude is a real number. We do not confuse a piece of gold with a real number.

Using the same word scalar in two ways, textbooks of linear algebra confuse two distinct algebraic structures. We will not perpetuate this bad practice. Rather, we will call each element in the field F a number, and will reserve the word scalar for an element in a scalar set.

Gibbs's error. The great American physicist Gibbs was an early developer of vector analysis. Page 1 of his textbook (Gibbs and Wilson, *Vector Analysis*, 1901) gave two definitions:

Definition: A vector is a quantity which is considered as possession *direction* as well as *magnitude*.

Definition: A scalar is a quantity which is considered as possessing *magnitude* but no direction.

Gibbs's distinction between vector and scalar is false. Unfortunately, textbooks have been copying these false statements to this day. We will not perpetuate Gibbs's error. Rather, for us a scalar set is simply a one-dimensional vector space. For example, parallel arrows form a scalar set. Any nonzero arrow in the set has both direction and magnitude. All real scalar sets are isomorphic to arrows in a line. All complex scalar sets are isomorphic to arrows in a plane.

Scientists tend to be too permissive in using the word scalar. Many call a set of quantities a set of scalars if the set maps to a number field. Many call temperature a scalar; for instance, see page 1 of Gibbs's textbook, or just see the Wikipedia entry on scalars. As we have explained, this usage is inconsistent with our definition of scalars. Temperatures are not additive to one another, or scalable by numbers, or proportional to one another. It is odd to call non-scalable quantities scalars.