2014 Drucker Medal Paper: A Derivation of the Theory of Linear Poroelasticity From Chemoelasticity

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The purpose of this brief paper is to present a new derivation of Biot's theory of linear poroelasticity (Biot, M., 1935, "Le Problemé de la Consolidation des Matiéres Argileuses Sous une Charge," Ann. Soc. Sci. Bruxelles, ***B55****, pp. 110–113; Biot, M., 1941, "General Theory of Three-Dimensional Consolidation," J. Appl. Phys.,* ***12****, pp. 155–164; and Biot, M., and Willis, D., 1957, "The Elastic Coefficients of the Theory of Consolidation," J. Appl. Mech.,* ***24****, pp. 594–601) in a modern thermodynamically consistent fashion, and show that it may be deduced as a special case of a more general theory of chemoelasticity.*

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1 Introduction

A theory for addressing coupled problems concerning fluid-saturated deformable porous solids with diffusive fluid transport is of importance for a variety of engineering applications. The classical linear isotropic poroelasticity theory which accounts for the influence of pore fluid diffusion on the quasi-static deformation of porous media is due to Biot [1–3]. For a good recent review of the theory, see Ref. [4]. Biot's theory is intended to represent the coupled deformation–diffusion response of a material which at a microscopic scale consists of a porous solid skeleton and a freely moving fluid in a fully connected pore space. Over the last 40 years, the theory has been widely applied to geotechnical problems [5–9], and it has also been applied to bone by many authors [10].

The purpose of this brief paper is to present a *new* derivation of the theory of linear poroelasticity in a modern thermodynamically consistent fashion,[1] and show that it may be deduced as a *special case* of a more general theory of linear chemoelasticity.[2] Following Biot, the fluid–solid mixture is treated as a single homogenized continuum body which allows for a mass flux of the fluid, and not as a multicomponent mixture as in the "theory of mixtures" [11,12].

In formulating the theory, we depart from the traditional formulations of poroelasticity which from the outset are based on the concept of "pore-pressure." It has long been recognized, going back to Gibbs [13]—and substantially later also by Biot [14,15] and Rudnicki [8]—that the appropriate concept to use is that of a *chemical potential*, μ, of the fluid molecules, as is now common in other developments of coupled deformation–diffusion problems in modern continuum mechanics [16–21]. As we shall discuss below, if the chemical potential μ is expressed in units of J/mol, and if the referential value of the molar volume of the fluid, $\Omega^f_{\rm R}$, is expressed in terms of m^3/mol, then a quantity

$$p \overset{\rm def}{=} \frac{\mu}{\Omega^f_{\rm R}} \qquad \text{which has units of N/m}^2$$

may be identified as a "pore-pressure," and used to recover a formulation which is identical in form to Biot's standard theory of linear poroelasticity.

In Secs. 2–6, we derive and summarize our thermodynamically consistent small deformation theory of chemoelasticity. Regarding the constitutive equations, the free energy per unit reference volume, ψ, will be taken to be a function of the small strain tensor $\boldsymbol{\epsilon}$, and a variable $c_{\rm R}$ which denotes the total number of moles of the fluid reckoned per unit volume of the reference body. Thermodynamic considerations will yield the restrictions that the free energy determines the Cauchy stress, $\boldsymbol{\sigma}$, and the chemical potential, μ, through the state relations

$$\boldsymbol{\sigma} = \frac{\partial\hat{\psi}(\boldsymbol{\epsilon}, c_{\rm R})}{\partial\boldsymbol{\epsilon}}, \qquad \mu = \frac{\partial\hat{\psi}_{\rm R}(\boldsymbol{\epsilon}, c_{\rm R})}{\partial c_{\rm R}} \tag{1.1}$$

Further, with $\mathbf{j}$ denoting the fluid molar flux, the constitutive equation for $\mathbf{j}$ will be taken to follow a generalized Fick's law [22]:

$$\mathbf{j} = -\mathbf{M}(\boldsymbol{\epsilon}, c_{\rm R})\nabla\mu \tag{1.2}$$

where $\mathbf{M}$ is a positive-semidefinite fluid mobility tensor.

In Sec. 7, we shall present a specialization of the general chemoelasticity theory given by Eqs. (1.1) and (1.2) for application to *linear poroelastic materials*. Instead of $c_{\rm R}$, the specialized constitutive equations will employ a variable

$$\zeta = \Omega^f_{\rm R}(c_{\rm R} - c_{\rm R0})$$

which represents a dimensionless measure of the *variation in the fluid content*, where $c_{\rm R0}$ is the initial value of $c_{\rm R}$. In this section, we also give an extended discussion of the relation of the theory developed here to Biot's classical linear theory of isotropic poroelasticity.

We close this paper in Sec. 8 by summarizing the specialized constitutive theory, and stating the complete coupled deformation–diffusion initial/boundary-value problem.

2 Kinematics and Force and Moment Balances

We consider a deformable body occupying a (fixed) region B. The deformed body is denoted by $\mathcal{B}$. We restrict attention to *isothermal* situations and *small deformations*, and since time scales

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[1] This statement is not intended to imply that the widely used constitutive equations for linear poroelasticity are thermodynamicaly inconsistent, but instead to reflect the fact that thermodynamics is seldom used consistently in their derivation.

[2] The linear chemoelasticity theory presented here is itself a specialization of the large deformation theory presented in by Gurtin et al. [16].

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associated with fluid diffusion are usually considerably longer than those associated with wave propagation, we *neglect all inertial effects*.

We take as the starting point the kinematical assumptions of the linear theory of elasticity. As in the theory of linear elasticity, $\mathbf{x}$ denotes a material point in B, ∇ and div denote the gradient and divergence with respect to $\mathbf{x}$. We use the following notation:

- $\mathbf{u}(\mathbf{x},t)$ displacement field of B
- $\nabla\mathbf{u}(\mathbf{x},t)$ displacement gradient in B
- $\boldsymbol{\epsilon}(\mathbf{x},t) = (1/2)(\nabla\mathbf{u}(\mathbf{x},t) + (\nabla\mathbf{u}(\mathbf{x},t))^{\top})$, strain in B;
- $\boldsymbol{\sigma}(\mathbf{x},t)$ stress
- $\mathbf{b}(\mathbf{x},t)$ (noninertial) body force field on B,
  and recall the following standard relations for force and moment balances from the theory of linear elasticity, which are also valid for the theory considered here

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\top} \tag{2.1}$$

3 Balance Law for the Diffusing Fluid

Throughout, we denote by P an arbitrary part of the reference body B with $\mathbf{n}$ the outward unit normal on ∂P. Let

$$c_{\mathrm{R}}(\mathbf{x},t) \tag{3.1}$$

denote the *total number of moles of fluid per unit reference volume*. Changes in c_{R} in a part P are brought about by diffusion across the boundary ∂P, which is characterized by a flux $\mathbf{j}(\mathbf{x},t)$, the number of moles of diffusing fluid measured per unit area per unit time, so that

$$-\int_{\partial \mathrm{P}} \mathbf{j}\cdot\mathbf{n}\,da$$

represents the number of moles of the diffusing fluid entering P across ∂P per unit time. The balance law for the diffusing fluid therefore takes the form

$$\dot{\overline{\int_{\mathrm{P}} c_{\mathrm{R}}\,dv}} = -\int_{\partial \mathrm{P}} \mathbf{j}\cdot\mathbf{n}\,da \tag{3.2}$$

for every part P. Bringing the time derivative in Eq. (3.2) inside the integral and using the divergence theorem on the integral over ∂P, we find that

$$\int_{\mathrm{P}} (\dot{c}_{\mathrm{R}} + \operatorname{div}\mathbf{j})dv = 0 \tag{3.3}$$

Since P is arbitrary, this leads to the following local balance:

$$\dot{c}_{\mathrm{R}} = -\operatorname{div}\mathbf{j} \tag{3.4}$$

4 Free-Energy Imbalance

We develop the theory within a framework that accounts for the first two laws of thermodynamics. For isothermal processes, the first two laws collapse into a single dissipation inequality which asserts that temporal changes in free energy of a part P be not greater than the power expended on P, plus the flux of energy carried into P.

Thus, let $\psi(\mathbf{x},t)$ denote the free energy density per unit reference volume. Then, neglecting inertial effects, the free-energy imbalance under isothermal conditions requires that for each part P of B.

$$\dot{\overline{\int_{\mathrm{P}} \psi\,dv}} \le \int_{\partial \mathrm{P}} \boldsymbol{\sigma}\mathbf{n}\cdot\dot{\mathbf{u}}\,da + \int_{\mathrm{P}} \mathbf{b}\cdot\dot{\mathbf{u}}\,dv - \int_{\partial \mathrm{P}} \mu\mathbf{j}\cdot\mathbf{n}\,da \tag{4.1}$$

where the left-hand side of Eq. (4.1) represents the rate of change of the free energy of the part P, and the right-hand side represents the power expended on P by the surface tractions and body force, plus the flux of energy carried into P by the flux $\mathbf{j}$ of diffusing fluid, where to characterize the energy carried into part P by fluid transport, we have introduced a *chemical potential field* $\mu(\mathbf{x},t)$. It is a *primitive quantity* that enters the theory through the manner in which it appears in the basic law expressing the balance of energy (cf. Gurtin et al. [16]).

Bringing the time derivative in Eq. (4.1) inside the integral and using the divergence theorem on the integrals over ∂P gives

$$\int_{\mathrm{P}} (\dot{\psi} - \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - (\operatorname{div}\boldsymbol{\sigma} + \mathbf{b})\cdot\dot{\mathbf{u}} + \mu\operatorname{div}\mathbf{j} + \mathbf{j}\cdot\nabla\mu)dv \le 0 \tag{4.2}$$

Thus, Eq. (4.2), balance of forces $(2.1)_1$, fluid balance (3.4), and the arbitrary nature of P together yield the following local form of the *free-energy imbalance*

$$\dot{\psi} - \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} - \mu\dot{c}_{\mathrm{R}} + \mathbf{j}\cdot\nabla\mu \le 0 \tag{4.3}$$

5 Constitutive Equations

Guided by the free-energy imbalance (4.3), we introduce the following constitutive response functions for the free energy ψ, the stress $\boldsymbol{\sigma}$, and the chemical potential μ, when the strain $\boldsymbol{\epsilon}$ and the fluid contentration c_{R} are known.

$$\left.\begin{aligned} \psi &= \hat{\psi}(\boldsymbol{\epsilon}, c_{\mathrm{R}}), \\ \boldsymbol{\sigma} &= \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}, c_{\mathrm{R}}), \\ \mu &= \hat{\mu}(\boldsymbol{\epsilon}, c_{\mathrm{R}}) \end{aligned}\right\} \tag{5.1}$$

To these constitutive equations, we append a simple Fick's law [22] for the fluid flux

$$\mathbf{j} = -\mathbf{M}\nabla\mu, \qquad \text{with} \qquad \mathbf{M} = \widehat{\mathbf{M}}(\boldsymbol{\epsilon}, c_{\mathrm{R}}) \tag{5.2}$$

the fluid mobility tensor.

5.1 Thermodynamic Restrictions.

The free energy imbalance (4.3) restricts the constitutive equations. By Eq. $(5.1)_1$

$$\dot{\psi} = \frac{\partial\hat{\psi}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial\boldsymbol{\epsilon}} : \dot{\boldsymbol{\epsilon}} + \frac{\partial\hat{\psi}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial c_{\mathrm{R}}}\dot{c}_{\mathrm{R}} \tag{5.3}$$

Hence, the free-energy imbalance (4.3) is equivalent to the requirement that

$$\begin{aligned} &\left[\frac{\partial\hat{\psi}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial\boldsymbol{\epsilon}} - \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}, c_{\mathrm{R}})\right] : \dot{\boldsymbol{\epsilon}} + \left[\frac{\partial\hat{\psi}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial c_{\mathrm{R}}} - \hat{\mu}(\boldsymbol{\epsilon}, c_{\mathrm{R}})\right]\dot{c}_{\mathrm{R}} \\ &\quad - \nabla\mu\cdot\widehat{\mathbf{M}}(\boldsymbol{\epsilon}, c_{\mathrm{R}})\nabla\mu \le 0 \end{aligned} \tag{5.4}$$

for all strain, fluid concentration, and chemical potential fields. Given any point $\mathbf{x}_0$ in B and any time t_0, it is possible to find a strain and a fluid concentration field such that

$$\boldsymbol{\epsilon},\ \dot{\boldsymbol{\epsilon}},\ c_{\mathrm{R}},\ \text{and}\ \dot{c}_{\mathrm{R}} \tag{5.5}$$

have arbitrarily prescribed values at $(\mathbf{x}_0, t_0)$. Granted this, the coefficients of $\dot{\boldsymbol{\epsilon}}$ and $\dot{c}_{\mathrm{R}}$ must vanish, for otherwise these rates may be chosen to violate the inequality (5.4). We therefore have the thermodynamic restrictions:

- The free energy determines the stress through the *stress relation*

$$\boldsymbol{\sigma} = \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}, c_{\mathrm{R}}) = \frac{\partial\hat{\psi}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial\boldsymbol{\epsilon}} \tag{5.6}$$

- The free energy determines the chemical potential through the *chemical potential relation*

$$\mu = \hat{\mu}(\boldsymbol{\epsilon}, c_{\mathrm{R}}) = \frac{\partial \hat{\psi}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial c_{\mathrm{R}}} \tag{5.7}$$

- The fluid flux satisfies the *fluid-transport inequality*

$$\nabla\mu \cdot \widehat{\mathbf{M}}(\boldsymbol{\epsilon}, c_{\mathrm{R}})\nabla\mu \geq 0 \qquad \text{for all } (\boldsymbol{\epsilon}, c_{\mathrm{R}}, \nabla\mu) \tag{5.8}$$

thus the mobility tensor $\mathbf{M}$ is positive semidefinite.

5.2 Consequences of the Thermodynamic Restrictions.

By Eqs. (5.3), (5.6), and (5.7), we have the *Gibbs relation*

$$\dot{\psi} = \boldsymbol{\sigma} : \dot{\boldsymbol{\epsilon}} + \mu \dot{c}_{\mathrm{R}} \tag{5.9}$$

while Eqs. (5.6) and (5.7) yield the *Maxwell relation*

$$\frac{\partial \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial c_{\mathrm{R}}} = \frac{\partial \hat{\mu}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial \boldsymbol{\epsilon}} \tag{5.10}$$

Let $\boldsymbol{\epsilon}(t)$ be a time-dependent strain tensor, let $c_{\mathrm{R}}(t)$ be a time-dependent fluid concentration, and write

$$\boldsymbol{\sigma}(t) = \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}(t),\, c_{\mathrm{R}}(t))$$

The chain-rule then yields the relation

$$\dot{\boldsymbol{\sigma}} = \frac{\partial \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial \boldsymbol{\epsilon}} \dot{\boldsymbol{\epsilon}} + \frac{\partial \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial c_{\mathrm{R}}} \dot{c}_{\mathrm{R}}$$

which suggests the introduction of two constitutive moduli:

- The fourth-order *elasticity tensor* $\mathbb{C}(\boldsymbol{\epsilon}, c_{\mathrm{R}})$ defined by

$$\mathbb{C}(\boldsymbol{\epsilon}, c_{\mathrm{R}}) = \frac{\partial \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial \boldsymbol{\epsilon}} = \frac{\partial^2 \hat{\psi}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial \boldsymbol{\epsilon}^2} \qquad (\text{at fixed } c_{\mathrm{R}}) \tag{5.11}$$

The elasticity tensor $\mathbb{C}$ represents elastic moduli under conditions in which c_{R} is held *constant*, that is, conditions under which the fluid is constrained from flowing in or out of the body during deformation. In the poroelasticity literature conditions under which c_{R} is constant are called *undrained* (cf. Rice and Cleary [5]), and accordingly we call $\mathbb{C}$ the *undrained elasticity tensor*. In practice, undrained conditions are approached for deformations which are rapid (though quasi-static) relative to the time-scale for fluid diffusion.

- The second-order *stress–chemistry tensor* $\mathbf{S}(\boldsymbol{\epsilon}, c_{\mathrm{R}})$ defined by

$$\mathbf{S}(\boldsymbol{\epsilon}, c_{\mathrm{R}}) = \frac{\partial \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial c_{\mathrm{R}}} = \frac{\partial^2 \hat{\psi}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial \boldsymbol{\epsilon} \partial c_{\mathrm{R}}} \tag{5.12}$$

For each $(\boldsymbol{\epsilon}, c_{\mathrm{R}})$, the elasticity tensor $\mathbb{C}(\boldsymbol{\epsilon}, c_{\mathrm{R}})$—a linear transformation that maps symmetric tensors to symmetric tensors—has the minor symmetry properties

$$C_{ijkl} = C_{jikl} = C_{ijlk} \tag{5.13}$$

as well as the major symmetry property

$$C_{ijkl} = C_{klij} \tag{5.14}$$

Further, since the stress $\boldsymbol{\sigma}$ is symmetric, we may conclude from Eq. (5.12) that the stress–chemistry modulus $\mathbf{S}(\boldsymbol{\epsilon}, c_{\mathrm{R}})$ is a symmetric tensor; this tensor measures the marginal change in stress due to a change in concentration holding the strain fixed.

Proceeding as above, let

$$\mu(t) = \hat{\mu}(\boldsymbol{\epsilon}(t), c_{\mathrm{R}}(t))$$

Then, by the chain rule, Eqs. (5.10) and (5.12)

$$\begin{aligned} \dot{\mu} &= \frac{\partial \hat{\mu}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial \boldsymbol{\epsilon}} : \dot{\boldsymbol{\epsilon}} + \frac{\partial \hat{\mu}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial c_{\mathrm{R}}} \dot{c}_{\mathrm{R}}, \\ &= \mathbf{S}(\boldsymbol{\epsilon}, c_{\mathrm{R}}) : \dot{\boldsymbol{\epsilon}} + \frac{\partial \hat{\mu}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial c_{\mathrm{R}}} \dot{c}_{\mathrm{R}} \end{aligned}$$

This suggests the introduction of

- a scalar *chemistry modulus* $\Lambda(\boldsymbol{\epsilon}, c_{\mathrm{R}})$ defined by

$$\Lambda(\boldsymbol{\epsilon}, c_{\mathrm{R}}) = \frac{\partial \hat{\mu}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial c_{\mathrm{R}}} \tag{5.15}$$

Unless specified otherwise, we assume that the chemistry modulus is strictly positive for all $(\boldsymbol{\epsilon}, c_{\mathrm{R}})$[3]

$$\Lambda(\boldsymbol{\epsilon}, c_{\mathrm{R}}) > 0 \tag{5.16}$$

5.3 Chemical Potential as Independent Variable.

For problems involving little or no fluid diffusion, the strain $\boldsymbol{\epsilon}$ and the concentration c_{R} are the natural choice of independent constitutive variables. However, for processes that occur over long time scales so that fluid diffusion is important, it is often preferable to replace constitutive dependence upon c_{R} by constitutive dependence upon the chemical potential μ.

Note from Eqs. (5.15) and (5.16) that we have assumed that

$$\frac{\partial \hat{\mu}(\boldsymbol{\epsilon}, c_{\mathrm{R}})}{\partial c_{\mathrm{R}}} > 0 \tag{5.17}$$

This allows us to conclude that, for each fixed $\boldsymbol{\epsilon}$, the relation

$$\mu = \hat{\mu}(\boldsymbol{\epsilon}, c_{\mathrm{R}}) \tag{5.18}$$

is smoothly invertible in c_{R}, so that

$$c_{\mathrm{R}} = \breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon}, \mu) \tag{5.19}$$

Let

$$\omega \overset{\text{def}}{=} \psi - c_{\mathrm{R}}\mu \tag{5.20}$$

define a *grand-canonical energy* per unit reference volume [16]. Then

$$\begin{aligned} \omega &= \breve{\omega}(\boldsymbol{\epsilon}, \mu) \\ &= \hat{\psi}(\boldsymbol{\epsilon}, \breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon}, \mu)) - \breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon}, \mu)\mu \end{aligned} \tag{5.21}$$

while Eq. (5.6) yields

$$\boldsymbol{\sigma} = \widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}, c_{\mathrm{R}}) \tag{5.22}$$

$$= \breve{\boldsymbol{\sigma}}(\boldsymbol{\epsilon}, \breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon}, \mu)) \tag{5.23}$$

Thus, bearing in mind that a "breve" denotes a function of $(\boldsymbol{\epsilon}, \mu)$ while a "hat" denotes a function of $(\boldsymbol{\epsilon}, c_{\mathrm{R}})$, we find, using Eqs. (5.6) and (5.7), that

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[3] This assumption does not hold for phase-separating materials.

$$\frac{\partial\breve{\omega}}{\partial\boldsymbol{\epsilon}}=\frac{\partial\hat{\psi}}{\partial\boldsymbol{\epsilon}}+\underbrace{\left(\frac{\partial\hat{\psi}}{\partial c_{\mathrm{R}}}-\mu\right)}_{=0}\frac{\partial\breve{c}_{\mathrm{R}}}{\partial\boldsymbol{\epsilon}}=\breve{\boldsymbol{\sigma}}$$

and

$$\frac{\partial\breve{\omega}}{\partial\mu}=\underbrace{\left(\frac{\partial\hat{\psi}}{\partial c_{\mathrm{R}}}-\mu\right)}_{=0}\frac{\partial\breve{c}_{\mathrm{R}}}{\partial\mu}-\breve{c}_{\mathrm{R}}=-\breve{c}_{\mathrm{R}}$$

The stress and the fluid concentration are therefore determined by the response function $\omega=\breve{\omega}(\boldsymbol{\epsilon},\mu)$ via the relations

$$\left.\begin{aligned}\boldsymbol{\sigma}=\breve{\boldsymbol{\sigma}}(\boldsymbol{\epsilon},\mu)&=\frac{\partial\breve{\omega}(\boldsymbol{\epsilon},\mu)}{\partial\boldsymbol{\epsilon}},\\ c_{\mathrm{R}}=\breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon},\mu)&=-\frac{\partial\breve{\omega}(\boldsymbol{\epsilon},\mu)}{\partial\mu}\end{aligned}\right\}\tag{5.24}$$

An immediate consequence of Eq. (5.24) is the *Gibbs relation*

$$\dot{\omega}=\boldsymbol{\sigma}:\dot{\boldsymbol{\epsilon}}-c_{\mathrm{R}}\dot{\mu}\tag{5.25}$$

and the Maxwell relation

$$\frac{\partial\breve{\boldsymbol{\sigma}}(\boldsymbol{\epsilon},\mu)}{\partial\mu}=-\frac{\partial\breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon},\mu)}{\partial\boldsymbol{\epsilon}}\tag{5.26}$$

The undrained elasticity and stress–chemistry tensors $\mathbb{C}(\boldsymbol{\epsilon},c_{\mathrm{R}})$ and $\mathbf{S}(\boldsymbol{\epsilon},c_{\mathrm{R}})$ have natural counterparts in the theory with chemical potential as independent variable; they are the *elasticity tensor*

$$\mathbb{C}^{\mathrm{dr}}(\boldsymbol{\epsilon},\mu)=\frac{\partial\breve{\boldsymbol{\sigma}}(\boldsymbol{\epsilon},\mu)}{\partial\boldsymbol{\epsilon}}\quad(\text{at fixed }\mu)\tag{5.27}$$

and the stress–chemical potential modulus

$$\mathbf{S}^{\mathrm{dr}}(\boldsymbol{\epsilon},\mu)=\frac{\partial\breve{\boldsymbol{\sigma}}(\boldsymbol{\epsilon},\mu)}{\partial\mu}\quad(\text{at fixed }\boldsymbol{\epsilon})\tag{5.28}$$

The elasticity tensor $\mathbb{C}^{\mathrm{dr}}$ represents the elastic moduli under conditions in which the chemical potential is held constant, and the fluid is *not constrained* from flowing in or out of the body during deformation. In the poroelasticity literature, such conditions are called *drained* (cf. Rice and Cleary [5]), and accordingly we call $\mathbb{C}^{\mathrm{dr}}$ the *drained elasticity tensor*. A "drained" response is attained for deformations that are slow relative to the time scale of fluid diffusion.

By Eqs. $(5.24)_1$ and (5.27)

$$\mathbb{C}^{\mathrm{dr}}(\boldsymbol{\epsilon},\mu)=\frac{\partial^2\breve{\omega}(\boldsymbol{\epsilon},\mu)}{\partial\boldsymbol{\epsilon}^2}\tag{5.29}$$

while the Maxwell relation (5.26) implies that

$$\mathbf{S}^{\mathrm{dr}}(\boldsymbol{\epsilon},\mu)=-\frac{\partial\breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon},\mu)}{\partial\boldsymbol{\epsilon}}\tag{5.30}$$

We now determine relations between these various material functions. Toward this end, we note, by Eq. (5.19), that we may relate the alternative descriptions of the stress in terms of $(\boldsymbol{\epsilon},c_{\mathrm{R}})$ and $(\boldsymbol{\epsilon},\mu)$ via

$$\breve{\boldsymbol{\sigma}}(\boldsymbol{\epsilon},\mu)=\widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon},\breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon},\mu))$$

Thus, by Eq. (5.28)

$$\begin{aligned}\mathbf{S}^{\mathrm{dr}}&=\frac{\partial\boldsymbol{\sigma}}{\partial\mu}\\&=\frac{\partial\widehat{\boldsymbol{\sigma}}}{\partial c_{\mathrm{R}}}\frac{\partial\breve{c}_{\mathrm{R}}}{\partial\mu}\end{aligned}$$

and, using Eqs. (5.12) and (5.15), we conclude that the stress–chemical potential and stress–chemistry moduli are related via

$$\mathbf{S}^{\mathrm{dr}}(\boldsymbol{\epsilon},\mu)=\frac{1}{\Lambda(\boldsymbol{\epsilon},c_{\mathrm{R}})}\mathbf{S}(\boldsymbol{\epsilon},c_{\mathrm{R}})\tag{5.31}$$

for $c_{\mathrm{R}}=\breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon},\mu)$.

The relation between the elasticity tensor $\mathbb{C}^{\mathrm{dr}}$ at fixed chemical potential to the elasticity tensor $\mathbb{C}$ at fixed concentration is based on computing the partial derivative

$$\begin{aligned}\mathbb{C}^{\mathrm{dr}}(\boldsymbol{\epsilon},\mu)&=\frac{\partial\breve{\boldsymbol{\sigma}}(\boldsymbol{\epsilon},\mu)}{\partial\boldsymbol{\epsilon}}\\&=\frac{\partial}{\partial\boldsymbol{\epsilon}}\Big(\widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon},\breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon},\mu))\Big)\end{aligned}$$

with respect to $\boldsymbol{\epsilon}$ holding μ fixed. Suppressing arguments and using components, this derivative of $\widehat{\boldsymbol{\sigma}}(\boldsymbol{\epsilon},\breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon},\mu))$ is given by

$$\frac{\partial(\widehat{\boldsymbol{\sigma}})_{ij}}{\partial\epsilon_{kl}}+\frac{\partial(\widehat{\boldsymbol{\sigma}})_{ij}}{\partial c_{\mathrm{R}}}\frac{\partial\breve{c}_{\mathrm{R}}}{\partial\epsilon_{kl}}$$

Thus, since the term

$$\frac{\partial(\widehat{\boldsymbol{\sigma}})_{ij}}{\partial c_{\mathrm{R}}}\frac{\partial\breve{c}_{\mathrm{R}}}{\partial\epsilon_{kl}}\quad\text{is the component form of}\quad\frac{\partial\widehat{\boldsymbol{\sigma}}}{\partial c_{\mathrm{R}}}\otimes\frac{\partial\breve{c}_{\mathrm{R}}}{\partial\boldsymbol{\epsilon}}$$

We find, with the aid of Eqs. (5.11), (5.12), (5.30), and (5.31), that

$$\mathbb{C}^{\mathrm{dr}}(\boldsymbol{\epsilon},\mu)=\mathbb{C}(\boldsymbol{\epsilon},c_{\mathrm{R}})-\frac{1}{\Lambda(\boldsymbol{\epsilon},c_{\mathrm{R}})}\mathbf{S}(\boldsymbol{\epsilon},c_{\mathrm{R}})\otimes\mathbf{S}(\boldsymbol{\epsilon},c_{\mathrm{R}})\tag{5.32}$$

for $c_{\mathrm{R}}=\breve{c}_{\mathrm{R}}(\boldsymbol{\epsilon},\mu)$; equivalently, in components, suppressing arguments

$$C^{\mathrm{dr}}_{ijkl}=C_{ijkl}-\frac{1}{\Lambda}S_{ij}S_{kl}\tag{5.33}$$

The identity (5.32) has two important consequences. First, given symmetric tensors $\mathbf{A}$ and $\mathbf{G}$

$$\mathbf{A}:(\mathbf{S}\otimes\mathbf{S})\mathbf{G}=(\mathbf{A}:\mathbf{S})(\mathbf{S}:\mathbf{G})\tag{5.34}$$

$$\qquad=\mathbf{G}:(\mathbf{S}\otimes\mathbf{S})\mathbf{A}\tag{5.35}$$

Thus, since $\mathbb{C}$ is symmetric and since the fourth-order tensor $\mathbf{S}\otimes\mathbf{S}$ is symmetric, the tensor $\mathbb{C}^{\mathrm{dr}}$ *is also symmetric*. Further, by Eqs. (5.32) and (5.34)

$$\mathbf{A}:\mathbb{C}(\boldsymbol{\epsilon},c_{\mathrm{R}})\mathbf{A}-\mathbf{A}:\mathbb{C}^{\mathrm{dr}}(\boldsymbol{\epsilon},\mu)\mathbf{A}=\frac{1}{\Lambda}(\mathbf{A}:\mathbf{S})^2$$

for any tensor $\mathbf{A}$, and since Λ is positive

$$\mathbf{A}:\mathbb{C}(\boldsymbol{\epsilon},c_{\mathrm{R}})\mathbf{A}>\mathbf{A}:\mathbb{C}^{\mathrm{dr}}(\boldsymbol{\epsilon},\mu)\mathbf{A}\quad\text{for all }\mathbf{A}\neq0\tag{5.36}$$

Thus $\mathbb{C}$ is positive-definite whenever $\mathbb{C}^{\mathrm{dr}}$ is positive definite. *We assume henceforth that both* $\mathbb{C}^{\mathrm{dr}}$ *and* $\mathbb{C}$ *are positive definite*.[4]

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[4] Positive-definiteness is a condition sufficient to ensure that these elasticity tensors are invertible.

Remark. The elasticity tensors $\mathbb{C}^{\mathrm{dr}}$ and $\mathbb{C}$ are directly analogous to the elasticity tensor at fixed temperature and the elasticity tensor at fixed entropy, respectively, in the theory of thermoelasticity (cf., e.g., Gurtin et al. [16])—an analogy which is well-known in the poroelasticity literature (cf., e.g., Rice and Cleary [5]; Lehner [4]).

6 Linear Chemoelasticity

Guided by the discussion in the previous section, we next consider a *linear* theory. With the strain–displacement relation given by

$$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\top}) \tag{6.1}$$

and using the strain $\boldsymbol{\epsilon}$ and the concentration c_{R} as the governing variables, we take the constitutive equations in the linear theory to be given by

$$\left.\begin{aligned}
&\psi = \mu_0(c_{\mathrm{R}} - c_{\mathrm{R}0}) + \frac{1}{2}\boldsymbol{\epsilon} : \mathbb{C}\boldsymbol{\epsilon} + (c_{\mathrm{R}} - c_{\mathrm{R}0})\mathbf{S} : \boldsymbol{\epsilon} + \frac{1}{2}\Lambda(c_{\mathrm{R}} - c_{\mathrm{R}0})^2,\\
&\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\epsilon} + \mathbf{S}(c_{\mathrm{R}} - c_{\mathrm{R}0}),\\
&\mu = \mu_0 + \mathbf{S} : \boldsymbol{\epsilon} + \Lambda(c_{\mathrm{R}} - c_{\mathrm{R}0}),\\
&\mathbf{j} = -\mathbf{M}\nabla\mu
\end{aligned}\right\} \tag{6.2}$$

where $\mathbb{C}, \mathbf{S}$, Λ, and $\mathbf{M}$ are, respectively, the undrained elasticity tensor, the stress–chemistry modulus, the chemistry modulus, and the fluid mobility tensor at the reference concentration $c_{\mathrm{R}0}$. Also, μ_0 is a reference chemical potential, and the reference configuration is taken to be stress-free.

By Eqs. (5.13), (5.14), and (5.36), the undrained elasticity tensor $\mathbb{C}$ is symmetric and positive-definite, and by Eq. (5.16) Λ is positive. Finally, by Eq. (5.8), the mobility tensor $\mathbf{M}$ is positive-semidefinite; we strengthen this by requiring that $\mathbf{M}$ be positive-definite

$$\mathbf{a}\cdot\mathbf{M}\mathbf{a} > 0 \quad \text{for all vectors } \mathbf{a} \neq \mathbf{0} \tag{6.3}$$

The basic equations of the linear theory of chemoelasticity consist of the strain–displacement relation (6.1), the constitutive equation (6.2), the local force balance

$$\operatorname{div}[\mathbb{C}\boldsymbol{\epsilon} + \mathbf{S}(c_{\mathrm{R}} - c_{\mathrm{R}0})] + \mathbf{b} = \mathbf{0} \tag{6.4}$$

and the local fluid balance

$$\dot{c}_{\mathrm{R}} = \operatorname{div}(\mathbf{M}\nabla\mu) \tag{6.5}$$

6.1 Isotropic Linear Chemoelasticity.

If the body is isotropic, then $\mathbb{C}, \mathbf{S}$, and $\mathbf{M}$ have the specific forms

$$\left.\begin{aligned}
\mathbb{C} &= 2G\mathbb{I}_{\mathrm{sym}} + \lambda\mathbf{1}\otimes\mathbf{1},\\
\mathbf{S} &= \beta\mathbf{1},\\
\mathbf{M} &= m\mathbf{1}
\end{aligned}\right\} \tag{6.6}$$

where $\mathbb{I}_{\mathrm{sym}}$ is the fourth-order identity tensor which maps tensors to their symmetric counterparts, and with G and λ *elastic moduli*, β the *stress–chemistry modulus*, and m the *fluid mobility*.[5]

Next, we determine the restrictions placed on the moduli G and λ by the requirement that the elasticity tensor $\mathbb{C}$ be positive-definite. Choose an arbitrary symmetric tensor $\mathbf{A}$ and let $\mathbf{A}'$ denote its deviatoric part

$$\mathbf{A}' = \mathbf{A} - \frac{1}{3}(\mathrm{tr}\ \mathbf{A})\mathbf{1}$$

Then $\mathrm{tr}\ \mathbf{A}' = 0$ and

$$|\mathbf{A}|^2 = \left(\mathbf{A}' + \frac{1}{3}(\mathrm{tr}\ \mathbf{A})\mathbf{1}\right) : \left(\mathbf{A}' + \frac{1}{3}(\mathrm{tr}\ \mathbf{A})\mathbf{1}\right) = |\mathbf{A}'|^2 + \frac{1}{3}(\mathrm{tr}\ \mathbf{A})^2$$

Thus, the positive-definiteness of $\mathbb{C}$ and $(6.6)_1$ requires that

$$\begin{aligned}
0 &< \mathbf{A} : \mathbb{C}\mathbf{A}\\
&= 2G|\mathbf{A}|^2 + \lambda(\mathrm{tr}\ \mathbf{A})^2\\
&= 2G|\mathbf{A}'|^2 + K(\mathrm{tr}\ \mathbf{A})^2
\end{aligned} \tag{6.7}$$

with

$$K = \lambda + \frac{2}{3}G \tag{6.8}$$

Choosing $\mathbf{A} = \mathbf{1}$ (so that $\mathrm{tr}\ \mathbf{A} = 3$ and $\mathbf{A}' = 0$) yields $K > 0$; choosing $\mathbf{A} = \mathbf{e}\otimes\mathbf{f} + \mathbf{f}\otimes\mathbf{e}$ with $\mathbf{e}$ and $\mathbf{f}$ orthonormal (so that $\mathrm{tr}\ \mathbf{A} = 0$ and $|\mathbf{A}'|^2 = 2$) yields $G > 0$. Thus, the elastic moduli G and λ satisfy

$$G > 0, \quad \lambda + \frac{2}{3}G > 0 \tag{6.9}$$

The scalars G and λ are generally referred to as *Lamé moduli*. In view of Eq. (6.8), the relation $(6.6)_1$ may alternatively be written in terms of the scalars G and K as

$$\mathbb{C} = 2G(\mathbb{I}_{\mathrm{sym}} - \frac{1}{3}\mathbf{1}\otimes 1) + K\mathbf{1}\otimes\mathbf{1} \tag{6.10}$$

Since $\mathbb{C}$ is the *undrained* elasticity tensor, G is the *undrained shear modulus*, while K is the *undrained bulk modulus*.

Further, since, by Eq. (6.3), $\mathbf{M}$ is positive-definite, the fluid mobility m must be positive

$$m > 0 \tag{6.11}$$

By Eqs. (6.2) and (6.6), the defining constitutive equations for an isotropic linear chemoelastic solid are

$$\left.\begin{aligned}
&\psi = \mu_0(c_{\mathrm{R}} - c_{\mathrm{R}0}) + G|\boldsymbol{\epsilon}|^2 + \frac{\lambda}{2}(\mathrm{tr}\ \boldsymbol{\epsilon})^2 + \beta(c_{\mathrm{R}} - c_{\mathrm{R}0})(\mathrm{tr}\ \boldsymbol{\epsilon})\\
&\qquad + \frac{1}{2}\Lambda(c_{\mathrm{R}} - c_{\mathrm{R}0})^2,\\
&\boldsymbol{\sigma} = 2G\boldsymbol{\epsilon} + \lambda(\mathrm{tr}\ \boldsymbol{\epsilon})\mathbf{1} + \beta(c_{\mathrm{R}} - c_{\mathrm{R}0})\mathbf{1},\\
&\mu = \mu_0 + \beta(\mathrm{tr}\ \boldsymbol{\epsilon}) + \Lambda(c_{\mathrm{R}} - c_{\mathrm{R}0}),\\
&\mathbf{j} = -m\nabla\mu
\end{aligned}\right\} \tag{6.12}$$

Granted (6.9), the stress–strain relation $(6.12)_2$ may be inverted to give

$$\boldsymbol{\epsilon} = \frac{1}{2G}\left(\boldsymbol{\sigma} - \frac{\lambda}{2G + 3\lambda}(\mathrm{tr}\ \boldsymbol{\sigma})\mathbf{1}\right) + \varkappa(c_{\mathrm{R}} - c_{\mathrm{R}0})\mathbf{1} \tag{6.13}$$

where

$$\varkappa \overset{\mathrm{def}}{=} -\frac{\beta}{2G + 3\lambda} \tag{6.14}$$

is the *coefficient of chemical expansion*. Thus, using Eq. (6.8), the stress–chemistry modulus β is related to the undrained bulk modulus K and the coefficient of chemical expansion $\varkappa$ by

$$\beta = -3K\varkappa \tag{6.15}$$

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[5] These results follow from standard representation theorems for isotropic functions.

Recall from Eq. (5.32) that the drained elasticity tensor $\mathbb{C}^{\mathrm{dr}}$ is related to the undrained elastic tensor $\mathbb{C}$ by

$$\mathbb{C}^{\mathrm{dr}}(\boldsymbol{\epsilon},\mu)=\mathbb{C}(\boldsymbol{\epsilon},c_{\mathrm{R}})-\frac{1}{\Lambda(\boldsymbol{\epsilon},c_{\mathrm{R}})}\mathbf{S}(\boldsymbol{\epsilon},c_{\mathrm{R}})\otimes\mathbf{S}(\boldsymbol{\epsilon},c_{\mathrm{R}}) \tag{6.16}$$

Thus, using Eqs. (6.7) and (6.15)

$$\mathbb{C}^{\mathrm{dr}}=2G\left(\mathbb{I}_{\mathrm{sym}}-\frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right)+K^{\mathrm{dr}}\mathbf{1}\otimes\mathbf{1} \tag{6.17}$$

where

$$K^{\mathrm{dr}}=K-\frac{\beta^2}{\Lambda}=K\left(1-\frac{9K\varkappa^2}{\Lambda}\right) \tag{6.18}$$

- The undrained and drained shear moduli are therefore identical, while the drained bulk modulus K^{dr} is related to the undrained bulk modulus K through Eq. (6.18).

Also, from Eqs. $(6.6)_2$ and (5.31)

$$\mathbf{S}^{\mathrm{dr}}=\frac{\beta}{\Lambda}\mathbf{1} \tag{6.19}$$

so that

$$\begin{aligned}\boldsymbol{\sigma}&=\mathbb{C}^{\mathrm{dr}}\boldsymbol{\epsilon}+\mathbf{S}^{\mathrm{dr}}(\mu-\mu_0),\\&=2G\boldsymbol{\epsilon}+\lambda^{\mathrm{dr}}(\mathrm{tr}\ \boldsymbol{\epsilon})\mathbf{1}+\frac{\beta}{\Lambda}(\mu-\mu_0)\mathbf{1}\end{aligned} \tag{6.20}$$

with

$$\lambda^{\mathrm{dr}}\overset{\mathrm{def}}{=}K^{\mathrm{dr}}-\frac{2}{3}G \tag{6.21}$$

a drained Lamé modulus. Also, from Eq. $(6.12)_3$

$$c_{\mathrm{R}}-c_{\mathrm{R}0}=-\frac{\beta}{\Lambda}(\mathrm{tr}\ \boldsymbol{\epsilon})+\frac{1}{\Lambda}(\mu-\mu_0) \tag{6.22}$$

Thus, the constitutive equations for isotropic linear chemoelasticity, with $\boldsymbol{\epsilon}$ and μ as independent variables, are given by

$$\left.\begin{aligned}&\omega=-c_{\mathrm{R}0}(\mu-\mu_0)+G|\boldsymbol{\epsilon}|^2+\frac{\lambda^{\mathrm{dr}}}{2}(\mathrm{tr}\ \boldsymbol{\epsilon})^2+\frac{\beta}{\Lambda}(\mu-\mu_0)\mathrm{tr}\ \boldsymbol{\epsilon}\\&\quad-\frac{1}{2}\frac{1}{\Lambda}(\mu-\mu_0)^2,\\&\boldsymbol{\sigma}=2G\boldsymbol{\epsilon}+\lambda^{\mathrm{dr}}(\mathrm{tr}\ \boldsymbol{\epsilon})\mathbf{1}+\frac{\beta}{\Lambda}(\mu-\mu_0)\mathbf{1},\\&c_{\mathrm{R}}=c_{\mathrm{R}0}-\frac{\beta}{\Lambda}(\mathrm{tr}\ \boldsymbol{\epsilon})+\frac{1}{\Lambda}(\mu-\mu_0),\\&\mathbf{j}=-m\nabla\mu\end{aligned}\right\} \tag{6.23}$$

Next, when B is homogeneous and isotropic, then G, λ^{dr}, β, Λ, and m are constants. In this case, since

$$2\mathrm{div}\boldsymbol{\epsilon}=\mathrm{div}(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\top})=\Delta\mathbf{u}+\nabla\mathrm{div}\,\mathbf{u}$$

and

$$\mathrm{div}[(\mathrm{tr}\ \boldsymbol{\epsilon})\mathbf{1}]=\mathrm{div}[(\mathrm{div}\ \mathbf{u})\mathbf{1}]=\nabla\mathrm{div}\ \mathbf{u}$$

the force balances (2.1) and $(6.23)_2$ yield

$$G\triangle\mathbf{u}+(\lambda^{\mathrm{dr}}+G)\nabla\mathrm{div}\ \mathbf{u}+\frac{\beta}{\Lambda}\nabla\mu+\mathbf{b}=\mathbf{0} \tag{6.24}$$

Further, the fluid balances (3.4) and $(6.23)_3$ yield

$$\frac{1}{\Lambda}\dot{\mu}=m\Delta\mu+\frac{\beta}{\Lambda}\mathrm{div}\dot{\mathbf{u}} \tag{6.25}$$

Remark. The system of coupled partial differential equations (6.24) and (6.25) for isotropic linear chemoelasticity are formally equivalent to those for the classical coupled theory of isotropic linear *thermoelasticity* (cf., e.g., Gurtin et al. [16]). In that context, μ denotes the *temperature*, $(1/\Lambda)>0$ is the *heat capacity*, and $m>0$ is the *thermal conductivity*. There, the term $(\beta/\Lambda)\nabla\mu$ arises from the *thermal stress*, and the term $(\beta/\Lambda)\mathrm{div}\ \dot{\mathbf{u}}$ corresponds to the *internal heating* due to the *dilatation rate*. In thermoelasticity, a simplifying approximation, which is often adopted to facilitate the solution of problems, is to neglect the coupling term $(\beta/\Lambda)\mathrm{div}\ \dot{\mathbf{u}}$ in the partial differential equation (6.25). Under this approximation, the resulting theory is referred to as the *weakly coupled theory* of isotropic linear thermoelasticity.

7 Isotropic Linear Poroelasticity

In this section, we specialize the linear chemoelasticity theory discussed in the previous section to model *linear poroelastic materials*. Thus, B now represents a deformable *porous* medium. For a porous body a "material point" $\mathbf{x}\in$ B is an infinitesimal element of volume which is assumed to be large enough to be representative of the material, but small enough so that the strains and stresses in this infinitesimal volume are adequately approximated to be homogeneous.

We begin by introducing some definitions essential to the theory of linear poroelasticity:

- Volume fraction of porosity:
  An infinitesimal volume element dv_{R} of the reference porous body B is mapped to a volume

  $$dv=Jdv_{\mathrm{R}}\quad\text{in }\mathcal{B} \tag{7.1}$$

  where, as is standard, for small deformations

  $$J\doteq 1+\mathrm{tr}\ \boldsymbol{\epsilon} \tag{7.2}$$

  Let dv_{pore} denote the volume of the connected pores in dv,[6] and let

  $$\phi\overset{\mathrm{def}}{=}\frac{dv_{\mathrm{pore}}}{dv}\quad\text{in }\mathcal{B} \tag{7.3}$$

  denote the *volume fraction of porosity* in the deformed body; note that $\phi\in[0,1)$.
  Using Eq. (7.1), we define a referential measure of the porosity by

  $$\phi_{\mathrm{R}}\overset{\mathrm{def}}{=}\frac{dv_{\mathrm{pore}}}{dv_{\mathrm{R}}}\equiv J\phi \tag{7.4}$$

  and denote its initial value by

  $$\phi_{\mathrm{R}0}\equiv\phi_0 \tag{7.5}$$

- Molar mass, mass density, molar volume of fluid:
  We denote the molar mass of the fluid by $\mathcal{M}^f$, its mass density by ρ^f, and its molar volume by $\Omega^f=\mathcal{M}^f/\rho^f$. Correspondingly, the *constant* referential molar volume Ω^f_{R} is the molar mass $\mathcal{M}^f$ divided by the constant referential mass density ρ^f_{R}.

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[6] Not occluded pores.

- Concentration of fluid:
  In poroelasticity, the fine-scale pores pace of the material is typically assumed to be fully saturated with the fluid.
  Under this assumption, the amount of fluid in terms of *number of moles of the fluid per unit volume* of the deformed body is

$$c = \frac{\phi}{\Omega^f} \quad \mathrm{mol/m^3} \quad \text{in } \mathcal{B} \tag{7.6}$$

  Then

$$c_{\mathrm{R}} \overset{\mathrm{def}}{=} Jc = \frac{\phi_{\mathrm{R}}}{\Omega^f} \quad \text{in B} \tag{7.7}$$

  represents the *number of moles of the fluid per unit reference volume*. The initial value of c_{R} is

$$c_{\mathrm{R}0} = \frac{\phi_{\mathrm{R}0}}{\Omega^f_{\mathrm{R}}} \equiv \frac{\phi_0}{\Omega^f_{\mathrm{R}}} \tag{7.8}$$

- Mass content of fluid:
  The referential fluid mass content is defined by

$$m^f_{\mathrm{R}} \overset{\mathrm{def}}{=} \mathcal{M}^f c_{\mathrm{R}} = \rho^f \Omega^f c_{\mathrm{R}} = \rho^f \phi_{\mathrm{R}} \tag{7.9}$$

  The initial value of m^f_{R} is

$$m^f_{\mathrm{R}0} \overset{\mathrm{def}}{=} \mathcal{M}^f c_{\mathrm{R}0} = \rho^f_{\mathrm{R}} \Omega^f_{\mathrm{R}} c_{\mathrm{R}0} = \rho^f_{\mathrm{R}} \phi_{\mathrm{R}0} \tag{7.10}$$

- Variation of fluid content:
  The dimensionless quantity

$$\begin{aligned} \zeta &\overset{\mathrm{def}}{=} \left(\frac{m^f_{\mathrm{R}} - m^f_{\mathrm{R}0}}{\rho^f_{\mathrm{R}}} \right), \\ &\equiv \frac{\mathcal{M}^f}{\rho^f_{\mathrm{R}}} (c_{\mathrm{R}} - c_{\mathrm{R}0}), \\ &\equiv \Omega^f_{\mathrm{R}} (c_{\mathrm{R}} - c_{\mathrm{R}0}) \end{aligned} \tag{7.11}$$

  represents a *normalized measure of the variation of the fluid mass content*. Consistent with the terminology in the poroelasticity literature, we call ζ the *variation of fluid content*.

The quantity ζ may alternatively be expressed as

$$\zeta \equiv \left(\frac{m^f_{\mathrm{R}} - m^f_{\mathrm{R}0}}{\rho^f_{\mathrm{R}}} \right) = \left(\frac{\rho^f}{\rho^f_{\mathrm{R}}} \right) \phi_{\mathrm{R}} - \phi_{\mathrm{R}0} \tag{7.12}$$

Hence, note that

— ζ accounts for a change in the volume fraction of pores, as well as a change in the density of the fluid.
Remark. The quantity

$$J^s \overset{\mathrm{def}}{=} 1 + \Omega^f_{\mathrm{R}} (c_{\mathrm{R}} - c_{\mathrm{R}0}) \tag{7.13}$$

represents a volumetric swelling-ratio due to the change in concentration of the fluid molecules in the body. Note that in general J is different from J^s due to additional mechanical deformation. Thus,

$$\zeta \overset{\mathrm{def}}{=} J^s - 1 = \Omega^f_{\mathrm{R}} (c_{\mathrm{R}} - c_{\mathrm{R}0}) \tag{7.14}$$

also represents a measure of volumetric strain caused by the insertion of the fluid molecules in the body.

7.1 Biot's Theory. In this section, we show that Biot's classical theory of linear isotropic poroelasticity [1–3], may be obtained as a special case of the linear isotropic chemoelasticity theory detailed in the previous section. Thus, consider a free energy function of the form

$$\psi = \mu_0 (c_{\mathrm{R}} - c_{\mathrm{R}_0}) + G|\boldsymbol{\epsilon}'|^2 + \frac{1}{2} K (\mathrm{tr}\ \boldsymbol{\epsilon})^2 - \alpha M (\mathrm{tr}\ \boldsymbol{\epsilon}) \zeta + \frac{1}{2} M \zeta^2 \tag{7.15}$$

For the free energy (7.15), Eqs. (5.6) and (5.7) give the stress and chemical potential as

$$\left. \begin{aligned} \boldsymbol{\sigma} &= 2G\boldsymbol{\epsilon} + (K - (2/3)G)(\mathrm{tr}\ \boldsymbol{\epsilon})\mathbf{1} - \alpha M \zeta \mathbf{1}, \\ \mu &= \mu_0 + \Omega^f_{\mathrm{R}} (-\alpha M (\mathrm{tr}\ \boldsymbol{\epsilon}) + M\zeta) \end{aligned} \right\} \tag{7.16}$$

Comparing Eqs. (7.15) and (7.16) to Eq. (6.12), the undrained elasticity tensor has the same form as discussed previously in Eq. (6.10), while the stress–chemistry modulus β and chemistry modulus Λ are given by

$$\beta = -\alpha M \Omega^f_{\mathrm{R}} \quad \text{and} \quad \Lambda = M (\Omega^f_{\mathrm{R}})^2 \tag{7.17}$$

respectively, where

- M is a Biot modulus and
- α is a Biot coefficient.

As before, positive-definiteness of the elasticity tensor $\mathbb{C}$ requires that

$$G > 0 \quad \text{and} \quad K > 0 \tag{7.18}$$

and positivity of the chemistry modulus Λ requires that

$$M > 0 \tag{7.19}$$

Next, recalling Eq. (6.18), the drained bulk modulus is given by

$$K^{\mathrm{dr}} \overset{\mathrm{def}}{=} K - \alpha^2 M \tag{7.20}$$

and positive-definiteness of $\mathbb{C}^{\mathrm{dr}}$ requires that

$$K^{\mathrm{dr}} > 0 \tag{7.21}$$

Hence, using $(\boldsymbol{\epsilon}, \mu)$ as the independent variables, Eqs. (6.23) and (7.17) give the constitutive equations for $\boldsymbol{\sigma}$ and ζ as

$$\left. \begin{aligned} \boldsymbol{\sigma} &= 2G\boldsymbol{\epsilon} + (K^{\mathrm{dr}} - (2/3)G)(\mathrm{tr}\ \boldsymbol{\epsilon})\mathbf{1} - \alpha \left(\frac{\mu - \mu_0}{\Omega^f_{\mathrm{R}}} \right) \mathbf{1}, \\ \zeta &= \alpha (\mathrm{tr}\ \boldsymbol{\epsilon}) + \frac{1}{M} \left(\frac{\mu - \mu_0}{\Omega^f_{\mathrm{R}}} \right) \end{aligned} \right\} \tag{7.22}$$

From Eq. (7.22)$_2$

$$\alpha \equiv \left. \frac{\partial \zeta}{\partial (\mathrm{tr}\ \boldsymbol{\epsilon})} \right|_{\mu \equiv \mathrm{constant}} \tag{7.23}$$

Thus, the Biot coefficient α determines the ratio of the change in the fluid content to the change in the macroscopic volumetric strain when the chemical potential μ is held constant. If $\alpha = 0$ then the fluid content does not change due to a prescribed volumetric strain, while if $\alpha = 1$ the variation in the fluid content is equal the volumetric strain. We assume that (cf. Eq. (A.16))

$$0 < \alpha \leq 1 \tag{7.24}$$

Finally, let

$$p \overset{\mathrm{def}}{=} \frac{\mu}{\Omega^f_{\mathrm{R}}} \tag{7.25}$$

define a *pore-pressure*. Then the constitutive equations (7.16) and (7.22) may be written as

$$\left.\begin{aligned}\boldsymbol{\sigma} &= 2G\boldsymbol{\epsilon} + (K - (2/3)G)(\mathrm{tr}\ \boldsymbol{\epsilon})\mathbf{1} - \alpha M\zeta\mathbf{1},\\ p - p_0 &= -\alpha M(\mathrm{tr}\ \boldsymbol{\epsilon}) + M\zeta\end{aligned}\right\} \tag{7.26}$$

and

$$\left.\begin{aligned}\boldsymbol{\sigma} &= 2G\boldsymbol{\epsilon} + (K^{\mathrm{dr}} - (2/3)G)(\mathrm{tr}\ \boldsymbol{\epsilon})\mathbf{1} - \alpha(p - p_0)\mathbf{1},\\ \zeta &= \alpha(\mathrm{tr}\ \boldsymbol{\epsilon}) + \frac{1}{M}(p - p_0)\end{aligned}\right\} \tag{7.27}$$

These constitutive equations are identical to the classical constitutive equations of Biot's linear theory of isotropic poroelasticity.[7]

The stress–strain relation $(7.27)_1$ may be equivalently written as

$$\boldsymbol{\sigma}_{\mathrm{eff}} = 2G\boldsymbol{\epsilon} + (K^{\mathrm{dr}} - (2/3)G)(\mathrm{tr}\ \boldsymbol{\epsilon})\mathbf{1} \tag{7.28}$$

where

$$\boldsymbol{\sigma}_{\mathrm{eff}} \stackrel{\mathrm{def}}{=} \boldsymbol{\sigma} + \alpha p\mathbf{1} \tag{7.29}$$

defines an *effective stress*. The Biot coefficient α is also known as the *effective stress coefficient* in the literature. The concept of effective stress is due to Terzaghi and Froölicj [23], and is of great importance in poroelasticity.

7.1.1 Fluid Flux. Recall from Eq. $(6.12)_4$, that for isotropic linear materials the fluid flux is given by

$$\mathbf{j} = -m\nabla\mu \quad \text{with} \quad m > 0 \tag{7.30}$$

Also, from Eq. (7.25) we have that $\mu = \Omega^f_{\mathrm{R}} p$, use of which in Eq. (7.30) gives the *molar fluid flux* as

$$\mathbf{j} = -(m\Omega^f_{\mathrm{R}})\mathrm{grad}\, p \quad (\mathrm{mol}/(\mathrm{m}^2\mathrm{s})) \tag{7.31}$$

The volumetric fluid flux is given by

$$\mathbf{q} \stackrel{\mathrm{def}}{=} \Omega^f_{\mathrm{R}}\mathbf{j} \quad (\mathrm{m}^3/(\mathrm{m}^2\mathrm{s})) \tag{7.32}$$

Using Eq. (7.31) in Eq. (7.32), gives Darcy's law [24]

$$\mathbf{q} = -k\ \mathrm{grad}\ p \tag{7.33}$$

where

$$k \stackrel{\mathrm{def}}{=} m(\Omega^f_{\mathrm{R}})^2 \tag{7.34}$$

is the *permeability* of the porous solid. Because the permeability of the fluid through a porous solid depends on the fluid viscosity and the geometry of the fluid pathways, in the poroelasticity literature k is often expressed as[8]

$$k \equiv \frac{k_{\mathrm{da}}}{\eta^f} \tag{7.35}$$

where

- η^f is the *viscosity* of the pore fluid (Pa·s) and
- k_{da} is a *permeability coefficient* with dimensions of length squared. Values of k_{da} are often given in *darcies* (1 darcy $= 10^{-12}$ m^2).

Hence, from Eqs. (7.34) and (7.35) the relation between the mobility m and the quantities $(k_{\mathrm{da}}, \eta^f)$, which are typically used in the poroelasticity literature, is

$$m = \frac{1}{(\Omega^f)^2}\left(\frac{k_{\mathrm{da}}}{\eta^f}\right) \tag{7.37}$$

8 Summary of the Theory of Linear Isotropic Poroelasticity

As derived here, the theory of isotropic linear poroelasticity relates the following basic fields:

$\mathbf{u}$	Displacement field
$\boldsymbol{\epsilon} = \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^\top)$	Strain
$\boldsymbol{\sigma}$	Cauchy stress
c_{R}	Referential fluid concentration
c_{R0}	Initial value of c_{R}
Ω^f_{R}	Referential molar volume of the fluid
$\zeta = \Omega^f_{\mathrm{R}}(c_{\mathrm{R}} - c_{\mathrm{R0}})$	Variation in fluid content
μ	Chemical potential
$\mathbf{j}$	Fluid molar flux

8.1 Constitutive Equations. The set of constitutive equations are:

- Equations for the stress and chemical potential

$$\left.\begin{aligned}\boldsymbol{\sigma} &= 2G\boldsymbol{\epsilon} + (K - (2/3)G)(\mathrm{tr}\ \boldsymbol{\epsilon})\mathbf{1} - \alpha M\zeta\mathbf{1},\\ \mu &= \mu_0 + \Omega^f_{\mathrm{R}}(-\alpha M(\mathrm{tr}\ \boldsymbol{\epsilon}) + M\zeta)\end{aligned}\right\} \tag{8.1}$$

- Fluid flux:
  The fluid molar flux is given by

$$\mathbf{j} = -m\,\mathrm{grad}\,\mu \tag{8.2}$$

To complete the constitutive model for a particular material the material parameters that need to be specified are

$$\left(G, K, \alpha, M, \Omega^f_{\mathrm{R}}, m\right)$$

together with a reference value μ_0 for the chemical potential. The Biot modulus M is related to the undrained bulk modulus K, the drained bulk modulus K^{dr}, and the Biot coefficient α by

$$M = \frac{K - K^{\mathrm{dr}}}{\alpha^2} \tag{8.3}$$

Also, the scalar mobility m is related to the viscosity of the pore-fluid η^f and the permeability coefficient k_{da} by

$$m = \frac{1}{(\Omega^f)^2}\left(\frac{k_{\mathrm{da}}}{\eta^f}\right) \tag{8.4}$$

---

[7] In the Appendix, we discuss an alternative form of the constitutive equation (7.26) in terms of a change in the porosity ϕ_{R}, rather than in terms of the variation of fluid content ζ.

[8] In the poroelasticity literature one also finds use of a quantity c called the *hydraulic diffusivity*. It is related to the permeability k and the other material parameters of the theory by

$$c \stackrel{\mathrm{def}}{=} \frac{k}{\alpha^2}\frac{(K - K^{\mathrm{dr}})(K^{\mathrm{dr}} + (4/3)G)}{K + (4/3)G} \tag{7.36}$$

8.2 Governing Partial Differential Equations, Boundary and Initial Conditions. The governing partial differential equations consist of:

(1) The local force balance

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \tag{8.5}$$

where the stress $\boldsymbol{\sigma}$ is given by Eq. (8.1)$_1$, and $\mathbf{b}$ is the non-inertial body force.

(2) The fluid balance equation is

$$\dot{c}_{\mathrm{R}} = -\operatorname{div}\mathbf{j} \tag{8.6}$$

with the fluid flux $\mathbf{j}$ given by Eq. (8.2), and the chemical potential μ by Eq. (8.1)$_2$.

With $\mathcal{S}_{\mathbf{u}}$ and $\mathcal{S}_{\mathbf{t}}$ denoting complementary subsurfaces of the boundary $\partial\mathrm{B}$ of the deformed body B, as boundary conditions we consider a pair of simple boundary conditions in which the displacement $\mathbf{u}$ is specified on $\mathcal{S}_{\mathbf{u}}$ and the surface traction on $\mathcal{S}_{\mathbf{t}}$

$$\left.\begin{aligned} \mathbf{u} &= \breve{\mathbf{u}} \quad \text{on} \quad \mathcal{S}_{\mathbf{u}} \times [0,T], \\ \boldsymbol{\sigma}\mathbf{n} &= \breve{\mathbf{t}} \quad \text{on} \quad \mathcal{S}_{\mathbf{t}} \times [0,T] \end{aligned}\right\} \tag{8.7}$$

With $\mathcal{S}_{\mu}$ and $\mathcal{S}_{\mathbf{j}}$ another pair of complementary subsurfaces of the boundary $\partial\mathcal{B}_t$, we also consider boundary conditions in which the chemical potential is specified on $\mathcal{S}_{\mu}$ and the spatial fluid flux on $\mathcal{S}_{\mathbf{j}}$

$$\left.\begin{aligned} \mu &= \breve{\mu} \quad \text{on} \quad \mathcal{S}_{\mu} \times [0,T], \\ \mathbf{j}\cdot\mathbf{n} &= \breve{j} \quad \text{on} \quad \mathcal{S}_{\mathbf{j}} \times [0,T] \end{aligned}\right\} \tag{8.8}$$

The initial data is taken as

$$\mathbf{u}(\mathbf{x},0) = \mathbf{0}, \quad \text{and} \quad \mu(\mathbf{x},0) = \mu_0 \quad \text{in} \quad \mathrm{B} \tag{8.9}$$

The coupled set of Eqs. (8.5) and (8.6), together with Eqs. (8.7)–(8.9), yields an initial/boundary-value problem for the displacement $\mathbf{u}(\mathbf{x},t)$ and the chemical potential $\mu(\mathbf{x},t)$.

Finally, the constitutive equations, governing partial differential equations, and the initial/boundary conditions may of course be equivalently stated in terms of the pore-pressure p, simply by replacing μ by $\mu/\Omega^f_{\mathrm{R}}$.

9 Concluding Remarks

We have presented a new derivation of Biot's theory of linear poroelasticity, and shown that it may be deduced as a special case of a more general theory of chemoelasticity. This correspondence between poroelasticity and chemoelasticity should be useful in generalizing the linear theory of poroelasticity to the nonlinear finite deformation regime—a regime in which the nonlinear theory of chemoelasticity [16] has already found recent success in modeling the response of elastomeric gels [18–21].

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Appendix: An Alternative Form of the Constitutive Equation (7.26) in Terms of the Change in Porosity: Estimates for the Material Parameters α, K, and M

In this section, we consider an alternative form of the constitutive equations in terms of the change in porosity. We also discuss how, under certain circumstances, the material constants α, K, and M may be estimated in terms of the bulk moduli of the fluid and solid constituents. Henceforth, for ease of notation, we write

$$p \equiv p - p_0$$

From Eq. (7.12)

$$\begin{aligned} \zeta &= \left(\frac{\rho^f}{\rho^f_{\mathrm{R}}}\right)\phi_{\mathrm{R}} - \phi_{\mathrm{R}0}, \\ &= (\phi_{\mathrm{R}} - \phi_{\mathrm{R}0}) + \left(\frac{\rho^f - \rho^f_{\mathrm{R}}}{\rho^f_{\mathrm{R}}}\right)\phi_{\mathrm{R}}, \\ &= (\phi_{\mathrm{R}} - \phi_{\mathrm{R}0}) + \left(\frac{\rho^f - \rho^f_{\mathrm{R}}}{\rho^f_{\mathrm{R}}}\right)(\phi_{\mathrm{R}0} + (\phi_{\mathrm{R}} - \phi_{\mathrm{R}0})) \end{aligned} \tag{A.1}$$

A "linearized" version of (A.1) which only retains terms to first order in the changes $(\rho^f - \rho^f_{\mathrm{R}})$ and $(\phi_{\mathrm{R}} - \phi_{\mathrm{R}0})$ gives

$$\zeta = (\phi_{\mathrm{R}} - \phi_{\mathrm{R}0}) + \left(\frac{\rho^f - \rho^f_{\mathrm{R}}}{\rho^f_{\mathrm{R}}}\right)\phi_{\mathrm{R}0} \tag{A.2}$$

For small changes in fluid density, we adopt a simple constitutive equation of the form

$$\frac{\rho^f - \rho^f_{\mathrm{R}}}{\rho^f_{\mathrm{R}}} = \frac{p}{K^f} \tag{A.3}$$

with p the pore-pressure and $K^f > 0$ a *bulk modulus for the fluid*. Use of Eq. (A.3) in Eq. (A.2) gives

$$\zeta = (\phi_{\mathrm{R}} - \phi_{\mathrm{R}0}) + \frac{\phi_{\mathrm{R}0}}{K^f}p \tag{A.4}$$

Substituting Eq. (A.4) in Eq. (7.27)$_2$, viz.,

$$\zeta = \alpha(\operatorname{tr}\boldsymbol{\epsilon}) + \frac{p}{M}$$

gives

$$(\phi_{\mathrm{R}} - \phi_{\mathrm{R}0}) = \alpha(\operatorname{tr}\boldsymbol{\epsilon}) + \frac{p}{M_b} \tag{A.5}$$

where we have introduced another Biot modulus M_b defined as

$$\frac{1}{M_b} \overset{\text{def}}{=} \frac{1}{M} - \frac{\phi_{\mathrm{R}0}}{K^f} \tag{A.6}$$

For later use, from Eqs. (A.6) and (7.20), note that

$$\frac{1}{M_b} \overset{\text{def}}{=} \frac{\alpha^2}{K - K^{\mathrm{dr}}} - \frac{\phi_{\mathrm{R}0}}{K^f} \tag{A.7}$$

Thus using Eq. (A.5), the constitutive equation (7.27) may be rewritten as

$$\left.\begin{aligned} &\boldsymbol{\sigma}' = 2G\boldsymbol{\epsilon}', \\ &\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma} = K^{\mathrm{dr}}(\mathrm{tr}\ \boldsymbol{\epsilon}) - \alpha p, \\ &p = -\alpha M_b(\mathrm{tr}\ \boldsymbol{\epsilon}) + M_b(\phi_{\mathrm{R}} - \phi_{\mathrm{R0}}) \end{aligned}\right\} \tag{A.8}$$

A1. Estimates for the Material Parameters α, K, and M. The discussion in this section is taken directly from Rice and Cleary [5] and Rice's unpublished notes. There is a simple but often applicable situation in which the constants α, K, and M may be estimated in terms of the bulk moduli of the fluid and solid constituents. Suppose that all the pore space is fluid infiltrated, and all the solid phase consists of material elements which respond isotropically to pure pressure states, with the same bulk modulus K^s. Suppose we simultaneously apply a pore-pressure

$$p = p^{\dagger} \tag{A.9}$$

as well as a macroscopic stress amounting to

$$\boldsymbol{\sigma} = -p^{\dagger}\mathbf{1}$$

at each point in the solid phase, so that each point in the solid part of the porous material is subjected to the same isotropic pressure

$$-\frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma} = p^{\dagger} \tag{A.10}$$

Because of the homogeneous state of pressure in the saturated porous material, the fluid phase could be replaced by the solid phase without any modification of the stress state. The medium behaves exactly as it was composed of single phase of bulk modulus K^s. This means that all linear dimensions of the material—*including those characterizing the void size*—reduce by the fractional amount $p^{\dagger}/3K^s$, causing the macroscopic volumetric strain

$$\mathrm{tr}\,\boldsymbol{\epsilon} = -\frac{p^{\dagger}}{K^s} \tag{A.11}$$

and change in porosity

$$\frac{\phi_{\mathrm{R}} - \phi_{\mathrm{R0}}}{\phi_{\mathrm{R0}}} = -\frac{p^{\dagger}}{K^s} \tag{A.12}$$

or

$$\phi_{\mathrm{R}} - \phi_{\mathrm{R0}} = \phi_{\mathrm{R0}}\left(-\frac{p^{\dagger}}{K^s}\right) \tag{A.13}$$

The stress–strain–pressure relation (A.8)$_2$, viz.,

$$\frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma} = K^{\mathrm{dr}}(\mathrm{tr}\ \boldsymbol{\epsilon}) - \alpha p$$

must be consistent with the special state just discussed, and by substituting Eqs. (A.9)–(A.11) in it we obtain

$$-p^{\dagger} = K^{\mathrm{dr}}\left(\frac{-p^{\dagger}}{K^s}\right) - \alpha p^{\dagger} \tag{A.14}$$

which upon rearranging yields the following estimate for the Biot coefficient

$$\alpha = 1 - \frac{K^{\mathrm{dr}}}{K^s} \tag{A.15}$$

Since $K^{\mathrm{dr}} \le K^s$, Eq. (A.15) implies that

$$0 < \alpha \le 1 \tag{A.16}$$

The parameter α will be near its upper limit $\alpha = 1$ for very porous materials since for such materials $K^{\mathrm{dr}} \ll K^s$.

Next, the constitutive equation (A.8)$_3$, viz.,

$$(\phi_{\mathrm{R}} - \phi_{\mathrm{R0}}) = \alpha(\mathrm{tr}\ \boldsymbol{\epsilon}) + \frac{p}{M_b}$$

must also be consistent with the special state just discussed, and by substituting Eqs. (A.9), (A.11), (A.12), and (A.7) in it we obtain

$$\phi_{\mathrm{R0}}\left(\frac{-p^{\dagger}}{K^s}\right) = \alpha\left(\frac{-p^{\dagger}}{K^s}\right) + \left(\frac{\alpha^2}{K - K^{\mathrm{dr}}} - \frac{\phi_{\mathrm{R0}}}{K^f}\right)p^{\dagger} \tag{A.17}$$

from which we obtain the following estimate for the undrained bulk modulus:

$$K = K^{\mathrm{dr}} + \frac{\alpha^2 K^s K^f}{K^s\phi_{\mathrm{R0}} + K^f(\alpha - \phi_{\mathrm{R0}})} \tag{A.18}$$

This equation relates K to K^{dr}, the porosity ϕ_{R0}, and the bulk moduli of the solid and fluid phases, K^s and K^f, respectively. As expected intuitively, when the fluid cannot flow out of the porous material, the porous material is stiffer so that

$$K > K^{\mathrm{dr}}$$

Finally, from Eq. (A.18) we obtain the following estimate for M

$$\frac{1}{M} = \frac{\phi_{\mathrm{R0}}}{K^f} + \frac{\alpha - \phi_{\mathrm{R0}}}{K^s} \tag{A.19}$$

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