Evaluation of Ductile/Brittle Failure Theory and Derivation of the Ductile/Brittle Transition Temperature

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A recently developed ductile/brittle theory of materials failure is evaluated. The failure theory applies to all homogeneous and isotropic materials. The determination of the ductile/brittle transition is an integral and essential part of the failure theory. The evaluation process emphasizes and examines all aspects of the ductile versus the brittle nature of failure, including the ductile limit and the brittle limit of materials' types. The failure theory is proved to be extraordinarily versatile and comprehensive. It even allows derivation of the associated ductile/brittle transition temperature. This too applies to all homogeneous and isotropic materials and not just some subclass of materials' types. This evaluation program completes the development of the failure theory.
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Introduction

The ductile/brittle transition for failure with all of its implications and ramifications is one of the most widely observed and universally acknowledged physical effects in existence. Paradoxically, it is also one of the least understood of all the physical properties and physical effects that are encountered in the world of materials applications. Critical judgments are made on the basis of experience only, purely heuristic and intuitive. The resolution of the ductile/brittle transition into a physically meaningful and useful mathematical form has always been problematic and elusive. It often has been suggested that such a development is highly unlikely. The rigorous answer to this question remains and continues as one of the great scientific uncertainties and challenges.

The long-time operational status of ductile/brittle behavior has reduced to a statement of the strain at failure in uniaxial tension. If the strain at failure is large, the material is said to be ductile. If the strain at failure is small, it is brittle. Loose and uncertain though this is, it could be general, even complete, if the world were one-dimensional. But the physical world is three-dimensional and in stress space it is six- or nine-dimensional. Even more complicating, some of the stress components are algebraic. So the problem is large and difficult, perhaps immensely large and immensely difficult.

To even begin to grapple with the ductile/brittle transition, one must first have a firm grasp upon a general and basic theory of failure. On further consideration, the two topics are seen to be inseparable. Any theory of failure that does not admit a full development of ductile versus brittle behavior is less than just incomplete. It is likely irrelevant and incorrect if it does not include the adjoining ductile/brittle delineation. In fact, this could be the number one test for the credibility of any particular failure criterion, does it admit a related and reasonable associated ductile/brittle formalization?

There is one exception to this quite dire state of affairs and it is the case of what are commonly called very ductile metals. The flow of dislocations embodies the essence of ductility and it has been a very active area of study for a great many years. The many related papers on the ductile/brittle transition in ductile metals generally examine the emission of dislocations at crack tips to see how local conditions can influence this. The most prominent piece of work is that of Rice and Thomson [1]. Further references in the particular field of dislocations and the ductile/brittle transition will be given later.

The direction to be followed here, however, is totally different from that mentioned above because all materials' types are to be considered here and not just ductile metals. Of course, there could be an argument to the effect that attention must be restricted to a single materials type. But a counter-argument is that there is not one theory of elasticity for metals, while a different theory of elasticity is needed for ceramics, etc. Failure theory and the ductile/brittle transition in its relationship to failure can be treated in a unified and general manner, just as elasticity theory can. The comprehensive failure theory derived by Christensen [2] integrates the failure approach with full account of the ductile/brittle transition. The failure theory is that for any and all homogeneous and isotropic materials and the ductile/brittle transition treatment applies to any isotropic materials type in any state of stress and not just ductile metals in uniaxial tension.

In Christensen [2], the theory of failure was evaluated by comparison with cases of highly recognized failure data, such as that of Taylor and Quinney [3]. The associated ductile/brittle theory was fully developed in Ref. [2], but it was not evaluated by detailed comparison with testing data because of the scarcity of such quality data. In this work, the major area of the evaluation of the ductile/brittle part of the general theory in Ref. [2] will be taken up. This further and final development of the ductile/brittle transition theory will be evaluated in much detail and considerable depth. This will be approached and treated after first outlining the overall ductile/brittle failure theory in the next section.

Following the evaluation section, the failure treatment will be generalized to nonisothermal conditions by deriving the related

ductile/brittle transition temperature. The ductile/brittle transition temperature is yet another aspect of failure that although supremely important has defied organized applicable theoretical development for general materials' types and not just for specific types of metals. It too remains as one of the major outstanding problems, approachable mainly on an empirical basis from testing results.

Probably, no major discipline has had more effort expended by more people with less to show for it than has this field of homogeneous materials failure. Under sufficiently high load conditions, failure is inevitable. In simple descriptive terms, ductile failure is a gradual, graceful, and progressive failure, but brittle failure can be a sudden, abrupt, and sometimes catastrophic event. The differences could not be more profound and consequential.

What is more, ductile versus brittle failure is not an on/off switch. There is a measured shift from one to the other just as there is with the glass transition temperature in polymers. A rational treatment and methodology for materials failure will be given and substantiated here, starting next.

The final section will comprise an overview of and the conclusion of the present extended materials' failure program.

The Ductile/Brittle Failure Theory

The failure theory is fully developed in Ref. [2]. For the background, motivation, derivation, and interpretation that reference should be consulted. It is important to see all of these developments, especially the derivation.

None of the controlling forms are empirical postulations, as is the usual approach. The theory itself will only be outlined here in terms of its essential elements so that it can be evaluated.

There are two separate and competitive failure criteria: the polynomial invariants criterion and the fracture criterion. These are stated below. The overall failure theory applies over the full range of materials having

$$0 \leq \frac{T}{C} \leq 1 \tag{1}$$

where T and C are the uniaxial tensile and compressive strengths. The theory applies to all full density, homogeneous, and isotropic materials, the basic materials of load-bearing structures.

Polynomial Invariants Failure Criterion. For $0 \leq (T/C) \leq 1$

$$\left(1 - \frac{T}{C}\right)\hat{\sigma}_{ii} + \frac{3}{2}\hat{s}_{ij}\hat{s}_{ij} \leq \frac{T}{C} \tag{2}$$

where $\mathbf{s}_{ij}$ is the deviatoric stress tensor. The stress is nondimensionalized by the uniaxial compressive failure stress as

$$\hat{\sigma}_{ij} = \frac{\sigma_{ij}}{C} \tag{3}$$

In principal stress space, the polynomial invariants criterion takes the form

$$\left(1 - \frac{T}{C}\right)(\hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3)$$
$$+ \frac{1}{2}\left[(\hat{\sigma}_1 - \hat{\sigma}_2)^2 + (\hat{\sigma}_2 - \hat{\sigma}_3)^2 + (\hat{\sigma}_3 - \hat{\sigma}_1)^2\right] \leq \frac{T}{C} \tag{4}$$

A second failure criterion is needed in a certain range of T/C values, which tends more toward the generally brittle types of materials. This failure criterion commences at $T/C = 1/2$, at which value the polynomial invariants criterion has certain special properties that coordinate with the fracture criterion to be stated next. This second and competitive criterion is that of the fracture behavior and it is stated in terms of principal stresses.

Fracture Criterion. For $0 \leq (T/C) \leq (T/2)$

$$\hat{\sigma}_1 \leq \frac{T}{C}$$
$$\hat{\sigma}_2 \leq \frac{T}{C} \tag{5}$$
$$\hat{\sigma}_3 \leq \frac{T}{C}$$

Thus, the fracture criterion only applies over the partial range of T/C values from 0 to 1/2.

Whichever failure criterion (2) or (5) specifies the more limiting failure stress values then that applies and controls the failure behavior.

This nondimensional failure theory is remarkably simple having only one parameter to be varied that of the T/C value. In the nondimensional specification (3), the compressive failure stress C must be used. Trying to effect nondimensionalization by using T would become degenerate.

The value of T/C is taken as the materials' type. The usual classes of materials can have overlapping T/C values. The polynomial invariants criterion forms a paraboloid in principal stress space. Its axis makes equal angles with the principal stress coordinate axes. The fracture criterion, when it applies, specifies planes normal to the principal stress axes, and these planes take cuts or slices out of the paraboloid. The paraboloid remains a paraboloid but with three flattened surfaces on it. The limiting case of the failure theory at $T/C = 1$ reduces to the Mises criterion.

Two simple but basic examples from these failure criteria will now be given. The first is that of the shear strength, S.

Shear Strength

$$\text{For} \quad \frac{T}{C} \geq \frac{1}{3}, \quad \hat{S}^2 = \frac{1}{3}\left(\frac{T}{C}\right) \quad S = \sqrt{\frac{TC}{3}} \tag{6}$$

$$\text{For} \quad \frac{T}{C} \leq \frac{1}{3}, \quad \hat{S} = \frac{T}{C} \quad S = T \tag{7}$$

The form (6) is required by the polynomial invariants criterion and the form (7) by the fracture criterion.

The second example is that of eqi-biaxial stress failure.

Eqi-Biaxial Failure Stress

$$\sigma_1 = \sigma_2 = \sigma$$
$$\sigma_3 = 0 \tag{8}$$

The polynomial invariants criterion gives

$$\hat{\sigma} = \frac{T}{C} - 1 \pm \sqrt{\left(\frac{T}{C}\right)^2 - \frac{T}{C} + 1} \tag{9}$$

The fracture criterion does not give any failure levels more critical than that of the polynomial invariants result in Eq. (9).

The Ductile/Brittle Transition. The additional capability of ductile versus brittle failure discrimination will now be brought into the failure theory framework. To be able to theoretically distinguish ductile failure from brittle failure would provide a large amplification in the power and usefulness of failure criteria. The failure theory of Ref. [2] did in fact include total coordination with the ductile/brittle transition. The failure criteria part of the general failure theory (Eqs. (1)–(5)) was fully evaluated in Ref. [2]. The main purpose of this paper is to evaluate the ductile/brittle characterization part of the general failure theory. This latter evaluation contained here will support and reinforce the entire failure theory.

For use with ductile/brittle considerations, let

$$\sigma_{ij}^{f}$$

designate the failure stresses from Eqs. (1) to (5). The associated ductile/brittle transition is specified by

$$\hat{\sigma}_{ii}^{f} = 3\frac{T}{C} - 1 \quad \text{D/B transition} \tag{10}$$

One of the uses of Eq. (10) will be to establish the value of T/C that places the material failure directly at the ductile/brittle transition for a specified stress state.

For materials and failure stress states not right at the ductile/brittle transition, the ductile versus brittle states of failure are specified by

$$\hat{\sigma}_{ii}^{f} < 3\frac{T}{3} - 1 \quad \text{ductile} \tag{11}$$

$$\hat{\sigma}_{ii}^{f} > 3\frac{T}{C} - 1 \quad \text{brittle} \tag{12}$$

Ductile/Brittle Transition for Uniaxial Stress. Two simple but again basic examples of the use of these ductile/brittle criteria will now be given. First, for uniaxial tension there follows that:

$$\hat{\sigma}_{11}^{f} = \frac{T}{C} \\ \hat{\sigma}_{22}^{f} = \hat{\sigma}_{33}^{f} = 0 \tag{13}$$

Thus,

$$\hat{\sigma}_{ii}^{f} = \frac{T}{C} \tag{14}$$

Substituting Eq. (14) into Eq. (10) gives

$$\frac{T}{C} = \frac{1}{2} \quad \text{D/B transition in uniaxial tension} \tag{15}$$

Similarly, for uniaxial compression, there is

$$\hat{\sigma}_{ii}^{f} = -1 \tag{16}$$

Substituting this into Eq. (10) gives

$$\frac{T}{C} = 0 \quad \text{D/B transition in uniaxial compression} \tag{17}$$

The two results ((15) and (17)) will be used later in the evaluation.

The unexpected thing in all this development is that the ductile/brittle transition is specified by only the T/C materials' type value. No empirical parameters are involved or needed. The ductile/brittle transition is represented by the particular plane (10) in the principal stress space that is normal to the axis of the polynomial invariants paraboloid. This plane divides the entire region into the ductile region versus the brittle region. It applies to regions controlled by either the polynomial invariants criterion or the fracture criterion.

The first term in the polynomial invariants criterion (2) is the first invariant of the stress tensor. The second term in Eq. (2) is the second invariant. The second term without the first term in Eq. (2) would be the usual ductile metals failure criterion based only upon distortional effects. It is the first term in Eq. (2) that brings in the ductile versus brittle effects. And, it is not surprising that this first term in the failure criterion (2) is also the controlling

form in the ductile/brittle transition specification (10). This first invariant of the stress tensor provides the "thermostat" that controls all the ductile/brittle characteristics, as well as many other things. This is the first clear and unmistakable indicator that this ductile versus brittle failure discrimination approach may be on the right track. The evaluation will push and probe much further.

Before turning to the next section on evaluation, a special value of this ductile/brittle failure theory should be noted. The theory is completely characterized by only two failure properties, T and C. Conventional thinking says that a general failure theory should require three or four parameters at least, if it even can be done at all. As studied and proved in Ref. [4], materials' failure represents the cessation of the linear elastic range of behavior, which itself is also composed by only two properties for isotropy. The relationship and balance between elasticity theory and failure theory is a remarkable physical tie that has remained hidden for so long but now can finally be recognized and exploited to great advantage [4], as it is here.

Evaluation of Ductile/Brittle Failure Theory

A significant part of this new failure theory was successfully evaluated in Ref. [2] using the very best available data on explicit failure cases performed in the laboratory under carefully controlled conditions. Virtually, all of the previous evaluations were independent of ductile versus brittle failure considerations. Now the attention is exclusively placed upon the ductile versus the brittle aspects of failure and how to test and verify the theoretical predictions for such ductile/brittle behaviors. This present evaluation will be even more demanding and intensive.

Critical quality ductile/brittle testing data comparable to that mentioned above do not appear to exist. An alternative approach for evaluating the ductile/brittle theory must be found and used. This will be accomplished through physical tests of the necessary consistency and compatibility of the ductile/brittle theory predictions.

First, a general picture of the full range of possible ductile/brittle behaviors will be constructed from the theory in the preceding section. Table 1 shows the values of T/C that specifies the occurrence of the ductile/brittle transition from Eq. (10) for the seven most basic stress states. The full results in Table 1 are from Eqs. (10) to (12) with typical results derived in Eqs. (15) and (17) as the specific examples for uniaxial tension and compression.

It is seen from Table 1 that there is a shift toward the benign condition of ductile failure for the compressive stress states. In contrast, the tensile states are much inclined toward the difficult occurrence of brittleness. While this does not explicitly prove or verify anything specific, it still is an eminently reasonable and rational general prediction of ductile/brittle behavior.

However, it will require much more specific and even critical conditions to accomplish the solid verification that is sought here. This will now be pursued through physical consistency tests.

Consistency Test 1: The Ductile/Brittle Transition in Uniaxial Tension. The time honored most important stress state is that of uniaxial tension. It is universally employed to determine

Table 1 The range of D/B transition behaviors

Stress state	T/C at D/B transition	
Eqi-triaxial compression		No failure
Eqi-biaxial compression		Always ductile
Uniaxial compression	$T/C = 0$	Otherwise ductile
Shear	$T/C = 1/3$	
Uniaxial tension	$T/C = 1/2$	
Eqi-biaxial tension	$T/C = 1$	Otherwise brittle
Eqi-triaxial tension		Always brittle

Table 2 The full range of isotropic materials T/C's with the D/B transition predicted at T/C = 1/2 for uniaxial tension

Materials type	T/C	Predicted D/B behavior in uniaxial tension
Aluminum	1	Perfectly ductile
Titanium	1	Perfectly ductile
Polyethylene	0.9	Extremely ductile
Polycarbonate	0.8	Very ductile
Epoxy	2/3	Ductile
Nickel and polystyrene	**1/2**	**D/B transition**
Cast iron	1/3	Brittle
Silicon carbide	1/5	Very brittle
Float glass	1/10	Extremely brittle
Dolomite	1/15	Extremely brittle
Some geological materials	1/50 to 1/100	Totally brittle

the properties of stiffness and strength. This will now be used to test the theoretical prediction of where the ductile/brittle transition is located for uniaxial tension, compared with real world experience.

Table 2 shows typical values of the T/C properties ratio for a very broad range of materials. All the major materials groups are represented in Table 2: ductile metals, brittle metals, polymers, ceramics, glasses, and geological materials. In some cases, a particular materials class has a broad range of T/C values, typical values are shown here. For example, the titanium is grade 1 and for polystyrene the T/C = 1/2 is that for untoughened forms of that particular polymer. This value of T/C = 1/2 is also about that for PMMA. The silicon carbide case is typical of a ceramic material. All of the individual entries in Table 2 are well representative of the materials' groups in which they reside.

The uniaxial tensile ductile/brittle behaviors shown in Table 2 are those from the ductile/brittle transition prediction (10) and the related ranges (11) and (12). For example, for epoxy the behavior is stated as being ductile at T/C = 2/3 since that value is closer to the ductile/brittle transition at T/C = 1/2 than it is to the perfectly ductile case at T/C = 1. The other predicted behaviors are similarly interpolated between the ductile/brittle transition and the two separate limits.

The most significant entry in Table 2 is that of the theoretically predicted existence of the ductile/brittle transition at T/C = 1/2 for simple tension. This prediction is in complete agreement and accordance with all intuitive, heuristic observations for the different materials' types. That the ductile/brittle transition at T/C = 1/2 is bracketed by ductile epoxy at T/C = 2/3 and brittle cast iron at T/C = 1/3 is especially supportive.

The composition of a table similar to that of Table 2 but for the solids forming elements of the periodic table gives completely similar and coordinating results. Gold, silver, and lead are near the ductile limit while silicon, carbon (diamond), and beryllium are near the brittle limit. Although the data are scattered, the elements nickel and cobalt appear to be at or near the ductile/brittle transition of T/C = 1/2.

There could not be a stronger verification of this failure theory than through the failure behavior in the uniaxial tensile failure perspective for all isotropic materials as in Table 2. This ductile/brittle transition prediction is consistent with the accumulated wisdom of use and experience for all materials over the whole span of technical history in addition to a massive amount of supporting data.

The success in predicting the ductile/brittle transition in uniaxial tension leads to the enlargement of the method in Ref. [2]. Specifically the failure number (Fn) was generated to predict the quantitative ductility level for any material in any state of stress. The failure number is defined by

$$\text{Fn} = \frac{1}{2}\left(3\frac{T}{C} - \hat{\sigma}_{ii}^{f}\right) \quad (18)$$

with

$$0 \le \text{Fn} \le 1 \quad (19)$$

When Fn in Eq. (18) generates a value larger than 1, it reverts to 1 and when it generates a value less than 0 it reverts to 0. From Eq. (18), Fn = T/C for uniaxial tension, the same as displayed in Table 2. The resulting values of Fn for any material in any stress state admit direct comparison with the range of materials in Table 2 for uniaxial tension to give an interpretation of the ductility level. See Ref. [2] for the derivation and general interpretation of Eq. (18). A complete failure number methodology is built up and based upon the significance of the results in Table 2. This is a further proof of the utility and versatility of the ductile/brittle failure theory.

Consistency Test 2: Simple Shear Ductile/Brittle Behavior. If at the beginning of examining materials' failure one was to speculate, a logical-related question might be: Which stress state is the most fundamental for characterizing failure, uniaxial tension, or shear? This intriguing question is not which test is easier to perform but which stress state is the most fundamental for failure. Probably more or even most investigators would opt for the choice of shear. The present failure theory establishes the opposite conclusion. Uniaxial tension is the most fundamental stress state but shear stress failure is definitely of great importance. It will be examined now.

From Table 1, it is seen that uniaxial tension and shear are the only two simple stress state cases with the ductile/brittle transition occurring inside the limits of T/C of 0 and 1. Unfortunately, there is not the kind of information and experience with shear failure to arrange a table for the case of shear like that of Table 2 for uniaxial tension. Still it is possible to make an evaluation on the shear stress failure prediction.

First, view shear stress as the two stresses of orthogonal tension and compression of equal magnitudes. In so far as ductile versus brittle behavior is concerned, the compressive stress component has an ameliorating effect upon the tensile stress component behavior. Thus, the shear stress state can be expected to tolerate a ductile/brittle transition down to a lower value of T/C than can uniaxial tension. Table 1 verifies this effect through the T/C = 1/3 prediction for shear versus T/C = 1/2 for uniaxial tension at the ductile brittle transition.

It is further helpful to show the behavior of the shear strength S versus T/C over its full range of values. The solution for S is given by Eqs. (6) and (7). The complete behavior is shown in Fig. 1. The ductile range versus the brittle range for shear stress comes from Eqs. (10) to (12).

The change of failure mode occurs at T/C = 1/3, the same as the change from ductile to brittle behavior. If it were not for the fracture criterion with its brittle behavior in Fig. 1, the behavior for S versus T would be very different. S would become unbounded compared with T as T/C →0. This would comprise physically
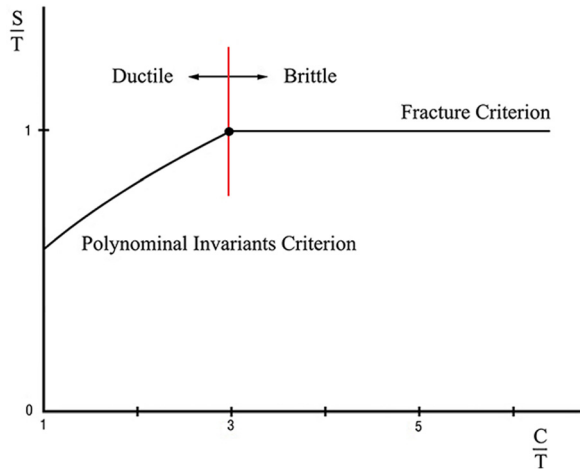
Fig. 1   Shear stress failure form

unrealistic behavior for shear stress at the brittle limit. Both T and S approach zero together as $T/C \rightarrow 0$.

The mention of the brittle limit brings up consistency test number 3.

Consistency Test 3: The Brittle Limit. There must exist a well-posed physical limit called the brittle limit at $T/C = 0$. This would complement the ductile limit at $T/C = 1$ which is the Mises criterion. The two limits would assure a complete treatment of all isotropic materials failure cases.

The brittle limit may not be easily accessible as an experimental and realizable case, but it must exist as a legitimate and consistent limit in the theoretical construct of the overall ductile/brittle failure theory.

Examination of the polynomial invariants failure criterion and the fracture criterion (Eqs. (1)–(5)) shows that they remain well posed and physically meaningful as $T/C \rightarrow 0$. The three-dimensional form of the brittle limit is as shown in Fig. 2. Also shown is the ductile/brittle transition from Eqs. (10) to (12). Even in the brittle limit, ductile behavior still remains possible so long as a sufficiently large component of hydrostatic pressure is present.

This is the only failure theory that admits a realistic brittle limit. The historic Mohr–Coulomb failure theory becomes degenerate as
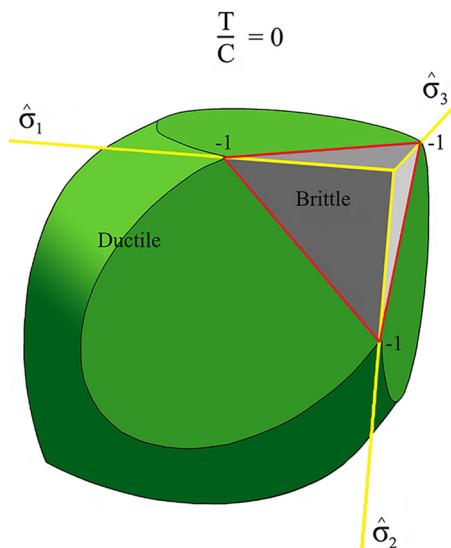

Fig. 2   The brittle limit

$T/C \rightarrow 0$. The existence of the brittle limit provides a very strong support for the general ductile/brittle failure theory.

Consistency Test 4: The Ductile Limit. Next, some complex stress states will be considered. Four starting cases will be considered, those of uniaxial tension and compression, shear, and eqi-biaxial tension. These four cases are the cases in Table 1 at or within the limits of $T/C = 0,1$. But each of these will be superimposed upon a hydrostatic state of stress. The posed problem is to determine in each of the four cases how much hydrostatic stress must be superimposed to bring a ductile limit, perfect $T/C = 1$ material to its ductile/brittle transition. Thus, each of the four cases is in a rather complex three-dimensional state of stress. This is the first ductile limit consistency test. There will be more.

The hydrostatic stress state has no effect on the failure stress level since this is a perfectly ductile $T/C = 1$ (Mises) material. But the superimposed hydrostatic stress state has a profound effect upon the ductile versus brittle nature of the failure.

The ductile/brittle transition (10) for a $T/C = 1$ material becomes

$$\hat{\sigma}_{ii}^f = 2 \quad \text{D/B transition at } \frac{T}{C} = 1 \qquad (20)$$

Illustrate the method for the special case of uniaxial tension. It follows that:

$$\hat{\sigma}_{ii}^f = \frac{T}{C} + 3\hat{\sigma}_h \qquad (21)$$

where σ_h is the hydrostatic stress (tensile or compressive). But for this material case with $T = C$, Eq. (21) becomes

$$\hat{\sigma}_{ii}^f = 1 + 3\hat{\sigma}_h \qquad (22)$$

Substituting Eq. (22) into Eq. (20) gives the result

$$\sigma_h = \frac{T}{3} \qquad (23)$$

So this is the hydrostatic stress needed to bring the $T/C = 1$ material in uniaxial tension to its ductile/brittle transition.

The other three cases follow similarly and all results are as shown in Table 3.

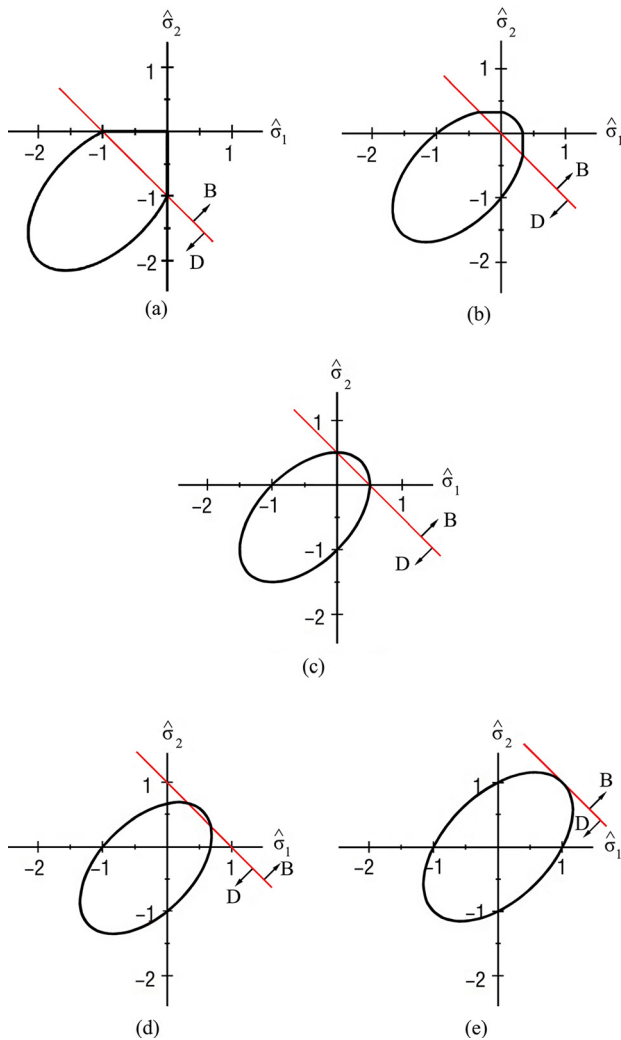Now examining case 4 in Table 3 that of uniaxial compression

$$\begin{aligned} \sigma_1 &= -C + T = 0 \\ \sigma_2 &= 0 + T = T \\ \sigma_3 &= 0 + T = T \end{aligned} \qquad (24)$$

It is seen that at the superimposed hydrostatic stress necessary to bring uniaxial compression to the ductile/brittle transition, it becomes identical to the eqi-biaxial tension case, which is already at the ductile/brittle transition with no superimposed hydrostatic stress needed.

This behavior passes the consistency test because the ductile/brittle transition equation (10) shows that cases 1 and 4 are both at the ductile/brittle transition, which they must be since they have identical stress states. Any other ductile/brittle transition form

Table 3   Hydrostatic stress needed to bring a $T/C = 1$ ductile limit material to its D/B transition

Case	Stress state	σ_h
1	Eqi-biaxial tension	0
2	Uniaxial tension	$T/3$
3	Shear	$2T/3$
4	Uniaxial compression	T

Fig. 3 Two-dimensional failure envelopes and the D/B transitions: (*a*) *T/C* = 0, (*b*) *T/C* = 1/3, (*c*) *T/C* = 1/2 (*d*) *T/C* = 2/3, and (*e*) *T/C* = 1

Table 4   D/B transitions for the 2D cases of Fig. 3

T/C	Stress state at D/B transition
0	Uniaxial compression
1/3	Shear
1/2	Uniaxial tension
2/3	Biaxial tension 2:1
1	Eqi-biaxial tension

The brittle limit is where no tensile stress component can be sustained. This is a fundamental precept. Figure 3(*a*) does show through the position of the ductile/brittle transition that with sufficient hydrostatic pressure, even a *T/C* = 0 material can deform and flow in a ductile manner.

The *T/C* = 1/3 case is shown in Fig. 3(*b*). It is typical for the case of cast iron. The state of shear stress failure is right at the position of the ductile/brittle transition. The failure of specimens of cast iron in torsion shows a spiral fracture pattern. The orientation of the failure pattern is about at 45 deg to the longitudinal axis. This reveals that the tensile principal stress component of the shear stress state controls the failure, consistent with the brittle behavior possibility at the ductile/brittle transition.

It is untoughened polystyrene and PMMA that are at or near the *T/C* = 1/2 case in Fig. 3(*c*). These materials are considered to be rather brittle, at least within the family of all glassy polymers. This further suggests that the ductile/brittle transition may at least in this case be fairly sharp since the state of simple tension is right at the ductile/brittle transition. For values of *T/C* > 1/2, the ductile/brittle transition exists only in the first quadrant of Fig. 3 and that completely changes the nature of failure for most stress states.

The next case is Fig. 3(*d*) for *T/C* = 2/3. Most aerospace grade epoxies have *T/C* in the range from 0.6 to 0.7 and have significant but not extreme ductility, such as occurs with aluminum. From Fig. 3(*d*) and the failure criteria (1)–(5) and the ductile/brittle criterion (10)–(12), it can be seen and shown that the ductile/brittle transition occurs at the two tensile biaxial stress states of

$$\sigma_1 = 2\sigma_2$$

and

$$\sigma_2 = 2\sigma_1$$

These two stress states in Fig. 3(*d*) are quite close to the state of uniaxial tension. This may indicate that the ductile/brittle transition is quite sharp in this case.

The final case in Fig. 3 is that of the perfectly ductile state at *T/C* = 1. This is representative of aluminum and all of the very ductile metals, such as silver, gold, and copper. At *T/C* = 1, all materials are universally considered to be extremely ductile and Fig. 3(*e*) is consistent with that behavior. Much testing has been done in the biaxial stress space of Fig. 3(*e*). But the exact condition of eqi-biaxial stress has not been actively studied experimentally. The definitive testing data of Taylor and Quinney [3] is for uniaxial tension plus superimposed shear. This combination does not come anywhere close to eqi-biaxial tension. This situation for the ductile/brittle transition in Fig. 3(*e*) at eqi-biaxial tension does leave some open questions as to its interpretation, it will be taken up further from a 3D point of view in consistency test 6.

These five cases of Fig. 3 are signal parts of the verification process for the continuum of failure modes for the full spectrum of materials' types.

They reveal distinctive but realistic features of the relation between ductile and brittle failure. They form a comprehensive account of the ductile/brittle failure characteristics as a function of the materials' type designation through the *T/C* value.

would not have given this result. These four cases form a closed loop of cases and not an open sequence of cases.

Consistency Test 5: Ductile/Brittle Transitions in 2D. There is much insight and understanding to be gained by examining two-dimensional plane stress failure envelopes as they progress from the brittle limit to the ductile limit. It is especially important to indicate and assess the exact position of the ductile/brittle transition in each particular case. The ductile/brittle planes (lines in 2D) must show a decisive and consistent pattern of change as the materials' type changes. Five cases will be given, those for *T/C* = 0, 1/3, 1/2, 2/3, and 1. These five materials cases are shown in Fig. 3. All of the graphics are computer generated from the ductile/brittle failure theory, no schematic renderings are involved.

The sequence of the stress states at the ductile/brittle transitions for the cases of Fig. 3 is given in Table 4.

It is seen that the ductile/brittle transitions in Fig. 3 and Table 4 form a logical sequence of stress states going from compressive to ever more tensile as the *T/C* ratio increases.

The 3D form of the brittle limit was shown in Fig. 2. In 2D plane stress, it is as given in Fig. 3(*a*). It is difficult to find reliable data at or near the brittle limit although some geological materials have *T/C* ratios in the range of 1/50 to 1/100. But it is clear that the brittle limit does exist, it is not just a vague concept.
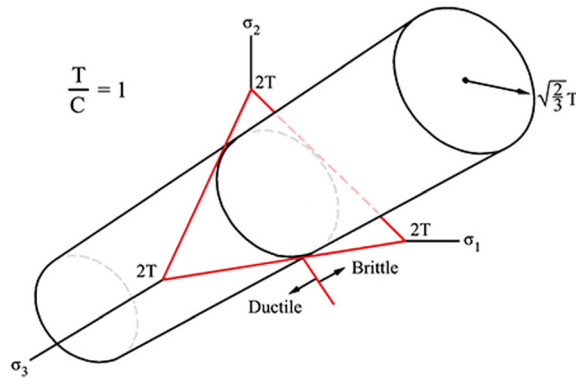
Fig. 4   The ductile limit at $T/C = 1$ with the D/B transition

Consistency Test 6: The Ductile Limit in 3D. As seen in Fig. 3(e), a very interesting situation arises as to the ductile/brittle interpretation of what is really happening with a perfectly ductile $T/C = 1$ material when placed in a state of eqi-biaxial tension and taken to failure.

The cases in Fig. 3 are all in 2D plane stress states. To better understand the meaning of the ductile/brittle transition in Fig. 3(e) for $T/C = 1$, it is necessary to also view this case in full 3D perspective. The $T/C = 1$ case is that of the Mises criterion and this is shown in Fig. 4 along with the ductile/brittle transition designation from Eq. (10). The Mises criterion is shown in the usual graphical form found in most mechanics of materials textbooks.

It is of relevance to understand the location of the ductile/brittle transition in Fig. 4. It is determined by the intercepts of the ductile/brittle transition plane with the coordinate axes, which is found to occur at $\hat{\sigma}_1 = \hat{\sigma}_2 = \hat{\sigma}_3 = 2$ from Eq. (10).

There is only one characteristic dimension for the Mises cylinder in Fig. 4 and it is that of its radius, r. One would expect the distance from the coordinate origin to the ductile/brittle transition plane to not only be the same order of magnitude as the radius of the Mises cylinder but also further expect it to be approximately of the same size. The comparison is as follows. The distance z to the ductile/brittle transition plane is

$$z = \frac{2}{\sqrt{3}}T = 1.15T \qquad (25)$$

while that for the radius of the Mises cylinder is

$$r = \sqrt{\frac{2}{3}}T = 0.817T \qquad (26)$$

Now with the ductile/brittle plane for a $T/C = 1$ material located as shown in Fig. 3(e), all 2D stress states are ductile except for that at eqi-biaxial tension which is right at the ductile/brittle transition. In Fig. 4 for all 3D stress states, there are stress states on both sides of the ductile/brittle transition although the extent of the Mises cylinder is unknown. The dividing line is still at eqi-biaxial tension. Thus, the crucial case needed for physical understanding is that of eqi-biaxial tension.

Is there a realistic means of producing a perfect state of eqi-biaxial tension? The answer is yes, and it is that of thin spherical pressure vessels. Such pressure vessels made from steel or titanium fail by brittle fracture involved with blowing out a plug or by major fragmentation. The only exception is when failure occurs at the weakening filling valves due to stress concentrations.

A graphic example of such brittle behavior is shown in the website www.FailureCriteria.com, Section VI. The plug was blown out with such momentum for the spherical, steel pressure vessel that it crippled the supporting structure. High-strength steel and titanium pressure vessels have $\sigma_1 = \sigma_2$ and $T/C = 1$. They do not

show major plasticity states of deformation but they do fail by explosive fragmentation. The collection of examples such as this one shows that the failure of perfectly ductile materials can and does occur with brittle behavior, consistent with the ductile/brittle predictions in Figs. 3(e) and 4. This is yet another fundamental corroboration of the general failure theory.

An interesting conclusion can be reached based upon the ductile/brittle behavior at the $T/C = 1$ end of the spectrum and the scarcity of corresponding ductile/brittle information at the other end of the materials' spectrum. This conclusion is that it is likely that the ductile/brittle transition is more defined and sharper in the $T/C > 1/2$ range of materials than it is in the $T/C < 1/2$ range.

Consistency Test 7: The Ductile/Brittle Transition Temperature. The purely mechanics based theory discussed up to here admits generalization to give rational and reasonable predictions of the ductile/brittle transition temperature for the full range of isotropic materials. This major development will be fully treated in the next section. At this point, it is stated as being another supporting consistency test, the details are provided in the next section.

Conclusions From Consistency Tests. This evaluation started with the ductile/brittle delineation of several of the most important stress states in Table 1. Then using physical consistency tests, the entire ductile/brittle theory was examined in great detail to obtain and illuminate the coordination of all aspects of failure. The native complexity is due to the interaction of all possible materials' types acted upon by any and all stress states. Only a truly comprehensive failure theory could unravel this convoluted interaction and not just for failure but for ductile versus brittle failure. In addition to seeing the ductile/brittle transition in all the most important forms of its existence, it was shown to be equally vital and necessary to understand the existence of the ductile limit and the brittle limit of the materials' types.

The stress state of uniaxial tension was shown to provide the key for quantifying all ductile versus brittle characteristics for all stress states. For uniaxial tension, the measure of the degree of ductility is directly given by the materials' type inherent T/C ratio. The full failure theory specifies the location of the ductile/brittle transition for any stress state as a function of the materials' type T/C. Alternatively, the converse is also outlined. A complete and physically consistent account of all these matters was the end result.

Another facet of understanding the ductile/brittle behavior is the fact that there is a range of failure behavior where for all materials' types all failure stress states are of ductile type and another range where for all materials' types all failure stress states produce brittle failure. From the ductile/brittle criteria (10)–(12) and from Eq. (1), it follows that for all failure stress states satisfying Eqs. (2)–(5):

$$\text{If} \quad \sigma_{11}^f + \sigma_{22}^f + \sigma_{33}^f > 2C, \quad \text{Always brittle}$$

and

$$\text{If} \quad \sigma_{11}^f + \sigma_{22}^f + \sigma_{33}^f < -C, \quad \text{Always ductile}$$

The assurance of ductility requires predominantly compressive stress states while the certainty of brittleness is associated with predominately tensile stress states. In between these two ranges of behavior is the third range where the real action lies. In this intermediate range, the failure behavior can be either ductile or brittle and each separate case must be evaluated individually using the full theory.

When combined with the previous evaluation of the failure theory in Ref. [2], this evaluation through consistency tests of the ductile/brittle aspects of the failure theory completes and finishes

the entire verification and validation proof for the new failure theory.

With this completion status for the mechanics based theory, attention can now be shifted into nonisothermal conditions in the next and final technical section.

The Ductile/Brittle Transition Temperature

Up until this point, all concerns have been with the purely mechanics based theory of ductility. How should ductility be defined, measured, verified, and utilized? These questions have been posed and answered.

With some success in that direction, there is now support and opportunity to venture into nonisothermal conditions. What will be approached and done will be far short of producing a general thermomechanical theory of ductile/brittle failure behavior. A more limited but still highly challenging effort will be made to extend the preceding mechanical theory into predicting the ductile/brittle transition temperatures for all isotropic materials.

The term "predicting" needs to be clarified. The goal is to develop the theoretical capability to predict the ductile/brittle transition temperature in terms of the relevant thermal and mechanical properties at any given ambient state. The ambient state of the material is set and determined by its chemical and physical constitution. It may be very ductile or very brittle or any varying degree in between depending upon its present state.

The problem posed is that of the classical problem, exactly what amount of temperature change is needed to bring the material to its ductile/brittle transition in uniaxial tension. With only a few exceptions, this problem is usually approached empirically by failure data generation under changing temperature conditions.

The most common empirical approach is to use Charpy or Izod impact tests to give an energy of fracture as a function of the temperature, over a considerable range of temperature change. The theoretical studies that have been done usually relate to particular metallic compositions. The prominent work of Rice and Thomson [1] has already been mentioned. Other typical works which focus explicitly on the ductile/brittle transition temperature are those of Petch [5], Heslop and Petch [6], Armstrong [7], Ashby and Embury [8], Hirsch and Roberts [9], Giannattasio and Roberts [10], and many others. These dislocation sourced mechanisms are most often for BCC metals and mainly relate to the temperature-controlled effects of plastic flow versus cleavage type fracture at the nanoscale.

The objective here is to obtain the theoretical prediction of the ductile/brittle transition temperature for all isotropic materials and not just a particular class of metals or a particular class of any other materials form. The approach will use the ductile/brittle theory specified through Eqs. (1)–(12). This theory allows the construction of the ductile/brittle transition as a function of an externally imposed pressure state. The extension into temperature dependence will occur through a thermomechanical relationship between temperature and pressure.

The ductile/brittle transition temperature will be posed as that for the state of uniaxial tension taken to failure. This is the standard and most common and most interesting condition for a fundamental statement of the ductile/brittle transition temperature.

To begin, it is necessary to specialize the general failure theory in Eqs. (1)–(12) to the case of uniaxial stress superimposed with a state of hydrostatic stress of positive or negative sign. Let p be the hydrostatic pressure but of algebraic character with positive p meaning negative hydrostatic stress and negative p being positive hydrostatic stress.

It is further necessary to treat the problem in two parts, that of $T/C \geq 1/2$ and that of $T/C \leq 1/2$. The former region will involve only the polynomial invariants failure criterion while the later could involve either the polynomial invariants criterion or the competitive fracture criterion.

Whichever it is to be must be determined.

Materials With $1/2 \leq T/C \leq 1$. In the materials' type range shown here, only the polynomial invariants failure criterion applies and not the fracture criterion. The problem of interest is that of uniaxial tension with superimposed pressure. This will open the door to the temperature variation problem of interest.

For uniaxial stress σ and pressure p, the failure criterion (4) gives

$$\hat{\sigma} = \frac{1}{2}\left[-\left(1 - \frac{T}{C}\right) \pm \left(1 + \frac{T}{C}\right)\sqrt{1 + 12\frac{\left(1 - \frac{T}{C}\right)}{\left(1 + \frac{T}{C}\right)^2}\hat{p}} \right] \quad (27)$$

The two signs correspond to uniaxial tension and compression. Relevance here is only for the tensile case. Pressure p is algebraic as already mentioned.

The interest here is with the ductile/brittle transition specified by Eq. (10).

The corresponding first invariant of the failure stress is given by

$$\hat{\sigma}_{ii}^f = -3\hat{p} + \hat{\sigma} \quad (28)$$

where $\hat{\sigma}$ is from Eq. (27). Now using Eqs. (27) and (28) in the ductile/brittle transition criterion (10) gives

$$\left(1 + \frac{T}{C}\right)\sqrt{1 + 12\frac{\left(1 - \frac{T}{C}\right)}{\left(1 + \frac{T}{C}\right)^2}\hat{p}} = 6\hat{p} + 5\frac{T}{C} - 1 \quad (29)$$

Solving expression (29) for $\hat{p}$ leads to a remarkably simple result for the ductile/brittle transition pressure

$$\hat{p}_{D/B} = \frac{1}{3}\left[1 - 3\frac{T}{C} + \sqrt{1 - 3\frac{T}{C} + 3\left(\frac{T}{C}\right)^2}\right] \quad (30)$$

At $T/C = 1/2$, relation (30) gives $\hat{p}_{D/B} = 0$ as it must since the material is already at the ductile/brittle transition for uniaxial tension. But at $T/C = 1$, the pressure needed to bring the material to the ductile/brittle transition is from Eq. (30)

$$p_{D/B} = -\frac{C}{3}, \quad \frac{T}{C} = 1 \quad (31)$$

Thus, a tensile hydrostatic stress of the size in Eq. (31) is required to bring the $T/C = 1$ material to its ductile/brittle transition in uniaxial tension. This result corresponds to case 2 in Table 3.

Now, it is necessary to relate pressure p to temperature. The uniaxial stress, strain, and temperature relation is given by

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} + \alpha(\theta - \theta_0) \quad (32)$$

where θ is the temperature, θ_0 is the ambient temperature, and α is the linear coefficient of thermal expansion. The corresponding dilatational form is given by

$$\varepsilon_{ii} = \frac{\sigma_{ii}}{3k} + 3\alpha(\theta - \theta_0) \quad (33)$$

where k is the bulk modulus. For pressure p, this can be rewritten as

$$\varepsilon_{ii} = -\frac{p}{k} + 3\alpha(\theta - \theta_0) \quad (34)$$

Next postulate the existence of the ductile/brittle transition temperature and further postulate the existence for it as a specific form or requirement corresponding to that of Eq. (30) for the ductile/brittle transition pressure. For the standard thermomechanical form (34) to be compatible with both of these ductile/brittle transition specifications, it is necessary that the volumetric strain on the left-hand side of Eq. (34) vanishes. Otherwise, there would be an extraneous strain associated with these two ductile/brittle transition requirements. This gives

$$\frac{p_{D/B}}{k} = 3\alpha\left(\theta_{D/B} - \theta_0\right) \tag{35}$$

To find the effect of temperature change on the ductile/brittle transition use Eq. (35) to replace $p_{D/B}$ by $\theta_{D/B}$ through

$$\hat{p}_{D/B} \to \frac{3\alpha k}{C}\left(\theta_{D/B} - \theta_0\right) \tag{36}$$

Relation (36) is the thermomechanical requirement to be used to relate the ductility dependence on pressure to the ductility dependence on temperature. This is certainly not a general thermomechanical theory of failure but it is a specific result that is sufficient for present purposes. Perhaps, this would be the first step in developing a general thermomechanical theory of failure.

The procedure is to replace $\hat{p}_{D/B}$ in the ductile/brittle transition result (30) to convert it to the corresponding result for temperature.

Using Eq. (36) in Eq. (30) gives the solution for the ductile/brittle transition temperature as

$$\theta_{D/B} = \theta_0 + \frac{C}{9\alpha k}\left[1 - 3\frac{T}{C} + \sqrt{1 - 3\frac{T}{C} + 3\left(\frac{T}{C}\right)^2}\,\right], \quad \frac{T}{C} \geq \frac{1}{2} \tag{37}$$

This final result thus shows that the ductile/brittle transition temperature depends upon the thermomechanical property α and two nondimensional mechanical property forms, T/C and C/k. This result will be checked against common observations, but first the other range of materials' types must be treated.

Materials With $0 \leq T/C \leq 1/2$. For materials with this range of T/C's, both the polynomial invariants failure criterion (2) and the fracture criterion (5) must be considered. For the stress states of uniaxial tension with superimposed hydrostatic pressure, it is found that the critical failure criterion to be used is that of fracture (5).

The stress states for σ and p take the form in Eq. (5) involving principal stresses as

$$\hat{\sigma} - \hat{p} \leq \frac{T}{C} \tag{38}$$

$$-\hat{p} \leq \frac{T}{C} \tag{39}$$

$$-\hat{p} \leq \frac{T}{C} \tag{40}$$

This gives (the same as in (28))

$$\hat{\sigma}_{ii}^f = -3\hat{p} + \hat{\sigma} \tag{41}$$

The ductile/brittle criterion (10) with Eq. (41) substituted into it becomes

$$-3\hat{p} + \hat{\sigma} - 3\frac{T}{C} + 1 = 0 \tag{42}$$

Using Eq. (38) in Eq. (42) gives the final result

$$\hat{p}_{D/B} = \frac{1}{2} - \frac{T}{C} \tag{43}$$

Then substituting this back into Eq. (42) yields

$$\hat{\sigma}_{D/B} = \frac{1}{2} \tag{44}$$

With knowledge from Eq. (43) that $\hat{p}$ is positive it follows that Eqs. (39) and (40) are satisfied to complete the process.

Relation (43) is the pressure required to bring the material to its ductile/brittle transition in uniaxial tension and the tensile failure stress is Eq. (44). Now to make the transition to temperature rather than pressure, the relation (36) will be employed and substituted into Eq. (43). This is the same procedure as in the previous case. This gives the solution for the ductile/brittle transition temperature as

$$\theta_{D/B} = \theta_0 + \frac{C}{6\alpha k}\left(1 - 2\frac{T}{C}\right), \quad \frac{T}{C} \leq \frac{1}{2} \tag{45}$$

The two results (37) and (45) cover the entire range of T/C's. As with relation (37), the form (45) gives the ductile/brittle transition temperature as the ambient state when $T/C = 1/2$ since that is the ductile/brittle transition in uniaxial tension.

Examples and Evaluation. First consider the two limits of $T/C = 1$ and $T/C = 0$. From Eqs. (37) and (45), there follows:

$$\theta_{D/B} = \theta_0 - \frac{T}{9\alpha k}, \quad \frac{T}{C} = 1 \tag{46}$$

and

$$\theta_{D/B} = \theta_0 + \frac{C}{6\alpha k}, \quad \frac{T}{C} = 0 \tag{47}$$

The limit (47) does not have immediate and obvious applications but the limit (46) is that of the ductile/brittle transition temperature for very ductile metals. This class of materials runs the range from steel and titanium to gold and silver. Over this range of ductile metals, the coefficient of thermal expansion, α, and the bulk modulus, k, do not vary by large amounts but the uniaxial strengths, $T = C$, do vary greatly. Thus, it is the uniaxial strength in that causes large variations. This variation can be as large as by a factor of 6 or 7. Gold is at the small end of the scale with the resulting ductile/brittle transition temperature for it being only a little less than that of the ambient temperature. The properties involved in Eq. (47) at the brittle limit $T/C = 0$ probably would require special interpretation.

Three typical examples for the prediction of the ductile/brittle transition temperature will be given. These will be for the particular cases of $T/C = 1$, 2/3, and 1/3 materials. The respective materials are high-strength steel, an epoxy thermoset, and gray cast iron. The first two predictions follow from Eq. (37) and the $T/C = 1/3$ case follows from Eq. (45).

The necessary properties for the three materials are assembled in Table 5. Although these properties vary somewhat for different compositions of the various alloys, these are generally in the proper range for the three materials' types. The ambient temperatures are at $\theta_0 = 25\,°$Celsius and all properties are at this temperature.

The predictions for the ductile/brittle transition temperatures are the bottom items shown in Table 5.

These predictions are not easily compared with data from typical impact tests, but these predictions are well within the range of practical experience for these materials. The ductile/brittle transition temperatures are not those of either very ductile or very brittle behavior, but rather are at the crucial intermediate stage of the

Table 5 D/B transition temperature predictions from Eqs. (37) and (45)

Property	Steel	Epoxy	Iron
T/C	1	2/3	1/3
C (MPa)	800	120	750
k (GPa)	150	3.7	170
$\alpha \times 10^{-6}$ (1/°C)	12	35	12
$\theta_{D/B}$ (°C)	−24.4	−18.5	45.4

transition. The predictions are quite remarkable in their consistency for the following reason. The properties that enter the formulas (37) and (45) include those of the coefficient of thermal expansion, the uniaxial tensile and compressive strengths, and the bulk modulus. In standard units, these properties for these materials vary by about 15 orders of magnitude and yet the predictions fall right in the proper range of practical experience for all three cases, which themselves represent a huge range of materials types. A less well-posed theory would probably be in error by many orders of magnitude just as the classical predictions of ideal strength are.

These are the first general predictions for the ductile/brittle transition temperatures that have ever been developed and substantiated. The formulas (37) and (45) are of general applicability. In some cases, the transition may be fairly sharp while in many other cases it would likely be diffuse and gradual.

Overall Completion and Conclusion

There have been many obstacles and false starts along the tortuous historical path of the development of failure theory for homogeneous and isotropic materials. One of the most significant difficulties has been that of sifting through the long-term maze of experimental data on failure. It is necessary to judge which data should be discarded, which is marginal but possibly helpful or marginal but possibly misleading, and which data constitutes gold standard, unshakably reliable failure data anchors.

The magnitude of the testing problem is best understood as follows. It is not possible to probe the necessary multidimensional stress space for failure without having multiaxial failure testing. But multiaxial failure testing (in contrast to elastic properties testing) is exquisitely sensitive to materials' quality, sample preparation, equipment design, and testing technique. It is the rule, not the exception, that the data of multiaxial failure testing inherently and especially includes large, sometimes extreme bias and scatter.

If that was not difficulty enough, the parallel state of confusion with the associated failure theory was even more acutely forbidding, difficult, and obscure. This general theoretical situation has been extensively examined and described previously [2]. Confronted by all these barriers and the pervasive state of negativity amounting to a presumption of impossibility, the present initiative set aside all previous attempts and started with a wholly new and unencumbered development program on failure.

The most intense attention and effort in the new program has been set upon securing the strongest possible theoretical foundation for the subject and then conducting the evaluations using only the very highest quality, time tested, and well-accepted experimental failure data. All this was and is supported by critical consistency evaluations as have been given here. The present evaluation of the ductile versus brittle criteria related to failure is the final piece in the overall plan. This broad gauge approach to failure characterization is the only one that could possibly succeed for such a long standing, classically difficult subject.

There is just one remaining and persistent question, even doubt, that must be given account. How could a macroscopic scale theory by itself accomplish all that is here shown to be so? The quite common view is that while macroscopic theories are utilitarian, if you require a fundamental understanding; the probe must go down in scale, extremely down in scale. It is true that many more, far more empirical excursions have been generated at the macroscopic scale than at any other scale but that certainly is not the fault of the scale itself.

The underlying truth and reality are that physically insightful conceptions, original and revealing synthesis, and rigorous mathematics can and does occur at all scales. There is no diminishing threshold of scale acceptability. There is absolute scale invariance in all these matters. The only relevant scale question to be resolved is which scale provides the tightest focus for any particular problem of interest.

The macroscopic failure of materials is a complex amalgam of many effects at all scales but especially those of interactive modes of failure, which are themselves seeded by defects at all scales, including the macroscopic scale. The macroscopic scale not only reveals itself but in the case of failure it also subsumes all smaller scales as well.

In final historical perspective, it is now apparent that the seemingly insurmountable enigma of the physical sources, mechanisms, and mathematical representations of materials failure has been unfolded and revealed by the always present discipline of the mechanics of continua. No special trappings or devices or subterfuges were involved. The straight, traditional, uncompromising, absolutely rigorous discipline of mechanics enabled the entire ductile/brittle failure theory development. Accordingly, all credit and all appreciation must go back to the scientific founders of mechanics: Newton, Hooke, Bernoulli(s), Euler, Lagrange, Navier, Cauchy, Maxwell, Timoshenko, and many others. Their theoretical creations always were and will always remain monumental. They are timeless.

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