NUMBER

Textbooks of linear algebra confuse two distinct algebraic structures: numbers and scalars. We eradicate this confusion by comparing the axioms of numbers and axioms of scalars. These notes specify numbers, and separate notes will specify scalars.

Axioms of Number Field

A set F is called a *number field* if the following conditions hold.

1. To every two elements α and β in F, there corresponds a unique element in F, written $\alpha+\beta$. The binary map, $+: F\times F \to F$, $(\alpha,\beta)\mapsto \alpha+\beta$, is called the *number-number addition*. That is, elements in F are *additive*, and the set F is *closed* under the addition.
2. $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ for every α, β and γ in F. That is, the scalar-scalar addition is *associative*.
3. There exists an element in F, written 0, such that $0+\alpha=\alpha$ for every α in F. That is, there exists an *identity element* for the number-number addition.
4. For every α in F, there exists an element γ in F, called the *negative element*, such that $\alpha+\gamma=0$. That is, numbers can *subtract*. We also write $\gamma=-\alpha$.
5. $\alpha+\beta=\beta+\alpha$ for every α and β in F. That is, the number-number addition is *commutative*.
6. To any two elements α and β in F there corresponds an element in F, written $\alpha*\beta$, or simply $\alpha\beta$. The binary map, $*: F\times F\to F$, $(\alpha,\beta)\mapsto\alpha\beta$, is called the number-number multiplication.
7. $(\alpha\beta)\gamma=\alpha(\beta\gamma)$ for every α, β and γ in F. That is the number-number multiplication is associative.
8. There exists an element in F, written 1, such that $1*\alpha=\alpha$ for every α in F. That is, there exists an *identity element* for the number-number multiplication.
9. For every $\alpha\neq 0$ in F, there exists an element γ in F, called the *inverse element*, such that $\alpha\gamma=1$. That is, numbers can *divide*. We also write $\gamma=1/\alpha$.
10. $\alpha\beta=\beta\alpha$ for every α and β in F. That is, the number-number multiplication is commutative.
11. $\gamma(\alpha+\beta)=\gamma\alpha+\gamma\beta$ for every α, β and γ in F. That is, the number-number multiplication *distributes* over the number-number addition.

Remarks. We call each element in the set F a *number*. Addition and multiplication are two distinct operations. Each operation turns two elements in F into an element in F. That is, each operation is a bilinear map, $F \times F \to F$. We say that the set F is *closed* under the two operations. We need to memorize nothing new: the two operations follow the usual arithmetic rules of addition, subtraction, multiplication, and division.

Examples of Number Fields

The smallest field. The definition of field explicitly mentions two elements 0 and 1. If we define addition by an unusual rule, 1 + 1 = 0, we can confirm that the set $\{0,1\}$ satisfies all the axioms of field. This set is the smallest field.

Integers. The set of integers is *not* a field. Here we assume the usual operations of addition and multiplication. The set of integers violates Axiom 9: the set of integers is *not* closed under division.

Rational, real, and complex numbers. The set of rational numbers Q is a field. The set of real numbers R is a field. The set of complex numbers C is a field. We follow the usual arithmetic rules of addition, subtraction, multiplication, and division.

The field of complex numbers contains the field of real numbers. The field of real numbers contains the field of rational numbers.

Irrational numbers. The set of irrational numbers is not a field. The set of irrational numbers is *not* closed under addition and multiplication. For example, the addition of the two irrational numbers $1+\sqrt{2}$ and $1-\sqrt{2}$ results in a rational number. The multiplication of the two irrational numbers $1+\sqrt{2}$ and $1-\sqrt{2}$ also gives a rational number.

R². The collection of all ordered pairs of real numbers is the Cartesian product, R^2. Each element in the set R^2 is an ordered pair of real numbers, (x,y). We do know how to add two elements in R^2, but we do not have a definition for the multiplication of two elements in R^2. Thus, R^2 is not a field.

Gold. Each element in a set is a piece of gold of some amount. We define the addition of two pieces in the set by melting them together, resulting in a piece in the set. However, we do not have a sensible definition of the multiplication that makes the multiplication of two pieces into another piece. This set is not a field.

Root. A set consists of all numbers of the form $\alpha+\beta\sqrt{2}$, where α and β are rational numbers. We adopt the usual definition of addition and multiplication for real numbers. This set is a number field. This field is contained in the field of real numbers.

More roots. A set consists of all numbers of the form $\alpha\sqrt{2}+\beta\sqrt{3}+\gamma\sqrt{5}$, where α, β and γ are rational numbers. We adopt the usual definition of addition and multiplication for real numbers. The multiplication of two elements in the set does not always give another element in the set. This set is not a number field.

Field of Complex Numbers

Complex numbers. The set of complex numbers is a field. Here we gather main properties complex numbers.

A complex number is an ordered pair of real numbers, (x,y), but has more structures. Write a complex number as

$$z=x+iy .$$

Here i is the *imaginary unit*, satisfying that

$$i^2=-1 .$$

For example, $2+3i$ is a complex number, 2 is a real number, and $3i$ is an imaginary number

Write the real and imaginary parts of a complex number as

$$x=\mathrm{Re}(z),$$

$$y=\mathrm{Im}(z).$$

For example,

$$\mathrm{Re}(2+3i)=2,$$

$$\mathrm{Im}(2+3i)=3.$$

Given a complex number $z=x+iy$, its complex conjugate is written as

$$\bar{z}=x-iy .$$

Note two identities:

$$\mathrm{Re}(z)=\frac{z+\bar{z}}{2},$$

$$\mathrm{Im}(z)=\frac{z-\bar{z}}{2i}.$$

The set of complex numbers obeys the axioms of field. The addition of two complex numbers works just as the addition of ordered pair of real numbers. For example,

$$(1+2i)+(3+5i)=4+7i\,.$$

The multiplication of two complex numbers works just as the multiplication of real numbers, but using the identity $i^2=-1$. For example,

$$(1+2i)(3+5i)=(1)(3)+(1)(5i)+(2i)(3)+(2i)(5i)=3+5i+6i-10=-7+11i\,.$$

The division of complex numbers works as follows:

$$\frac{x+iy}{u+iv}=\frac{(x+iy)(u-iv)}{(u+iv)(u-iv)}=\frac{(xu+yv)+i(yu-xv)}{u^2+v^2}\,.$$

The set of complex numbers as the Cartesian product of the set of real numbers. A complex number is an ordered pair of real numbers. Thus, the set of complex numbers, *C*, inherits all properties of the Cartesian product of the set of real number, R^2.

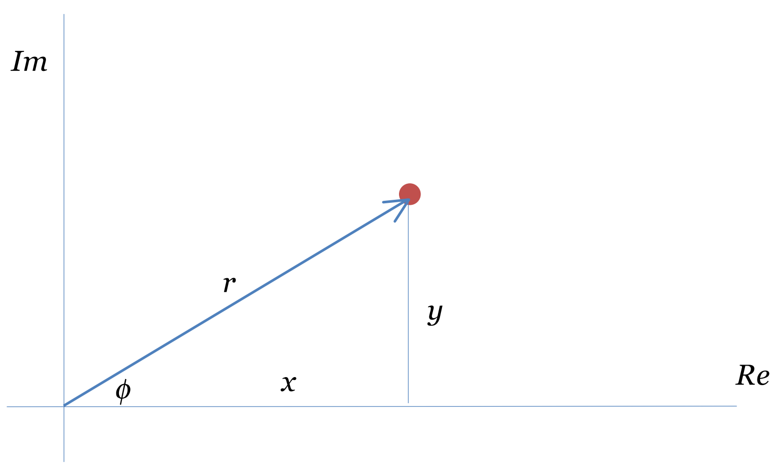
In a plane, use the horizontal axis to represent the real part, and the vertical axis to represent the imaginary part.

- Each complex number corresponds to a point in the plane, or the arrow from the origin to the point.
- The addition of two complex numbers follows the rule of parallelogram.
- The multiplication of a complex number z and a real number α gives another complex number αz. When $\alpha>0$, the two complex numbers z and αz have the same direction, and their lengths differ by a factor of α.

Polar representation. Write

$$z=x+iy\,.$$

Here z is a complex number, x is its real part, and y is its imaginary part. The ordered pair (x,y) is a point in the plane R^2. The arrow from the origin to the point (x,y) represents the complex number z.



The length of the arrow is

$$r = \sqrt{x^2 + y^2} \, ,$$

The length is called the *modulus* of the complex number z. The angle from the x-axis to the arrow, ϕ, is called the *phase* of the complex number z.

The definition of the trigonometric functions gives that

$$\cos\phi = \frac{x}{r}, \quad \sin\phi = \frac{y}{r} .$$

Consequently, we can rewrite $z = x + iy$ as

$$z = r\left(\cos\phi + i\sin\phi\right) .$$

This expression is called the *polar representation* of the complex number.

Recall Euler's formula:

$$\exp\left(i\phi\right) = \cos\phi + i\sin\phi \, .$$

Thus, we can also write the polar representation as

$$z = r\exp\left(i\phi\right) .$$

Example. Calculate the real and imaginary parts of the number 2^{3i}.

$$\begin{aligned} 2^{3i} &= \exp\left(\log\left(2^{3i}\right)\right) \quad \text{The two functions exp and log are inverse to each other} \\ &= \exp\left(3i\log 2\right) \quad \text{Use a property of the log function} \\ &= \cos\left(3\log 2\right) + i\sin\left(3\log 2\right) \quad \text{Use Euler's formula} \end{aligned}$$

The real part is $\cos\left(3\log 2\right)$, and the imaginary part is $\sin\left(3\log 2\right)$.

Multiplication of two complex numbers. The set of complex numbers C differs from R^2 in an important way: C is a field, but R^2 is not. The multiplication of two complex numbers is still a complex number. By contrast, we do not have a sensible definition of multiplication of two elements in R^2.

Let us examine the multiplication of two complex numbers in their polar representations. Consider two complex numbers:

$$z_1 = r_1 \exp\left(i\phi_1\right),$$

$$z_2 = r_2 \exp\left(i\phi_2\right).$$

Their multiplication is

$$z_1 z_2 = r_1 r_2 \exp\left(i\phi_1 + i\phi_2\right).$$

Here we have used a property of the exponential function:

$$\exp\left(i\phi_1\right)\exp\left(i\phi_2\right) = \exp\left(i\phi_1 + i\phi_2\right).$$

Thus, the multiplication of two complex numbers results in a complex number, whose magnitude is the product of the magnitudes of the two complex numbers, and whose phase is the addition of the phases of the two complex numbers.