An approximate JKR solution for a general contact, including rough contacts

M. Ciavarella

Politecnico di BARI. Center of Excellence in Computational Mechanics. Viale Gentile 182, 70126 Bari. Mciava@poliba.it

Abstract

In the present note, we suggest a simple closed form approximate solution to the adhesive contact problem under the so-called JKR regime. The derivation is based on generalizing the original JKR energetic derivation assuming calculation of the strain energy in adhesiveless contact, and unloading at constant contact area. The underlying assumption is that the contact area distributions are the same as under adhesiveless conditions (for an appropriately increased normal load), so that in general the stress intensity factors will not be exactly equal at all contact edges. The solution is simply that the indentation is $\delta = \delta_1 - \sqrt{2wA'/P''}$ where w is surface energy, δ_1 is the adhesiveless indentation, A' is the first derivative of contact area and P'' the second derivative of the load with respect to δ_1. The solution only requires macroscopic quantities, and not very elaborate local distributions, and is exact in many configurations like axisymmetric contacts, but also sinusoidal waves contact and correctly predicts some features of an ideal asperity model used as a test case and not as a real description of a rough contact problem. The solution permits therefore an estimate of the full solution for elastic rough solids with Gaussian multiple scales of roughness, which so far was lacking, using known adhesiveless simple results. The result turns out to depend only on rms amplitude and slopes of the surface, and as in the fractal limit, slopes would grow without limit, tends to the adhesiveless result – although in this limit the JKR model is inappropriate. The solution would also go to adhesiveless result for large rms amplitude of roughness h_{rms}, irrespective of the small scale details, and in agreement with common sense, well known experiments and previous models by the author.

Key words:

1. Introduction

Exact solution to adhesive problems are very scarse. Bradley (1932) and Derjaguin (1934) obtained the adhesive force between two *rigid* spheres, equal to $2\pi Rw$, where w is the work of adhesion, and R is the radius of the sphere. Then, JKR (Johnson Kendall and Roberts 1971) developed the first exact theory for elastic bodies, namely spheres, assuming adhesive forces occur entirely within the contact area, obtaining 3/4 of the Bradley pull-off value, and independence on the elastic modulus which seem to indicate that the result would be corresponding to the rigid Bradley limit. The result was even more surprising when Derjaguin-Muller-Toporov (DMT) developed their elastic theory (Derjaguin *et al.*, 1975) which seemed to indicate the same pull-off value of Bradley rather that JKR. Tabor brilliantly solved the dilemma, indicating transition from rigid to JKR depends on the Tabor parameter (Tabor, 1977)

$$\mu = \left(\frac{Rw^2}{E^{*2} a_0^3} \right)^{1/3} \tag{1}$$

where a_0 is the range of attraction of adhesive forces, close to atomic distance for crystals, and E^* the plane strain elastic modulus.

The JKR regime therefore is only valid for large Tabor parameters (especially if instability at jump-into contact is required accurately, see Ciavarella *et al.*2017). JKR permits to find many solutions easily by superposition of contact and crack solutions (see Johnson, 1995), whereas the original JKR energetic method has been less popular except of course the adhesion problem can be formulated in elaborate numerical algorithms by minimization methods (Carbone *et al.*, 2015). Particularly the problem of rough surfaces has seen significant effort in the last 40 years or so (Fuller & Tabor, 1975, Persson, 2002, Pastewka & Robbins, 2014, Persson and Scaraggi, 2014, Ciavarella, 2015, Afferrante *et al.*, 2015, Ciavarella and Papangelo, 2017a,b,c, Ciavarella *et al.*, 2017, 2018, Ciavarella, 2017a,b), but no simple theories exist which permit to estimate the JKR regime, including negative loads and pull-off, except for Fuller & Tabor (1975) asperity theory, which however has been questioned by Pastewka & Robbins (2014), and certainly contains many

strong approximations inherent in the asperity model. In the DMT regime, a very simple solution was given by Ciavarella (2017a) with a "bearing-area" model, which turned out to give very reasonable fit of the Pastewka & Robbins (2014) pull-off data, whereas some discrepancy was remarked about the area-slope "stickiness" criterion with their own pull-off data. Persson (2002) is aimed at the JKR regime, seems perfectly reversible, is quite complex and anyway it is probably not valid in unloading as shown in the plots in Persson and Scaraggi (2014) which only show the positive load regime – and also as explained in details by Carbone et al. (2015) who have constructed for 1D profiles, loading and unloading curves and PSD (Power Spectrum Density) of the deformed profile closely follows a power law predicted by Persson's theory, but not on unloading. In particular, they explain why Persson's theory is not adequate for adhesion. Moving to the DMT approximation of Persson and Scaraggi (2014), the contact is assumed to be split into "repulsive" contact areas and "attractive" contact areas, and no effect of tensile tractions occurs so there is a simple convolution of separation of the repulsive solution with the force-separation law — however, the DMT approximation leads to large errors even in the simple case of a sphere or a cylinder (Ciavarella, 2017b), and it is unclear what happens for rough contacts where many further approximations are made. In any case, the solution remains numerical and not simple in this case either.

All these models are purely theoretical or numerical. Experimental studies typically rely on spherical geometry like Fuller & Tabor (1975).

JKR (1971) originally derived an energetic method which could serve as an approximate solution to a much more general contact case, not just including halfspace geometries but really anything for which we know the adhesionless solution. We shall therefore generalize the JKR model to arbitrary contact geometry, in an approximate sense, in the present paper.

2. The model

We need to consider the total potential energy of the system comprising elastic strain energy, surface energy, and [when load P is prescribed] potential energy of the applied force.

The elastic strain energy can be determined by devising the original JKR loading scenario leading to the required final state and calculating the work done during loading. Such scenario is suggested by the superposition in comprising the two steps (see Fig.1)

- (i) "repulsive" loading without adhesive forces until the contact area is a given value (this is the load path $\overline{OA}$ as in the original JKR paper), followed by

- (ii) rigid-body displacement at constant total contact area A (this is load path $\overline{AB}$ as in the original JKR paper) of an unknown amount which we shall find by a minimization procedure of the total potential, like in the classical Griffith crack problem. In Irwin's equivalent procedure, this unloading could be prescribed until the required stress-intensity factor (SIF) is achieved at the contact edge. Here, we can only fulfill this requirement in an "average" sense, because the SIF at the indivual contact edges will differ. But it is not convenient to evaluate the individual SIFs and make their average, as this would require in general a cumbersome procedure. Similarly, a precise solution should minimize over many variables, which are the position and size and number of the contact spots (Carbone *et al.*, 2015)

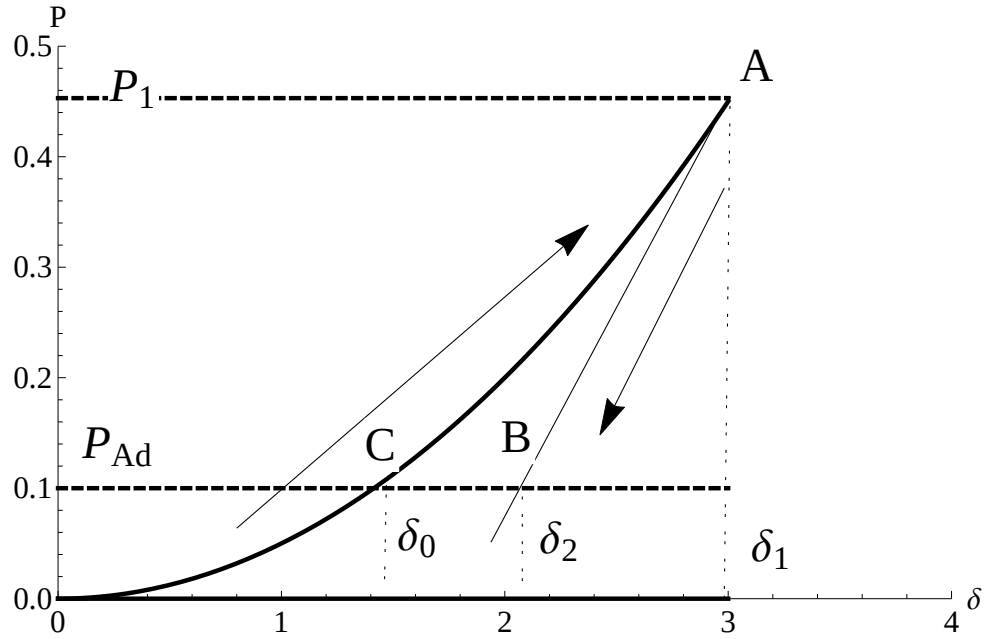


Fig.1. The loading scenario. (i) "repulsive" loading without adhesive forces until the contact area is a given value (load path $\overline{OA}$ as in the original JKR paper); (ii) rigid-body displacement at constant total contact area A (load path $\overline{AB}$ as in the original JKR paper)

In the present paper, we repeat this process in general, and not just for the spherical Hertzian contact as in the original JKR paper. During phase (i), we can integrate the load-displacement curve to get the elastic strain energy

$$U_1(\delta_1) = \int_0^{\delta_1} P(\delta)\, d\delta \tag{2}$$

while in the second phase (ii) we have an unknown downloading to $\delta = \delta_2$

$$U_2(\delta_1, \delta) = \int_{\delta_1}^{\delta} P_A(\delta)\, d\delta \tag{3}$$

where we write $P_A(\delta)$ for unloading as we keep the contact area A fixed.

We then obtain the total elastic strain energy is

$$U(\delta_1, \delta) = U_1(\delta_1) - U_2(\delta_1, \delta) \tag{4}$$

However, since the contact area is fixed in the second phase, the load really decreases linearly so the Taylor series expansion can be exactly truncated to first order

$$P_{Ad}(\delta) = P_1 + \left(\frac{\partial P}{\partial \delta}\right)_{\delta_1} (\delta - \delta_1) \tag{5}$$

and hence the work done is

$$U_2(\delta_1, \delta) = \int_{\delta_1}^{\delta} P_{Ad}(\delta)\, d\delta = P_1(\delta - \delta_1) + \left(\frac{\partial P}{\partial \delta}\right)_{\delta_1} \frac{(\delta - \delta_1)^2}{2} \tag{6}$$

2.1. Displacement control

The surface energy is $-Aw$, and hence if the displacement is prescribed, the total potential energy is

$$\Pi(\delta_1, \delta) = U(\delta_1, \delta) - Aw \tag{7}$$

In practice, we would need to write Π as a function of the contact area A since the equilibrium position is then determined by the condition

$$\frac{\partial \Pi}{\partial A} = 0 \tag{8}$$

However, this can be rewritten obviously using the chain's rule, as

$$\frac{\partial U}{\partial \delta_1}\frac{\partial \delta_1}{\partial A} = w \tag{9}$$

Now, trivially we find

$$\frac{\partial U_1}{\partial \delta_1} = P_1$$

$$\frac{\partial U_2}{\partial \delta_1} = -P_1 + \frac{\partial P_1}{\partial \delta_1}(\delta - \delta_1) + \frac{\partial \left[\left(\frac{\partial P}{\partial \delta}\right)_{\delta_1}\right]}{\partial \delta_1}\frac{(\delta - \delta_1)^2}{2} - \left(\frac{\partial P}{\partial \delta}\right)_{\delta_1}(\delta - \delta_1)$$

Summing the contribution and cancelling the equal terms, this reduces to

$$\frac{\partial^2 P}{\partial \delta_1^2}\frac{(\delta - \delta_1)^2}{2} - w\frac{\partial A}{\partial \delta_1} = 0$$

which solves into

$$\delta = \delta_1 - \sqrt{2w\frac{\partial A}{\partial \delta_1}/\frac{\partial^2 P}{\partial \delta_1^2}} \tag{10}$$

as we have choosen only the physical solution for $\delta < \delta_1$. This (10) is the quite general approximate solution, valid *for any arbitrary contact problem.* Notice that it only requires macroscopic quantities, and not very elaborate local distributions.

A solution under force control will not differ, but the stability condition (which involves the second derivative of the potential) will be different. However, we don't need to describe this trivial extension.

2.2. Check for Hertzian contact, and other validations

For Hertzian contact,

$$\frac{\pi a^2}{\pi R} = \frac{A}{\pi R} = \delta_1, \quad P = \frac{4}{3}E^*\sqrt{R}\delta_1^{3/2},$$

and hence from our general result (10)

$$\delta = \delta_1 - \sqrt{\frac{2w\pi\sqrt{R}}{E^*}\delta_1^{1/2}} \tag{11}$$

which reduces to the known form of the exact JKR solution (Johnson et al, 1971), when translated back into an expression for indentation vs contact radius

$$\delta = \frac{a^2}{R} - \sqrt{\frac{2w\pi}{E^*}a}$$

This exact coincidence of the proposed result (10) with the JKR solution would also occur for any axisymmetric contact case (see Popov and Heß, 2015), since obviously the contact area remains circular, including the case of a waviness when contact remains compact, like in Guduru (2007), which shows a possible large enhancement of adhesion with respect to the smooth sphere case. As in this case there is no approximation in our calculation with respect to any axisymmetric configuration, the SIF will be constant at circular contact edge by construction, and there is no need to further test these cases.

Another case where the solution would be exact is the sinusoidal contact of Johnson (1995), since here the contact area is by simmetry defined by a single parameter, and a fortiori the SIF will be equal for the configurations described by this single parameter. Hence, there is no check the result in this case either. For 2D sinusoidal contact, instead, even of equal wavelengths, an error may appear when contact area is large, since the contacts will not be circular. A comparison with an asperity model follows.

2.3. Comparison - asperity models

Real surfaces do not satisfy the approximation of asperity models, because of geometrical errors in the description of the roughness, and also because of neglecting interaction effects. However, in an ideal case of a true set of independent asperities, if our result were exactly valid, one should be able to obtain the same result by the classical approach of superposing results for each individual asperity, and our direct equation. Therefore, this comparison is a valid test case to check the effect of a distribution of contact spots of different sizes.

We take for simplicity the exponential distribution of heights $\phi = \frac{1}{\sigma_s} \exp\left(-\frac{z_s}{\sigma_s}\right)$ $(z_s > 0)$, which is good enough for our test. Area and load are each proportional to number of asperities in contact $n = N \exp\left(-\frac{u}{\sigma_s}\right)$, where N is their total number. Here, σ_s is a scale parameter of the heights. Hence, the ratio of area to load is constant in the classical repulsive case

$$A = \pi R \sigma_s n = \pi R \sigma_s N \exp\left(-\frac{u_1}{\sigma_s}\right) \tag{12}$$

$$P = n E^* \left(\sigma_s^3 R\right)^{1/2} \sqrt{\pi} = E^* \left(\sigma_s^3 R\right)^{1/2} \sqrt{\pi} N \exp\left(-\frac{u_1}{\sigma_s}\right) \tag{13}$$

where R is radius of the asperity. Hence,

$$\frac{\partial A}{\partial \delta_1} = -\frac{\partial A}{\partial u} = \pi R N \exp\left(-\frac{u_1}{\sigma_s}\right) \tag{14}$$

$$\frac{\partial^2 P}{\partial \delta_1^2} = \frac{\partial^2 P}{\partial u^2} = E^* \frac{R^{1/2}}{\sigma_s^{1/2}} \sqrt{\pi} N \exp\left(-\frac{u_1}{\sigma_s}\right) \tag{15}$$

and our equation (10) gives

$$\delta = \delta_1 - \sqrt{2w \frac{\pi R^{1/2} \sigma_s^{1/2}}{E^* \sqrt{\pi}}} \tag{16}$$

Now, the indentation and mean separation are in the relationship

$$\delta - \delta_1 = u_1 - u \tag{17}$$

and hence we have obtained the repulsive load vs actual adhesive gap.

Now, we return to (5) which here will be

$$P_{Ad} = P_1 + \left(-\frac{\partial P}{\partial u_1}\right)(u_1 - u) \tag{18}$$

giving

$$P_{Ad} = P_1 \left(1 - \sqrt{\theta_{\exp}}\right) \tag{19}$$

where $\theta_{\exp}$ is the same as that predicted by Fuller Tabor in the exponential form if we take DMT instead of JKR for the individual asperity (Ciavarella & Papangelo 2017d),

$$\theta_{\exp} = 2\sqrt{\pi} \frac{R^{1/2}}{\sigma_s^{3/2}} l_a \tag{20}$$

and hence the contact is either always tensile or compressive, a result which is exactly as obtained in asperity model of Fuller and Tabor (1975) in the exponential form (Ciavarella & Papangelo 2017d). However, the asperity model would give $P_{Ad} = P_1 (1 - \theta_{\exp})$ and hence the adhesive load is different because of the square root. Notice however that we have obtained a result which is non-hysteretic but is close to the DMT asperity model which therefore is closer to the unloading regime of a JKR solution than to a loading one. This gives us some confidence that the results for a more general problem will be of some validity. In particular, the transition from sticky

to non-sticky seems correctly predicted but the quantitative results seems to be that for low adhesion, the effect of adhesion may be overestimated, while for high adhesion, underestimated. But some level of approximation is inevitable in all models of complex adhesive problems, and particularly in the case of roughness, where no exact solution is known even just in the case without adhesion.

3. Application to rough contacts - Persson's theory

In applying the model to a random rough surface, we need a theory to estimate the variation of contact area, and of load, with indentation: the terms $\frac{\partial A}{\partial \delta_1}, \frac{\partial^2 P}{\partial \delta_1^2}$ in (10). Persson (2007) gives a mean repulsive pressure σ_{rep} vs mean separation u law which, for the practical case of self-affine surfaces of low fractal dimension ($D \simeq 2.2$ is a value of common experience, Persson *et al.*, 2014), assumes a simple asymptotic form which is sufficiently valid for not too large σ_{rep}

$$\frac{\sigma_{rep}}{E^*} \simeq \frac{3}{8\gamma} q_0 h_{rms} \exp\left(\frac{-u}{\gamma h_{rms}}\right) \tag{21}$$

where $\gamma \simeq 0.45$, q_0 is the smallest wavevector in the self-affine process where the power law Power Spectrum starts, and h_{rms} is the rms amplitude of roughness. Notice that we have corrected the multiplier in agreement with numerical findings of Papangelo *et al.*(2017). Hence, for low fractal dimensions, the result does not depend on fine-scale details of the surface.

Persson (2001) then suggests for the proportion of actual contact at a given nominal pressure which, after a more recent corrective factor of Putignano et al (2012) has been included, reads

$$\frac{A_{rep}}{A_0} = \mathrm{erf}\left(\frac{\sqrt{\pi}}{2}\frac{\sigma_{rep}}{\sigma_{rough}}\right) \tag{22}$$

where $\sigma_{rough} = E^* h'_{rms}/2$ where h'_{rms} is the rms slope of the surface, and σ_{rep} can be estimated as a function of u from (21).

We can combine the two Persson's results (21,22) to find

$$\frac{A_{rep}}{A_0} = \mathrm{erf}\left[\frac{\sqrt{\pi}}{2}\frac{E^*}{\sigma_{rough}}\frac{3}{8\gamma} q_0 h_{rms} \exp\left(\frac{-u}{\gamma h_{rms}}\right)\right] \tag{23}$$

from which, putting

$$\alpha = \frac{\sqrt{\pi}}{2}\frac{E^*}{\sigma_{rough}}\frac{3}{8\gamma}q_0 h_{rms} \tag{24}$$

we obtain

$$\frac{\partial}{\partial u}\frac{A_{rep}}{A_0} = -\frac{2\alpha}{\gamma h_{rms}\sqrt{\pi}}\exp\left(-\alpha^2\exp\left(-\frac{2u}{\gamma h_{rms}}\right) - \frac{u}{\gamma h_{rms}}\right) \tag{25}$$

whereas

$$\frac{\partial}{\partial u}\frac{\sigma_{rep}}{E^*} \simeq -\frac{3}{8\gamma^2}q_0\exp\left(\frac{-u}{\gamma h_{rms}}\right) \quad , \quad \frac{\partial^2}{\partial u^2}\frac{\sigma_{rep}}{E^*} \simeq \frac{3}{8\gamma^3}q_0\frac{1}{h_{rms}}\exp\left(\frac{-u}{\gamma h_{rms}}\right) \tag{26}$$

Finally, noticing that the indentation and mean separation are in the relationship (17), we need to change sign to the derivatives, and then we can substitute in the general solution.

Let us introduce the length $l_a = w/E^*$ as an alternative measure of adhesion, and we can use again (21), such that $\left(\log\frac{8\gamma^2\sigma_{rep}}{3E^*q_0}\right) \simeq -u_1$ to show after some algebra that our general result (11) leads to

$$\begin{aligned} u &= u_1 + \sqrt{\frac{4\gamma}{\sqrt{\pi}}\frac{l_a h_{rms}}{h'_{rms}}\exp\left(-\frac{3\pi}{8\gamma}\frac{q_0 h_{rms}}{h'^2_{rms}}\frac{\sigma_{rep}}{E^*}\right)} \\ &= -\log\frac{8\gamma^2\sigma_{rep}}{3E^*q_0} + \sqrt{\frac{4\gamma}{\sqrt{\pi}}\frac{l_a h_{rms}}{h'_{rms}}\exp\left(-\frac{3\pi}{8\gamma}\frac{q_0 h_{rms}}{h'^2_{rms}}\frac{\sigma_{rep}}{E^*}\right)} \end{aligned} \tag{27}$$

This solution is now written in terms of actual adhesive mean separation, as a function of the repulsive pressure.

It is clear that in the fractal limit, this solution tends to the adhesiveless result, because $h'_{rms} \to \infty$. But it would also go to adhesiveless for large h_{rms} because of the term in the exponential.

To find pull-off, we need to elaborate more on the solution. We have to return to the definition of the load after unloading at constant contact area, which now for rough surfaces will read

$$\frac{\sigma_{Ad}}{E^*} = \frac{\sigma_{rep}(u_1)}{E^*} + \left(\frac{1}{E^*}\frac{\partial\sigma_{rep}}{\partial u}\right)_{u_1}(u - u_1) \tag{28}$$

Also, from (21) $\left(\frac{1}{E^*}\frac{\partial \sigma_{rep}}{\partial u}\right)_{u_1} = -\frac{3}{8\gamma^2}q_0 \exp\left(\frac{-u_1}{\gamma h_{rms}}\right)$ and hence

$$\frac{\sigma_{Ad}}{E^*} = \frac{\sigma_{rep}(u_1)}{E^*} - \frac{3}{8\gamma^2}q_0 \exp\left(\frac{-u_1}{\gamma h_{rms}}\right)(u-u_1) \tag{29}$$

$$= \frac{3}{8\gamma}q_0 h_{rms} \exp\left(\frac{-u_1}{\gamma h_{rms}}\right)\left[1 - \frac{1}{\gamma h_{rms}}(u-u_1)\right] \tag{30}$$

and finally using (27) for $(u-u_1)$

$$\frac{\sigma_{Ad}(u_1)}{E^*} = \frac{3}{8\gamma}q_0 h_{rms} \exp\left(\frac{-u_1}{\gamma h_{rms}}\right) \times \left[1 - \frac{1}{\gamma h_{rms}}\sqrt{\frac{4\gamma}{\sqrt{\pi}}\frac{l_a h_{rms}}{h'_{rms}} \exp\left(-\left(\frac{3}{8\gamma}\right)^2 \pi \frac{(q_0 h_{rms})^2}{h'^2_{rms}} \exp\left(\frac{-u_1}{\gamma h_{rms}}\right)\right)}\right] \tag{31}$$

which gives the mean pressure with adhesion as a function of the "adhesion-less separation" u_1 which is only the value needed to obtain a given contact area: but spanning all values of u_1, we can find anyway the entire solution, as we can use (27) to find the actual u, and all the other quantities are known. We can also rewrite the solution introducing a constant a_0 of the order of atomic spacing to obtain a non dimensional version.

In particular, but we shall see more problematic in terms of accuracy, is to push the solution to find the minimum value of the adhesive stress, and hence pull-off

$$(\sigma_{Ad})_{po} = \min \sigma_{Ad}(u_1) \tag{32}$$

not because the minimum requires a numerical root finder, but because it turns out in practise that this approximation leads to either always tensile forces, or always compressive. This is due to the fact that we have used the simplest approximation of the Persson's solution, at small pressures (21), whereas the full solution would require a much more elaborate form, which we leave for further studies, when $u_1 \to 0$, and the repulsive pressure tends to very high values. In fact the only possible simple estimate for pull-off is

at zero gap

$$\frac{\sigma_{Ad}(0)}{E^*} = \frac{3}{8\gamma} a_0 q_0 \frac{h_{rms}}{a_0} \times \left[1 - \frac{1}{\gamma h_{rms}/a_0} \sqrt{ \frac{4\gamma}{\sqrt{\pi}} \frac{(l_a/a_0)(h_{rms}/a_0)}{h'_{rms}} \exp\left(- \left(\frac{3}{8\gamma} \right)^2 \pi \frac{\left(a_0 q_0 \frac{h_{rms}}{a_0} \right)^2}{h'^2_{rms}} \right) } \right] \tag{33}$$

and obviously this is tensile only if

$$\gamma h_{rms}/a_0 < \frac{4}{\sqrt{\pi}} \frac{(l_a/a_0)}{h'_{rms}} \exp\left(- \left(\frac{3}{8\gamma} \right)^2 \pi \frac{\left(a_0 q_0 \frac{h_{rms}}{a_0} \right)^2}{h'^2_{rms}} \right) \tag{34}$$

which is a quite restrictive (implicit) condition on the rms amplitude, which we can approximate for not too small h'_{rms} (notice that the numerator in the exponential term is an apparent "slope" at small wavevectors and hence is much smaller than the denominator)

$$h_{rms}/a_0 < (h_{rms}/a_0)_{th} = \frac{4}{\sqrt{\pi}\gamma} \frac{(l_a/a_0)}{h'_{rms}} \tag{35}$$

3.1. Comparison with Pastewka-Robbins (PR) criterion

PR stickiness criterion is obtained in the original paper (eqt.10), in the form

$$\frac{h'_{rms}\Delta r}{\kappa_{rep} l_a} \left[\frac{h'_{rms} d_{rep}}{4\Delta r} \right]^{2/3} < \pi \left(\frac{3}{16} \right)^{2/3} \simeq 1.03 \tag{36}$$

where Δr is range of attractive forces, and d_{rep} is a characteristic diameter of repulsive contact areas, which they estimate as $d_{rep} = 4h'_{rms}/h''_{rms}$ and finally $\kappa_{rep} \approx 2$. In order to incorporate their choice of truncated potentials, the range of attraction is easily obtained from Suppl.Inf. of PR paper to be $\Delta r/a_0 = \sqrt{24 l_a/a_0}$. For the Lennard-Jones situations ($l_a/a_0 = 0.05$), $\Delta r \approx a_0$ and grouping the variables using the Nayak bandwidth parameter $\alpha_N = \frac{m_0 m_4}{m_2^2}$, where m_n are the moments of order n in the random process, we can restate (36) as

$$\frac{h_{rms}}{a_0} < \sqrt{\alpha_N} \left(\frac{2 l_a}{a_0 h'_{rms}} \right)^{3/2} \tag{37}$$

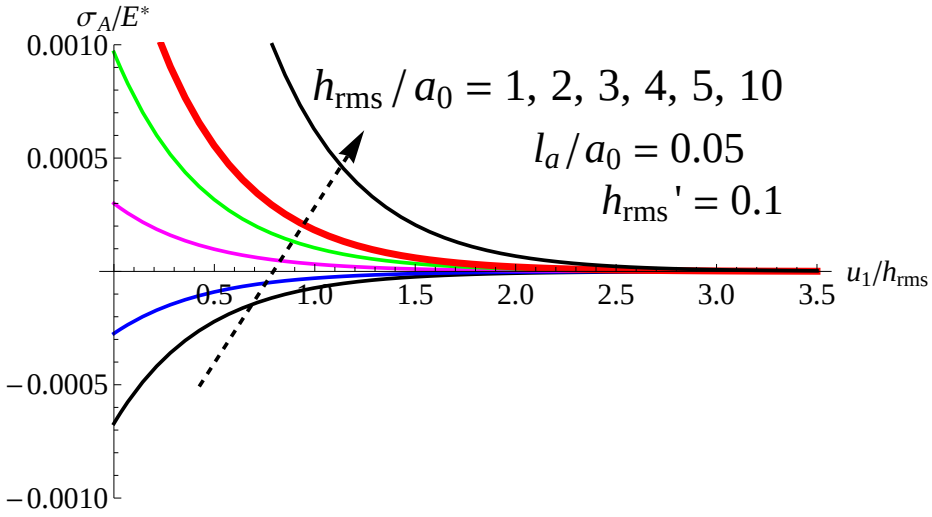
and for example, for $l_a/a_0 = 0.05$, $h'_{rms} = 0.1$, our criterion (35) gives $(h_{rms}/a_0)_{th} = \frac{4}{\sqrt{\pi}0.45}\frac{0.05}{0.1} = 2.5$ while PR's one $(h_{rms}/a_0)_{th} = \sqrt{\alpha_N}$, and therefore they seem to coincide for low bandwidth parameter, while at large bandwidths, it would be important to further check results. It is remarkable that we obtained with a very simple asperity model with exponential distribution of heights (Ciavarella, 2017c)

$$\frac{h_{rms}}{a_0} < 0.33 \left(\frac{l_a}{a_0 h'_{rms}} \right)^{3/2} \tag{38}$$

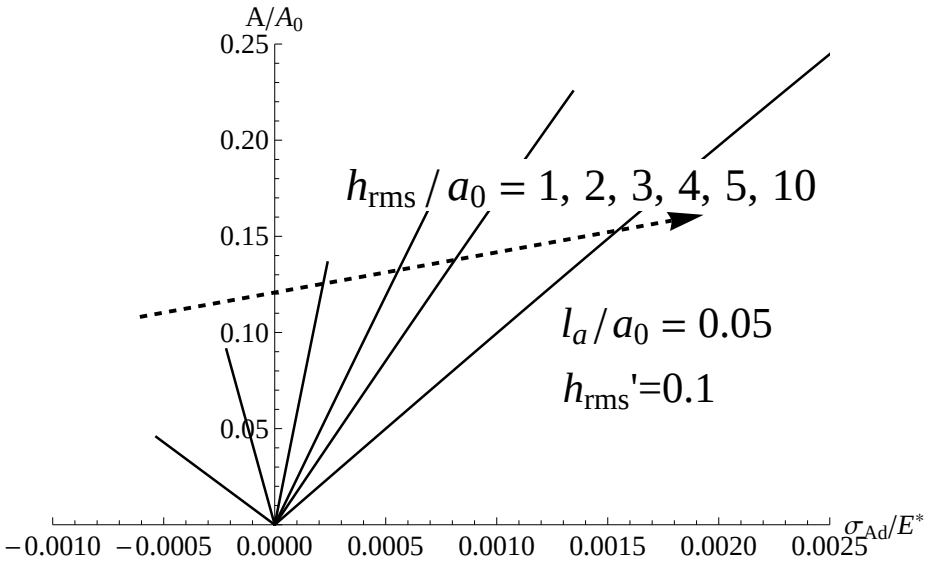
which is remarkably close both qualitatively and quantitatively to PR parameter (37) at low bandwidths: therefore, while all criteria seem to qualitatively give similar results in the limit case of low bandwidth, the details differ at large bandwidth, and in this case, there remains some uncertainty also because PR simulations show a threshold for stickiness which is not corresponding to their own data on pull-off as discussed in various previous papers (Ciavarella 2017a,b,c, Ciavarella & Papangelo 2017b, Ciavarella & Papangelo 2017c).

4. Examples

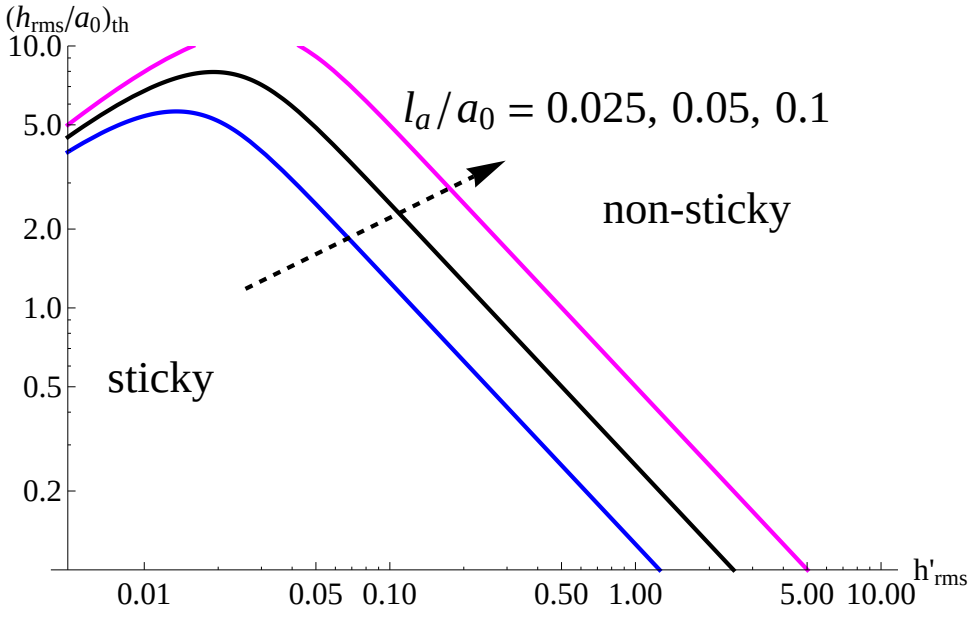
Let us consider a self-affine surface with power law PSD $A\left|\mathbf{q}\right|^{-2(1+H)}$ for wavevectors $q_0 < |\mathbf{q}| < q_s$ ($q = 2\pi/\lambda$) and zero otherwise (pure power-law). The surfaces have Hurst exponent $H = 0.8$, and $\lambda_0 = 2048a_0$, where a_0 could be an atomic spacing, but more in general here enters only as a normalizing factor for the energy of adhesion which we define as $l_a/a_0 = 0.05$ when we imitate the Lennard-Jones potential (see also Pastewka-Robbins (2014)). For slopes we use $h'_{rms} = 0.1$ unless otherwise indicated (and notice that there is a minimum level of slope for a given rms amplitude as we have fixed the smallest wavevector). Using (31), we obtain some example results.


σ_A/E^*
0.0010
0.0005
−0.0005
−0.0010
0.5
1.0
1.5
2.0
2.5
3.0
3.5
u_1/h_{rms}
$h_{rms}/a_0 = 1, 2, 3, 4, 5, 10$
$l_a/a_0 = 0.05$
$h_{rms}' = 0.1$

(a)


A/A0
0.25
0.20
0.15
0.10
0.05
−0.0010
−0.0005
0.0000
0.0005
0.0010
0.0015
0.0020
0.0025
σ_{Ad}/E^*
$h_{rms}/a_0 = 1, 2, 3, 4, 5, 10$
$l_a/a_0 = 0.05$
$h_{rms}'=0.1$

(b)



(c)

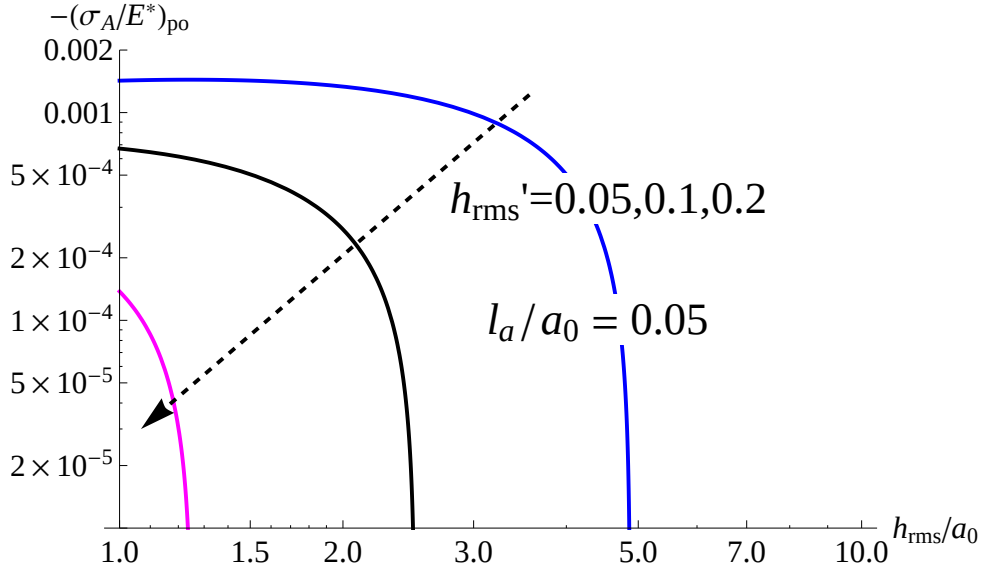
Fig.2. (a) The adhesive mean stress vs the mean gap u_1 and (b) fraction area over the nominal one A/A_0 : for various levels of h_{rms}/a_0 for adhesion level $l_a/a_0 = 0.05$ and rms slopes $h'_{rms} = 0.1$. (c) threshold "sticky"-"non-sticky" in terms of h_{rms}/a_0 as a function of h'_{rms} and for various adhesion levels $l_a/a_0 = 0.025, 0.05, 0.1$.

Fig.2a,b show that with increasing rms amplitude, the curves become increasingly less adhesive as expected, for a given $h'_{rms} = 0.1$. The curves in Fig.2b show a linear trend both in the "unsticky" and "sticky" cases. Notice that the contact area, even at $u_1 = 0$ are still quite small. Both these effects are due, again, to having used the asymptotic simple Persson solution at small pressures.

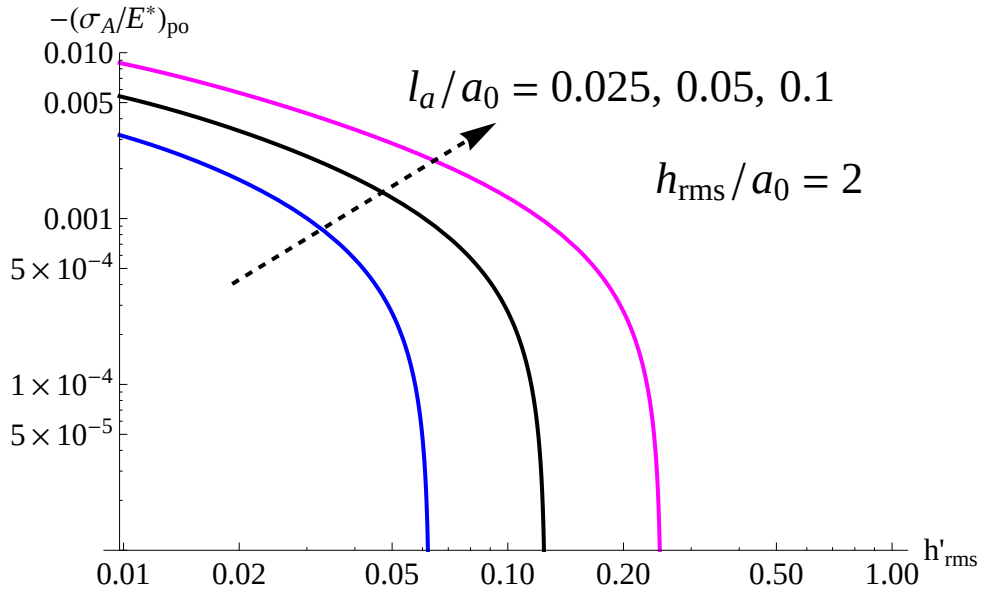
Fig.2c shows that the "threshold" for stickiness $(h_{rms}/a_0)_{th}$ is, apart from an initial range in which we don't have a reliable result since the slopes are too small to apply the Persson's result for self-affine processes (we have too small bandwidth), generally there is a very good power law regime

Fig.3 show the decay of the crude estimates of pull-off (estimated as the value at zero gap) with rms amplitude, for various $h'_{rms} = 0.05, 0.1, 0.2$. This is qualitatively in agreement with previous results (Ciavarella, 2017a). However, in the present case, there is a dependence also on the slopes h'_{rms}, whereas Ciavarella (2017a) only involved the rms amplitudes h_{rms}, which seemed to fit better the case of the Pastewka-Robbins (2014) simulations.

However, this may simply mean that being those simulations concerned with roughness at nanoscale with very low Tabor parameter (of the order of 1), the DMT "bearing-area" model of Ciavarella (2017a) which only involved the rms amplitudes h_{rms}, is a better model for this case. Unfortunately, it is difficult to find in the literature accurate solutions of the JKR problem with roughness, except for (Carbone et al., 2015) which however, are rather limited to very few results.



(a)



(b)

Fig.3. A very crude estimate of pull-off value. (a) as a function of h_{rms}/a_0 and $l_a/a_0 = 0.05$ while $h'_{rms} = 0.05, 0.1, 0.2$. (b) for a given $h_{rms}/a_0 = 2$ and as a function of h'_{rms} for various values of $l_a/a_0 = 0.025, 0.05, 0.1$

5. Conclusion

We have provided a very simple model for JKR regime of a general contact, within the approximation that contact areas with adhesion are the same as contact area under adhesiveless conditions but with an appropriately increased load. This has led to a very simple solution, which only needs the variation of the contact area and load with indentation in the adhesiveless problem, for which powerful analytical or numerical methods are available for many users, whereas adhesive codes are much more complex, particular those involving highly non-linear force laws like Lennard-Jones. The solution gives exact results in axisymmetric cases, sinusoidal case, and similar results in an ideal asperity model, resembling closely to the DMT model of Fuller and Tabor which therefore is similar to the unloading prediction of a JKR asperity model. It seems to give reasonable results even for the very complex problem of JKR adhesion for rough surfaces, for which no analytical previous result is known, at negative loads. However, it should be borne in mind that the JKR limit is increasingly inappropriate at small scales. Near pull-off, we have provided only the crudest estimates, since the more accurate values should require the non-asymptotic form of the Persson's adhesiveless solution. The quantitative comparison with an ideal test case using an asperity model (which does not represent an actual random roughness as we know) shows that the transition from sticky to non-sticky seems correctly predicted, although the effect of adhesion may be overestimated at low surface energy, while underestimated at high ones.

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