FRAME INDIFFERENCE

Rheological behavior is independent of rigid-body motion. The bumpy airplane makes us dizzy. We sense the acceleration of the airplane relative to the ground. By contrast, the rheological behavior of materials seems to be independent of rigid-body translation and rotation. The elastic modulus of the wing remains the same when the airplane rolls, yaws, and pitches. So does the viscosity of gasoline.

Here is the fundamental hypothesis: the rheological behavior of materials is unaffected by rigid-body motion of all kinds. We focus on the consequences of the hypothesis. We construct variables invariant with respect to rigid-body motion. Later we will use these variables to construct rheological models invariant with respect to rigid-body motion.

Frame-indifference. In the Euclidean space, a frame of reference consists of an origin and three base vectors. We can attach a frame to a spot on the ground, and attach another frame to a spot in the airplane. The two frames are rigid, but translate and rotate relative to each other. A variable independent of rigid-body motion is independent of the choice of the frame, and vice versa. Such a variable is known as a frame-indifferent variable.

Separation is frame-indifferent. The separation between two places in the Euclidean space is a frame-indifferent vector. The separation is the mother of all frame-indifferent variables. They have different fathers—time, energy, entropy, electric charge, as well as quantity of atoms, molecules and colloids of every species. They are scalars, and are frame-indifferent. We will watch the separation and the scalars breed other frame-indifferent variables.

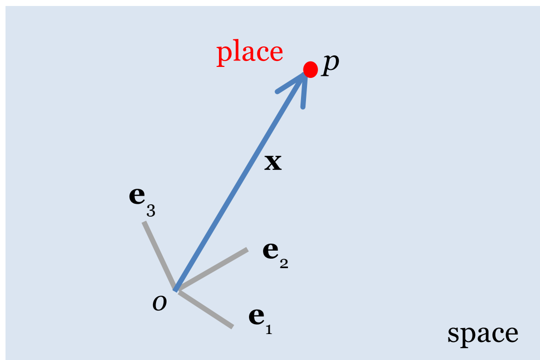
Relative velocity is frame-sensitive. Even though the separation is frame-indifferent, its rate—the relative velocity—is frame-sensitive. In general, the rate of a frame-indifferent variable is frame-sensitive. This makes us question what we mean by rate, and generalize its definition to allow frame-indifferent rates.

Frames of Reference

We adopt a usual model of space and time. The model preserves the *isotropy and homogeneity of space*, as well as the *uniform flow of time*. We will consider speeds much below the speed of light, and will regard space and time as independent. Specifically, we will not invoke the Minkowski metric for the combined vector space of separations and durations (Taylor and Wheeler 1992).

Space. A space is a set of objects called *places*. From one place to another place we draw an arrow, which we call *separation*. The separations between all pairs of places constitute a three-dimensional vector space, to which we add an *inner product*. Such a set of places is called a *Euclidean space*.

Any place serves as an *origin* of the Euclidean space. Any three separations not on the same plane form a *basis* of the vector space associated with the Euclidean space. Denote the origin by o and the basis by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We will always use *orthonormal basis*, $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.



Let p be a place in the Euclidean space. Denote the separation from place o to place p by $\mathbf{x}$, and call this separation the *radial vector*. The radial vector is a linear combination of the three base vectors, $\mathbf{x} = x_i \mathbf{e}_i$. The coefficients x_i are called the *coordinates* of the place p relative to the origin and the basis.

Time. We call another set of objects *times*. From one time to another time we draw an arrow, which we call *duration*. The durations between all pairs of times constitute a one-dimensional vector space, to which we add no further structure.

Any time serves as an *origin of time*. The duration between any two times serves as a base vector of the one-dimensional vector space of durations. We call the base vector a *unit of time*. Denote the origin of time by α, and the unit of time by t_{unit}.

Let β be a time, and t be the duration between the time β and the origin of time α. The set of all durations is a one-dimensional vector space, so that the duration between the time β and the origin α is linear in the unit of time, $t = st_{unit}$. The numerical coefficient s represents the time β relative to the origin α and the unit t_{unit}.

Frame. The origin o and the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of space, together with the origin α and the unit t_{unit} of time, constitute a *frame of reference*, or *frame* for brevity.

The choice of frame is arbitrary. For example, the birth of Confucius marks an origin of time, and the duration of his mother's pregnancy serves as a unit of time. The place of his birth serves as the origin of space, and the vector from the place of his birth to three kingdoms he served set up a basis. He traveled far. The four places are surely not on the same plane.

Frame Indifference. We will always use duration, which is independent of the choice of origin of time. We do not distinguish the duration of time and its numerical value. If we change the unit of time, we just multiply the answer by the factor of conversion.

We will concentrate on the change of the origin and basis of space. Here is the fundamental hypothesis: the rheological behavior of materials is independent of rigid-body motion. Independence of rigid-body motion is equivalent to the independence of the choice of frame. You can choose a frame of any kind, so long as it is *rigid*. (As Henry Ford would say, you can choose a car of any color, so long as it is black.)

Whether the rheological behavior of a material is frame-indifferent is ultimately settled by experience. We do know that the rheological behavior of most materials is at least insensitive to rigid-body motion.

Here we focus on the *consequence* of the hypothesis. In particular, we develop frame-independent variables, such as stress, strain, rate of stress, and rate of strain. Later we will use these variables to construct frame-indifferent rheological models.

This line of reasoning is developed in the classic papers by Oldroyd (1950) and Noll (1955), as well as in textbooks by, e.g., Holzapfel (2000), Nemat-Nasser (2004), Gurtin, Fried and Anand (2010), Irgens (2014). For historical perspectives—and opinions—see Prager (1961), Trusedell and Noll (2004), and Tanner and Walter (1998).

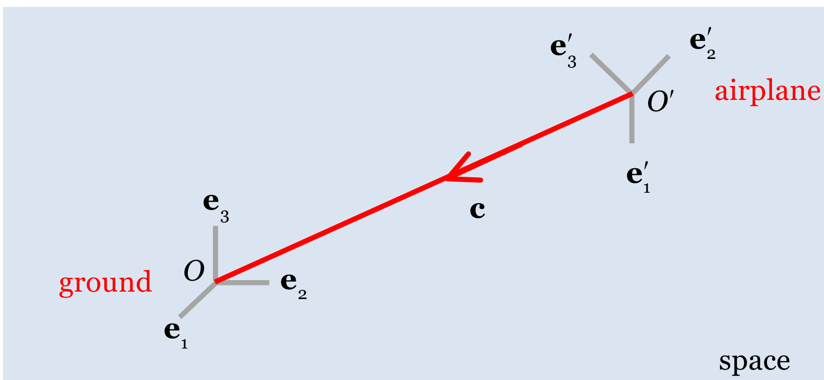
Two frames. To display frame-sensitivity of a variable, we will need at least two frames. We attach the frame of origin o and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to the ground. We attach another frame, of origin o' and basis $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, to an airplane. The new basis is also orthonormal, $\mathbf{e}'_a \cdot \mathbf{e}'_b = \delta_{ab}$.

Frame translation. Let $\mathbf{c}$ be the vector from place o' to place o. This vector represents the translation of the airplane relative to the ground, and is called the *frame translation*. When the airplane translates relative to the ground, the frame translation changes with time, $\mathbf{c}(t)$.

Frame rotation. Relative to the ground, the airplane rolls, yaws and pitches. Designate the cosine of the angle between a base vector attached to the airplane and a base vector attached to the ground by

$$Q_{ai} = \mathbf{e}'_a \cdot \mathbf{e}_i .$$

As a convention, the first index of Q_{ai} indicates a base vector attached to the airplane, and the second index indicates a base vector attached to the ground. To help reading, we use the beginning letters in the alphabet for the airplane frame, and use the middle letters like i and j in the alphabet for the ground frame. The matrix $\mathbf{Q}$ represents the rotation of airplane frame relative to the ground frame, and is called the *frame rotation*. When the two frames rotate relative to each other, the frame rotation changes with time, $\mathbf{Q}(t)$.



A base vector attached to the airplane, $\mathbf{e}'_a$, is a linear combination of the base vectors attached to the ground, $\mathbf{e}'_a = Q_{ai}\mathbf{e}_i$. Similarly, a base vector attached to the ground, $\mathbf{e}_i$, is a linear combination of the base vectors attached to the airplane, $\mathbf{e}_i = Q_{ai}\mathbf{e}'_a$. The two expressions together give that $\mathbf{e}'_a = Q_{ai}Q_{bi}\mathbf{e}'_b$. Any vector is a unique linear combination of the base vectors, so that

$$Q_{ai}Q_{bi} = \delta_{ab} , \quad \mathbf{Q}\mathbf{Q}^T = \mathbf{I} .$$

That is, the frame rotation $\mathbf{Q}$ is an *orthogonal matrix*.

Frame spin. The frame rotation $\mathbf{Q}(t)$, as well as the frame translation $\mathbf{c}(t)$, is *relative* between the two frames. The airplane rolls and flies. The Earth spins and quakes.

Relative to the ground, the airplane rotates according to $\mathbf{e}'_a(t) = Q_{ai}(t)\mathbf{e}_i$. Differentiating with respect to time, we get

$$\dot{\mathbf{e}}'_a = \dot{Q}_{ai}\mathbf{e}_i .$$

Here we fix the ground basis, and rotate the airplane basis. We call this rate the *in-frame rate*. In taking such a time derivative, we must specify which frame is fixed. Recall that $Q_{ai} = \mathbf{e}'_a \cdot \mathbf{e}_i$ is the cosine of the angle between two base vectors, so that $\dot{Q}_{ai} = dQ_{ai}/dt$ is the rate of the cosine of the angle.

Recall that $\mathbf{e}_i = Q_{bi}\mathbf{e}'_b$, and write above as

$$\dot{\mathbf{e}}'_a = \dot{Q}_{ai}Q_{bi}\mathbf{e}'_b .$$

Designate the matrix by

$$\mathbf{S} = \dot{\mathbf{Q}}\mathbf{Q}^T , \quad S_{ab} = \dot{Q}_{ai}Q_{bi} .$$

Thus, S_{ab} is $\dot{\mathbf{e}}'_a$ projected on $\mathbf{e}'_b$, namely,

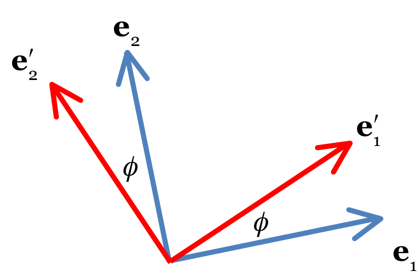
$$S_{ab} = \dot{\mathbf{e}}'_a \cdot \mathbf{e}'_b , \quad \dot{\mathbf{e}}'_a = S_{ab}\mathbf{e}'_b .$$

The matrix $\mathbf{S}$ represents the rate of rotation of the airplane relative to the ground, with the components projected onto the airplane basis. The matrix $\mathbf{S}$ is called the *frame spin*.

Differentiating $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ with respect to time, we obtain that $\dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\dot{\mathbf{Q}}^T = 0$. Thus, the matrix $\mathbf{S} = \dot{\mathbf{Q}}\mathbf{Q}^T$ is anti-symmetric. We can also write several other expressions

$$\dot{\mathbf{Q}} = \mathbf{S}\mathbf{Q} , \quad \dot{Q}_{ai} = S_{ab}Q_{bi} .$$

$$\dot{\mathbf{Q}}^T = -\mathbf{Q}^T\mathbf{S} , \quad \dot{Q}_{ai} = -S_{ba}Q_{bi} .$$



Spin around an axis. Consider a special case. At a given time, the two frames coincide, but the airplane rotates relative to the ground around $\mathbf{e}_3$. Let the angle of rotation be $\phi(t)$. The two sets of base vectors relate as

$$\mathbf{e}'_1 = \mathbf{e}_1\cos\phi + \mathbf{e}_2\sin\phi ,$$

$$\mathbf{e}'_2 = -\mathbf{e}_1\sin\phi + \mathbf{e}_2\cos\phi ,$$

$$\mathbf{e}'_3 = \mathbf{e}_3 .$$

Consequently, the frame rotation is

$$\mathbf{Q} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The frame spin is

$$\mathbf{S} = \dot{\mathbf{Q}}\mathbf{Q}^T = \dot{\phi}\begin{bmatrix} -\sin\phi & \cos\phi & 0 \\ -\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \dot{\phi}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This result corresponds to that

$$\dot{\mathbf{e}}'_1 = \dot{\phi}\mathbf{e}'_2, \quad \dot{\mathbf{e}}'_2 = -\dot{\phi}\mathbf{e}'_1, \quad \dot{\mathbf{e}}'_3 = 0 .$$

The airplane frame spins relative to the ground frame at the rate $\dot{\phi}$.

Three frames. We attach the third frame, of origin o'' and basis $\mathbf{e}''_1, \mathbf{e}''_2, \mathbf{e}''_3$, to a bird. The bird basis is also orthonormal, $\mathbf{e}''_p \cdot \mathbf{e}''_q = \delta_{pq}$. Define the frame rotation of the bird frame relative to the airplane frame by $R_{pa} = \mathbf{e}''_p \cdot \mathbf{e}'_a$. The bird basis relate to the airplane basis as $\mathbf{e}''_p = R_{pa}\mathbf{e}'_a$ and $\mathbf{e}'_a = R_{pa}\mathbf{e}''_p$.

Write

$$\dot{\mathbf{e}}''_p = \dot{R}_{pa}\mathbf{e}'_a = \dot{R}_{pa}R_{qa}\mathbf{e}''_q .$$

Here we fix the airplane basis. The spin of the bird relative to the airplane is

$$T_{pq} = \dot{R}_{pa}R_{qa}, \qquad \mathbf{T} = \dot{\mathbf{R}}\mathbf{R}^T .$$

The bird rotates relative to the ground as $\mathbf{e}''_p = R_{pa}Q_{ai}\mathbf{e}_i$. The spin of the bird relative to the ground is

$$\mathbf{Z} = (\mathbf{RQ})^{\cdot}(\mathbf{RQ})^T = (\dot{\mathbf{R}}\mathbf{Q} + \mathbf{R}\dot{\mathbf{Q}})\mathbf{Q}^T\mathbf{R}^T = \mathbf{T} + \mathbf{RSR}^T .$$

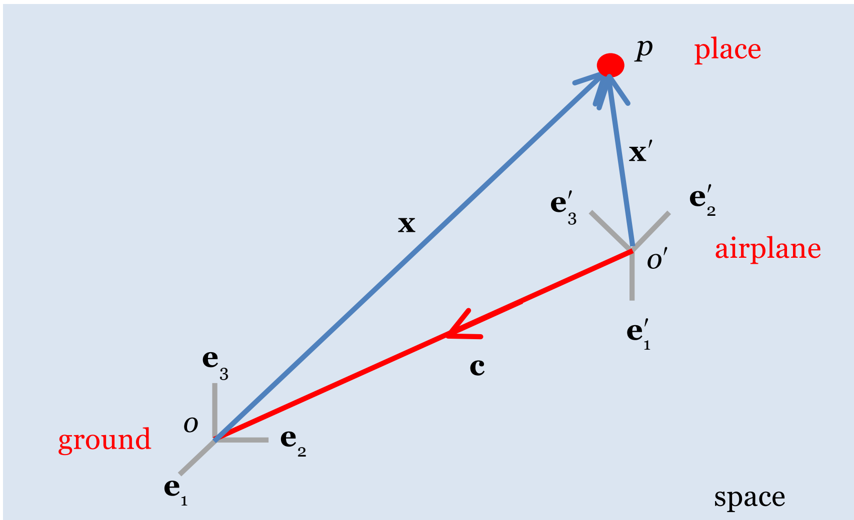
Here $\mathbf{Z}$ is a matrix relative to the bird basis. On the right-hand side, $\mathbf{T}$ is the spin of the bird relative to the airplane, $\mathbf{S}$ is the spin of the airplane relative to the ground, and $\mathbf{RSR}^T$ projects this spin to the bird basis.

Place and Separation

Place. Let p be a place in the Euclidean space. Let $\mathbf{x}$ be the radial vector from the origin o to place p, and $\mathbf{x}'$ be the radial vector from the origin o' to place p. The three places, p, o and o', from a triangle. The sides of the triangle are the vectors and **c**, $\mathbf{x}'$ and **x**. The three vectors obey the rule of summation in vector algebra:

$$\mathbf{x}' = \mathbf{x} + \mathbf{c} .$$

The radial vectors $\mathbf{x}'$ and $\mathbf{x}$ are relative to the origins of the two frames, and frame-sensitive.



Coordinates of a place. The radial vector **x** is a linear combination of the base vectors, $\mathbf{x} = x_i \mathbf{e}_i$, where x_i are the coordinates of the place of the particle relative to the ground frame. Recall that $\mathbf{e}_i = Q_{ai}\mathbf{e}'_a$. Write $\mathbf{x} = Q_{ai} x_i \mathbf{e}'_a$. Now **x** is the radial vector of the place p from the ground origin, but $Q_{ai}x_i$ are the components of the vector **x** relative to the airplane basis.

The radial vector $\mathbf{x}'$ is a linear combination of the base vectors, $\mathbf{x}' = x'_a \mathbf{e}'_a$, where x'_a are the coordinates of the place p relative to the airplane frame. Similarly write $\mathbf{c} = c'_a \mathbf{e}'_a$, where c'_a are the components of the frame translation **c** relative to the airplane frame.

Write the vector sum in two forms:

$$\mathbf{x}' = \mathbf{x} + \mathbf{c} , \quad x'_a = Q_{ai} x_i + c'_a .$$

The two equations have the same meaning. One equation uses the vectors, and the other uses the components relative to the airplane basis.

When **c** = **0**, the above equations reduce to

$$\mathbf{x}' = \mathbf{x} , \quad x'_a = Q_{ai} x_i .$$

The radial vector is independent of the choice of frame among frames of the same origin.

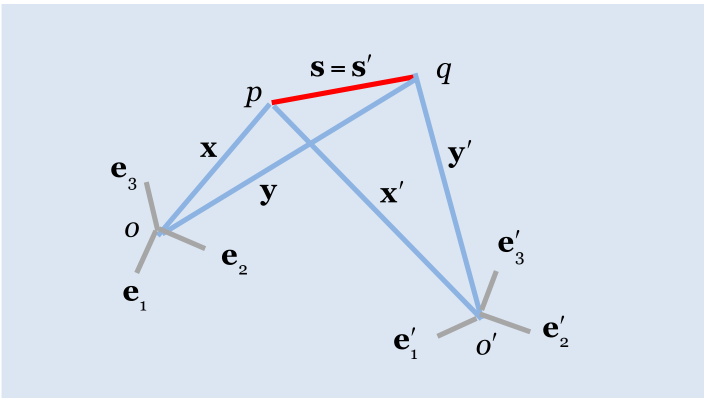
Do not confuse a vector with its components. Most authors write the equation $x'_a = Q_{ai}x_i + c'_a$ as $\mathbf{x}' = \mathbf{Q}\mathbf{x} + \mathbf{c}$, which is inconsistent with the rule in vector algebra, $\mathbf{x}' = \mathbf{x} + \mathbf{c}$. The inconsistency results from letting $\mathbf{x}$ stand for two things, first as the column of the components of a vector in $\mathbf{x}' = \mathbf{Q}\mathbf{x} + \mathbf{c}$, and then as a vector itself in $\mathbf{x}' = \mathbf{x} + \mathbf{c}$. This usage leads to confusion when we have two frames.

To avoid the confusion, we let $\mathbf{x}$ stand for the radial vector of the particle relative to the origin *o*. The same vector $\mathbf{x}$ has two sets of components: x_i relative to the ground basis, and $Q_{ai}x_i$ relative to the airplane basis. We will list important equations using vectors and their components, side by side, such as

$$\mathbf{x}' = \mathbf{x} + \mathbf{c}, \quad x'_a = Q_{ai}x_i + c'_a.$$

The first equation is evident in looking at the figure; we should not abandon this fundamental equation. The second equation helps computation.

Separation. Let *p* and *q* be two places in the Euclidean space. The separation $\mathbf{s}$ between the two places has nothing to do with any frame. This much is evident when we picture the space, the two places, the separation, and two frames. The frame-indifference of separation is consistent with the definition of the Euclidean space: the set of all separations form a vector space. This definition needs no frame.



Separation in two frames. So far as the separation is concerned, the frames are superfluous. But it helps to learn to connect frame-indifferent variables to frame-sensitive ones. In the ground frame, the two places *p* and *q* have the radial vectors $\mathbf{x}$ and $\mathbf{y}$, and separation $\mathbf{s} = \mathbf{y} - \mathbf{x}$. In the airplane frame, the two places *p* and *q* have radial vectors $\mathbf{x}'$ and $\mathbf{y}'$, and separation $\mathbf{s}' = \mathbf{y}' - \mathbf{x}'$. The two radial vectors $\mathbf{x}$ and $\mathbf{x}'$ of the place *p* are different, and the two radial vectors $\mathbf{y}$ and $\mathbf{y}'$ of the place r are different. The separation between the two places p and q is the same in the two frames, $\mathbf{y}' - \mathbf{x}' = \mathbf{y} - \mathbf{x}$, namely, $\mathbf{s}' = \mathbf{s}$.

Components of separation. The separation $\mathbf{s}$ is a vector, and is a linear combination of the base vectors, $\mathbf{s} = s_i \mathbf{e}_i$, where s_i are the components of the separation relative to the ground basis. Similarly, the separation is a linear combination of the base vectors, $\mathbf{s}' = s'_a \mathbf{e}'_a$, where s'_a are the components of the separation relative to the airplane basis.

Recall that $\mathbf{e}_i = Q_{ai}\mathbf{e}'_a$, and write $\mathbf{s} = Q_{ai} s_i \mathbf{e}'_a$. The following equations are equivalent:

$$\mathbf{s}' = \mathbf{s}, \quad s'_a = Q_{ai} s_i .$$

Both equations say that the separation is a frame-indifferent vector. The equation $s'_a = Q_{ai} s_i$ transforms the components of the separation in the ground frame to the components of the separation in the airplane frame. The transformation holds because the separation is a frame-indifferent vector, $\mathbf{s}' = \mathbf{s}$. Conversely, if the components in two frames transform according to $s'_a = Q_{ai} s_i$, the vector is frame-indifferent.

Velocity and Relative Velocity

Particle. We call a small piece of material a *particle*. For example, we can picture a particle as a bird. At time t, the bird occupies a place $p(t)$ in the Euclidean space. Let $\mathbf{x}(t)$ be the radial vector from the ground origin o to the bird, and $\mathbf{x}'(t)$ be the radial vector from the airplane origin o' to the bird. As time flows, the bird moves from place to place, forming a *trajectory*. The trajectory is a subset of the Euclidean space, and is described by either the function $\mathbf{x}(t)$ or $\mathbf{x}'(t)$.

As discussed above, at any given time t, the three places, p, o and o', from a triangle. The sides of the triangle are the vectors and $\mathbf{c}$, $\mathbf{x}'$ and $\mathbf{x}$. The three vectors obey the rule of summation in vector algebra:

$$\mathbf{x}' = \mathbf{x} + \mathbf{c}, \quad x'_a = Q_{ai} x_i + c'_a .$$

Velocity. In the ground frame, the coordinates of the bird change at the rates:

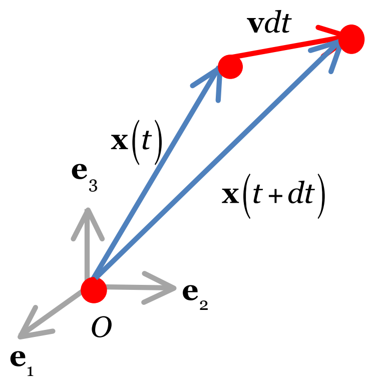
$$\dot{x}_i = \frac{x_i(t + dt) - x_i(t)}{dt} .$$

The velocity of the bird relative to the ground frame is $\mathbf{v} = \dot{x}_i \mathbf{e}_i$. Here we fix the ground basis, and allow the components relative to the ground basis to change in time. Thus, velocity is the in-frame rate of radial vector.

Similarly, in the airplane frame, the coordinates of the bird change at the rates:

$$\dot{x}'_i = \frac{x'_i(t+dt) - x'_i(t)}{dt}.$$

The velocity of the bird relative to the airplane frame is $\mathbf{v}' = \dot{x}'\mathbf{e}'_i$. Here we fix the ground basis, and allow the components relative to the ground basis to change in time.



Relate velocities in two frames. Differentiating $x'_a = Q_{ai} x_i + c'_a$ with respect to time, we get

$$\dot{x}'_a = Q_{ai}\dot{x}_i + \dot{Q}_{ai} x + \dot{c}'_a.$$

Recall that $\dot{Q}_{ai} = S_{ab} Q_{bi}$, and we write the above equation as

$$\dot{x}'_a = Q_{ai}\dot{x}_i + S_{ab} Q_{bi} x_i + \dot{c}'_a, \quad \mathbf{v}' = \mathbf{v} + \mathbf{S}\mathbf{x} + \dot{\mathbf{c}}.$$

Recall that $Q_{bi} x_i$ is the component of the vector $\mathbf{x}$ projected on the base vector $\mathbf{e}'_b$. Here $-\dot{\mathbf{c}}$ is the velocity of the airplane origin relative to the ground origin, and $\mathbf{S}$ is the spin of the airplane relative to the ground. Think of a bird in the cabin of the airplane. The velocity of the bird in the airplane frame is different from the velocity of the bird in the ground frame. Velocity of the bird is frame-sensitive.

When the two frames are stationary relative to each other, $\dot{\mathbf{c}} = 0$ and $\dot{\mathbf{Q}} = 0$, the above equations reduce to $\dot{x}'_a = Q_{ai}\dot{x}_i$ and $\mathbf{v}' = \mathbf{v}$. The velocity is independent of the choice of frame among frames stationary relative to each other.

Relative velocity. Now think of two flying birds. Is their relative velocity frame-sensitive? To answer this question, we must define the relative velocity in each frame.

In the ground frame, the components of the separation between the two birds change with time, $s_i(t)$. The components of the separation change at the rates:

$$\dot{s}_i = \frac{s_i(t+dt) - s_i(t)}{dt}.$$

In the ground frame, define the relative velocity between the birds by

$$\mathbf{g} = \dot{s}_i \mathbf{e}_i.$$

Here we fix the ground basis, and allow the components of the separation relative to the ground basis to change in time. Thus, the relative velocity is the in-frame rate of the separation.

In the airplane frame, the components of the separation between the two birds change at the rates, $s'_a(t)$. The components of the separation change at the rates:

$$\dot{s}'_a = \frac{s'_a(t+dt) - s'_a(t)}{dt}.$$

In the airplane frame, define the relative velocity between the birds by

$$\mathbf{g}' = \dot{s}'_a \mathbf{e}'_a.$$

Here we fix the airplane basis, and allow the components of the separation relative to the airplane basis to change in time.

Relative velocity is frame-sensitive. The separation between the birds is frame-indifferent, so that the components of the separation in the airplane frame relate to those in the ground frame as

$$s'_a = Q_{ai} s_i.$$

Differentiating this equation with respect to time, we get

$$\dot{s}'_a = Q_{ai}\dot{s}_i + \dot{Q}_{ai} s_i.$$

Recall that $\dot{Q}_{ai} = S_{ab} Q_{bi}$, and that $Q_{bi} s_i$ is the component of the vector $\mathbf{s}$ projected on the base vector $\mathbf{e}'_b$. We write the above equation as

$$\mathbf{g}' = \mathbf{g} + \mathbf{S}\mathbf{s}, \quad \dot{s}'_a = Q_{ai}\dot{s}_i + S_{ab} Q_{bi} s_i$$

The relative velocity is frame-sensitive! This frame-sensitivity is caused by the spin $\mathbf{S}$ of the airplane relative to the ground. Even when both birds are stationary in the airplane, a spin of the airplane makes their relative velocity nonzero in the ground frame.

When the two frames translate but not rotate relative to each other, the spin vanishes, and the above equations reduce to $\mathbf{g}' = \mathbf{g}$ and $g'_a = Q_{ai} g_i$. The relative velocity is independent of the choice of frame among frames that translate relative to one another.

Velocity Gradient, Rate of Deformation, and Rate of Rotation

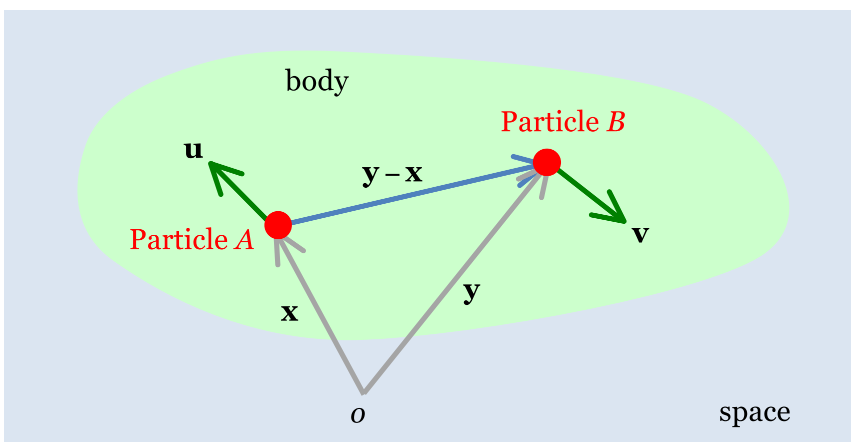
Body. A collection of particles constitutes a *body*. We label the particles as *A*, *B*,... As time flows, each particle in the body moves along a trajectory, and the body deforms.

At time *t*, particle *A* occupies a place of radial vector $\mathbf{x}(A,t)$. The function $\mathbf{x}(A,t)$ describes the deformation of the body. The vector is a linear combination of the base vectors:

$$\mathbf{x}(A,t) = x_i(A,t)\mathbf{e}_i ,$$

where $x_i(A,t)$ are the coordinates of the place occupied by particle *a* at time *t*. The velocity of the particle relative to the ground frame is

$$\mathbf{v}(A,t) = \frac{\partial x_i(A,t)}{\partial t}\mathbf{e}_i .$$



Velocity gradient. At a given time, particle *A* occupies a place of radial vector $\mathbf{x}$ and moves at velocity $\mathbf{u}$, while particle *B* occupies a place of radial vector $\mathbf{y}$ and moves at velocity $\mathbf{v}$. The two particles have the separation $\mathbf{y}-\mathbf{x}$ and the relative velocity $\mathbf{v}-\mathbf{u}$.

Define the *velocity gradient* $\mathbf{L}$ as the map that maps the separation of two particles to their relative velocity:

$$\mathbf{v}-\mathbf{u} = \mathbf{L}(\mathbf{y}-\mathbf{x}).$$

When **L** is a linear map and is independent of the choice of the two particles, the body is said to undergo a *homogeneous deformation*. All quantities are for the same time *t*, and here for brevity we have dropped time *t* in all expressions. In general, the velocity gradient depends on time, $\mathbf{L}(t)$.

The set of all separations form a vector space. The set of all relative velocities form another vector space. In linear algebra, a linear map between two vector spaces is an example of *tensor*. The separation is a frame-independent vector, but the relative velocity is a frame-sensitive vector. Consequently, the velocity gradient is a frame-sensitive tensor.

Components of velocity gradient. The separation and the relative velocity are linear combinations of the base vectors:

$$\mathbf{y}-\mathbf{x}=(y_j-x_j)\mathbf{e}_j,$$

$$\mathbf{v}-\mathbf{u}=(v_i-u_i)\mathbf{e}_i.$$

The base vector $\mathbf{e}_j$ itself is the separation between two places. Then $\mathbf{L}(\mathbf{e}_j)$ maps this separation to the relative velocity of the particles at the two places at the time. The relative velocity is a linear combination of the base vectors:

$$\mathbf{L}(\mathbf{e}_j)=L_{ij}\mathbf{e}_i,$$

where L_{ij} are the components of the velocity gradient relative to the ground frame. The first index of L_{ij} indicates the direction of the component of the relative velocity, and the second index indicates the direction of the base vector.

Because the velocity gradient is a linear map, write

$$\mathbf{L}(\mathbf{y}-\mathbf{x})=(y_j-x_j)\mathbf{L}(\mathbf{e}_j)=L_{ij}(y_j-x_j)\mathbf{e}_i.$$

A comparison of the above gives

$$v_i-u_i=L_{ij}(y_j-x_j).$$

Velocity gradient in two frames. At a given time, particles *A* and *B* have a separation $\mathbf{s}=\mathbf{y}-\mathbf{x}$ in the ground frame, and a separation $\mathbf{s}'=\mathbf{y}'-\mathbf{x}'$ in the airplane frame. At the same time, the two particles move at a relative velocity $\mathbf{g}=\mathbf{v}-\mathbf{u}$ in the ground frame, and a relative velocity $\mathbf{g}'=\mathbf{v}'-\mathbf{u}'$ in the airplane frame. The separation is frame-indifferent, $\mathbf{s}'=\mathbf{s}$, but the relative velocity is frame-sensitive, $\mathbf{g}'=\mathbf{g}+\mathbf{S}\mathbf{s}$.

Write the velocity gradient as linear maps between the separation and the relative velocity in the two frames:

$$\mathbf{g}=\mathbf{L}\mathbf{s}, \quad \mathbf{g}'=\mathbf{L}'\mathbf{s}'.$$

A comparison of the above equations gives that

$$\mathbf{L}'\mathbf{s}=(\mathbf{L}+\mathbf{S})\mathbf{s}.$$

This equation holds for every pair of particles in the body, so that

$$\mathbf{L}' = \mathbf{L} + \mathbf{S},$$

Consequently, the velocity gradient is frame-sensitive. For a given body under a homogeneous deformation, its velocity gradients in the two frames differ by the spin of one frame relative to the other.

Components of velocity gradient in two frames. The separation is frame-indifferent, and the relative velocity is frame-sensitive:

$$s'_b = Q_{bi} s_i, \quad g'_a = Q_{ai} g_i + S_{ab} Q_{bi} s_i$$

Write $g_i = L_{ij} s_j$, where L_{ij} are the components of the velocity gradient relative to the ground basis. Similarly write $g'_a = L'_{ab} s'_b$, where L'_{ab} are the components of the velocity gradient relative to the airplane basis. A combination of these expressions gives that

$$L'_{ab} s'_b = \left(Q_{ai} Q_{bj} L_{ij} + S_{ab} \right) s'_b .$$

This equation holds for separation for every pair of particles, so that

$$\mathbf{L}' = \mathbf{L} + \mathbf{S}, \quad L'_{ab} = Q_{ai} Q_{bj} L_{ij} + S_{ab} .$$

Rate of deformation. Define the rate of deformation by the symmetric part of the velocity gradient:

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T \right).$$

Recall that the frame spin $\mathbf{S}$ is an anti-symmetric matrix. The symmetric part of the equation $\mathbf{L}' = \mathbf{L} + \mathbf{S}$ gives that

$$\mathbf{D}' = \mathbf{D}, \quad D'_{ab} = Q_{ai} Q_{bj} D_{ij} .$$

The rate of deformation is frame-indifferent.

Rate of rotation. Define the rate of rotation by the anti-symmetric part of the velocity gradient:

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T \right).$$

The anti-symmetric part of the $\mathbf{L}' = \mathbf{L} + \mathbf{S}$ gives that

$$\mathbf{W}' = \mathbf{W} + \mathbf{S}, \quad W'_{ab} = Q_{ai} Q_{bj} W_{ij} + S_{ab} .$$

The rate of rotation is frame-sensitive. For a given body, its rate of rotation in the airplane frame is the sum of its rate of rotation in the ground frame and the spin of the airplane frame relative to the ground frame.

The rate of rotation of a body, $\mathbf{W}$, and the rate of rotation of a frame, $\dot{\mathbf{Q}}$, are two distinct objects.

Breeding Frame-Indifferent Variables

The separations between all pairs of places constitute a three-dimensional vector space with an inner product. This single vector space, together with several scalar sets, generates all vectors and tensors in rheology. Some of them are frame-indifferent, others frame-sensitive.

The linear algebra of the world (Suo 2014). The algebraic structure of the world consists of the following building blocks:

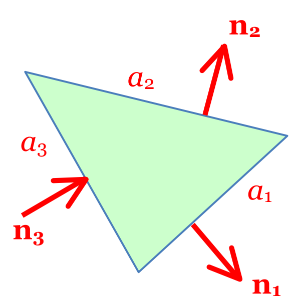
- One number field: real numbers
- One vector space: separations between places
- Many scalar sets: time, entropy, energy, charge, as well as the quantity of atoms, ions, molecules and colloids of every species

We then construct linear maps of various kinds:

- *scalar to scalar* (e.g., entropy to energy gives temperature, charge to energy gives electric potential, quantity of a species to energy gives chemical potential.)
- *scalar to vector* (e.g., time to separation gives velocity)
- *vector to scalar* (e.g., separation to energy is force, separation to electric potential is electric field, area vector to quantity of species per unit time to flux.)
- *vector to vector* (e.g., area vector to force to stress, separation to velocity gives velocity velocity.)

A choice of basis for each scalar set gives a unit to the scalar. We fix the bases of all the scalar sets. A choice of a basis of the vector space of separations gives the bases for all other vectors and tensors, including velocity, force, electric field, and stress.

We now use this algebraic structure to construct frame-indifferent variables, including stress, rate of deformation, and rate of stress. Various measures of strain are defined in this way in the notes on finite deformation (Suo 2013).



Planar regions. Area is a frame-indifferent scalar. We can construct a frame-indifferent vector, which we call the *area vector*. Let $\mathbf{a} = a\mathbf{n}$ represent a planar region of area a, normal to the unit vector $\mathbf{n}$. We next confirm that the set of all planar regions constitute a vector space. In linear algebra, a set is a vector space if

1. $\beta\mathbf{a}$ is in the set for every number β and every $\mathbf{a}$ in the set, and
2. $\mathbf{a}_1 + \mathbf{a}_2$ is in the set for every $\mathbf{a}_1$ and $\mathbf{a}_2$ in the set

The object $-\mathbf{a}$ represents a planar region of area a, also normal to the unit vector $\mathbf{n}$. Let β be a positive number. Thus, the object $\beta\mathbf{a}$ represents a planar region of area βa normal to the unit vector $\mathbf{n}$. Taken together, we have confirmed that, for every planar region $\mathbf{a}$ and every number β, the product $\beta\mathbf{a}$ is also a planar region.

Next consider two planar regions represented by $\mathbf{a}_1$ and $\mathbf{a}_2$. Because the shapes of the planar regions do not affect the definition of the flux and stress, we may choose the two regions as rectangular regions. The sum $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$ represents another planar region. The three planar regions form the surfaces of a prism. The cross section of the prism is a triangle. If the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ point toward the exterior of the prism, $\mathbf{n}_3$ points toward the interior of the prism. Thus, the equation $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$ is just the vector sum of separations in disguise. The area vectors are frame-indifferent vectors.

Flux. Define heat flux as a map that maps the area vector of a planar region to the energy per unit time crossing this region:

$$P = \mathbf{J}(\mathbf{a}),$$

where P is the flow of energy per unit time, and $\mathbf{J}$ is the heat flux. Because both the energy flow P and the area vector $\mathbf{a}$ are frame indifferent, so is the flux $\mathbf{J}$.

We next show that this function is a linear map between a vector space and a scalar set. The vector space consists of the area vectors. The scalar set consists of flows per unit time.

In linear algebra, a function that maps one vector to a scalar set is a linear map if

1 $\mathbf{J}(\beta\mathbf{a}) = \beta\mathbf{J}(\mathbf{a})$ for every number β and every vector $\mathbf{a}$, and

2 $\mathbf{J}(\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{J}(\mathbf{a}_1) + \mathbf{J}(\mathbf{a}_2)$ for any vectors.

A linear map that maps a vector to a scalar is a vector.

Let β be a positive number. Because the body is in a homogeneous state, the energy crossing the planer region $\beta\mathbf{a}$ is linear in β:

$$\mathbf{J}(\beta\mathbf{a}) = \beta\mathbf{J}(\mathbf{a}).$$

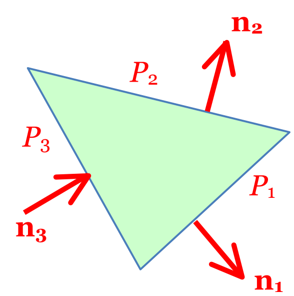
Consider a thin slice of the body. Let $\mathbf{a}$ be one face of the slice, and $-\mathbf{a}$ be the other face of the slice. In each case, the unit vector normal to the face points outside the slice. The energy crossing the two faces are $\mathbf{J}(\mathbf{a})$ and $\mathbf{J}(-\mathbf{a})$. The conservation of energy requires that

$$\mathbf{J}(-\mathbf{a}) = -\mathbf{J}(\mathbf{a}).$$

The combination of the above statements shows that the function obeys

$$\mathbf{J}(\beta\mathbf{a}) = \beta\mathbf{J}(\mathbf{a})$$

for every vector $\mathbf{a}$ and every number β.



Next consider two planar regions $\mathbf{a}_1$ and $\mathbf{a}_2$. Once again, because the shapes of the two regions do not affect the definition of the stress, we choose the two regions as rectangular regions. The sum $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$ is another planar region. The three planar regions form the surfaces of a prism. The cross section of the prism is shown in the figure. If the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ point toward the exterior of the prism, $\mathbf{n}_3$ points toward the interior of the prism. The energy crossing the three faces of the prism are $P_1 = \mathbf{J}(\mathbf{a}_1)$, $P_2 = \mathbf{J}(\mathbf{a}_2)$ and $P_3 = \mathbf{J}(-\mathbf{a}_3)$. The prism is a free-body diagram. The forces acting on the three faces are balanced, $P_3 + P_1 + P_2 = 0$, so that

$$\mathbf{J}(\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{J}(\mathbf{a}_1) + \mathbf{J}(\mathbf{a}_2).$$

This equation holds for any planar regions.

We have confirmed that the function $P = \mathbf{J}(\mathbf{a})$ is a linear map that maps a vector (the area vector) to a scalar set (the flow of energy per unit time). In linear algebra, such a linear map is called a vector. Because both the flow of energy and the planar resigns are frame-indifferent, the flux is a frame-indifferent tensor.

Similarly, one can show that the flux of the quantity of every conserved species is a frame-indifferent vector.

Components of flux. The preceding definition is independent the choice of the basis of the vector space. We next choose an orthonormal basis

$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Consider a unit cube of the body. Crossing the face $\mathbf{e}_i$ of the unit cube is the energy per unit time $\mathbf{J}(\mathbf{e}_i)$. This quantity is a scalar. Write

$$\mathbf{J}(\mathbf{e}_i) = J_i,$$

where J_i are the three components of the flux to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Write the area vector in terms of its components, $\mathbf{a} = a_i \mathbf{e}_i$. Note that

$$P = \mathbf{J}(\mathbf{a}) = \mathbf{J}(a_i \mathbf{e}_i) = a_i \mathbf{J}(\mathbf{e}_i) = a_i J_i.$$

Thus, the energy flow is the inner product of the area vector and the flux.

Flow density. Consider a planar region of area a, normal to a unit vector $\mathbf{n}$. Crossing the planar region is a flow of energy per unit time, P. Define the *flow density* by

$$j = \frac{P}{a}.$$

The body is in a homogeneous state. The flow density is independent of the shape and the area of the region, but depends on the direction $\mathbf{n}$ of the region.

We have just established that the flux is a linear map, so that

$$P = \mathbf{J}(\mathbf{a}) = \mathbf{J}(a\mathbf{n}) = a\mathbf{J}(\mathbf{n}),$$

A comparison of the two equations gives that

$$j = \mathbf{J}(\mathbf{n}), \quad j = J_i n_i.$$

Force. Define force as the linear map that maps the displacement to the change in energy:

$$U = \mathbf{f}(\mathbf{s}).$$

Energy is a scalar, and is frame-indifferent. The displacement is equivalent to the separation, $\mathbf{s}$, and is a frame-indifferent vector. For this equation to hold for all displacement, the force must be frame-indifferent. The algebra is similar to that of flux.

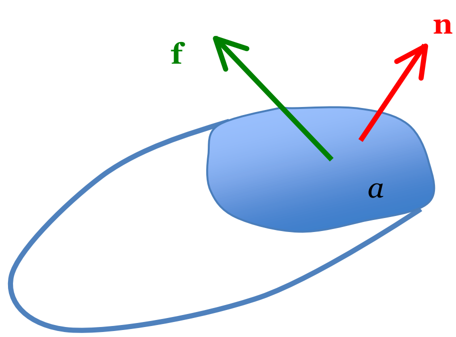
Similarly, electric potential is a scalar, and the electric field is the gradient of the electric potential. This means that the electric field is a linear map that maps separation between places to the difference in the electric potential. Consequently, the electric field is a frame-indifferent vector.

Traction. Consider a planar region of area a, normal to a unit vector $\mathbf{n}$. Acting on the planar region is a force $\mathbf{f}$. In general, the force has components normal and tangential to the plane. Define the traction $\mathbf{t}$ by the force acting on the planar region divided by the area of the region:

$$\mathbf{t} = \frac{\mathbf{f}}{a}.$$

The body is in a homogeneous state. The traction is independent of the shape and the area of the region, but depends on the direction $\mathbf{n}$ of the region. For instance, for the chewing gum in tension, a plane not normal to the axial direction will have both normal and shear traction.

Traction is commonly called a vector. This designation is wrong. Tractions acting on various planar regions form a set. This set, however, is not a vector space. Tractions do not obey the rule of vectors: the sum of tractions acting on two planar regions, in general, does not give traction acting on another planar region.



Stress. For a body in a homogeneous state, how many independent quantities do we need to calculate traction in a planar region of any direction? The answer is six. To see this answer requires a deeper generalization than the definition of traction.

Consider a planar region, of area a, normal to a unit vector $\mathbf{n}$. The product $a\mathbf{n}$ represents the planar region as a vector, written as $\mathbf{a} = a\mathbf{n}$. The force $\mathbf{f}$ acting on the planar region depends on both the area and direction of the region. We write this relation as a function:

$$\mathbf{f} = \mathbf{T}(\mathbf{a}).$$

The input of the function $\mathbf{T}$ is the vector $\mathbf{a}$ representing a planar region, and the output of the function is another vector $\mathbf{f}$ representing the force acting on the planar region. The function $\mathbf{T}$ is called a *state of stress*, or stress for brevity.

Stress is a frame-indifferent tensor. The set of all area vectors is a vector space, and the set of forces is another vector space. We next show that function $\mathbf{f} = \mathbf{T}(\mathbf{a})$ is a linear map.

In linear algebra, a function that maps one vector space to another vector space is a linear map if

1. $\mathbf{T}(\beta\mathbf{a}) = \beta\mathbf{T}(\mathbf{a})$ for every number β and every vector $\mathbf{a}$, and
2. $\mathbf{T}(\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{T}(\mathbf{a}_1) + \mathbf{T}(\mathbf{a}_2)$ for any vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

Let β be a positive number. Because the body is in a homogeneous state, the force acting on the planer region $\beta\mathbf{a}$ is linear in β:

$$\mathbf{T}(\beta\mathbf{a}) = \beta\mathbf{T}(\mathbf{a}).$$

Consider a thin slice of the body. Let $\mathbf{a}$ be one face of the slice, and $-\mathbf{a}$ be the other face of the slice. In each case, the unit vector normal to the face points outside the slice. The forces acting on the two faces are $\mathbf{T}(\mathbf{a})$ and $\mathbf{T}(-\mathbf{a})$. The balance of forces acting on the slice requires that

$$\mathbf{T}(-\mathbf{a}) = -\mathbf{T}(\mathbf{a}).$$

The combination of the above statements shows that the function obeys

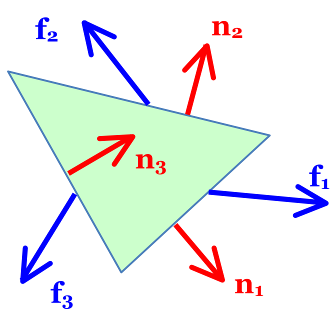
$$\mathbf{T}(\beta\mathbf{a}) = \beta\mathbf{T}(\mathbf{a})$$

for every vector $\mathbf{a}$ and every number β.

Consider two planar regions $\mathbf{a}_1$ and $\mathbf{a}_2$. Once again, because the shapes of the two regions do not affect the definition of the stress, we choose the two regions as rectangular regions. The sum $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$ is another planar region. The three planar regions form the surfaces of a prism. The cross section of the prism is shown in the figure. If the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ point toward the exterior of the prism, $\mathbf{n}_3$ points toward the interior of the prism. The forces acting on the three faces of the prism are $\mathbf{f}_1 = \mathbf{T}(\mathbf{a}_1)$, $\mathbf{f}_2 = \mathbf{T}(\mathbf{a}_2)$ and $\mathbf{f}_3 = \mathbf{T}(-\mathbf{a}_3)$. The prism is a free-body diagram. The forces acting on the three faces are balanced, $\mathbf{f}_3 + \mathbf{f}_1 + \mathbf{f}_2 = \mathbf{o}$, so that

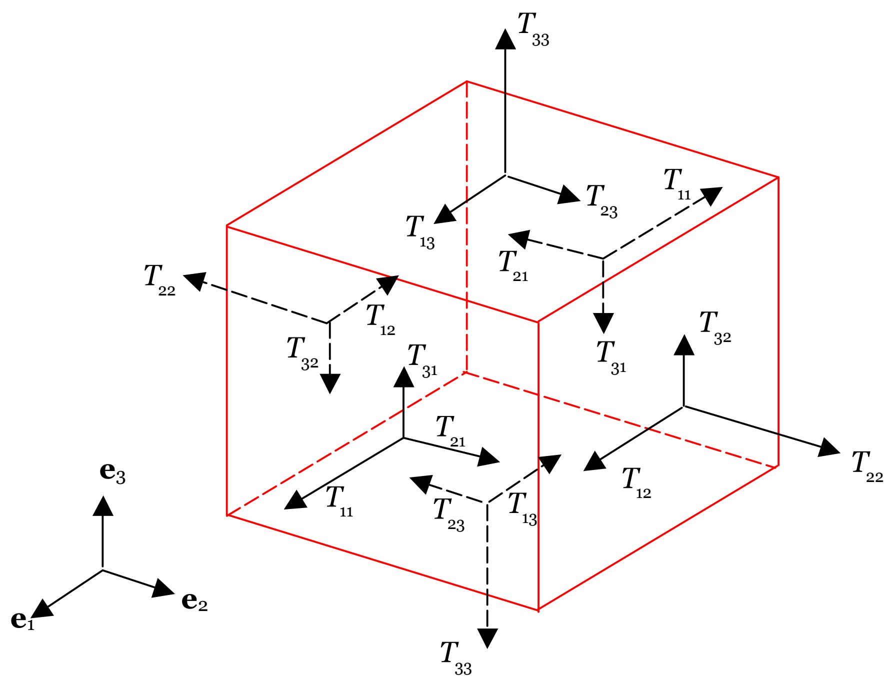
$$\mathbf{T}(\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{T}(\mathbf{a}_1) + \mathbf{T}(\mathbf{a}_2).$$

This equation holds for any planar regions.



We have confirmed that the function $\mathbf{f} = \mathbf{T}(\mathbf{a})$ is a linear map that maps one vector (the area vector) to another vector (the force). In linear algebra, such a linear map is called a tensor. In mechanics, we call this linear map the stress.

Because both the force and the planar resigns are frame-indifferent vectors, the stress is a frame-indifferent tensor.



Components of stress. The preceding definition is independent the choice of the basis of the vector space. We next choose an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Consider a unit cube of the body. Acting on the face $\mathbf{e}_j$ of the unit cube is the force $\mathbf{T}(\mathbf{e}_j)$. This force is a vector, which is also a linear combination of the three base vectors:

$$\mathbf{T}(\mathbf{e}_j) = T_{ij}\mathbf{e}_i,$$

where T_{ij} are the three components of the force relative to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The force acting on the unit cube on the face whose normal is $-\mathbf{e}_1$ is given by $\mathbf{T}(-\mathbf{e}_1) = -\mathbf{T}(\mathbf{e}_1)$. This algebra is consistent with a physical requirement: the balance of the forces acting on the unit requires that the two forces acting on each pair of parallel faces of the unit cube be equal in magnitude and opposite in direction.

The nine quantities T_{ij} are the components of stress. The first index indicates the direction of the force, and the second index indicates the direction of the vector normal to the face. Consider a piece of the body of a fixed number of molecules. In the current state, the piece is in the shape of a unit cube, with faces parallel to the coordinate planes. Acting on each face is a force. Denote by T_{ij} the component *i* of the force acting on the face of the cube normal to the axis *j*.

We adopt the following sign convention. When the outward normal vector of the face points in the positive direction of axis *j*, we take T_{ij} to be positive if the component *i* of the force points in the positive direction of axis *i*. When the outward normal vector of the face points in the negative direction of the axis *j*, we take T_{ij} to be positive if the component *i* of the force points in the negative direction of axis *i*.

Using the summation convention, we write the above three expressions as

$$\mathbf{T}\left(\mathbf{e}_j\right) = T_{ij}\mathbf{e}_i .$$

The nine components of the stress can be listed as a matrix:

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

The first index indicates the row, and the second the column.

The balance of moment acting on the cube gives that $T_{ij} = T_{ji}$.

Traction and stress. Define the traction $\mathbf{t}$ by the force $\mathbf{f}$ acting on the planar region divided by the area a of the region:

$$\mathbf{t} = \frac{\mathbf{f}}{a} .$$

We have just established that the stress is a linear map, so that

$$\mathbf{f} = \mathbf{T}\left(\mathbf{a}\right) = \mathbf{T}\left(a\mathbf{n}\right) = a\mathbf{T}\left(\mathbf{n}\right),$$

where a is the area of the planar region, and $\mathbf{n}$ is the unit vector normal to the planar region. A comparison of the two equations gives that

$$\mathbf{t} = \mathbf{T}\left(\mathbf{n}\right), \quad t_i = T_{ij}n_j .$$

This relation connects the traction on a plane to the stress.

In-Frame Rate and Co-Rotational Rate

Frame-indifferent vector. The algebra of separation is applicable to all frame-indifferent vectors. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis attached to a spot in the ground, and $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$ be another orthonormal basis attached to a spot in the airplane. The two sets of base vectors relate through a frame rotation: $\mathbf{e}_a' = Q_{ai}\mathbf{e}_i$ and $\mathbf{e}_i = Q_{ai}\mathbf{e}_a'$.

Let $\mathbf{f}$ and $\mathbf{f}'$ be two vectors. For the time being, the two vectors are unrelated. Write

$$\mathbf{f} = f_i \mathbf{e}_i, \quad \mathbf{f}' = f'_a \mathbf{e}'_a,$$

where f_i are the components of the vector $\mathbf{f}$ relative to the ground basis, and f'_a are the components of the other vector $\mathbf{f}'$ relative to the airplane basis. Because $\mathbf{e}_i = Q_{ai}\mathbf{e}'_a$, e can also write

$$\mathbf{f} = Q_{ai} f_i \mathbf{e}'_a, \quad \mathbf{f}' = f'_a \mathbf{e}'_a,$$

where $Q_{ai} f_i$ are the components of the vector $\mathbf{f}$ relative to the airplane basis.

If $\mathbf{f}' = \mathbf{f}$ holds for all frames, so does $f'_a = Q_{ai} f_i$. Conversely, if $f'_a = Q_{ai} f_i$ holds for all frame, so does $\mathbf{f}' = \mathbf{f}$. Thus, the following two statements are equivalent:

$$\mathbf{f}' = \mathbf{f}, \quad f'_a = Q_{ai} f_i.$$

They both mean that the vector is frame-indifferent. We do not write $f'_a = Q_{ai} f_i$ as $\mathbf{f}' = \mathbf{Q}\mathbf{f}$.

In-frame rate of a frame-indifferent vector. The separation between two particles is a frame-indifferent vector. The in-frame rate of the separation, the relative velocity of the two particles, however, is frame-sensitive. This observation turns out to be general.

A frame-indifferent vector transforms between the two frames according to

$$\mathbf{f}' = \mathbf{f}, \qquad f'_a = Q_{ai} f_i.$$

In the ground frame, we have measured the components of the force as functions of time, $f_i(t)$, and they change at the rates:

$$\dot{f}_i = \frac{f_i(t+dt) - f_i(t)}{dt}.$$

Define a new quantity called the *in-frame rate* relative to the ground:

$$\partial_{in}\mathbf{f} = \dot{f}_i \mathbf{e}_i.$$

Here we fix the ground basis, and allow the components in the ground basis to change in time.

We follow the same procedure in the airplane frame. In the airplane frame, we have measured the components of the same force as functions of time, $f'_i(t)$, and they change at the rates:

$$\dot{f}'_i = \frac{f'_i(t+dt) - f'_i(t)}{dt}.$$

Define another new quantity called the *in-frame rate* relative to the airplane:

$$\partial_{in}\mathbf{f}' = \dot{f}'_a\mathbf{e}'_a .$$

Here we fix the airplane basis, and allow the components in the airplane frame to change in time.

Differentiating $f'_a = Q_{ai}f_i$ with respect to time, we get

$$\dot{f}'_a = Q_{ai}\dot{f}_i + \dot{Q}_{ai}f_i .$$

Recall that $\dot{Q}_{ai} = S_{ab}Q_{bi}$, and we write the above equation as

$$\dot{f}'_a = Q_{ai}\dot{f}_i + S_{ab}Q_{bi}f_i , \quad \partial_{in}\mathbf{f}' = \partial_{in}\mathbf{f} + \mathbf{Sf} .$$

Recall that $Q_{bi}f_i$ is the component of the force projected on the base vector $\mathbf{e}'_b$. The in-frame rate is frame-sensitive. For example, in the airplane you press a spring with a constant force. The rate of force vanishes in the airplane frame. However, in the ground frame, the components of the force change with time when the airplane rotates relative to the ground.

Do not confuse vector with its components. In the literature, the relation $\mathbf{f}' = \mathbf{f}$ is commonly written as $\mathbf{f}' = \mathbf{Qf}$. This practice in effect let $\mathbf{f}'$ stand for the components of the force relative to airplane basis, and $\mathbf{f}$ stand for the components of the force relative to ground basis. If $\mathbf{f}'$ and $\mathbf{f}$ stand for components relative to the two frames, then $\partial_{in}\mathbf{f} = \dot{\mathbf{f}}$ and $\partial_{in}\mathbf{f}' = \dot{\mathbf{f}}'$. In such notation, the relation $\partial_{in}\mathbf{f}' = \partial_{in}\mathbf{f} + \mathbf{Sf}$ is written as $\dot{\mathbf{f}}' = \mathbf{Q}\dot{\mathbf{f}} + \mathbf{SQf}$. The last equation is the same as $\dot{f}'_a = Q_{ai}\dot{f}_i + S_{ab}Q_{bi}f_i$.

Most authors write the in-frame rate of a vector as $\dot{\mathbf{f}}$. This notation might be confusing: it may suggest that $\dot{\mathbf{f}} = \dot{f}_i\mathbf{e}_i + f_i\dot{\mathbf{e}}_i$. We prefer to write the in-frame rates explicitly as $\partial_{in}\mathbf{f} = \dot{f}_i\mathbf{e}_i$ and $\partial_{in}\mathbf{f}' = \dot{f}'_a\mathbf{e}'_a$. In each case, we fix the basis when taking the rates of the components.

Examples of in-frame rate of vectors. We have seen examples of the in-frame rate. Velocity is the in-frame rate of radial vector:

$$\mathbf{v} = \partial_{in}\mathbf{x}, \quad v_i = \dot{x}_i .$$

The radial vector is frame-sensitive, and transforms as

$$\mathbf{x}' = \mathbf{x} + \mathbf{c} , \quad x'_a = Q_{ai}x_i + c'_a .$$

The velocity is also frame-sensitive, and transforms as

$$\mathbf{v}' = \mathbf{v} + \mathbf{Sx} + \dot{\mathbf{c}} , \quad \dot{x}'_a = Q_{ai}\dot{x}_i + S_{ab}Q_{bi}x_i + \dot{c}'_a .$$

The relative velocity is the in-frame rate of separation:

$$\mathbf{g} = \partial_{in}\mathbf{s}, \quad g_i = \dot{s}_i .$$

The separation is frame-indifferent, and transforms as

$$\mathbf{s}' = \mathbf{s} , \quad s'_a = Q_{ai}s_i .$$

The relative velocity is frame-sensitive, and transforms as

$$\mathbf{g}' = \mathbf{g} + \mathbf{S}\mathbf{s}, \quad \dot{s}'_a = Q_{ai}\dot{s}_i + S_{ab}Q_{bi}s_i .$$

We have also examined the in-frame rate:

$$\dot{\mathbf{e}}'_a = \dot{Q}_{ai}\mathbf{e}_i .$$

Here $\dot{\mathbf{e}}'_a$ is the rate of a base vector in the airplane frame taken while the ground frame is fixed.

Co-rotational rate of a vector. A frame-indifferent vector transforms from one frame to another according to

$$\mathbf{f}' = \mathbf{f}, \quad f'_a = Q_{ai}f_i .$$

The in-frame rate of the frame-indifferent vector is frame-sensitive, and transforms according to

$$\partial_{in}\mathbf{f}' = \partial_{in}\mathbf{f} + \mathbf{S}\mathbf{f}, \quad \dot{f}'_a = Q_{ai}\dot{f}_i + S_{ab}Q_{bi}f_i .$$

Recall that

$$\mathbf{S} = \mathbf{W}' - \mathbf{W}, \quad S_{ab} = W'_{ab} - W_{ab} .$$

Replacing $\mathbf{S}$ with $\mathbf{W}' - \mathbf{W}$ in $\partial_{in}\mathbf{f}' = \partial_{in}\mathbf{f} + \mathbf{S}\mathbf{f}$, we get

$$\partial_{in}\mathbf{f}' - \mathbf{W}'\mathbf{f}' = \partial_{in}\mathbf{f} - \mathbf{W}\mathbf{f}, \quad \dot{f}'_a - W'_{ab}f'_b = Q_{ai}\dot{f}_i - W_{ab}Q_{bi}f_i .$$

This equation says that the combination

$$\partial_{in}\mathbf{f} - \mathbf{W}\mathbf{f}, \quad \dot{f}_i - W_{ip}f_p$$

is frame-indifferent. Define the *co-rotational rate* by

$$\partial\mathbf{f} = \partial_{in}\mathbf{f} - \mathbf{W}\mathbf{f}. \quad \partial f_i = \dot{f}_i - W_{ip}f_p$$

The co-rotational rate of the frame-indifferent vector is frame-indifferent.

Co-rotational rate of separation. Given a frame, of origin *O* and basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, measure the radial vector of a particle in the frame, $\mathbf{x} = x_i\mathbf{e}_i$. The velocity of the particle is the in-frame rate of the radial vector:

$$\mathbf{u} = \dot{x}_i\mathbf{e}_i, \quad \mathbf{u} = \partial_{in}\mathbf{x} .$$

Now consider a body undergoing homogeneous deformation. At a given time, one particle occupies a place of radial vector $\mathbf{x}$ and moves at a velocity $\mathbf{u}$, and another particle occupies a place of radial vector $\mathbf{y}$ and moves at a velocity $\mathbf{v}$. The separation of the two particles, $\mathbf{s} = \mathbf{y} - \mathbf{x}$, is a frame-indifferent vector, but the relative velocity, $\partial_{in}\mathbf{s} = \mathbf{v} - \mathbf{u}$, is a frame-sensitive vector. The co-rotational rate of the separation is

$$\partial\mathbf{s} = \partial_{in}\mathbf{s} - \mathbf{W}\mathbf{s} .$$

The separation is frame-indifferent, and so is its co-rotational rate.

The velocity gradient $\mathbf{L}$ is defined as the linear map that maps the separation to the relative velocity:

$$\partial_{in}\mathbf{s} = \mathbf{L}\mathbf{s} .$$

Thus, the velocity gradient is the in-frame rate of separation. Combining the above two expression, we obtain that

$$\partial\mathbf{s} = \mathbf{D}\mathbf{s} .$$

That is, the rate of deformation is the linear map that maps the separation to the co-rotational rate of the separation.

Frame-indifferent tensor. We can extend the above ideas to tensors. A second-rank tensor is a linear map between two vector spaces. Let $\mathbf{A}$ and $\mathbf{A}'$ be two tensors. For the time being, the two tensors are unrelated. In the two frames, write

$$\mathbf{A}\left(\mathbf{e}_j\right) = A_{ij}\mathbf{e}_i , \quad \mathbf{A}'\left(\mathbf{e}'_b\right) = A'_{ab}\mathbf{e}'_a$$

where A_{ij} are the components of tensor $\mathbf{A}$ relative to the ground basis, and A'_{ab} are the components of the tensor $\mathbf{A}'$ relative to the airplane basis.

We can, of course, write the tensor A in terms of its components in the airplane frame. The two sets of base vectors are related as $\mathbf{e}'_b = Q_{bj}\mathbf{e}_j$ and $\mathbf{e}_i = Q_{ai}\mathbf{e}'_a$. The following identity holds:

$$\mathbf{A}\left(\mathbf{e}'_b\right) = \mathbf{A}\left(Q_{bj}\mathbf{e}_j\right) = Q_{bj}\mathbf{A}\left(\mathbf{e}_j\right) = Q_{bj}A_{ij}\mathbf{e}_i = Q_{ai}Q_{bj}A_{ij}\mathbf{e}'_a .$$

We rewrite the above equations using the airplane basis:

$$\mathbf{A}\left(\mathbf{e}'_b\right) = Q_{ai}Q_{bj}A_{ij}\mathbf{e}'_a , \quad \mathbf{A}'\left(\mathbf{e}'_b\right) = A'_{ab}\mathbf{e}'_a .$$

If $\mathbf{A}' = \mathbf{A}$ holds for all frames, so does $A'_{ab} = Q_{ai}Q_{bj}A_{ij}$. Conversely, if $A'_{ab} = Q_{ai}Q_{bj}A_{ij}$ holds for all frames, so does $\mathbf{A}' = \mathbf{A}$. Thus, the following two statements are equivalent:

$$\mathbf{A}' = \mathbf{A} , \quad A'_{ab} = Q_{ai}Q_{bj}A_{ij} .$$

They both mean that the tensor is frame-indifferent. We do not write expression $A'_{ab} = Q_{ai}Q_{bj}A_{ij}$ as $\mathbf{A}' = \mathbf{Q}\mathbf{A}\mathbf{Q}^T$.

In-frame rate of a tensor. In the ground frame, we have measured the components of a tensor as functions of time, $A_{ij}\left(t\right)$, and they change at the rates:

$$\dot{A}_{ij} = \frac{A_{ij}\left(t+dt\right) - A_{ij}\left(t\right)}{dt} .$$

Define the in-frame rate relative to the ground $\partial_{in}\mathbf{A}$ as a linear map:

$$\left(\partial_{in}\mathbf{A}\right)\left(\mathbf{e}_j\right) = \dot{A}_{ij}\mathbf{e}_i .$$

Here we fix the ground basis, and allow the components in the ground basis to change in time.

We follow the same procedure in the airplane frame. In the airplane frame, we have measured the components of the tensor as functions of time, $A'_{ij}(t)$, and they change at the rates:

$$\dot{A}'_{ij} = \frac{A'_{ij}(t+dt) - A'_{ij}(t)}{dt}.$$

Define the in-frame rate relative to the airplane $\partial_{in}\mathbf{A}'$ as a linear map

$$(\partial_{in}\mathbf{A}')(\mathbf{e}'_b) = \dot{A}_{ab}\mathbf{e}'_a.$$

Here we fix the airplane basis, and allow the components in the airplane frame to change in time.

Differentiating $A'_{ab} = Q_{ai}Q_{bj}A_{ij}$ with respect to time, we get

$$\dot{A}'_{ab} = Q_{ai}Q_{bj}\dot{A}_{ij} + \dot{Q}_{ai}Q_{bj}A_{ij} + Q_{ai}\dot{Q}_{bj}A_{ij}.$$

Note the identities:

$$\dot{Q}_{ai} = S_{ac}Q_{ci}, \quad \dot{\mathbf{Q}} = \mathbf{SQ}.$$

We obtain that

$$\dot{A}'_{ab} = Q_{ai}Q_{bj}\dot{A}_{ij} + S_{ac}Q_{ci}Q_{bj}A_{ij} + Q_{ai}S_{bc}Q_{cj}A_{ij}.$$

Recall that the frame spin is anti-symmetric, $S_{bc} = -S_{cb}$. The above expression is the same as

$$\partial_{in}\mathbf{A}' = \partial_{in}\mathbf{A} + \mathbf{SA} - \mathbf{AS}.$$

The in-frame rate of a frame-indifferent tensor is, in general, frame-sensitive. The in-frame rate of force is frame-sensitive. For example, in the airplane you press a spring with a constant force. The rate of force vanishes in the airplane frame. However, in the ground frame, the components of the force change with time when the airplane rotates relative to the ground.

Co-rotational rate of a tensor. A frame-indifferent tensor transforms from one frame to another according to

$$\mathbf{A}' = \mathbf{A}, \quad A'_{ab} = Q_{ai}Q_{bj}A_{ij}.$$

The in-frame rate of the frame-indifferent tensor is frame-sensitive, and transforms according to

$$\partial_{in}\mathbf{A}' = \partial_{in}\mathbf{A} + \mathbf{SA} - \mathbf{AS}, \quad \dot{A}'_{ab} = Q_{ai}Q_{bj}\dot{A}_{ij} + S_{ac}Q_{ci}Q_{bj}A_{ij} + Q_{ai}S_{bc}Q_{cj}A_{ij}.$$

Recall that

$$\mathbf{S} = \mathbf{W}' - \mathbf{W}, \quad S_{ab} = W'_{ab} - W_{ab}.$$

Replacing **S** with $\mathbf{W}' - \mathbf{W}$ in $\partial_{in}\mathbf{f}' = \partial_{in}\mathbf{f} + \mathbf{Sf}$, we get

$$\partial_{in}\mathbf{A}' - \mathbf{W}'\mathbf{A}' + \mathbf{A}'\mathbf{W}' = \partial_{in}\mathbf{A} - \mathbf{WA} + \mathbf{AW},$$

or

$$\dot{A}'_{ab} - W'_{ap}A'_{pb} - W'_{bp}A'_{ap} = Q_{ai}Q_{bj}\dot{A}_{ij} - W_{ac}Q_{ci}Q_{bj}A_{ij} - W_{bc}Q_{ai}Q_{cj}A_{ij}.$$

This equation says that the combination

$$\partial_{in}\mathbf{A} - \mathbf{WA} + \mathbf{AW}, \quad \dot{A}_{ij} - W_{ip}A_{pj} - W_{jp}A_{ip}$$

is frame-indifferent. Define the *co-rotational rate* by

$$\partial\mathbf{A} = \partial_{in}\mathbf{A} - \mathbf{WA} + \mathbf{AW}, \quad \partial A_{ij} = \dot{A}_{ij} - W_{ip}A_{pj} - W_{jp}A_{ip}.$$

The co-rotational rate of the frame-indifferent tensor is frame-indifferent.

One can similarly define co-rotational rate of higher order tensors. For example, consider a third order, frame-indifferent tensor. In the two frames, write

$$\mathbf{A}' = \mathbf{A}, \quad A'_{abc} = Q_{ai}Q_{bj}Q_{ck}A_{ijk}.$$

The co-rotational rate is

$$\partial A_{ijk} = \dot{A}_{ijk} - W_{ip}A_{pjk} - W_{jp}A_{ipk} - W_{kp}A_{ijp}.$$

We do not attempt to rewrite this expression using the notation of matrices.

Other Frame-Indifferent Rates

Given a frame-indifferent vector, one can define different frame-indifferent rates. Here are two examples

$$\partial_{in}\mathbf{h} - \mathbf{Lh},$$

$$\partial_{in}\mathbf{h} - \mathbf{Wh} + \mathrm{tr}(\mathbf{D})\mathbf{h}.$$

Each example is a sum of the co-rotational rate $\partial_{in}\mathbf{h} - \mathbf{Wh}$ and terms involving frame-indifferent variables.

Similarly, given a frame-indifferent tensor, one can define different frame-indifferent rates. Here are two examples:

$$\partial_{in}\mathbf{A} - \mathbf{LA} - \mathbf{AL}^T,$$

$$\partial_{in}\mathbf{A} + \mathbf{L}^T\mathbf{A} + \mathbf{AL}.$$

Each example is a sum of the co-rotational rate $\partial_{in}\mathbf{A} - \mathbf{WA} + \mathbf{AW}$ and terms involving only frame-indifferent variables.

Insignificance of alternative rates. The choice of these rates has no experimental consequence, so long as we remember to convert from one rate to another. For example, a rheological model of viscoelasticity and elastoplasticity relates rate of stress $\partial\mathbf{T}$, to stress $\mathbf{T}$ and rate of deformation $\mathbf{D}$:

$$\partial\mathbf{T} = \mathbf{F}(\mathbf{T},\mathbf{D}).$$

Any extra terms involving $\mathbf{T}$ and $\mathbf{D}$ added to the co-rotational rate are superfluous, as they can be put in the function on the right-hand side. What is significant is that frame-indifferent variables be used to construct the rheological model, so that the model is unaffected by any rigid-body motion.

Product Rules

Vectors and tensors are linear maps. It is desirable to use rates that obey product rules. In-frame rates obey product rules. Co-rotational rate also obey product rules. Other frame-indifferent rates may not follow some of the product rules.

In-frame rates obey product rules. For example, consider a tensor $\mathbf{A}$ that linearly maps a vector $\mathbf{h}$ to a vector $\mathbf{g}$:

$$\mathbf{g} = \mathbf{A}\mathbf{h}.$$

Write the tensor and the vectors in terms of components in the ground frame:

$$\mathbf{g} = g_i \mathbf{e}_i, \quad \mathbf{A}\left(\mathbf{e}_j\right) = A_{ij}\mathbf{e}_i, \quad \mathbf{h} = h_j \mathbf{e}_j.$$

Note the identity:

$$\mathbf{A}\mathbf{h} = \mathbf{A}\left(h_j \mathbf{e}_j\right) = h_j \mathbf{A}\left(\mathbf{e}_j\right) = A_{ij} h_j \mathbf{e}_i.$$

A comparison of the above expressions shows that

$$g_i = A_{ij} h_j.$$

Differentiating with respect to time, we get

$$\dot{g}_i = \dot{A}_{ij} h_j + A_{ij} \dot{h}_j.$$

Recall the definition of the in-frame rates, and we obtain that

$$\partial_{in}\left(\mathbf{A}\mathbf{h}\right) = \left(\partial_{in}\mathbf{A}\right)\mathbf{h} + \mathbf{A}\left(\partial_{in}\mathbf{h}\right).$$

Thus, the in-frame rates follow the product rule.

Product rule $\partial\left(\mathbf{g}\cdot\mathbf{h}\right) = \left(\partial\mathbf{g}\right)\cdot\mathbf{h} + \mathbf{g}\cdot\left(\partial\mathbf{h}\right)$. The two vectors $\mathbf{g}$ and $\mathbf{h}$ always obey the product rule for the in-frame rates:

$$\partial_{in}\left(\mathbf{g}\cdot\mathbf{h}\right) = \left(\partial_{in}\mathbf{g}\right)\cdot\mathbf{h} + \mathbf{g}\cdot\left(\partial_{in}\mathbf{h}\right).$$

Recall that $\partial\mathbf{g} = \partial_{in}\mathbf{g} - \mathbf{W}\mathbf{g}$ and $\partial\mathbf{h} = \partial_{in}\mathbf{h} - \mathbf{W}\mathbf{h}$. The product $\mathbf{g}\cdot\mathbf{h}$ is a scalar, so that its in-frame rate is the same as the co-rotational rate, $\partial_{in}\left(\mathbf{g}\cdot\mathbf{h}\right) = \partial\left(\mathbf{g}\cdot\mathbf{h}\right) = d\left(\mathbf{g}\cdot\mathbf{h}\right)/dt$. Insert these expressions, and we get

$$\partial\left(\mathbf{g}\cdot\mathbf{h}\right) = \left(\partial\mathbf{g}\right)\cdot\mathbf{h} + \mathbf{g}\cdot\left(\partial\mathbf{h}\right) + \left(\mathbf{W}\mathbf{g}\right)\cdot\mathbf{h} + \mathbf{g}\cdot\left(\mathbf{W}\mathbf{h}\right).$$

Because $\mathbf{W}$ is anti-symmetric, the last two terms cancel each other, and we obtain the product rule for the inner product of two vectors.

Incidentally, $\partial_{in}\mathbf{h} - \mathbf{L}\mathbf{h}$ is also a frame-indifferent rate, but it does not obey the above product rule.

Use product rule to define frame-indifferent rate of vector. We often define a frame-indifferent vector as a linear map from a frame-indifferent to a scalar. For example, the force is the linear map from displacement to change in energy, $U = \mathbf{f} \cdot \mathbf{s}$. Suppose we have adopted the co-rotational rates for one vector $\partial \mathbf{s} = \partial_{in} \mathbf{s} - \mathbf{W}\mathbf{s}$. The rate of a scalar is the same as the co-rotational rate, $\partial_{in} U = \partial U = dU/dt$. We now wish to define a rate of vector, $\partial \mathbf{f}$, such that it obeys the product rule:

$$\partial(\mathbf{f} \cdot \mathbf{s}) = (\partial \mathbf{f}) \cdot \mathbf{s} + \mathbf{f} \cdot (\partial \mathbf{s}).$$

In this expression, every term other than $\partial \mathbf{f}$ is known to be frame-indifferent. Consequently, the rate $\partial \mathbf{f}$ satisfying this product rule must also be frame-indifferent.

The in-frame rates always obey the product rule,

$$\partial_{in}(\mathbf{f} \cdot \mathbf{s}) = (\partial_{in} \mathbf{f}) \cdot \mathbf{s} + \mathbf{f} \cdot (\partial_{in} \mathbf{s}).$$

The difference between the in-frame and the co-rotational product rules is

$$0 = (\partial \mathbf{f} - \partial_{in} \mathbf{f}) \cdot \mathbf{s} + \mathbf{f} \cdot (-\mathbf{W}\mathbf{s}).$$

Because $\mathbf{W}$ is anti-symmetric, $\mathbf{f} \cdot (-\mathbf{W}\mathbf{s}) = (\mathbf{W}\mathbf{f}) \cdot \mathbf{s}$. The above equation becomes that

$$0 = (\partial \mathbf{f} - \partial_{in} \mathbf{f} + \mathbf{W}\mathbf{f}) \cdot \mathbf{s}.$$

For this equation to hold for all vector **s**, we get

$$\partial \mathbf{f} = \partial_{in} \mathbf{f} - \mathbf{W}\mathbf{f}.$$

Thus, once we adopt the co-rotational rate for one vector, and insist on the product rule, the rate of the other vector is the co-rotational rate.

Product rule $\partial(\mathbf{A}\mathbf{h}) = (\partial \mathbf{A})\mathbf{h} + \mathbf{A}(\partial \mathbf{h})$. Let $\mathbf{A}$ be a tensor and $\mathbf{h}$ be a vector. Their in-frame rates obey the product rule:

$$\partial_{in}(\mathbf{A}\mathbf{h}) = (\partial_{in} \mathbf{A})\mathbf{h} + \mathbf{A}(\partial_{in} \mathbf{h}).$$

Recall that $\partial \mathbf{h} = \partial_{in} \mathbf{h} - \mathbf{W}\mathbf{h}$ and $\partial \mathbf{A} = \partial_{in} \mathbf{A} - \mathbf{W}\mathbf{A} + \mathbf{A}\mathbf{W}$. The product $\mathbf{A}\mathbf{h}$ is a vector, so that $\partial(\mathbf{A}\mathbf{h}) = \partial_{in}(\mathbf{A}\mathbf{h}) - \mathbf{W}\mathbf{A}\mathbf{h}$. A combination of the above gives that

$$\partial(\mathbf{A}\mathbf{h}) = (\partial \mathbf{A})\mathbf{h} + \mathbf{A}(\partial \mathbf{h}).$$

Use product rule to define frame-indifferent rate of tensor. We often define a frame-indifferent tensor as a linear map between two frame-indifferent vectors, $\mathbf{g} = \mathbf{A}\mathbf{h}$. For example, the stress is the linear map between the area vector of a planner region and the force acting on the planar region. Suppose we have adopted the co-rotational rates for the two vectors $\partial \mathbf{g} = \partial_{in} \mathbf{g} - \mathbf{W}\mathbf{g}$ and $\partial \mathbf{h} = \partial_{in} \mathbf{h} - \mathbf{W}\mathbf{h}$. We now wish to define a rate of tensor, $\partial \mathbf{A}$, such that it obeys the product rule:

$$\partial(\mathbf{A}\mathbf{h}) = (\partial \mathbf{A})\mathbf{h} + \mathbf{A}(\partial \mathbf{h}).$$

In this expression, every term other than $\partial \mathbf{A}$ is known to be frame-indifferent. Consequently, all $\partial \mathbf{A}$ satisfying this product rule must also be frame-indifferent.

The in-frame rates always obey the product rule,

$$\partial_{in}(\mathbf{Ah}) = (\partial_{in}\mathbf{A})\mathbf{h} + \mathbf{A}(\partial_{in}\mathbf{h}).$$

The difference between the in-frame and the co-rotational product rules is

$$\partial(\mathbf{Ah}) - \partial_{in}(\mathbf{Ah}) = (\partial\mathbf{A} - \partial_{in}\mathbf{A})\mathbf{h} + \mathbf{A}(\partial\mathbf{h} - \partial_{in}\mathbf{h}).$$

Inserting $\partial(\mathbf{Ah}) = \partial_{in}(\mathbf{Ah}) - \mathbf{WAh}$ and $\partial\mathbf{h} = \partial_{in}\mathbf{h} - \mathbf{Wh}$, we obtain that

$$(\partial\mathbf{A} - \partial_{in}\mathbf{A} + \mathbf{WA} - \mathbf{AW})\mathbf{h} = \mathbf{0}.$$

For this equation to hold for all vector **h**, we get

$$\partial\mathbf{A} = \partial_{in}\mathbf{A} - \mathbf{WA} + \mathbf{AW}.$$

Thus, once we adopt the co-rotational rate for the two vectors, and insist on the product rule, the rate of tensor is the co-rotational rate.

Product rule $\partial(A_{ij}B_{ij}) = (\partial A_{ij})B_{ij} + A_{ij}(\partial B_{ij})$. Let **A** and **B** be two second-rank tensors. Their components obey the product rule:

$$\frac{d}{dt}(A_{ij}B_{ij}) = \dot{A}_{ij}B_{ij} + A_{ij}\dot{B}_{ij}.$$

The difference between the two types of product rules is

$$0 = (\partial A_{ij} - \dot{A}_{ij})B_{ij} + A_{ij}(\partial B_{ij} - \dot{B}_{ij}).$$

Inserting $\partial A_{ij} = \dot{A}_{ij} - W_{ip}A_{pj} - W_{jp}A_{ip}$ and $\partial B_{ij} = \dot{B}_{ij} - W_{ip}B_{pj} - W_{jp}B_{ip}$, we get

$$(W_{ip}A_{pj} + W_{jp}A_{ip})B_{ij} + A_{ij}(W_{ip}B_{pj} + W_{jp}B_{ip}) = 0.$$

We can confirm that this equation holds because **W** is anti-symmetric, $W_{ip} = -W_{pi}$.

As a special case, the rate of the second invariant of a tensor is

$$\frac{d}{dt}(A_{ij}A_{ij}) = 2A_{ij}\partial A_{ij}.$$

We will use this product rule in the model of strain hardening.

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