Homework Assignment 3

Course: ECI 289F (Fall Quarter, 2008)

Due: **December 3, 2008**

Problem 1

Consider the function

$$u(x) = \psi(x) + \delta(x), \quad x \in [-\pi, \pi] \tag{1}$$

where

$$\psi(x) = \exp(-ax^2)\cos(bx), \quad x \in [-\pi, \pi], \tag{2}$$
$$\delta(x) = c(1 + dx^2)\cos(x - 0.5), \quad x \in [-\pi, \pi], \tag{3}$$

where $\delta(x)$ is a "perturbation function", $a = 1.3$, $b = 2.9$, $c = -0.15$, and $d = 0.25$. Let the domain be discretized by seven nodes that are located at: $x_1 = -\pi$, $x_2 = -2$, $x_3 = -1$, $x_4 = 0$, $x_5 = 1$, $x_6 = 2$, $x_7 = \pi$. Now, consider the following approximations:

1. FEM: $u^{FE}(x) = \sum_{i=1}^{7} \phi_i^{FE}(x) u_i$
2. MLS: $u^{MLS}(x) = \sum_{i=1}^{7} \phi_i^{MLS}(x) u_i$ with $\mathbf{p} = \{1,\, x,\, \psi(x)\}^T$. Use a quartic weight function and choose appropriate nodal support sizes in your computations.
3. PUFEM: $u^{PUFEM}(x) = \sum_{i=1}^{7} \phi_i^{FE}\{u_i + a_i\psi(x)\}$.

For FEM and PUFEM, find the best least-squares solution for the coefficients by solving the following problem:

$$\min \sum_{j=1}^{np} [u^h(x_j) - u(x_j)]^2, \tag{4}$$

where np are the number of points chosen in $[-\pi, \pi]$. Pick $np = 50$ or $np = 100$ equi-spaced points. For Case 2, use MLS basis functions that can reproduce all functions in $\mathbf{p}$, with $u_i = u(x_i)$. The functions $\psi(x)$, $\delta(x)$, and $u(x)$ are shown in Fig. 1 within $\Omega = [-3, 3]$.

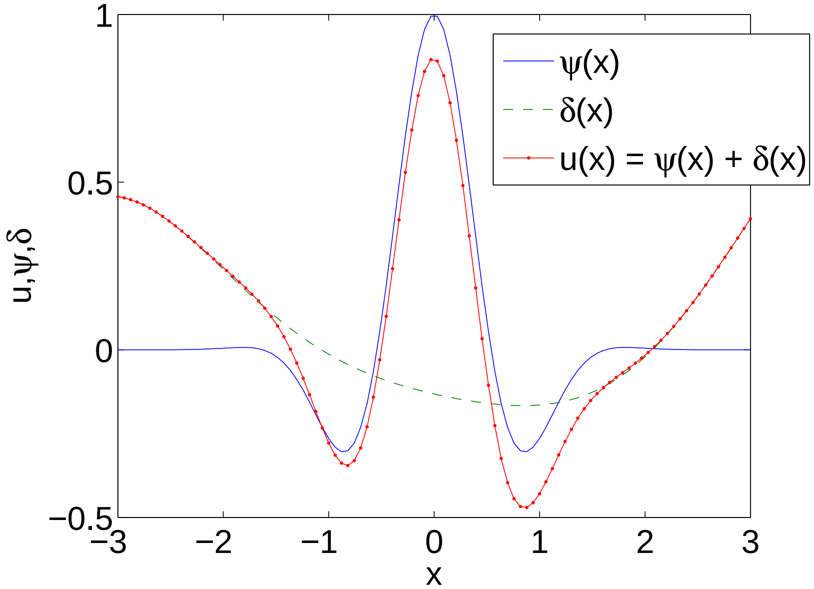

Figure 1: Plots of functions (problem 1).

Problem 2

Consider the following boundary-value problem:

$$u''(x) = \exp(x), \quad x \in (0,1) \tag{5a}$$
$$u(0) = 1, \quad u'(1) = \exp(1), \tag{5b}$$

where exp is the exponential function. The exact solution of the above problem is: $u(x) = \exp(x)$. Let the domain be discretized by one element (two nodes). We write the partition of unity finite element approximation as

$$u^h(x) = \sum_{i=1}^{2} \phi_i(x) \left\{ u_i + \psi(x) a_i \right\}, \tag{6}$$

where the enrichment function $\psi(x) = \exp(x)$ is chosen and both nodes are enriched. To render $u^h(x)$ kinematically admissible, we set $u_1 = 1 - a_1$ so that $u^h(0) = 1$ (must satisfy essential boundary conditions). Consider the functional $\Pi[u]$ that corresponds to the above BVP ($\delta\Pi = 0$ would lead to the variational/weak form). Now, substitute the trial function and its derivative in $\Pi[u]$ so that $\Pi[u^h] \equiv \Pi(a_1, u_2, a_2)$.

1. By minimizing $\Pi(a_1, u_2, a_2)$ with respect to the three coefficients, show that you obtain $u_1 = 0$, $u_2 = 0$, $a_1 = 1$, and $a_2 = 1$, and hence $u^h(x) = u(x)$ (exact solution is obtained). The algebra will be simplified if you first take the derivative of Π

with respect to a_1, u_2, and a_2, and then compute the definite integrals. To cross-check your calculation, here's an interim result that you should obtain—in the first equation, the term $\dfrac{\partial}{\partial a_1} \int_0^1 e^x u^h(x)\, dx = 1/4(5 - 4e + e^2)$.

2. Instead of the 'correct' enrichment function, if $\psi(x) = \exp(2x)$ is used, then obtain the corresponding numerical solution and compare it to the exact solution on a plot. Proceed as you did for the earlier case. Once you solve for the coefficients, you can substitute them in Eq. (6) to obtain the numerical solution.

Problem 3

Consider the following boundary-value problem:

$$-u''(x) = f(x), \quad x \in (-1, 1) \tag{7a}$$

$$u(-1) = 0, \quad u'(1) = -\frac{3}{2}, \tag{7b}$$

where $f(x)$ is chosen such that the exact solution is:

$$u(x) = \psi(x) - \frac{x^2}{2} - \frac{x}{2}, \quad \psi(x) = \exp(-\alpha x^2), \tag{8}$$

and $\alpha >> 1$ is a constant. The partition of unity finite element (PUFE) approximation is written as

$$u^h(x) = \sum_{i \in I} \phi_i(x) u_i + \sum_{i \in J} \phi_i(x) \psi(x) a_i, \tag{9}$$

where $\phi_i(x)$ are piece-wise linear finite element basis functions, $\psi(x)$ is an enrichment function, and a_i are additional nodal degrees of freedom. Use $\alpha = 1000$ and $\alpha = 10000$ in the numerical computations. To solve the BVP, use the following:

1. Linear finite elements.

2. PUFE approximation given in Eq. (9). For the PUFEM, the set J ($J \subset I$) consists of nodes that are enriched. Use a mesh such that a node is located at $x = 0$ (even number of elements). Run the problem with the minimal set J (just one node). On increasing the number of nodes in J, are you able to compute the numerical solution for the above values of α. Why or why not?

For one coarse grid and one fine grid, plot the FE and PUFE solutions for u and u'. Also compute the relative L_2 error norm, $E = ||u - u^h||_2/||u||_2$, with increasing number of nodes (decreasing mesh spacing, h).

Tasks

Prepare a short report with particular emphasis on the key observations and findings for each problem. Include numerical results (derivation, plots, etc.) that will support your findings and conclusions.

Project Topics (Incomplete List)

Here are a few additional problems that you can pursue in the area of meshfree and partition-of-unity methods. You can pick one of the following, or an alternative one of your choosing:

(a) Implement any meshfree method for Euler-Bernoulli beams.

(b) Implement a 2D MLS code for scattered data approximation.

(c) Solve the Poisson equation ($-\nabla^2 u = f$) with zero Dirichlet boundary conditions using the EFG or max-ent meshfree method.

(d) Use the partition of unity finite element method with Heaviside enrichment to model a crack that is aligned with the boundary of the element (crack lies along an element edge and originates as well as terminates at a node). The domain is a unit square and use rectangular four-node elements to construct the finite element basis functions.

(e) Study potential improvements in the condition number of the linear system when enriched bases are used (via orthogonalization of the enriched bases for instance).

(f) Use of partition of unity finite elements to solve (i) a boundary layer problem that arises in fluids, (ii) a problem in solid/geo/fracture-mechanics that admits singular solution for the stress or strain fields, (iii) wavefunctions for the harmonic oscillator, or (iv) eigenfrequencies of a rod or beam (will need higher-order consistency) under Dirichlet boundary conditions.

For (b) and (c), the domain is a square, which is discretized by a set of nodes (can use a mesh generator, a random number generator, etc.). For the data approximation/surface fitting problem, you can test your code on a few functions $u(\mathbf{x})$ of your choice (there is a paper by Franke that has some examples). For the PDE application, you can pick any BVP with a closed-form solution and any of the methods discussed in class to impose the essential boundary conditions for meshfree methods. With the PUFE code, you will be able to demonstrate (starting point) how crack modeling can be performed independent of the finite element mesh. To keep it simple, assume the square plate is loaded under uniaxial tension. You will obtain a displacement solution that is discontinuous across the crack, and with mesh refinement, the stresses in the vicinity of the crack-tip will increase. For this case, you can compare your PUFE solution to a standard FE solution where the crack is explicitly meshed—both solutions will be identical.