NONLINEAR VISCOSITY

Nonlinear viscosity. A purely viscous fluid has no memory. When the state of stress changes, the rate of deformation changes instantly, with no delay. A model of viscosity specifies a relation between the state of stress and the rate of deformation. Water is viscous: the state of stress is linear in the rate of deformation.

Ice is also viscous, but the relation between the state of stress and the rate of deformation is nonlinear. The flow of ice contributes to the dynamics of glaciers. Nonlinear viscosity prevails in most materials. Think of ice creams, skin creams, greases, toothpastes, chocolates, polymers, ceramics, and metals.

Why do we study nonlinear viscosity? We study nonlinear viscosity for its own sake, and for the insight into more complex rheological behavior. Rheology is a study of unity, as well as diversity. Viscosity, plasticity and viscoplasticity have the same microscopic origin: atoms and molecules change neighbors. The continuum theories of viscosity, plasticity and viscoplasticity have an identical structure (Prager 1961). These theories must feed into the theories of more complex rheological behavior, such as elastoplasticity and viscoelasticity (Reiner 1945).

Nonlinear viscosity highlights, in purest forms, many salient features of rheology: deformation of arbitrary magnitude, dissipation of energy, use of invariants, use of convex functions, and concern over the uniqueness of solution.

Homogeneous deformation of a small piece. Inhomogeneous deformation of a body. In a test such as pure shear, a material flows by *homogeneous deformation*. The test determines the relation between the shearing stress and the rate of shear, known as the *flow curve*.

The flow curve, together with the balance of forces and compatibility of deformation, governs the flow in a pipe, squeezing of a film, and lubrication between parts in a machine. In these examples, the body of fluid flows by *inhomogeneous deformation*. We regard the body as a sum of many small pieces. Each small piece undergoes a homogeneous deformation. In these examples, each small piece deforms by shear, just as that in the test of pure shear. The shearing flows of the small pieces constitute the inhomogeneous flow of the body.

In a more complex flow, each small piece still undergoes a homogeneous deformation, but the rate of deformation is a tensor of all its components. To analyze inhomogeneous deformation in general, we need a relation between the rate of deformation and arbitrary state of stress.

The second-invariant model of viscosity. A particularly popular relation between the state of stress and the rate of deformation is the *second-invariant model of viscosity*, also known as the generalized Newton's model of viscosity, or the J_2 model. The model assumes that the state of deviatoric stress equals a scalar times the rate of deformation, and that the scalar is a function of the second invariant of the rate of deformation. The scalar function of a scalar is fixed by using a flow curve (i.e., a relation between the stress and the rate of deformation), measured by subject the material to a simple test, such as pure shear and uniaxial tension. The model predicts all components of the rate of deformation under any state of stress, or the other way around. The second-invariant model readily accommodates viscoplasticity, i.e., the Bingham fluids.

Power-law creep. Ilyushin theorem. Very viscous fluids flow very slowly. They *creep*. Neglecting the effect of inertia, the balance of forces, as well as the compatibility of geometry, gives linear equations. The only nonlinearity comes from the flow curve. When the flow curve obeys the power law, creeping flows obey a scaling relation (Ilyushin 1946).

Theoretical properties of the second-invariant model. The second-invariant model has several significant theoretical properties. The model, introduced to plasticity by von Mises (1928), is a special case of the theory of dissipation function (Rayleigh 1871). The model satisfies the thermodynamic inequality. The model has a convex dissipation function and a convex flow potential. (We briefly outline the mathematics of convex functions.) The solution to the boundary-value problem of any creeping flow is unique (Hill 1956). The boundary-value problems obey variational principles.

Linear Viscosity

Linear, isotropic, incompressible, viscous fluid. Let us briefly review linear viscosity. Consider a fluid deforming homogeneously, at a state of stress, σ_{ij}, and at a rate of deformation, D_{ij}. The mean stress is $\sigma_m = \sigma_{kk}/3$, and the deviatoric stress is $s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$. By a linear, isotropic, incompressible, viscous fluid we mean a model specified by

$$D_{kk} = 0,$$

$$s_{ij} = 2\eta D_{ij},$$

So long as η, the viscosity, is independent of the rate of deformation, the relation between the deviatoric stress and the rate of deformation is *linear*.

Given a state of stress, σ_{ij}, the model predicts all components of the rate of deformation, D_{ij}. Given a rate of deformation, D_{ij}, the model predicts all components of the deviatoric stress, s_{ij}, but not the mean stress, σ_m. The model says that superimposing a hydrostatic stress does not affect deformation.

The expression $D_{kk} = 0$ ensures *incompressibility*. When we change the basis of the space, the individual components of the rate of deformation change, but all the coefficients in the equation $D_{kk}=0$ remain unchanged. Such an equation describes an isotropic behavior.

We next examine the expression $s_{ij}=2\eta D_{ij}$. When we change the basis of the space, the individual components of the deviatoric stress and those of the rate of deformation change, but the coefficient η remains unchanged. Consequently, the equation $s_{ij}=2\eta D_{ij}$ also describes an isotropic behavior.

The expression $D_{kk}=0$ represents a single equation:

$$D_{11}+D_{22}+D_{33}=0.$$

The expression $s_{ij}=2\eta D_{ij}$ represents six equations:

$$\sigma_{12}=2\eta D_{12},$$
$$\sigma_{23}=2\eta D_{23},$$
$$\sigma_{31}=2\eta D_{31},$$
$$\sigma_{11}-\frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}=2\eta D_{11},$$
$$\sigma_{22}-\frac{\sigma_{22}+\sigma_{33}+\sigma_{11}}{3}=2\eta D_{22},$$
$$\sigma_{33}-\frac{\sigma_{33}+\sigma_{11}+\sigma_{22}}{3}=2\eta D_{33}.$$

The sum of the last three equations gives an identity, 0 = 0. Thus, the last three equations consist of only two independent equations. The two types of relations, $D_{kk}=0$ and $s_{ij}=2\eta D_{ij}$, consist of a total of six independent linear equations between the twelve components of stress and rate of deformation.

Thermodynamic inequality. The state of stress, σ_{ij}, is due to an externally applied force. When the fluid flows at the rate of deformation, D_{ij}, the applied force changes its potential energy at the rate $-\sigma_{ij}D_{ij}V$, where V is the volume of the fluid.

The fluid is incompressible, and is in thermal equilibrium with a heat bath of a fixed temperature. Once the values of the two thermodynamic properties—the temperature and the volume—are fixed, the fluid is in a fixed thermodynamic state, with all its thermodynamic properties fixed, even as the fluid flows. This statement applies to a purely viscous fluid: its thermodynamic state is unaffected by the amount of deformation. In particular, the Helmholtz free energy of the fluid is constant.

The fluid and the applied force together constitute a composite thermodynamic system. The composite system exchanges energy with the heat bath, held at a fixed temperature. The Helmholtz free energy of the composite system sums over its parts (i.e., the fluid and the applied force): $0 - \sigma_{ij} D_{ij} V$. Thermodynamics requires that the Helmholtz free energy of the composite system should never increase:

$$\sigma_{ij} D_{ij} \geq 0 .$$

During the isothermal flow, the fluid does not change its own Helmholtz free energy, but converts the potential energy of the applied force to the energy in the heat bath. The fluid, per unit volume, per unit time, dissipates energy $\sigma_{ij} D_{ij}$, a scalar which we call the *rate of dissipation*.

In an incompressible fluid, the rate of dilation vanishes, $D_{kk} = 0$, so that

$$\sigma_{ij} D_{ij} = s_{ij} D_{ij} .$$

Only the deviatoric stress dissipates energy; the mean stress does not. The thermodynamic inequality becomes that

$$s_{ij} D_{ij} \geq 0 .$$

This thermodynamic inequality applies when an incompressible, viscous fluid flows at a fixed temperature. The inequality holds without requiring linearity and isotropy.

The model of linear viscosity satisfies the thermodynamic inequality. For a linear, isotropic, incompressible, viscous fluid, the rate of dissipation is

$$s_{ij} D_{ij} = 2\eta D_{ij} D_{ij} .$$

Note that the scalar $D_{ij} D_{ij}$ is positive-definite. The model satisfies the thermodynamic inequality for arbitrary rate of deformation if and only if the viscosity is non-negative:

$$\eta \geq 0 .$$

This thermodynamic inequality places a familiar constraint on the model of linear viscosity. For instance, in a shearing flow, the inequality simply says that the direction of the shearing stress must be in the same direction of the rate of shear.

Inhomogeneous deformation. When a body of material undergoes inhomogeneous deformation, we regard the body as a sum of many small pieces. Each small piece underdoes homogeneous deformation, and the relation between the state of stress and the rate of deformation specifies the rheology of material. Different pieces in the body communicate through the compatibility of deformation and the balance of forces. The three ingredients translate to the following equations.

Rheology of material. The model of linear viscosity relates the state of stress and the rate of deformation:

$$D_{kk} = 0,$$

$$s_{ij} = 2\eta D_{ij},$$

where the deviatoric stress is $s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$, and the mean stress is $\sigma_m = \sigma_{kk}/3$.

Compatibility of deformation. Let $\mathbf{x}$ be the coordinate of a place in space, t be the time, and $v_i(\mathbf{x},t)$ be the velocity of a small piece of fluid at the place $\mathbf{x}$ and time t. The compatibility of deformation relates the rate of deformation to the field of velocity:

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

This relation between the rate of deformation and velocity holds for deformation of any magnitude. This relation is used in the theories of viscosity, plasticity, and viscoplasticity (Prager 1961). The relation resembles that between the strain and the displacement in elasticity of infinitesimal deformation.

Balance of forces. Let $b_i(\mathbf{x},t)$ be the body force per unit volume, and ρ be the mass per unit volume. For an incompressible fluid, ρ is constant. Inside the body, the balance of forces relates the stress to the body force and the inertial force:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right).$$

The right-hand side is the inertial force per unit volume.

Let $n_i(\mathbf{x},t)$ be the unit vector normal to a small part of the surface of the body. Let $t_i(\mathbf{x},t)$ be the traction, i.e., the force per unit area acting on the surface. On the surface of the body, the balance of forces relates the stress to the traction:

$$\sigma_{ij} n_j = t_i.$$

Boundary conditions. We divide the surface of the body into two parts. On one part of the surface, S_v, we prescribe the velocity of the fluid. On the other part of the surface, S_t, we prescribe the traction. These boundary conditions, along with the equations listed above, constitute a boundary-value problem that governs the inhomogeneous deformation in the body.

Creep. Very viscous fluids flow very slowly. They *creep*. Neglecting inertia, we balance forces by

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 .$$

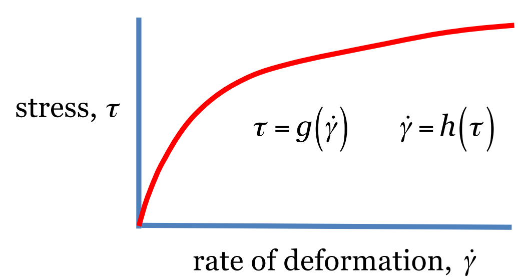
In an entertaining talk, Purcell (1977) described some of the salient features of creep. He focused on linear viscosity, but much of his description also applies to nonlinear viscosity.

What need be done for nonlinear viscosity? All our work on linear viscosity pays off when we study nonlinear viscosity. We only need modify a single equation, $s_{ij} = 2\eta D_{ij}$; all other equations remain the same. As we will see later, even the equation $s_{ij} = 2\eta D_{ij}$ remains intact in a most commonly used model of nonlinear viscosity: all we will really do is to make the viscosity depend on the rate of deformation. Let us first look at nonlinear viscosity in flows of two special types, shear and extension.

Nonlinear Viscosity of Shear

Shearing flow curve. We test a material under shear, and measure the relation between the shearing stress τ and the rate of shear $\dot{\gamma}$. We plot the experimental data as a $\tau - \dot{\gamma}$ curve, known as the *flow curve*. The fluid is linearly viscous if the flow curve is a straight line from the origin, shear-thickens if the flow curve bends above a straight line, and shear-thins if the flow curve bends below a straight line. The slope of the flow curve defines the viscosity. For a nonlinear flow curve, the viscosity depends on the rate of shear.

In many cases the flow curve is monotonic, providing a one-to-one relation between the shearing stress and the rate of shear. We represent the flow curve as a function $\tau = g(\dot{\gamma})$, or a function $\dot{\gamma} = h(\tau)$.



We often fit the experimentally measured flow curve to a power law:

$$\tau = A\dot{\gamma}^N ,$$

where A and N are parameters used to fit the experimentally measured flow curve. The power law is also written as

$$\dot{\gamma} = \left(\frac{\tau}{A} \right)^n ,$$

where $n = 1/N$. For example, a representative values for ice is $n = 3$ (Glen 1955).

Flow in a pipe. A fluid flows in a pipe of a circular cross-section, radius a and length l, subject to pressure p_H and p_L at the two ends of the pipe. The flow is invariant along the length of the pipe, but is inhomogeneous in the radial direction: the shear is large at the wall of the pipe, and vanishes at the centerline of the pipe. This inhomogeneous flow involves shear only.

Scaling analysis. When the fluid flows steadily in the pipe, the velocity of the fluid v is invariant along the length of the pipe, but varies from point to point in each cross section of the pipe. The volume of fluid crossing any section of the pipe per unit time defines the *rate of flow*. The rate of relates to the velocity of the fluid as

$$Q = \int v dA .$$

The integral extends over the cross-sectional.

The pressure in the fluid varies linearly along the length of the pipe, but is constant within each cross section of the pipe. Denote the gradient of pressure by

$$G = \frac{p_H - p_L}{l} .$$

Because the flow is invariant along the length of the pipe, the length and the difference in pressure affect the flow through the gradient of pressure, G.

We now relate the rate of flow, Q, to the gradient of pressure, G. The field in the fluid is governed by a boundary-value problem of a single length scale, a, and of no time scale. The rate of deformation scales with the ratio Q / a^3, and the shearing stress scales with Ga. The two quantities relate to each other through a function:

$$\frac{Q}{a^3} = F(Ga) .$$

This scaling analysis shows how the radius of the pipe scales the G-Q relation.

When the fluid is linearly viscous, of viscosity η, we write the above scaling relation more explicitly as

$$\frac{Q}{a^3} \propto \frac{Ga}{\eta} .$$

The scaling analysis leaves the pre-factor undetermined. In this case, the pre-factor is a dimensionless constant. The same scaling relation also holds for a pipe of cross section of any shape. In that case, a is a length characteristic of the cross section, and the pre-factor is a dimensionless function of ratios that specify the shape of the cross section.

When the fluid obeys the power law, $\dot{\gamma} = \left(\tau / A\right)^n$, we write the scaling relation as

$$\frac{Q}{a^3} \propto \left(\frac{Ga}{A}\right)^n .$$

This scaling relation can be justified using the Ilyushin theorem, which we will study later. The scaling analysis does not determine the pre-factor, which should be a dimensionless function of n and ratios that specify the shape of the cross section. The rate of flow of a shear-thinning fluid (n > 1) increases more rapidly with the radius of the pipe and the gradient of pressure than that of a linearly viscous flow.

We next analyze the inhomogeneous flow using the three ingredients.

Balance of forces. Picture a part of the fluid, a cylinder of radius r and length l, in a free-body diagram. On the surface of the cylinder, the shearing stress has a constant magnitude, τ, pointing in the direction that balances the difference in pressure at the two ends of the cylinder:

$$2\pi r l \tau = \pi r^2 \left(p_H - p_L\right).$$

Rearranging, we obtain that

$$\tau = \frac{r\left(p_H - p_L\right)}{2l}.$$

Thus, the balance of forces determines the distribution of stress in the fluid. Such a boundary-value problem is *statically determinate*. For a thin and long cylinder, $r << l$, the shearing stress is much smaller than the difference in pressure. The small shearing stress over a larger area balances the large difference in pressure over a small area, a situation known as *shear lag*.

Write the above expression as

$$\tau = \frac{rG}{2}.$$

The shearing stress is linear in the radial distance, vanishes at the centerline of the pipe, and is largest at the wall of the pipe.

Compatibility of deformation. The velocity of each fluid particle directs in the axial direction. The flow is inhomogeneous, and the velocity varies with the radial distance, $v\left(r\right)$. We assume that the fluid at the wall of the pipe does not slip, $v\left(a\right) = 0$. The gradient of velocity gives the rate of shear:

$$\dot{\gamma} = -\frac{dv}{dr}.$$

When the high pressure is at the left end of the pipe, the fluid flows toward the right, and the rate of shear $\dot{\gamma}$ points toward the left. Here we use this set of directions as positive directions for these quantities. This ad hoc sign convention for the rate of shear differs from the commonly used sign convention. In this ad hoc convention, we need to place a negative sign in the above expression.

Rheology of fluid. The fluid flows in the pipe by inhomogeneous deformation, but each small piece of the fluid undergoes homogeneous deformation. We describe the homogeneous deformation of the small piece by using a rheological model—that is, a relation between the rate of shear and the shearing stress:

$$\dot{\gamma} = h(\tau).$$

For example, we will consider separately a linearly viscous fluid, $\dot{\gamma} = \tau / \eta$, and a nonlinearly viscous fluid, $\dot{\gamma} = (\tau / A)^n$.

Mixing the three ingredients. First consider a linearly viscous fluid, $\dot{\gamma} = \tau / \eta$. Because the shearing stress is linear in radial distance, $\tau = Gr/2$, the rate of shear is also linear in radial distance,

$$\dot{\gamma} = \frac{Gr}{2\eta}.$$

Integrating $\dot{\gamma} = -dv/dr$ and using the boundary condition $v(a) = 0$, we find that

$$v(r) = \frac{G}{4\eta}\left(a^2 - r^2\right).$$

The velocity is quadratic in the radial distance. For the pipe of circular cross section, the rate of flow relates to the velocity of fluid as

$$Q = \int_0^a v(r) 2\pi r \, dr.$$

The rate of flow is

$$Q = \frac{\pi a^4 G}{8\eta}.$$

Next consider a power-law fluid, $\dot{\gamma} = (\tau / A)^n$. The shearing stress is still linear in radial distance, $\tau = Gr/2$. The rate of shear is

$$\dot{\gamma} = \left(\frac{Gr}{2A}\right)^n.$$

The velocity is

$$v(r) = \left(\frac{G}{2A}\right)^n \frac{a^{n+1} - r^{n+1}}{n+1}.$$

For a shear-thinning fluid, $n > 1$, the rate of shear is negligible when the shearing stress is small, and increases nonlinearly with the shearing stress. Because the shearing stress itself is linear in r, deformation is negligible near the centerline of the pipe, but is severe near the wall of the pipe. The fluid moves like a plug. This behavior should be contrasted with parabolic distribution of velocity for a linearly viscous fluid. The rate of flow is

$$Q = \left(\frac{Ga}{2A}\right)^n \frac{\pi a^3}{n+3} .$$

Use *G*-*Q* curve to determine $\tau - \dot{\gamma}$ curve. The flow in a pipe has long been used as an experimental setup to determine the shearing flow curve (Coleman, Markovitz and Noll 1966; Markovitz 1968; Sutera and Skalak 1993). Given a fluid and a pipe, we can measure the *G*-*Q* curve experimentally. We can then use the *G*-*Q* curve to determine the $\tau - \dot{\gamma}$ curve.

Recall the definition of the rate of flow,

$$Q = \int_0^a v(r) 2\pi r \, dr .$$

Integrate by part, and we obtain that

$$Q = \pi r^2 v(r) \Big|_0^a - \pi \int_0^a r^2 \frac{dv(r)}{dr} dr .$$

The first term vanishes. Recall the three ingredients: $\tau = rG/2$, $\dot{\gamma} = -dv/dr$, and $\dot{\gamma} = h(\tau)$. Change the variable of integration from r to τ, and we obtain that

$$Q = \frac{8\pi}{G^3} \int_0^{\tau_w} \tau^2 h(\tau) d\tau ,$$

where $\tau_w = aG/2$ is the shearing stress at the wall of the pipe. The above expression calculates the *G*-*Q* relation for any given flow curve $h(\tau)$.

We now wish to do the opposite: use the experimentally determined *G*-*Q* relation to deduce the flow curve $h(\tau)$. Recall the fundamental theorem of calculus: integration and differentiation are inverse operations. Write $G = 2\tau_w / a$, and rescale the experimentally measured *G*-*Q* curve to a $\tau_w - Q$ curve. This curve gives the product QG^3 as a function of τ_w. Differentiating the above integral with respect to τ_w, we obtain that

$$h(\tau_w) = \frac{1}{8\pi (\tau_w)^2} \frac{d(QG^3)}{d\tau_w} .$$

This expression converts the experimentally determined *G*-*Q* curve to the flow curve $h(\tau)$.

Nonlinear Viscosity of Extension

Extensional flow curve. Often it is more convenient to test a material under uniaxial tension. We represent the experimentally measured curve of the tensile stress σ and the rate of extension $\dot{\varepsilon}$ as a function:

$$\dot{\varepsilon} = f(\sigma).$$

For example, we may fit the flow curve under tension to power law:

$$\dot{\varepsilon} = \left(\frac{\sigma}{K}\right)^n,$$

where n and K are parameters used to fit the flow curve measured in the tensile test.

Creep of a rod. A material obeys the power law:

$$\dot{\varepsilon} = \left(\frac{\sigma}{K}\right)^n.$$

A rod of the material, initial length L and initial cross-sectional area A, is subject to a hanging weight P. We wish to calculate the length of the rod as a function of time, $l(t)$.

By definition, the rate of extension is

$$\dot{\varepsilon} = \frac{dl(t)}{ldt}.$$

As the rod elongates, the cross-sectional area of the rod also changes with time, $a(t)$. The material is taken to be incompressible, so that

$$a(t)l(t) = AL.$$

By definition, the stress is

$$\sigma = \frac{P}{a}.$$

Mixing the above equations, we get

$$\frac{dl}{ldt} = \left(\frac{Pl}{ALK}\right)^n.$$

This is an ordinary differential equation for the function $l(t)$. Integrating, we obtain that

$$\left(\frac{L}{l}\right)^n = 1 - n\left(\frac{P}{AK}\right)^n t.$$

Growth of a cavity. In a state of hydrostatic stress σ_{appl}, an incompressible material does not deform. However, the material will deform if it contains a cavity. We neglect the surface energy of the cavity, and assume that the surface of the cavity is traction-free. Consequently, near the cavity the material is not in a state of hydrostatic stress, and the material deforms. Far away from the cavity, the material is still in the state of hydrostatic stress σ_{appl}, and the material does not deform. This applied stress will cause the cavity to enlarge. The deformation in the material is inhomogeneous.

Scaling analysis. We assume the cavity to be spherical. At time *t*, the radius of cavity is $a(t)$, and grows at the rate $da(t)/dt$. The rate of deformation in the circumferential direction at the surface of the cavity is

$$\frac{da}{adt}.$$

At any time, the field in the material is governed by a boundary-value problem of a single length scale, *a*, and of no time scale. Consequently, the rate of deformation at the surface of cavity is a function of the stress applied far away from the cavity:

$$\frac{da}{adt} = F\left(\sigma_{appl}\right).$$

Consequently, the cavity expands exponentially in time:

$$a(t) = a(0)\exp\left[tF\left(\sigma_{appl}\right)\right].$$

The scaling analysis enables us to reach this remarkable result with very little work.

The field in the material is of spherical symmetry, so that the non-vanishing fields are the radial component of stress σ_r, circumferential components of stress σ_θ, and the radial component of the velocity *v*. For the cavity at a given radius *a*, all these fields are functions of radial coordinate *r*. We next determine the field by using the three ingredients.

Compatibility of deformation. At a given time, the velocity of the surface of the cavity is $v(a) = da/dt$, and the volume of material per unit time crossing the surface of the cavity is $4\pi a^2(da/dt)$. The volume of material per unit time crossing the spherical surface of radius *r* is $4\pi r^2 v(r)$. Incompressibility of the material requires that the two flows be equal, $4\pi r^2 v(r) = 4\pi a^2(da/dt)$, giving the velocity field:

$$v(r) = \frac{a^2}{r^2}\frac{da}{dt}.$$

The velocity decays as a function of the distance *r*. Thus, the compatibility of deformation determines the distribution of velocity in the fluid. Such a boundary-value problem is *kinematically determinate*. The rate of deformation in the radial direction is $D_r = dv/dr$, or

$$D_r = -2\frac{a^2}{r^3}\frac{da}{dt}.$$

The rate of deformation in the radial direction at the surface of the cavity is

$$D_a = -2\frac{da}{adt}.$$

The rates of deformation in the circumferential directions are $D_\theta = v/r$. The material contracts in the radial direction and extends in the circumferential directions.

Balance of forces. Consider a hemispherical shell, inner radius *r* and outer radius *r* + *dr*. The base of the hemispherical shell is an annulus lying in the plane normal to the *z* axis. The radial stress on the outer surface of the shell is $\sigma_r(r+dr)$, resulting in a force in the positive *z* direction, $\pi(r+dr)^2\sigma_r(r+dr)$. The radial stress on the inner surface is $\sigma_r(r)$, resulting in a force in the negative *z* direction, $\pi a^2\sigma_r(r)$. The circumferential stress acting on base of the hemispherical shell gives a force in the negative *z* direction, $(2\pi r dr)\sigma_\theta(r)$. Balancing these forces acting on the hemispherical shell, we obtain that

$$\frac{d\sigma_r}{dr} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0.$$

Rheology of material. Consider a nonlinear, viscous material. Tested under uniaxial tensile stress, the stress relates to the rate of deformation as

$$\sigma = f(\dot{\varepsilon}).$$

Each small piece of material around the cavity is in a state of triaxial stress, $(\sigma_r,\sigma_\theta,\sigma_\theta)$, and a state of triaxial rate of deformation, $(D_r,-D_r/2,-D_r/2)$. Superimposing a state of hydrostatic stress does not affect the flow. Superposing a state of hydrostatic stress $(-\sigma_\theta,-\sigma_\theta,-\sigma_\theta)$ on the original state $(\sigma_r,\sigma_\theta,\sigma_\theta)$, we obtain a state of uniaxial compressive stress, $(\sigma_r-\sigma_\theta,0,0)$, and keep the state of deformation the same, $(D_r,-D_r/2,-D_r/2)$. Consequently, each small piece of fluid deforms in a homogeneous state equivalent to that caused by a uniaxial compressive stress. We assume that the rheological model is symmetric with respect to tension and compression. In the relation $\sigma = f(\dot{\varepsilon})$, we replace σ with $-(\sigma_r-\sigma_\theta)$, and $\dot{\varepsilon}$ with $-D_r$, so that

$$-(\sigma_r-\sigma_\theta) = f(|D_r|).$$

Mixing the three ingredients. Combining the balance of forces and the rheology of material, we obtain that

$$\frac{d\sigma_r}{dr} = \frac{2}{r} f\left(\left|D_r\right|\right).$$

The boundary conditions are $\sigma_r(a) = 0$ and $\sigma_r(\infty) = \sigma_{appl}$. Integration gives that

$$\sigma_{appl} = 2\int_a^\infty f\left(\left|D_r\right|\right)\frac{dr}{r}.$$

Recall the expression for the rate of deformation in the radial direction,

$$D_r = \frac{a^3}{r^3} D_a.$$

We use this expression to change the variable in the above integral from *r* to $\left|D_r\right|$:

$$\sigma_{appl} = \frac{2}{3}\int_0^{\left|D_a\right|} f\left(\left|D_r\right|\right)\frac{d\left|D_r\right|}{\left|D_r\right|}.$$

This expression relates the applied stress σ_{appl} to the rate of expansion $\left|D_a\right|$ at the surface of the cavity.

Consider a power-law material, $\sigma = K\left(\dot{\varepsilon}\right)^N$. Inserting $f\left(\left|D_r\right|\right) = K\left|D_r\right|^N$ to the above integral, we obtain that

$$\sigma_{appl} = \frac{2K}{3N}\left(\frac{2}{a}\frac{da}{dt}\right)^N.$$

Integrating with respect to time *t*, we obtain that

$$a(t) = a(0)\exp\left[\frac{t}{2}\left(\frac{3N\sigma_{appl}}{2K}\right)^{\frac{1}{N}}\right].$$

The cavity enlarges exponentially in time.

Moving-boundary problem. As the cavity in a body of fluid enlarges, the surface of the body moves. At a given time, the surface is known, and we solve the boundary-value problem. We then use the velocity field to update the surface for a small increment of time. We repeat the procedure through many small increments of time. For a spherical cavity, the cavity enlarges but remains spherical, so that the surface is fully characterized by a single parameter, the radius of the cavity. In general, when the cavity changes shape, the problem becomes more challenging (Budianskey, Hutchinson and Slutsky 1982).

Collapse of a cavity. For a small cavity, the surface energy of the cavity tends to cause the cavity to shrink. Let γ be the surface energy per unit area. The surface energy causes a Laplace stress,

$$\sigma_r(a) = \frac{2\gamma}{a}.$$

We assume that the body is stress-free far away from the cavity, $\sigma_r(\infty) = 0$. Following the above steps, we get

$$\frac{2\gamma}{a} = \frac{2K}{3N}\left(-\frac{2}{a}\frac{da}{dt}\right)^N.$$

The ordinary differential equation determines $a(t)$.

Second-Invariant Fluid

Nonlinear, isotropic, incompressible, viscous fluid. We have tested a material under shear, and measured the curve between the stress and the rate of deformation, $\tau = g(\dot{\gamma})$. What can we do with the curve? We can compare the curve with that of another material. We can study the microscopic origins for the values of *A* and *N* (Ashby and Frost, 1982). We can even use the data to solve some boundary-value problems. But to solve boundary-value problems in general, we will need to have a relation between the state of stress and rate of deformation in all types of flow, not just shear.

Isotropy of the material implies that the relation $\tau = g(\dot{\gamma})$ applies in all shearing directions:

$$\sigma_{12} = g(2D_{12}),$$
$$\sigma_{23} = g(2D_{23}),$$
$$\sigma_{31} = g(2D_{31}).$$

The experimental data, $\tau = g(\dot{\gamma})$, in general do not predict the relation between the tensile stress and the rate of extension.

By a viscose fluid here we mean a model in which the deviatoric stress is a unique function of the rate of deformation. In general, we need to determine functions of five independent variables:

$$s_{ij} = s_{ij}(D_{11}, D_{22}, D_{23}, D_{31}, D_{12}).$$

The functions are nonlinear if the viscous behavior is nonlinear. Here we have dropped the dependence on D_{33}; due to incompressibility, D_{33} is not an independent quantity, $D_{33} = -D_{11} - D_{22}$.

A brute-force method to determine these functions of five variables is to measure them experimentally. This method is impractical. A function of one variable corresponds to a curve, a function of two variables corresponds to many curves on a page, a function of three variables corresponds to many pages of a book, and a function of four variables corresponds to many books in a library. A function of five variables will require many libraries.

We need to construct a model to reduce the number of experiments. We wish to modify the multiaxial model of viscosity in a single aspect. The new model will accommodate the nonlinear relation between the state of stress and the rate of deformation, but will preserve incompressibility and isotropy. The condition of incompressibility remains the same:

$$D_{kk} = 0 .$$

We need to learn how to construct models that preserve isotropy.

Invariant of a vector. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis of a Euclidean space. Any vector **u** in the Euclidean space is a linear combination of the base vectors:

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 .$$

We say that u_1, u_2, u_3 are the components of the vector **u** relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We are familiar with the geometric interpretations of these ideas. The vector **u** is an arrow in the space. The basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ consists of three unit vectors normal to one another. The components u_1, u_2, u_3 are the projection of the vector **u** on to the three unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Once a vector **u** is given in the Euclidean space, the vector itself does not change if we choose another basis. However, the components u_1, u_2, u_3 do change if we choose another basis. We know the rule of the transformation of the components of the same vector relative to two bases.

The sum $u_i u_i$ does not have any free index, and is a scalar. When a new basis is used, the components u_1, u_2, u_3 change, but $u_i u_i$ remains invariant. This invariant has a familiar geometric interpretation: $\sqrt{u_i u_i}$ is the length of the vector **u**. The length of the vector is invariant when the basis changes.

Invariants of a tensor. Let **A** be a second-rank tensor, and A_{ij} be the components of the tensor relative to an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The tensor is symmetric, so that $A_{ij} = A_{ji}$. The components of the tensor form three scalars:

$$A_{ii}, \quad A_{ij}A_{ij}, \quad A_{ij}A_{jk}A_{ki} .$$

We form a scalar by combining the components of the tensor in a way that makes all indices dummy. The three scalars are independent of the choice of the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and are known as the invariants of the tensor **A**.

Invariants of stress tensor. A state of stress is a physical fact, independent of how we choose a basis. Once we choose a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, we picture a unit cube in the fluid, with the faces of the cube on the coordinate planes. Forces acting on the faces of the cube define the components of the stress, σ_{ij}, relative to the basis. Whereas the state of stress does not depend on the choice of the basis, the components of the stress do.

Some combinations of the components are invariants, independent of the choice of basis. For example, the trace of the stress tensor, σ_{kk}, is an invariant. The flow of an incompressible fluid is unaffected when we superimpose any hydrostatic stress. Define the deviatoric stress, $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$, and write the other two invariants of the stress tensor as

$$s_{ij}s_{ij}, \quad s_{ij}s_{jk}s_{ki}.$$

Each invariant is a scalar measure of the state of stress. The two invariants are sometimes designated as $J_2 = s_{ij}s_{ij}/2$ and $J_3 = s_{ij}s_{jk}s_{ki}/3$.

Invariants of the rate-of-deformation tensor. For an incompressible material, the trace of the deformation gradient vanishes,

$$D_{kk} = 0.$$

We write the other two invariants of the rate of deformation as

$$D_{ij}D_{ij}, \quad D_{ij}D_{jk}D_{ki}.$$

Each invariant is a scalar measure of the rate of deformation. The two invariants are sometimes designated as $I_2 = D_{ij}D_{ij}/2$ and $I_3 = D_{ij}D_{jk}D_{ki}/3$.

The second-invariant fluid. We have tested a fluid under shear, and measured the curve between the shearing stress and the rate of shear, $\tau = g(\dot{\gamma})$. Can we use this shearing flow curve to predict the relation between the stress and the rate of deformation under a load other than shear? The answer is no, in general. However we fudge, the prediction will disagree with the experimental data for some states of stress. But the desire to use the curve $\tau = g(\dot{\gamma})$ to predict the rate of deformation under an arbitrary state of stress is so strong that we do it by making assumptions. Fudge we do.

The most widely used model relies on two assumptions. First, the model takes the form:

$$D_{kk} = 0,$$

$$s_{ij} = 2\eta D_{ij}.$$

Second, the viscosity η depends on the rate of deformation in a particular way—the viscosity is a function of the second invariant, $\eta\left(D_{ij}D_{ij}\right)$. The use of the invariant preserves isotropy. The use of only one invariant simplifies the model. The model ensures incompressibility.

We fit the function $\eta\left(D_{ij}D_{ij}\right)$ to the experimentally measured flow curve under shear. We then use the model to calculate the deviatoric stress for any given rate of deformation.

Fit the second-invariant model to experimental data. Under pure shear, the rate of deformation is

$$\left[D_{ij}\right] = \begin{bmatrix} 0 & \dot{\gamma}/2 & 0 \\ \dot{\gamma}/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The second invariant is

$$D_{ij}D_{ij} = D_{12}D_{12} + D_{21}D_{21} = \frac{1}{2}\dot{\gamma}^2 .$$

Given a general rate of deformation D_{ij}, calculate the equivalent rate of shear according to

$$\dot{\gamma}_e = \sqrt{2D_{ij}D_{ij}} .$$

The equivalent rate of shear coincides with the applied rate of shear under the pure shear condition, and is a scalar measure of the rate of deformation under a general rate of deformation.

The equivalent rate of shear $\dot{\gamma}_e$ is just a way to write the second invariant $D_{ij}D_{ij}$. Consequently, we can write the second-invariant model by

$$s_{ij} = 2\eta\left(\dot{\gamma}_e\right)D_{ij} .$$

That is, the model requires that the viscosity be a function of the equivalent rate of shear. Under the pure shear condition, the second invariant model reduces to

$$\tau = \eta\left(\dot{\gamma}\right)\dot{\gamma} .$$

We have tested the fluid under shear, and measured the curve between the stress and the rate of deformation, $\tau = g\left(\dot{\gamma}\right)$. Fitting to the experimentally measured flow curve, the second-invariant model takes the form

$$s_{ij} = \frac{2g\left(\dot{\gamma}_e\right)}{\dot{\gamma}_e}D_{ij} .$$

The second-invariant model uses the flow curve $\tau = g(\dot{\gamma})$ measured under shear to predict the relation between stress and rate of deformation for all types of flow. The model achieves unusual economics: buy one, and get everything else for free.

Power-law creep. As an example, suppose we have fit the experimentally determined flow curve to the power law:

$$\tau = A\dot{\gamma}^N .$$

For the fluid at an arbitrary rate of deformation D_{ij}, we calculate the equivalent rate of shear $\dot{\gamma}_e = \sqrt{2D_{ij}D_{ij}}$, and the second-invariant model predicts the deviatoric stress:

$$s_{ij} = 2A(\dot{\gamma}_e)^{N-1} D_{ij} .$$

Predict the flow curve under uniaxial tension. We can use the second-invariant model to predict the deviatoric stress under any rate of deformation. As an example, consider a material under uniaxial tension at stress σ and the rate of extension $\dot{\varepsilon}$. The material is at a rate of deformation

$$[D_{ij}] = \begin{bmatrix} \dot{\varepsilon} & 0 & 0 \\ 0 & -\dot{\varepsilon}/2 & 0 \\ 0 & 0 & -\dot{\varepsilon}/2 \end{bmatrix}.$$

The equivalent rate of shear is

$$\dot{\gamma}_e = \sqrt{2D_{ij}D_{ij}} = \sqrt{2\left[(\dot{\varepsilon})^2 + \left(\frac{\dot{\varepsilon}}{2}\right)^2 + \left(\frac{\dot{\varepsilon}}{2}\right)^2\right]} = \sqrt{3}\dot{\varepsilon} .$$

The material is in the state of stress

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The mean stress is $\sigma_m = \sigma / 3$. The deviatoric stress is

$$[s_{ij}] = \begin{bmatrix} 2\sigma/3 & 0 & 0 \\ 0 & -\sigma/3 & 0 \\ 0 & 0 & -\sigma/3 \end{bmatrix}.$$

The second-invariant model $s_{ij} = 2A(\dot{\gamma}_e)^{N-1} D_{ij}$ reduces to

$$\frac{2\sigma}{3} = 2A\left(\sqrt{3}\dot{\varepsilon}\right)^{N-1} \dot{\varepsilon} .$$

This expression predicts the flow curve under uniaxial tension. Of course, there is no reason to expect this prediction to agree exactly with the flow curve measured experimentally in a uniaxial tensile test. It is just a prediction of a model.

Thermodynamic inequality. The thermodynamic inequality $s_{ij}D_{ij} \geq 0$ applies when an incompressible, viscous fluid flows at a fixed temperature. The inequality holds without requiring linearity and isotropy. The second-invariant model satisfies the thermodynamic inequality for arbitrary state of flow. Note that $s_{ij}D_{ij} = \eta(\dot{\gamma}_e)D_{ij}D_{ij}$. The second invariant $D_{ij}D_{ij}$ is positive definite. Thus, $s_{ij}D_{ij} \geq 0$ provided $g \geq 0$. The latter means that, under shear, the shearing stress is in the direction of the shearing deformation.

Stress as independent variable. We can also require that the viscosity be a function of the second invariant of deviatoric stress. Form the equation $s_{ij} = 2\eta D_{ij}$ we obtain that

$$s_{ij}s_{ij} = 4\left[\eta\left(D_{ij}D_{ij}\right)\right]^2 D_{ij}D_{ij}.$$

The right-hand side is $\left[g(\dot{\gamma}_e)\right]^2$. So long as the function $g(\dot{\gamma}_e)$ is monotonic, the above equation is a one-to-one relation between the second invariant of stress, $s_{ij}s_{ij}$, and the second invariant of rate of deformation, $D_{ij}D_{ij}$. We can make the viscosity as a function of the second invariant of the deviatoric stress, $\eta(s_{ij}s_{ij})$. The second-invariant model is also known as the J_2 model.

Under the pure shear condition, the deviatoric stress is

$$\left[s_{ij}\right] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The second invariant is

$$s_{ij}s_{ij} = s_{12}s_{12} + s_{21}s_{21} = 2\tau^2.$$

Define the equivalent shearing stress by

$$\tau_e = \sqrt{\frac{1}{2}s_{ij}s_{ij}}.$$

The equivalent shearing stress is just another way to write the second invariant of deviatoric stress, and reproduces the applied shearing stress under the pure shear condition.

The second-invariant model gives the following recipe. Test a material in shear to measure the relation between the shearing stress and the rate of shear,

$\dot{\gamma} = h(\tau)$. Given an arbitrary state of stress σ_{ij}, calculate mean stress $\sigma_m = \sigma_{kk}/3$, the deviatoric stress $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$, and the equivalent shearing stress $\tau_e = \sqrt{s_{ij}s_{ij}/2}$. The model predict all components of the rate of deformation:

$$D_{ij} = \frac{h(\tau_e)}{2\tau_e} s_{ij}.$$

We have presented the model using the second invariant of stress and using the second invariant of rate of deformation. The two methods give the same model. The two methods use the same experimental data, and give identical predictions. Indeed, the equivalent shearing stress relates to the equivalent rate of shear through the flow curve:

$$\tau_e = g(\dot{\gamma}_e),$$

$$\dot{\gamma}_e = h(\tau_e).$$

Fit the second-invariant model to the experimental data measured under uniaxial tension. Often it is more convenient to test a material under uniaxial tension. We represent the experimentally measured curve of the tensile stress σ and the rate of extension $\dot{\varepsilon}$ as a function:

$$\dot{\varepsilon} = f(\sigma).$$

We next use the data measured under uniaxial tension to fix the second-invariant model.

Under the uniaxial tension, the stress tensor is

$$[\sigma_{ij}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The mean stress is $\sigma_m = \sigma/3$. The deviatoric stress is

$$[s_{ij}] = \begin{bmatrix} 2\sigma/3 & 0 & 0 \\ 0 & -\sigma/3 & 0 \\ 0 & 0 & -\sigma/3 \end{bmatrix}.$$

The second invariant of the deviatoric stress is

$$s_{ij}s_{ij} = \frac{2}{3}\sigma^2.$$

Define the equivalent tensile stress (or the von Mises equivalent stress) by

$$\sigma_e = \sqrt{\frac{3}{2}s_{ij}s_{ij}}.$$

The equivalent stress is yet another way to write the second invariant. Under uniaxial tension, the equivalent stress coincides with the applied stress.

We can write the second-invariant model by

$$D_{ij} = \frac{s_{ij}}{2\eta(\sigma_e)}.$$

That is, the model requires that the viscosity be a function of the equivalent stress. The second-invariant model gives the following recipe. Test a material in uniaxial tension, and measure the relation between the stress and the rate of deformation, $\dot{\varepsilon} = f(\sigma)$. Given an arbitrary state of stress σ_{ij}, calculate the deviatoric stress $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$, and calculate the equivalent stress, $\sigma_e = \sqrt{3s_{ij}s_{ij}/2}$. The second-invariant model predicts the rate of deformation as

$$D_{ij} = \frac{3f(\sigma_e)}{2\sigma_e} s_{ij}.$$

This expression makes the viscosity as a function of the second invariant, and reproduces the flow curve $\dot{\varepsilon} = f(\sigma)$ measured under a uniaxial tensile test.

As an example, suppose that we have tested the material under uniaxial tensile stress σ, and fit the measured rate of extension to a power law:

$$\dot{\varepsilon} = \left(\frac{\sigma}{K}\right)^n.$$

For the material under a multiaxial state of stress σ_{ij}, the second-invariant model predicts the rate of deformation:

$$D_{ij} = \frac{3}{2}\left(\frac{\sigma_e}{K}\right)^{n-1} \frac{s_{ij}}{K}.$$

Predict the rate of deformation under biaxial stresses. Consider a thin membrane of the same material subject to biaxial stresses σ_1 and σ_2. We have fit the second-invariant model to the flow curve under uniaxial tension. We now wish to predict the rates of extension D_1 and D_2 under biaxial stresses. Under the biaxial stresses, the stress tensor is

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The mean stress is

$$\sigma_m = \frac{\sigma_1 + \sigma_2}{3}.$$

The deviatoric stress is

$$[s_{ij}] = \begin{bmatrix} (2\sigma_1 - \sigma_2)/3 & 0 & 0 \\ 0 & (2\sigma_2 - \sigma_1)/3 & 0 \\ 0 & 0 & -(\sigma_1 + \sigma_2)/3 \end{bmatrix}.$$

The equivalent stress is

$$\sigma_e = \sqrt{\frac{3}{2} s_{ij} s_{ij}} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2}.$$

Consequently, the second-invariant model predicts the rates of extension:

$$D_1 = \left(\frac{\sigma_e}{K}\right)^{n-1} \frac{1}{K}\left(\sigma_1 - \frac{\sigma_2}{2}\right),$$

$$D_2 = \left(\frac{\sigma_e}{K}\right)^{n-1} \frac{1}{K}\left(\sigma_2 - \frac{\sigma_1}{2}\right).$$

Ilyushin Theorem

Scaling relation of power-law creep. Consider a boundary value problem characterized by a length a and load σ_{appl}. Ilyushin (1946) noted that the fields of stress, rate of deformation, and velocity take the form

$$\sigma_{ij} = \sigma_{\text{appl}} \hat{\sigma}_{ij}\left(\frac{\mathbf{x}}{a}, n\right),$$

$$D_{ij} = \left(\frac{\sigma_{\text{appl}}}{A}\right)^n \hat{D}_{ij}\left(\frac{\mathbf{x}}{a}, n\right),$$

$$v_i = a\left(\frac{\sigma_{\text{appl}}}{A}\right)^n \hat{v}_i\left(\frac{\mathbf{x}}{a}, n\right).$$

Here $\hat{\sigma}_{ij}$, $\hat{D}_{ij}$ and $\hat{v}_i$ stand for dimensionless functions.

Thus, when a body is subject to an applied stress σ_{appl}, the field of stress is linear in σ_{appl}, and the field of rate of deformation and the field of velocity are proportional to $(\sigma_{\text{appl}})^n$. The Ilyushin theorem determines the dependence of the fields on the applied stress. The distribution of the fields still need be determined by solving the boundary-value problem.

The proof of this theorem is simple: the above form satisfies all the governing equations. The boundary-value problems are governed by the following equations. For highly viscous material, we often neglect the effect of inertia. We also often neglect body force. Under these conditions, the balance of forces takes the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 .$$

Compatibility of deformation requites that

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

The incompressibility condition requires that

$$D_{k,k} = 0$$

These partial differential equations are linear.

The uniaxial stress-strain curve is fit to the power law:

$$\dot{\gamma} = \left(\tau / A \right)^n .$$

The second-invariant model predicts that

$$D_{ij} = \left(\frac{\tau_e}{A} \right)^{n-1} \frac{s_{ij}}{2A} .$$

Drift of a rigid particle in a linearly viscous liquid. For a rigid spherical particle drifting in a Newtonian fluid, Stokes obtained the following relation:

$$v = \frac{f}{6\pi a \eta} ,$$

where v is the velocity of the particle, f the force applied to the particle, a the radius of the particle, and η the viscosity of the fluid.

Stokes obtained this result by solving the partial differential equation. However, the main aspect of the result can be appreciated as follows. The problem has only one length scale, the radius of the particle, a. The rate of deformation around the particle scales with v/a. The stress around the partical scales with f / a^2. For a linearly viscous liquid, the rate of deformation is proportional to the stress, so that

$$\frac{v}{a} \propto \frac{f}{a^2 \eta} .$$

Thus, without solving the boundary-value problem, we have obtained a scaling relation between the velocity and the force. The scaling relation leaves a numerical factor undetermined.

Drift of a rigid particle in a power-law creeping liquid. Now consider a rigid spherical particle drifting in a nonlinear viscous fluid. The fluid creeps according to the power law. When the liquid is under the pure shear condition, the rate of deformation $\dot{\gamma}$ relates to the stress τ as

$$\dot{\gamma} = \left(\frac{\tau}{A}\right)^n ,$$

where A and n are parameters used to fix the experimental data. Use the Ilyushin theorem, we obtain the following scaling relation between the velocity of the particle and the force applied on the particle:

$$\frac{v}{a} \propto \left(\frac{f}{Aa^2}\right)^n .$$

The scaling relation leaves a numerical factor undetermined.

Similar power-law scaling appears in the flow in a pipe and the growth of a cavity.

The merit of power law. The power-law model characterizes a material using two parameters, A and n. The model is versatile enough to describe many materials, and the solutions to boundary-value problems have a simple form identified by Ilyushin. Many boundary-value problems have been solved (e.g., Tanner 2000; Irgens 2014; Budiansky, Hutchinson and Slutsky 1982).

In his original paper, Ilyushin used the power law to describe the stress-strain relation of a nonlinearly elastic material. For finite deformation, the relation between strain and displacement is nonlinear. To save his theorem, he limited himself to small elastic deformation.

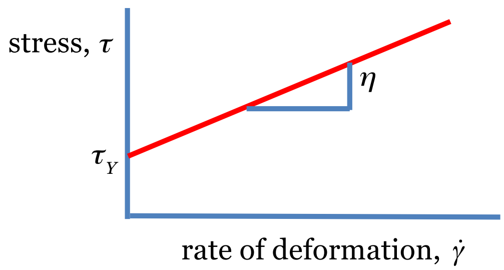
The power law has also been used to describe strain-hardening plastic deformation of metals. In many cases plastic strain is large enough to neglect elastic strain, but small enough to linearize the relation between strain and displacement. Also, when the components of stress increase proportionally, the plastic stress-strain relation is the same as nonlinearly elastic stress-strain relation. In such cases, the Ilyushin theorem applies. In particular, it has been used to describe the fundamental solutions in nonlinear fracture mechanics (Rice and Rosengren, 1968; Hutchinson 1968; He and Hutchinson 1981).

If you ever have a chance to solve an original nonlinear problem, ask if you can use the power law to describe the material, and use the Ilyushin theorem to organize the solution to the boundary-value problem.

Viscoplasticity

Yield stress. Bingham fluid. In many cases a material creeps negligibly when the stress is small, and creeps appreciably when the stress is large. This shear-thinning behavior is often modeled by a power law. Bingham (1922) proposed an alternative model:

$$\dot{\gamma} = \begin{cases} 0, & \text{for } \tau < \tau_Y \\ \dfrac{\tau - \tau_Y}{\eta}, & \text{for } \tau > \tau_Y \end{cases}$$



The model characterizes a fluid using two parameters: the yield stress τ_Y and the viscosity η. The combination of yield stress and viscous flow is called viscoplasticity.

The merit of modeling the shear-thinning behavior using the yield stress or using the power law is debatable (Barnes 1999).

Viscoplastic flow in a pipe. Consider a Bingham fluid flowing in a pipe of circular cross section, radius a. The balance of forces gives the distribution of the shearing stress:

$$\tau = \frac{Gr}{2}.$$

The shearing stress is linear in the radial distance from the centerline of the pipe, and reaches the maximum at the wall of the pipe. The fluid does not flow when the shearing stress at the wall is below the yield stress,

$$\frac{Ga}{2} < \tau_Y.$$

We next assume that the gradient of pressure is large enough,

$$\frac{Ga}{2} > \tau_Y,$$

so that the fluid near the wall of the pipe flows. Let radius b be determined by

$$\frac{Gb}{2} = \tau_Y.$$

The fluid within the radius b does not flow. The rate of shear is

$$\dot{\gamma} = \begin{cases} 0, & \text{for } r < b \\ \dfrac{\tau - \tau_Y}{\eta}, & \text{for } r > b \end{cases}$$

Assume the no-slip boundary condition, $v(a) = 0$. The velocity profile is given by

$$v(r) = -\int_a^r \dot{\gamma}\, dr .$$

Integrating, we obtain that

$$v(r) = \begin{cases} \dfrac{G}{4\eta}\left[(a-b)^2 - (r-b)^2\right], & b < r < a \\ \dfrac{G}{4\eta}(a-b)^2, & 0 < r < b \end{cases}$$

As the fluid in the annulus $b < r < a$ shears, the fluid in the cylinder of radius b moves as a rigid body.

Viscoplasticity under multiaxial stress. The Bingham model provides a specific relation between the stress and the rate of deformation under shear. The relation can be generalized to multiaxial loading using the second-invariant model.

Given an arbitrary state of stress σ_{ij}, calculate the deviatoric stress $s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$, calculate the equivalent shearing stress, $\tau_e = \sqrt{s_{ij}s_{ij}/2}$, and calculate the rate of deformation:

$$D_{ij} = \begin{cases} 0, & \text{for } \tau_e < \tau_Y \\ \left(\dfrac{\tau_e - \tau_Y}{\eta}\right)\dfrac{s_{ij}}{2\tau_e}, & \text{for } \tau_e > \tau_Y \end{cases}$$

This viscoplastic model satisfies the thermodynamic condition $s_{ij}D_{ij} \geq 0$ for arbitrary state of stress.

von Mises yield condition. The equation

$$\frac{1}{2}s_{ij}s_{ij} = (\tau_Y)^2 ,$$

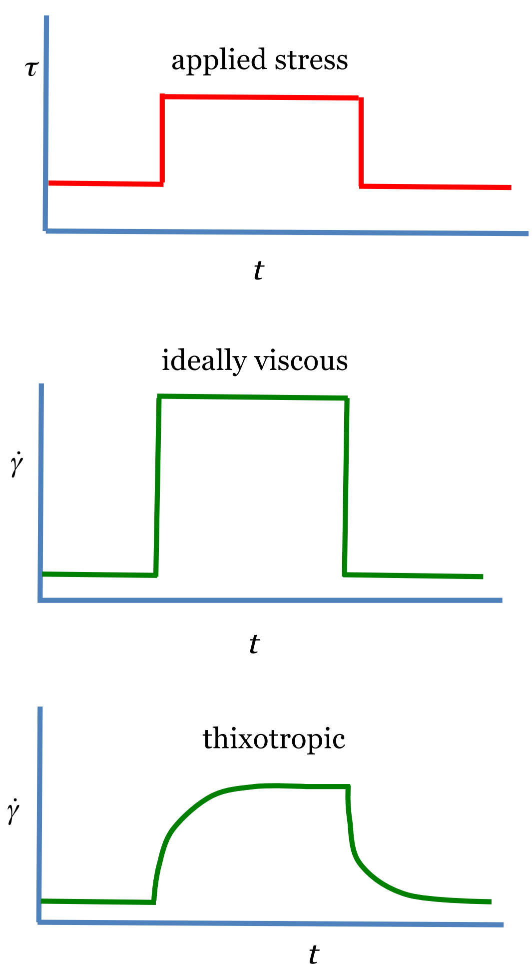
is due to von Mises (1913). This equation uses the yield stress measured under shear to predict the yield condition under any type of stress. The prediction uses the second invariant of deviatoric stress. We will study the yield condition in more detail later.

Thixotropy

Dynamic microstructure. The flow of a material requires that its constituting particles change neighbors. A liquid of small molecules, such as water, does not form any microstructure. The flow of such a small-molecule liquid directly connects to the change of neighbors of molecules.

By contrast, a crystal such as ice and a metal has a microstructure—a forest of dislocations. The flow of the crystal still requires that atoms change neighbors, but the change is mediated by the change of the microstructure. The microstructure is dynamic: dislocations glide, climb, nucleate, and annihilate. Consequently, the crystal ages with time, and hardens by deformation.

Thixotropy. In a suspension, colloidal particles also form a microstructure at a length scale larger than the size of individual particles. When an applied stress causes the suspension to flow, the deformation changes the microstructure. When we apply the stress for some time, the microstructure and the rate of deformation will reach a steady state.



When we suddenly step up the applied stress, the microstructure will change, and so does the rate of deformation. After some time, the microstructure and the rate of deformation will reach a new steady state.

When we suddenly step down the applied stress to the initial level, the microstructure will change again, and so does the rate of deformation. After some time, the microstructure and the rate of deformation will recover the initial steady state.

Following a sudden change in stress, if the microstructure changes rapidly, the material changes its rate of deformation to a new level instantly, with no delay. In this case the model of ideal viscosity is adequate.

If the microstructure changes slowly, however, the model of ideal viscosity is inadequate. When the stress changes, the material changes its rate of deformation to a new level gradually. This behavior is termed *thixotropy*.

Microstructure and deformation co-evolve. A large number of models have been developed to describe thixotropy (Mewis and Wagner 2009). Such a model typically describes the state microstructure using a parameter ξ. The rate of deformation depends on both the stress and the state of microstructure:

$$\dot{\gamma} = h(\tau, \xi).$$

The microstructure evolves according to

$$\frac{d\xi}{dt} = f(\tau, \xi)$$

These two equations evolve the deformation and the microstructure simultaneously.

Normal Stresses

The second-invariant model sometimes fails. The prediction of the second-invariant model often disagrees with experimental data. For example, once we fit the model to the experimental data measured under shear, we can use the model to predict the relation between stress and rate of deformation in other types flow. It is not surprising that the prediction disagrees with experimental data. Often we disregard the discrepancy, but sometimes we cannot. Let us look at a particular set of experimental observations.

Shearing can cause normal stresses. When a nonlinear, isotropic, incompressible, viscous fluid is subject to a rate of shear, in the absence of all other components of the rate of deformation, the second-invariant model predicts that the fluid only develops shearing stress. This prediction is wrong. Experiments show that the fluid also develops normal stresses.

Let the rate of shear be $D_{12} = D_{21} = \dot{\gamma}/2$; all other components of the rate of deformation vanish. We can determine the shearing stress as a function of the rate of shear:

$$\sigma_{12} = g(\dot{\gamma}).$$

Reversing the direction of the shear will also reverse the shearing stress, so that $g(-\dot{\gamma}) = -g(\dot{\gamma})$; it is an odd function. For an isotropic fluid, the symmetry precludes shearing stresses in other directions, $\sigma_{23} = \sigma_{31} = 0$.

The isotropy, however, does not preclude normal stresses. The flow of an incompressible fluid is unaffected by the superimposition of a state of hydrostatic stress. We subtract the normal stress in every direction by σ_{22}, the fluid is now subject to normal stress $\sigma_{11} - \sigma_{22}$ in direction 1, and normal stress $\sigma_{33} - \sigma_{22}$ in direction 3. The two normal stresses are also functions of the rate of shear:

$$\sigma_{11} - \sigma_{22} = N_1(\dot{\gamma}),$$
$$\sigma_{22} - \sigma_{33} = N_2(\dot{\gamma}),$$

Reversing the direction of the shear will not affect the normal stresses, so that $N_1(-\dot{\gamma}) = N_1(\dot{\gamma})$ and $N_2(-\dot{\gamma}) = N_2(\dot{\gamma})$; they are even functions. Experiments show that the normal stress effects are pronounced in viscoelastic liquids; the first normal stress difference is larger in magnitude than the second normal stress difference (Barnes, Hutton, Walters 1989).

The normal stresses induced in the shearing flow are responsible for several spectacular experimental observations, such as rod climbing and die swelling (Boger and Walters 1993).

Reiner-Rivlin fluid. In the second invariant model, $s_{ij} = 2\eta D_{ij}$, the viscosity is taken to be a function of the second invariant. We can of course assume that the viscosity is a function of second and third invariants, $\eta(D_{ij}D_{ij}, D_{ij}D_{jk}D_{ki})$. This model is more general, but requires more experimental data to fit the function. Besides, the two-invariant model still predicts that a rate of shear generates no normal stresses.

A fluid deforms in a homogeneous state, at the stress σ_{ij} and the rate of deformation D_{ij}. Assume that the fluid is isotropic, and that the state of stress is a function of the state of rate of deformation. The most general form of the function is

$$\sigma_{ij} = a\delta_{ij} + bD_{ij} + cD_{ik}D_{kj},$$

where *a*, *b* and *c* are functions of the three invariants $D_{ii}, D_{ij}D_{ij}, D_{ij}D_{jk}D_{ki}$. This result is due to Reiner (1945) and Rivlin (1948).

We further assume that the fluid is incompressible, $D_{kk} = 0$, and that superimposing a hydrostatic stress does not affect deformation. Let the mean stress be $\sigma_m = \sigma_{kk}/3$, and the deviatoric stress be $s_{ij} = \sigma_{ij} - \sigma_m\delta_{ij}$. Modify the above relation between the stress and the rate of deformation as

$$s_{ij} = \alpha D_{ij} + \beta\left(D_{ik}D_{kj} - \frac{1}{3}D_{mk}D_{km}\delta_{ij}\right),$$

where α and β are functions of the two remaining invariants, $D_{ij}D_{ij}, D_{ij}D_{jk}D_{ki}$. Because the deviatoric stress is traceless, we make the right-hand side traceless.

Reiner-Rivlin model fails to predict the form of the normal stress effect observed in experiment. Now consider a fluid subject to a rate of shear $\dot{\gamma}$. The rate of deformation is

$$\left[D_{ij}\right]=\begin{bmatrix} 0 & \dot\gamma/2 & 0 \\ \dot\gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The product is

$$\left[D_{ik}D_{kj}\right]=\begin{bmatrix} 0 & \dot\gamma/2 & 0 \\ \dot\gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & \dot\gamma/2 & 0 \\ \dot\gamma/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} \dot\gamma^2/4 & 0 & 0 \\ 0 & \dot\gamma^2/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and also,

$$\frac{1}{3}D_{mk}D_{km}=\frac{1}{6}\dot\gamma^2$$

The Reiner-Rivlin model predicts that

$$\sigma_{12}=\alpha\dot\gamma/2$$

$$\sigma_{23}=0$$

$$\sigma_{31}=0$$

$$\sigma_{11}-\sigma_m=\frac{\beta}{12}\dot\gamma^2$$

$$\sigma_{22}-\sigma_m=\frac{\beta}{12}\dot\gamma^2$$

$$\sigma_{33}-\sigma_m=-\frac{\beta}{6}\dot\gamma^2$$

The model predicts the differences in the normal stresses:

$$\sigma_{11}-\sigma_{22}=0$$

$$\sigma_{22}-\sigma_{33}=-\frac{1}{4}\beta\dot\gamma^2$$

The model predicts zero first normal-stress difference, but finite second normal-stress difference. The prediction disagrees with experimental observations. The Reiner-Rivlin model is a general mathematical model of viscosity, but the model does not predict experimentally observed form of normal stresses. The normal stress effect is pronounced in viscoelastic liquids, and cannot be captured by a model of pure viscosity.

Reiner-Rivlin model struggles to satisfy the thermodynamic inequality. Let us return to the Reiner-Rivlin model:

$$\sigma_{ij}=a\delta_{ij}+bD_{ij}+cD_{ik}D_{kj}.$$

The external force does work at the rate

$$\sigma_{ij}D_{ij}=aD_{kk}+bD_{ij}D_{ij}+cD_{ik}D_{kj}D_{ji}.$$

For an incompressible fluid, $D_{kk} = 0$. The second invariant is nonnegative for all rates of deformation, $D_{ij}D_{ij} \geq 0$.

The third invariants, however, can be either positive or negative. If we choose c as a constant, the model will violate the thermodynamic inequality $\sigma_{ij}D_{ij} \geq 0$ for some rates of deformation. If we choose $c = \zeta D_{ij}D_{jk}D_{ki}$ with ζ as a positive constant, then the Reiner-Rivlin model satisfies the thermodynamic inequality for all rates of deformation.

A Function of Many Variables

Rheological models are often phrased in terms of a function of many variables. The theory of a function of many variables is a subject where geometry, algebra and analysis meet. The three branches of mathematics use distinct dialects, which can be a source of confusion, as well as inspiration. We need to translate concepts from one dialect to another.

A function of many variables. Let V be an n-dimensional vector space and S be a scalar set. Let Q be a function that maps a vector $\mathbf{v}$ in V to a scalar s in S:

$$s = Q(\mathbf{v}).$$

We often write the vector $\mathbf{v}$ in terms of its components, $(v_1,...,v_n)$, and write the function as $Q(v_1,...,v_n)$.

Level set. For a given scalar a in S, the equation

$$Q(\mathbf{v}) = a$$

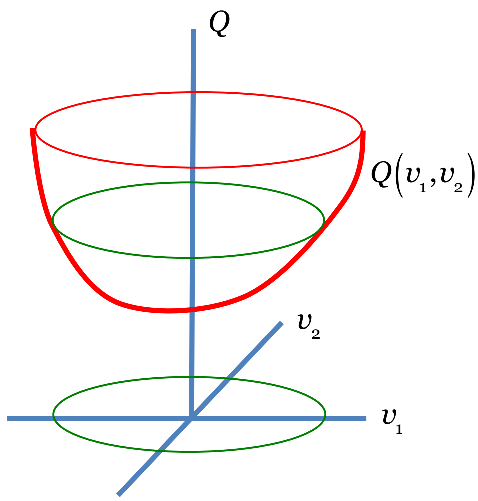
defines a subset of vectors in V, called a *level set*.

We can represent a level set graphically when n = 2. Let V be a two-dimensional vector space. We use two vectors in V as a basis. We use the two vectors as the axes of the plane. Each point in the plane corresponds to a vector in the V. For a vector $\mathbf{v}$, let its components relative to the basis be v_1 and v_2.

Write the function as $Q(v_1,v_2)$. We erect Q as the third axes. In the three-dimensional space, the function $Q(v_1,v_2)$ is a surface.

Given a scalar a in S, the equation $Q = a$ represents a plane in the three dimensional space at the height a. The equation $Q(v_1,v_2) = a$ represents the intersection of the plane $Q = a$ and the surface $Q = Q(v_1,v_2)$. The intersection is a curve parallel to the plane (v_1, v_2). We can move this curve vertically to the plane (v_1, v_2), and label this curve by the value *a*. This curve is a level set.

We can repeat this procedure for another value *b*, and add the curve $Q(v_1, v_2) = b$ to the plane (v_1, v_2). This procedure leads to a contour plot of the function $Q(v_1, v_2)$.



Gradient. Once again consider a function that maps a vector to a scalar:

$$s = Q(\mathbf{v}).$$

When the vector changes from $\mathbf{v}$ to $\mathbf{v} + d\mathbf{v}$, the scalar changes from $Q(\mathbf{v})$ to $Q(\mathbf{v} + d\mathbf{v})$. According to calculus, for any small change $d\mathbf{v}$,

$$Q(\mathbf{v} + d\mathbf{v}) - Q(\mathbf{v}) = \frac{\partial Q(\mathbf{v})}{\partial v_1} dv_1 + ... + \frac{\partial Q(\mathbf{v})}{\partial v_n} dv_n .$$

The *n* partial derivatives

$$\frac{\partial Q(\mathbf{v})}{\partial v_1}, ..., \frac{\partial Q(\mathbf{v})}{\partial v_n}$$

are components of another vector, called the *gradient*. Denote the gradient by

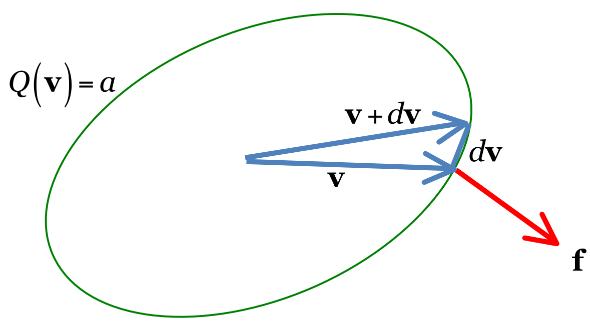
$$f_i = \frac{\partial Q(v_1, ..., v_n)}{\partial v_i}.$$

Given a vector **v**, the gradient **f** is known.

Write the change in the value of the function as $dQ = Q(\mathbf{v} + d\mathbf{v}) - Q(\mathbf{v})$. We translate the above result in analysis into a statement using words in geometry and algebra: the change in the scalar, dQ, is the inner product of the gradient $\mathbf{f}$ and the vector $d\mathbf{v}$, namely,

$$dQ = f_1 dv_1 + \ldots f_n dv_n .$$

Level set and gradient. Consider a special change $d\mathbf{v}$ such that the function has the same value at the two vectors, that keeps the value of Q unchanged—that is, $Q(\mathbf{v} + d\mathbf{v}) = Q(\mathbf{v})$, or dQ = 0. This vector $d\mathbf{v}$ is tangent to a level set $Q(\mathbf{v}) = a$. The expression $dQ = f_1 dv_1 + \ldots f_n dv_n$ says that the inner product of the gradient $\mathbf{f}$ and the vector tangent to the level set $d\mathbf{v}$ vanishes. Consequently the gradient of $\mathbf{f}$ is a vector normal to the level set passing through the vector $\mathbf{v}$.



Legendre transform. Given a function $Q(v_1, \ldots, v_n)$, and given a vector $(v_1, \ldots, v_n)$, the definition

$$f_i = \frac{\partial Q(v_1, \ldots, v_n)}{\partial v_i}$$

let us calculate the vector $(f_1, \ldots f_n)$. This procedure takes a single function $Q(v_1, \ldots, v_n)$ into n functions $f_i(v_1, \ldots, v_n)$. Recall an identify in calculus:

$$dQ = f_1 dv_1 + \ldots f_n dv_n .$$

Define a new scalar by

$$R = v_1 f_1 + \ldots + v_n f_n - Q .$$

The scalar R is called the Legendre transform of Q. A combination of the above two relations gives that

$$dR = v_1 df_1 + \ldots + v_n df_n .$$

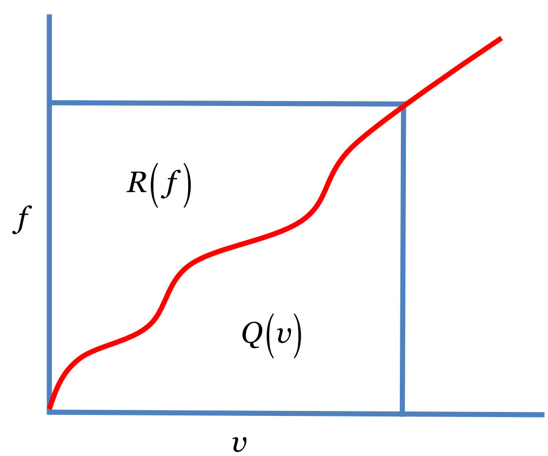
Invertible Legendre transform. We next assume that the functions $f_i(v_1,...,v_n)$ are *invertible*. That is, associated with a vector $(f_1,...f_n)$ is a unique vector $(v_1,...,v_n)$. Under this assumption of inevitability, the definition of the Legendre transform, $R = v_1 f_1 + ... + v_n f_n - Q$, allows us to write R as a function $R(f_1,...f_n)$. According to calculus,

$$dR = \frac{\partial R(f_1,...,f_n)}{\partial f_1} df_1 + ... + \frac{\partial R(f_1,...,f_n)}{\partial f_n} df_n .$$

Comparing this identity to $dR = v_1 df_1 + ... + v_n df_n$, we get

$$v_i = \frac{\partial R(f_1,...,f_n)}{\partial f_i}.$$

So long as the functions $f_i(v_1,...,v_n)$ are invertible, the function $Q(v_1,...,v_n)$ and its Legendre transform $R(f_1,...f_n)$ act symmetrically. The Legendre transform of the $R(f_1,...f_n)$ is $Q(v_1,...,v_n)$.



Legendre transform in a diagram. For a function of a single variable, its Legendre transform has a graphic interpretation.

Let $Q(v)$ be a function of a single variable. The gradient of the function is

$$f = \frac{dQ(v)}{dv}.$$

This relation defines a function $f(v)$. Note that

$$dQ = fdv .$$

Define the Legendre transform of the function $Q(v)$ by

$$R = fv - Q .$$

Note that

$$dR = vdf .$$

Provided $f(v)$ is a one-to-one function, the definition $R = fv - Q$ leads to a function $R(f)$. In this case, the relation $dR = vdf$ gives that

$$v = \frac{dR(f)}{df} .$$

So long as the *f*-*v* relation is one-to-one, the function $Q(v)$ and its Legendre transform $R(f)$ act symmetrically.

A relation between *f* and *v* corresponds to a curve in the plane with *f* and *v* as axes. The *f*-*v* relation is one-to-one if and only if the curve is *monotonic*. To simplify the description, we use a monotonically increasing curve, and place the origin of the coordinates on the curve. For given values of *v* and *f*, the function $Q(v)$ is the area between the curve and the *v*-axis, the function $R(f)$ is the area between the curve and the *f*-axis, and the product *fv* is the area of the rectangle. The Legendre transform means the obvious geometric relation between the three areas:

$$R(f) + Q(v) = fv .$$

Such a diagram is commonly used in mechanics, and $Q(v)$ and $R(f)$ are said to be *complementary functions*.

Convex Functions

Monotonically increasing function. For a smooth function of a singe variable, $f(v)$, the following three statements are equivalent:

- The function is a monotonically increasing function, i.e., $(u-v)[f(u)-f(v)] > 0$ for all unequal *u* and *v*.
- The derivative of the function is positive, $df(v)/dv > 0$.
- The indefinite integral of the function, $Q(v) = \int f(v)dv$, is convex.

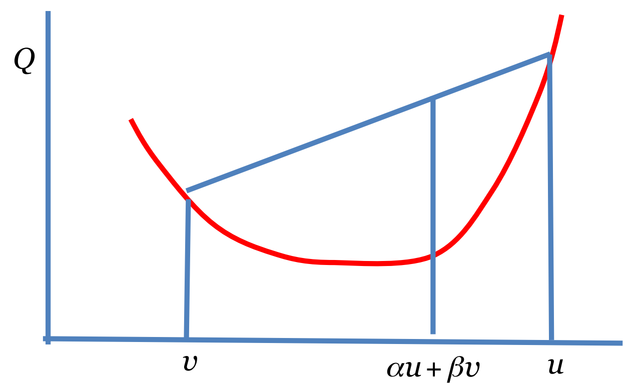
We next generalize these properties to a function of many variables.

Convex function. A function $Q(\mathbf{v})$ is called *convex* if

$$Q(\alpha \mathbf{u} + \beta \mathbf{v}) \le \alpha Q(\mathbf{u}) + \beta Q(\mathbf{v})$$

for all vectors **u** and **v** and for all positive numbers satisfying $\alpha + \beta = 1$. The function is called *strictly convex* if the equality holds only when **u** = **v**.

Unless otherwise noted, we will only consider strictly convex functions, and will not repeat the word "strictly". This definition formalizes the meaning of the word "convex". Two well-known textbooks on convex functions are Rockafellar (1970) and Roberts and Varberg (1973). We next list several theorems with partial proofs.



Theorem A. A differentiable function $Q(\mathbf{v})$ is convex if and only if

$$(u_i - v_i)\frac{\partial Q(\mathbf{v})}{\partial v_i} \le Q(\mathbf{u}) - Q(\mathbf{v}) \qquad \text{(A)}$$

for all unequal vectors **u** and **v**. When the function has a single variable, this theorem has a graphic interpretation.

Proof. If $Q(\mathbf{v})$ is convex, then for $0 < \alpha < 1$,

$$Q[\alpha \mathbf{u} + (1-\alpha)\mathbf{v}] < \alpha Q(\mathbf{u}) + (1-\alpha) Q(\mathbf{v})$$

for all unequal vectors **u** and **v**. Rearranging, we get

$$\frac{Q[\mathbf{v} + \alpha(\mathbf{u} - \mathbf{v})] - Q(\mathbf{v})}{\alpha} < Q(\mathbf{u}) - Q(\mathbf{v}).$$

The limit $\alpha \to 0$ gives inequality (A).

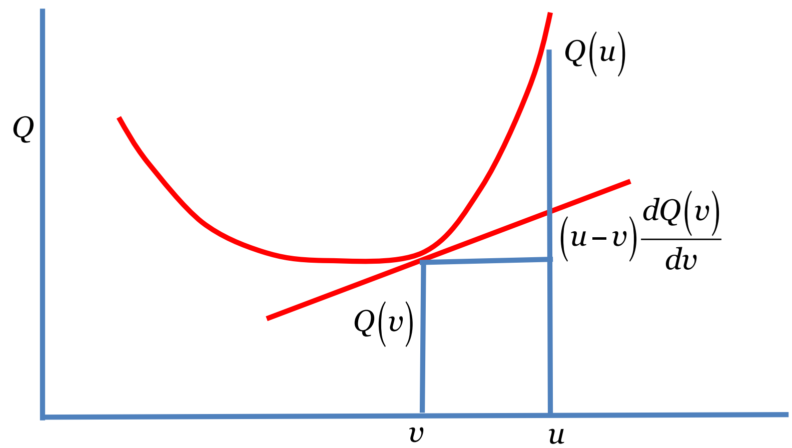
We next prove the converse is also true. Let $\mathbf{w} = \alpha \mathbf{u} + (1-\alpha)\mathbf{v}$. Write

$$Q(\mathbf{w}) = \alpha\left[Q(\mathbf{w}) + (u_i - w_i)\frac{\partial Q(\mathbf{w})}{\partial w_i}\right] + (1-\alpha)\left[Q(\mathbf{w}) + (v_i - w_i)\frac{\partial Q(\mathbf{w})}{\partial w_i}\right]$$

Using inequality (A), we get

$$Q(\mathbf{w}) < \alpha Q(\mathbf{u}) + (1-\alpha) Q(\mathbf{v}),$$

proving that the function $Q(\mathbf{v})$ is convex.



Theorem B. A differentiable function $Q(\mathbf{v})$ is convex if and only if

$$(u_i - v_i)\left[\frac{\partial Q(\mathbf{u})}{\partial u_i} - \frac{\partial Q(\mathbf{v})}{\partial v_i}\right] > 0. \quad \text{(B)}$$

for all unequal vectors **u** and **v**. This theorem generalizes the definition of a monotonically increasing function of a single variable.

Proof. Suppose the function $Q(\mathbf{v})$ is convex. We apply Theorem A twice:

$$(u_i - v_i)\frac{\partial Q(\mathbf{v})}{\partial v_i} < Q(\mathbf{u}) - Q(\mathbf{v}),$$

$$(v_i - u_i)\frac{\partial Q(\mathbf{u})}{\partial u_i} < Q(\mathbf{v}) - Q(\mathbf{u}).$$

Adding the two inequalities, we obtain (B).

Theorem C. A differentiable function $Q(\mathbf{v})$ is convex if and only if the matrix

$$\frac{\partial^2 Q(\mathbf{v})}{\partial x_i \partial x_j}$$

is positive-definite.

Theorem D. For a convex function $Q(\mathbf{v})$, the expression

$$f_i = \frac{\partial Q(\mathbf{v})}{\partial v_i}$$

is a one-to-one map between vector **v** and vector **f**.

Proof. If the function Q had the same gradient at two unequal vectors **u** and **v**, we would get

$$\frac{\partial Q(\mathbf{u})}{\partial u_i} = \frac{\partial Q(\mathbf{v})}{\partial v_i}.$$

This relation violates inequality (B).

Theorem E. The Legendre transform of a convex function is also convex.

Dissipation Function

Rayleigh's model of viscosity. Consider a system of n generalized coordinates. The time derivative of each generalized coordinate defines the associated generalized velocity. Associated with this generalized velocity is a generalized viscous force. Thus, the system has n generalized velocities, $v_1,...,v_n$, and n generalized viscous forces, $f_1,...,f_n$. A general model of viscosity is to write each force as a function of the n velocities:

$$f_1 = f_1(v_1,...,v_n)$$

$$......$$

$$f_n = f_n(v_1,...,v_n)$$

These functions are nonlinear if the viscous behavior is nonlinear.

Rayleigh (1871) introduced a model of viscosity. He assumed that a function $Q(v_1,...,v_n)$ exists, such that the viscous forces relate to the velocities as

$$f_i = \frac{\partial Q(v_1,...,v_n)}{\partial v_i}.$$

He called $Q(v_1,...,v_n)$ the *dissipation function*.

Rayleigh's model of viscosity is more restricted than the general model of viscosity, but is simpler. His model only requires a single scalar function of n variables, $Q(v_1,...,v_n)$. Later research has shown that Rayleigh's model has remarkable mathematical properties. In most applications, Rayleigh's model has prevailed over any other models.

Forces as independent variables. Rayleigh's model of viscosity uses the n velocities as independent variables. Alternatively, we can use the n forces as independent variables.

Start with a scalar function of n generalized forces, $R(f_1,...,f_n)$. Assume that each velocity is the partial derivative of the creep potential with respect to the force associated with the velocity:

$$v_i = \frac{\partial R(f_1,...,f_n)}{\partial f_i}.$$

The function $R(f_1,...,f_n)$ is known as the creep potential, flow potential, or plastic potential.

Now we have two models, one based on a function of velocities, and the other based on a function of forces. Are the two models the same? Here we see the power of mathematics. We can answer this question without worrying whether either model is useful in practice. So long as the dissipation function $Q(v_1,...,v_n)$ is convex, its Legendre transform defines the creep potential $R(f_1,...,f_n)$. Furthermore, the convexity ensures that the model is a one-to-one map between viscous forces and velocities

Thermodynamic inequality. The viscous forces dissipate power $f_i v_i$. Thermodynamics requires that the dissipation be positive-definite:

$$f_i v_i \geq 0.$$

The equality holds only when all velocities vanish.

In Rayleigh's model of viscosity, the thermodynamic inequality takes the form

$$\frac{\partial Q(v_1,...,v_n)}{\partial v_i} v_i \geq 0.$$

This inequality restricts the choice of the dissipation function. For the dissipation to be positive-definite, the angle between the two vectors must be acute. The shape of the surface is restricted to ensure that the force vector and the velocity vector form an acute angle.

If $Q(v_1,...,v_n)$ is a convex function, and if the origin is inside the level set of the dissipation function, Rayleigh's model of viscosity satisfies the thermodynamic inequality.

The thermodynamic inequality, however, does not requires that the dissipation function be convex. For example, in the case of a system of a single degree of freedom, the relation between the viscous force and the velocity may not be monotonic. Such a system can satisfy the thermodynamic inequality, but the dissipation function is no longer convex.

Linear viscosity. Rayleigh restricted himself to linear viscosity—that is, the viscous forces are linear in velocities. In this case, the dissipation function is quadratic in the generalized velocities:

$$Q = \frac{1}{2} H_{ij} v_i v_j ,$$

where the matrix H_{ij} represents generalized viscosities, and is independent of the velocities. We call H_{ij} the viscosity matrix. The anti-symmetric part of the matrix does not affect the value of Q. We assume that the matrix is symmetric, $H_{ij} = H_{ji}$. Rayleigh's model, $f_i = \partial Q(v_1,...,v_n) / \partial v_i$, gives

$$f_i = H_{ij} v_j .$$

The viscous forces are linear in the velocities.

The viscous forces dissipate power

$$f_i v_i = H_{ij} v_i v_j .$$

The power is positive definite if and only if the matrix H_{ij} is positive-definite.

Thus, for Rayleigh's model of linear viscosity, the three statements are equivalent: the dissipation function is positive-definite, the dissipation function is convex, and the model satisfies the thermodynamic inequality.

Linear fluidity. When the matrix H_{ij} is positive-definite, it is certainly nonsingular. We can invert the matrix and obtain

$$v_j = G_{ij} f_i .$$

The matrix G_{ij} represents the generalized fluidities. When the matrix H_{ij} is symmetric and positive-definite, so is the matrix G_{ij}.

Introduce a scalar

$$R = \frac{1}{2} G_{ij} f_i f_j .$$

This scalar is a function of the viscous forces, $R(f_1,...,f_n)$, and is called the creep potential. We confirm that

$$v_i = \frac{\partial R(f_1,...,f_n)}{\partial f_i} .$$

We can confirm that the two functions satisfy

$$Q + R = f_i v_i .$$

The two functions $Q(v_1,...,v_n)$ and $R(f_1,...,f_n)$ are Legendre transform of each other.

A nonlinear model that satisfies the thermodynamic inequality. We construct a model by assuming that the dissipation function depends on the velocities through a single variable, a positive-definite quadratic form of the velocities: $Q(\alpha)$ with $\alpha = H_{ij}v_i v_j$. Inserting this dissipation function into $f_i = \partial Q(v_1,...,v_n)/\partial v_i$, we get

$$f_i = \beta H_{ij} v_j ,$$

where $\beta = 2dQ(\alpha)/d\alpha$. The factor β represents the nonlinearity of the model. So long as $Q(\alpha)$ is monotonically increasing function, $\beta > 0$, and the power dissipation

$$f_i v_i = \beta H_{ij} v_j v_i$$

is positive-definite.

A system of many degrees of freedom commonly consists of multiple dissipative elements, represented by nonlinear dashpots. We require each dashpot to satisfy the thermodynamic inequality.

Dissipation Function in Multiaxial Stress State

Apply Rayleigh's model to construct a nonlinear creep model under multiaxial stress state. Start with a scalar function of stress, $F(\sigma_{11},...,\sigma_{12})$. The function maps a tensor to a scalar. Call the function the creep potential. Assumes that each component of the rate-of-deformation tensor is a partial derivative of the creep potential with respect to the component of stress associated with the component of the rate of deformation:

$$D_{ij} = \frac{\partial F(\sigma_{11},...,\sigma_{12})}{\partial \sigma_{ij}} .$$

For an incompressible, isotropic material, the creep potential is a function of the two invariants of the deviatoric stress, $F(J_2, J_3)$. For simplicity, we drop the dependence on J_3 and assume that the creep potential is a function of a single variable, $F(J_2)$. According to Rayleigh's model, the rate of deformation is

$$D_{ij} = \frac{\partial F(J_2)}{\partial \sigma_{ij}} = \frac{dF(J_2)}{dJ_2}\frac{\partial J_2}{\partial \sigma_{ij}} .$$

Recall that $J_2 = s_{ij}s_{ij}/2$, so that $\partial J_2 / \partial \sigma_{ij} = s_{ij}$. The model becomes that

$$D_{ij} = \frac{s_{ij}}{2\eta},$$

where the viscosity is a function of the second invariant, $\eta(J_2)$. This procedure recovers the second-invariant model.

The second invariant $J_2 = s_{ij}s_{ij}/2$ is a convex function of the deviatoric stress. So long as $F(J_2)$ is a monotonically increasing function of J_2, F is a convex function of the deviatoric stress.

Flow potential. Adopting the second-invariant model, we can obtain an explicit expression for the flow potential. Suppose that we have tested material under pure shear, and obtain the relation between the rate of shear and the shearing stress, $\dot{\gamma} = h(\tau)$. We wish to use the flow curve $\dot{\gamma} = h(\tau)$ to express the flow potential $F(J_2)$.

When the material is subject to a general state of stress σ_{ij}, the mean stress is $\sigma_m = \sigma_{kk}/3$, the deviatoric stress is $s_{ij} = \sigma_{ij} - \sigma_m \delta_{ij}$, and the equivalent shearing stress is

$$\tau_e = \sqrt{\frac{1}{2}s_{ij}s_{ij}}.$$

This definition is just another way to write the second invariant of the deviatoric stress, and reduces to the shearing stress when the material is under pure shear. The second-invariant model gives the relation between the deviatoric stress and the rate of deformation:

$$D_{ij} = \frac{h(\tau_e)}{2\tau_e}s_{ij}.$$

The material is incompressible, $D_{kk} = 0$. Write

$$dF = D_{ij}d\sigma_{ij} = D_{ij}ds_{ij} = \frac{h(\tau_e)}{2\tau_e}s_{ij}ds_{ij} = h(\tau_e)d\tau_e.$$

Thus, the dissipation function is

$$F = \int_0^{\tau_e} h(\tau_e)d\tau_e.$$

For a power-law fluid under pure shear,

$$\dot{\gamma} = \left(\frac{\tau}{A}\right)^n.$$

The dissipation function is

$$F(\tau_e) = \frac{A}{n+1}\left(\frac{\tau_e}{A}\right)^{n+1}.$$

This relation expresses the flow potential as a function of the equivalent stress, i.e., a function of the second invariant.

Dissipation function. We can also use the rate of deformation as the independent variable. We characterize the material using a dissipation function $W(\mathbf{D})$, in the sense that the stress relates to the rate of deformation by

$$\sigma_{ij} = \frac{\partial W(\mathbf{D})}{\partial D_{ij}}.$$

We next adopt the second-invariant model and obtain an explicit expression for the dissipation function. Suppose that we have tested material under uniaxial tension, and obtain the relation between the tensile stress and the rate of extension, $\sigma = f(\dot{\varepsilon})$. Under the uniaxial tensile stress, the rate of deformation is

$$[D_{ij}] = \begin{bmatrix} \dot{\varepsilon} & 0 & 0 \\ 0 & -\dot{\varepsilon}/2 & 0 \\ 0 & 0 & -\dot{\varepsilon}/2 \end{bmatrix}$$

The second invariant is

$$D_{ij}D_{ij} = (\dot{\varepsilon})^2 + \left(\frac{\dot{\varepsilon}}{2}\right)^2 + \left(\frac{\dot{\varepsilon}}{2}\right)^2 = \frac{3}{2}(\dot{\varepsilon})^2.$$

For a fluid in a generate state of flow D_{ij}, we define an equivalent rate of extension by

$$\dot{\varepsilon}_e = \sqrt{\frac{2}{3}D_{ij}D_{ij}}.$$

This definition is just another way to write the second invariant, and reduces to the rate of extension when the fluid is subject to a uniaxial tensile stress. The second-invariant model gives the relation between the deviatoric stress and the rate of deformation:

$$s_{ij} = \frac{2f(\dot{\varepsilon}_e)}{3\dot{\varepsilon}_e}D_{ij}.$$

The fluid is incompressible, $D_{kk} = 0$. Write

$$dW = \sigma_{ij}dD_{ij} = s_{ij}dD_{ij} = \frac{2f(\dot{\varepsilon}_e)}{3\dot{\varepsilon}_e}D_{ij}dD_{ij} = f(\dot{\varepsilon}_e)d\dot{\varepsilon}_e.$$

Thus, the dissipation function is

$$W(\dot{\varepsilon}_e) = \int_0^{\dot{\varepsilon}_e} f(\dot{\varepsilon}_e) d\dot{\varepsilon}_e .$$

For a power-law fluid under uniaxial tension,

$$\sigma = K(\dot{\varepsilon})^N .$$

The dissipation function is

$$W(\dot{\varepsilon}_e) = \frac{K}{N+1}(\dot{\varepsilon}_e)^{N+1} .$$

This relation expresses the dissipation function as a function of the equivalent rate of extension, i.e., a function of the second invariant.

Convex dissipation function. Boundary-value problems of nonlinearly viscous materials have nice properties if we assume that the dissipation function $W(\mathbf{D})$ is convex (Hill 1956). That is,

$$W(\alpha\mathbf{D} + \beta\mathbf{D}^*) \le \alpha W(\mathbf{D}) + \beta W(\mathbf{D}^*)$$

for all $\mathbf{D}$ and $\mathbf{D}^*$ and for all numbers satisfying $\alpha + \beta = 1$, $\alpha > 0$ and $\beta > 0$. The equality holds only when $\mathbf{D} = \mathbf{D}^*$.

Theorem A is written as

$$W(\mathbf{D}^*) - W(\mathbf{D}) > (D_{ij}^* - D_{ij}) \frac{\partial W(\mathbf{D})}{\partial D_{ij}}$$

for any two states $\mathbf{D}^* \ne \mathbf{D}$.

Theorem B is written as

$$(D_{ij} - D_{ij}^*) \left[\frac{\partial W(\mathbf{D})}{\partial D_{ij}} - \frac{\partial W(\mathbf{D}^*)}{\partial D_{ij}^*} \right] > 0 ,$$

or

$$(D_{ij} - D_{ij}^*)(\sigma_{ij} - \sigma_{ij}^*) > 0 .$$

Second-invariant model has a convex dissipation function. The second-invariant model specifies the dissipation function $W(\mathbf{D})$ by composition:

$$W = k(I_2) ,$$

$$I_2 = D_{ij} D_{ij} .$$

We fit the function $k(I_2)$ to the flow curve under a simple test, such as pure shear test and uniaxial tensile test. So long as the flow curve monotonically increases, so does the function $k(I_2)$. The second invariant is a convex function, $I_2 = D_{ij}D_{ij}$. Consequently, $W(\mathbf{D})$ is a convex function.

Uniqueness of Solution

Boundary-value problems. The equations for the boundary-value problems are as follows. We characterize the rheology of the material using a dissipation function $W(\mathbf{D})$, such that the stress relates to the rate of deformation by

$$\sigma_{ij} = \frac{\partial W(\mathbf{D})}{\partial D_{ij}}.$$

The compatibility of deformation relates the rate of deformation to the velocity as

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

For highly viscous material, we often neglect the effect of inertia. We also often neglect body force. Under these conditions, the balance of forces takes the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0.$$

We prescribe boundary conditions on the surface of the body. The surface of the body is divided into two parts. On one part of the surface, S_v, the velocity of the fluid is prescribed. One the other part of the surface, S_t, the traction is prescribed:

$$\sigma_{ij} n_j = t_i.$$

Principle of virtual power. Let D_{ij}^a and v_{ij}^a satisfy

$$D_{ij}^a = \frac{1}{2}\left(\frac{\partial v_i^a}{\partial x_j} + \frac{\partial v_j^a}{\partial x_i}\right).$$

They need not satisfy any other equations of the boundary-value problem. They are known as *virtual rate of deformation* and *virtual velocity*.

Let σ_{ij}^b satisfy

$$\frac{\partial \sigma_{ij}^b}{\partial x_j} + b_i = 0,$$

in the body, and

$$\sigma_{ij}^b n_j = t_i$$

on the surface. The field σ_{ij}^b need not satisfy any other equations of the boundary-value problem. The field is known as the *virtual stress*.

The two sets of fields satisfy an identity:

$$\int D_{ij}^a \sigma_{ij}^b dV = \int v_i^a b_j dV + \int v_i^a t_j dA .$$

The identity is known as the *principle of virtual power*. The proof invokes the divergence theorem:

$$\int D_{ij}^a \sigma_{ij}^b dV = \int \frac{\partial v_i^a}{\partial x_j} \sigma_{ij}^b dV$$

$$= \int \frac{\partial \left(v_i^a \sigma_{ij}^b \right)}{\partial x_j} dV - \int v_i^a \frac{\partial \sigma_{ij}^b}{\partial x_j} dV$$

$$= \int v_i^a \sigma_{ij}^b n_j dV - \int v_i^a \frac{\partial \sigma_{ij}^b}{\partial x_j} dV$$

Inserting the equations of force balance, we get the principle of virtual power.

Uniqueness of solution. Consider a creeping flow of a body of fluid. The rheology of the fluid is characterized by a convex dissipation function $W(\mathbf{D})$. At a given time, the shape of the body is known. The fluid is subject to body force $b_i(\mathbf{x})$ in the volume, traction $t_i(\mathbf{x})$ on part of the surface S_t, and velocity $v_i(\mathbf{x})$ on the other part of the surface S_v. All the equations do not explicitly contain time, and form a boundary-value problem that governs the velocity field. Will the solution to this boundary-value problem be unique?

Suppose that two distinct solutions exist, denoted as $\left(\sigma_{ij}, D_{ij}, v_i\right)$ and $\left(\sigma_{ij}^*, D_{ij}^*, v_i^*\right)$. The relation between the rate of deformation and the velocity is linear:

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

Consequently, the fields $D_{ij} - D_{ij}^*$ and $v_i - v_i^*$ satisfy the equation of compatibility, and $v_i - v_i^*$ vanishes on S_v.

For the creeping flow, the effect of inertial is neglected, and the balance of forces gives leads to linear equations:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0$$

in the body and

$$\sigma_{ij} n_j = t_i ,$$

on S_t. Consequently, the field $\sigma_{ij} - \sigma^*_{ij}$ satisfies the equations of force balance, with vanishing body force and vanishing traction on S_t.

Applying the principle of virtual power, we obtain that

$$\int \left(D_{ij} - D^*_{ij} \right) \left(\sigma_{ij} - \sigma^*_{ij} \right) dV = 0 .$$

This equation violates the inequality $\left(D_{ij} - D^*_{ij} \right) \left(\sigma_{ij} - \sigma^*_{ij} \right) > 0$ for the convex dissipation function.

Variational Principles

Variational principle. Let $\mathbf{v}$ be a field that satisfies the prescribed velocity on S_v, and calculate

$$D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

Such a field $\mathbf{v}$ is known as a virtual velocity, and the field $\mathbf{D}$ is known as the associated virtual rate of deformation. Define

$$\Pi(\mathbf{v}) = \int W(\mathbf{D}) dV - \int b_i v_i \, dV - \int t_i v_i \, dA .$$

The last integral is over S_t. The functional $\Pi(\mathbf{v})$ maps a field of virtual velocity to a scalar. The scalar has the unit of power.

Theorem. Of all fields virtual velocity $\mathbf{v}$, the field of actual velocity minimizes the functional $\Pi(\mathbf{v})$.

Proof. Recall theorem A for the convex function $W(\mathbf{D})$. For any unequal tensors, we have

$$W(\mathbf{D}^*) - W(\mathbf{D}) > \left(D^*_{ij} - D_{ij} \right) \frac{\partial W(\mathbf{D})}{\partial D_{ij}} .$$

Now we designate $(\mathbf{v}, \mathbf{D})$ as the actual solution, and $(\mathbf{v}^*, \mathbf{D}^*)$ as the virtual field. Using Theorem A, we obtain that

$$\Pi(\mathbf{v}^*) - \Pi(\mathbf{v}) = \int \left[W(\mathbf{D}^*) - W(\mathbf{D}) \right] dV - \int b_i \left(v^*_i - v_i \right) dV - \int t_i \left(v^*_i - v_i \right) dA$$

$$> \int \left(D^*_{ij} - D_{ij} \right) \frac{\partial W(\mathbf{D})}{\partial D_{ij}} dV - \int b_i \left(v^*_i - v_i \right) dV - \int t_i \left(v^*_i - v_i \right) dA$$

The last expression vanishes according to the principle of virtual power. Thus, $\Pi(\mathbf{v}^*) - \Pi(\mathbf{v}) > 0$.

Growth of a cavity. As an example of using the variational principle, consider again the growth of a cavity in a power-law material. The material is incompressible. We will not apply hydrostatic tension remote from the cavity. Instead, we apply radial compression of magnitude σ_{appl} on the surface of the cavity.

The incompressibility requires that the radial velocity take the form

$$v = \frac{a^2}{r^2}\frac{da}{dt}.$$

The radial and the circumferential rates of deformation are

$$D_r = -2\frac{a^2}{r^3}\frac{da}{dt}, \quad D_\theta = \frac{a^2}{r^3}\frac{da}{dt}.$$

The equivalent rate of extension is

$$\dot{\varepsilon}_e = \left|D_r\right| = 2\frac{a^2}{r^3}\frac{da}{dt}.$$

The dissipation function is

$$W = \frac{K}{N+1}\left|D_r\right|^{N+1}.$$

The functional is

$$\Pi = \int_a^\infty \frac{K}{N+1}\left(2\frac{a^2}{r^3}\frac{da}{dt}\right)^{N+1} 4\pi r^2 dr - 4\pi a^2 \sigma_{appl}\frac{da}{dt}.$$

Integrating, we obtain that

$$\Pi = 4\pi \frac{K\left(2a^2\right)^{N+1}}{3N\left(N+1\right)a^{3N}}\left(\frac{da}{dt}\right)^{N+1} - 4\pi a^2 \sigma_{appl}\frac{da}{dt}.$$

The functional reduces to a function of single variable, the velocity at the surface of the cavity, da/dt. Minimizing Π by taking differentiation with respect to da/dt, we obtain that

$$\sigma_{appl} = \frac{2K}{3N}\left(\frac{2}{a}\frac{da}{dt}\right)^N.$$

This result recovers that obtained by solving the differential equations.

Complementary variational principle. We describe the rheology of a fluid by a flow potential, $F\left(\sigma_{11},...,\sigma_{12}\right)$, such that

$$D_{ij} = \frac{\partial F\left(\sigma_{11},...,\sigma_{12}\right)}{\partial \sigma_{ij}}.$$

We will abbreviate the flow potential as $F\left(\sigma\right)$.

Let σ_{ij} satisfy

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0,$$

in the body, and

$$\sigma_{ij} n_j = t_i$$

on S_t. The field σ_{ij} need not satisfy any other equations of the boundary-value problem. The field is known as the *virtual stress*.

Define

$$\Phi(\sigma) = \int F(\sigma) dV - \int t_i v_i dA .$$

The last integral is over S_v. The functional $\Phi(\sigma)$ maps a field of virtual stress to a scalar. The scalar has the unit of power.

Theorem. Of all virtual fields of stress, the field of actual stress minimizes the functional $\Phi(\sigma)$.

Proof. Recall theorem A for the convex function $F(\sigma)$. For any unequal tensors, we have

$$F(\sigma^*) - F(\sigma) > (\sigma^*_{ij} - \sigma_{ij}) \frac{\partial F(\sigma)}{\partial \sigma_{ij}} .$$

Now we designate $(\sigma_{ij}, v_i, D_{ij})$ as the actual field, and σ^*_{ij} as a virtual field. Using Theorem A, we obtain that

$$\Phi(\sigma^*) - \Phi(\sigma) = \int [F(\sigma^*) - F(\sigma)] dV - \int (t^*_i - t_i) v_i dA$$

$$> \int (\sigma^*_{ij} - \sigma_{ij}) \frac{\partial F(\sigma)}{\partial \sigma_{ij}} dV - \int (t^*_i - t_i) v_i dA$$

The last expression vanishes according to the principle of virtual power. Thus, $\Phi(\sigma^*) - \Phi(\sigma) > 0$.

References

H.A. Barnes, The yield stress. Journal of Non-Newtonian Fluid Mechanics 81, 133-178 (1999).

H.A. Barnes, J.F. Hutton, K. Walters. An Introduction to Rheology. Elsevier 1989.

E.C. Bingham, Fluidity and Plasticity. McGraw-Hill, 1922.

D.V. Boger and K. Walters. Rheological Phenomena in Focus. Elsevier, 1993.

B. Budiansky, J.W. Hutchinson, S. Slutsky. Void Growth and Collapse in Viscous Solids. In Mechanics of Solids edited by H. G. Hopkins and M. J. Sewell, Pergamon Press, 13-45 (1982). PDF file online: http://www.seas.harvard.edu/hutchinson/papers/362.pdf

B.D. Coleman, H. Markovitz, W. Noll. Viscometric Flows of Non-Newtonian Fluids. Springer, 1966.

H.J. Frost and M.F. Ashby, Deformation-Mechanism Maps: The Plasticity and Creep of Metals and Ceramics. Pergamon Press, 1982.

J.W. Glen, The creep of polycrystalline ice. Proceedings of the Royal Society of London 228, 519-538 (1955).

R. Hill, New horizons in the mechanics of solids. Journal of the Mechanics and Physics of Solids 5, 66-74 (1956).

M.Y. He and J.W. Hutchinson, The penny-shaped crack and the plane strain crack in an infinite body of power-law material. Journal of Applied Mechanics 48, 830-840 (1981). http://www.seas.harvard.edu/hutchinson/papers/360.pdf

J.W. Hutchinson, Singular behavior at the end of a tensile crack in a hardening material. Journal of the Mechanics and Physics of Solids **16**, 13-31 (1968). http://www.seas.harvard.edu/hutchinson/papers/312.pdf

F. Irgens, Rheology and Non-Newtonian Fluids. Springer, 2014.

AA Ilyushin, The theory of small elastic-plastic deformations. Prikadnaia Matematika i Mekhanika, PMM 10, 347-356 (1946).

H. Markovitz, The emergence of rheology. Physics Today 21, 23-30 (1968).

J. Mewis and N.J. Wagner, Thixotropy. Advances in Colloid and Interface Science 147-148, 214-227 (2009).

W. Prager, Introduction to Mechanics of Continua. Ginn and Company, 1961.

E.M. Purcell, Life at low Reynolds number. American Journal of Physics 45, 3-11 (1977).

Rayleigh, Some general theorems relating to vibration. Proceedings of London Mathematical Society s1-4, 357-368 (1871).

M. Reiner, A Mathematical Theory of Dilatancy. American Journal of Mathematics 67, 350-362 (1945).

M. Reiner, A classification of rheological properties. Journal of Scientific Instruments 22, 127-129 (1945)

J.R. Rice and G.F. Rosengren, Plane-strain deformation near a crack tip in a power-law hardening material. Journal of the Mechanics and Physics of Solids **16**, 1-12 (1968). http://esag.harvard.edu/rice/016_RiceRosengren_CrackSing_JMPS68.pdf

R.S. Rivlin, The hydrodynamics of non-Newtonian fluids. Proceedings of the Royal Society of London 193, 260-281 (1948).

R.T. Rockafellar, Convex Analysis. Princeton University Press, 1970.

A.W. Roberts and D.E. Varberg, Convex Functions. Academic Press, 1973.

S.P. Sutera and R. Skalak, The history of Poiseuille's law. Annual Review of Fluid Mechanics 25, 1-19 (1993).

R.I. Tanner, Engineering Rheology, 2nd edition. Oxford University Press, 2000.

R. von Mises, Mechanik der festen Koper im plastisch deformation. Zustand. Nachr. Ges. Wiss. Gottingen, 582-592 (1913)

R. von Mises, Mechanik der plastischen Formänderung von Kristallen, Zeitschrift für Angewandte Mathematik und Mechanik 8, 161-185 (1928)