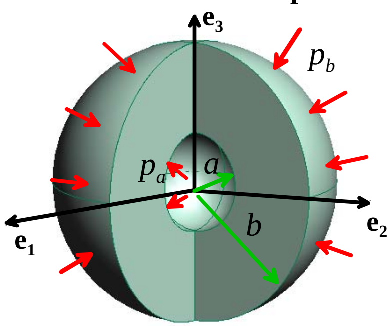
ENGM 922 Exam prep

- # Hydrostatic Pressure:

Pressurized hollow sphere



Assume that

- No body forces act on the sphere ➡ $\sigma_{ij,i} = 0$ ⇒ in spherical coord

$$\frac{\partial}{\partial r}\left(\sigma_{rr} + \sigma_{r\theta} + \sigma_{r\phi}\right) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left(\sigma_{\theta r} + \sigma_{\theta\theta} + \sigma_{\theta\phi}\right) = 0$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\sigma_{\phi r} + \sigma_{\phi\theta} + \sigma_{\phi\phi}\right) = 0$$

- The sphere has uniform temperature
- The inner surface $R=a$ is subjected to pressure p_b
- The outer surface $R=b$ is subjected to pressure p_b

Constitutive relation: $\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$ ➡

$$\boldsymbol{\sigma} = \begin{bmatrix} \lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi}\right)+2\mu\varepsilon_{rr} & 0 & 0 \\ 0 & \lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi}\right)+2\mu\varepsilon_{\theta\theta} & 0 \\ 0 & 0 & \lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi}\right)+2\mu\varepsilon_{\phi\phi} \end{bmatrix}$$

Hydrostatic σ state

Given the θ & ϕ symmetry of the hydrostatic state only $u_r \neq 0$ & the 3 =librium eq's become:

$$\frac{\partial}{\partial r}\left(\lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi}\right)+2\mu\varepsilon_{rr}\right) = 0$$

0 as well due to symmetry

$$\frac{1}{r}\frac{\partial}{\partial\theta}\left(\lambda u_{r,r}\right) = 0$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\lambda u_{r,r}\right) = 0$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

The simplistic ε – u relation to the left is not so straight forward for coord sys other than cartesian:

$$\epsilon_{RR} = \frac{\partial u_R}{\partial R}, \qquad e_{\theta\theta} = \frac{1}{R}\left(\frac{\partial u_\theta}{\partial\theta} + u_R\right)$$

$$\epsilon_{\phi\phi} = \frac{1}{R\sin\theta}\left(\frac{\partial u_\phi}{\partial\phi} + (\sin\theta)u_R + (\cos\theta)u_\theta\right)$$

Applying the ε – u relations for spherical coord's the radial momentum =librium eq becomes:

$u_\theta = 0$ $u_\phi = 0$ $u_\theta = 0$

$$\frac{\partial}{\partial r}\left(\lambda\left(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi}\right)+2\mu\varepsilon_{rr}\right) = (\lambda+2\mu)\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial}{\partial r}\left[\frac{\lambda}{r}\left(\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)+\frac{1}{\sin\theta}\left(\frac{\partial u_\phi}{\partial\phi}+u_r\sin\theta+u_\theta\cos\theta\right)\right)\right] = (\lambda+2\mu)\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial}{\partial r}\left[\frac{\lambda}{r}(2u_r)\right] = 0$$

$$(\lambda+2\mu)\frac{\partial^2 u_r}{\partial r^2} + 2\lambda\left(\frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}\right) = 0$$

Hydrostatic Pressure (con'd):

$$(\lambda+2\mu)\frac{\partial^2 u_r}{\partial r^2}+2\lambda\left(\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)=0 \quad \mu=\frac{E}{2(1+\nu)},\quad \lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \frac{E}{(1+\upsilon)(1-2\upsilon)}\left[(1-\upsilon)\frac{\partial^2 u_r}{\partial r^2}+2\upsilon\left(\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)\right]=0$$

note: $\frac{d^2u}{dR^2}+\frac{2}{R}\frac{du}{dR}-\frac{2u}{R^2}=\frac{d}{dR}\left\{\frac{1}{R^2}\frac{d}{dR}\left(R^2u\right)\right\}$

http://solidmechanics.org/

In Dr. Allan Bower's text, **Applied Mechanics of Solids**, from Brown Univ. he managed to get rid of the coeff terms dependent on the Poisson's ratio that prevent me from obtaining the simpler ordinary diff eq which would allow me to resolve u(r) via integration. There must be some obscure assumption or relation that I am not aware of which would allow me to proceed per Dr. Bower's direction.

Adopting the governing diff eq per Dr. Bower's text: $\frac{\partial^2 u_r}{\partial r^2}+\left(\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)=\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\left(r^2u_r\right)}{\partial r}\right)=0$ soln via direct integration $u_r=\frac{1}{r^2}\left(\frac{A}{3}r^3+B\right)$

Resolve A & B via application of BC's:

$$\sigma_{rr}(r=a)=(\lambda+2\mu)\frac{\partial^2 u_r}{\partial r^2}\bigg|_{r=a}+\frac{\partial}{\partial r}\left[\frac{\lambda}{r}(2u_r)\right]\bigg|_{r=a}=P_a$$

$$\sigma_{rr}(r=b)=(\lambda+2\mu)\frac{\partial^2 u_r}{\partial r^2}\bigg|_{r=b}+\frac{\partial}{\partial r}\left[\frac{\lambda}{r}(2u_r)\right]\bigg|_{r=b}=P_b$$

$$(\lambda+2\mu)\frac{6B}{a^4}+2\lambda\left[\frac{1}{a}\left(A-2\frac{B}{a^3}\right)+\frac{1}{a^2}\left(Aa+\frac{B}{a^2}\right)\right]=P_a$$

$$(\lambda+2\mu)\frac{6B}{b^4}+2\lambda\left[\frac{1}{b}\left(A-2\frac{B}{b^3}\right)+\frac{1}{b^2}\left(Ab+\frac{B}{b^2}\right)\right]=P_b$$

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4.1.3 General solution to the spherically symmetric linear elasticity problem

Our goal is to solve the equations given in Section 4.1.2 for the displacement, strain and stress in the sphere. To do so,

1. Substitute the strain-displacement relations into the stress-strain law to show that

$$\begin{bmatrix} \sigma_{RR} \\ \sigma_{\theta\theta} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 2\nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \dfrac{du}{dR} \\ \dfrac{u}{R} \end{bmatrix} - \frac{E\alpha\Delta T}{1-2\nu} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2. Substitute this expression for the stress into the equilibrium equation and rearrange the result to see that

$$\frac{d^2u}{dR^2} + \frac{2}{R}\frac{du}{dR} - \frac{2u}{R^2} = \frac{d}{dR}\left\{\frac{1}{R^2}\frac{d}{dR}\left(R^2 u\right)\right\} = \frac{\alpha(1+\nu)}{(1-\nu)}\frac{d\Delta T}{dR} - \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\rho_0 b(R)$$

Given the temperature distribution and body force this equation can easily be integrated to calculate the displacement *u*. Two arbitrary constants of integration will appear when you do the integral these must be determined from the *boundary conditions* at the inner and outer surface of the sphere. Specifically, the constants must be selected so that either the displacement or the radial stress have prescribed values on the inner and outer surface of the sphere.

In the following sections, this procedure is used to derive solutions to various boundary value problems of practical interest.