NUMBER

We wish to eradicate the confusion between numbers and scalars in textbooks of linear algebra. We will define scalars in separate notes, and will define numbers here. Before talking about numbers, we recall the basics of set, map, and operation.

Set

A set and its elements. A collection of objects is called a *set*. Each object in the set is called an *element*.

An object a either belongs to a set A or not. If the object a belongs to the set A, we translate the sentence as $a \in A$. Thus, the symbol "$\in$" is read "belongs to", or "is an element of". If the object a does not belong to the set A, we translate the sentence as $a \notin A$.

The set that contains no element is called the *empty set*. A set that contains at least one element is called a *nonempty set*.

We will use words "set", "collection", "family", and "class" interchangeably, but prefer the shortest word "set". We will also use words "element", "object", and "member" interchangeably. The flexibility in usage is nice when we speak of "a family of sets", and of "a member in a family". Compare these phrases to "a set of sets", and "an element in a family".

Specify a set by specifying its elements. We can specify a set by listing its elements between braces, $\{ \}$. The notation

$$\{\text{Daniel, Michael}\}$$

denotes a set of two elements, Daniel and Michael. The notation

$$\{\text{wine, cheese, cracker}\}$$

denotes a set of three elements: wine, cheese, and cracker.

Listing all elements of a set is impossible if the set contains too many elements. The notation

$$\{1,2,3,...\}$$

denotes the set of positive integers, which has infinite number of elements. In writing this way, we assume that the reader knows what we mean. If we do not wish to make this assumption, we may specify the set by using plain English. We say, "Let I be the set of positive integers".

We can also specify a set by giving a rule that determines its membership. We write $\{x|p\}$, which is read "the set of x such that property p holds". For example, we write

$$\{x|x \text{ is positive real number}\}.$$

This notation is read "The set of x such that x is positive real number".

You may want to say something about repeated elements.

Relations between sets. Let A and B be two sets. The *union* of the two sets, written as $A \cup B$, is the set of all elements that belong to either A or B.

The *intersection* of the two sets, written as $A \cap B$, is the set of all elements that belong to both A and B. The two sets are called *disjoint* when they have no element in common—that is, when $A \cap B$ is the empty set.

If every element of A is also an element of B, we say that A is a *subset* of B, and write $A \subset B$, or $B \supset A$. If in addition there is at least one element in B but not in A, we say that A is a *proper subset* of B.

We can represent the logical relations between sets by the Venn diagram.

Ordered list, tuple. An ordered list of n elements is called an n-tuple. We write an n-tuple by listing its elements between parentheses, $(\ )$. The notation

$$(6,1,7,4,9,5,3,7,8,9)$$

denotes a 10-tuple of an ordered list of ten numbers. The 10-tuple happens to represent a telephone number. In defining a tuple, the order of the elements is significant. By contrast, in defining a set, the order of elements is insignificant.

Cartesian product. Let A and B be two sets. Any two elements $a \in A$ and $b \in B$ form an *ordered pair*, written as (a,b). The collection of all such ordered pairs is called the Cartesian product of the two sets A and B, written as $A \times B$. Thus, the Cartesian product is defined as

$$A \times B = \{(a,b) | a \in A \text{ and } b \in B\}.$$

For example, let $A = \{\text{male, female}\}$ and $B = \{\text{dog, cat, pig}\}$The Cartesian product of the two sets is

$$A \times B = \{(\text{male, dog}), (\text{male, cat}), (\text{male, pig}), (\text{female, dog}), (\text{female, cat}), (\text{female, pig})\}$$

We can similarly define Cartesian product of more than two sets.

We can, of course, form Cartesian product of a set and itself. Let S be a set. We write $S \times S$ as S^2, $S \times S \times S$ as S^3, etc. Thus, S^n stands for the set of n-tuples of elements in the set S, namely,

$$S^n = \{(x_1,...,x_n) | x_1 \in S,...,x_n \in S\}.$$

Map

Definition. A map f is a rule that associates each element x in a set X to an element y in another set Y. We write the map in several ways:

$$y = f(x),$$

$$x \mapsto f(x),$$

$$f : X \to Y,$$

Sometimes "target space".

$$X \xrightarrow{f} Y .$$

We call the input x to a map the argument, and the output y of a map the value. We call the set X the domain of the map, and the set Y the codomain of the map.

Remarks. To define a map, we must specify a domain, a codomain, and a rule to associate each element in the domain to an element in the codomain.

We use the words "map", "mapping", and "function" interchangeably, and prefer the shortest word "map". On occasions, we may replace the word "map" with the word "functional", "form", "transformation", "operator", or "operation".

Perhaps it should be emphasized that the associated element in the codomain is unique but the elements in the domain can, in general, be mapped to the same element in codomain.

Usually, for a "functional" the codomain is the set of real numbers.

Image. Let A be a subset of the domain X. The elements in Y associated with all elements in A constitute a subset in the codomain Y. We call this subset the image of A under f, and write the subset as $f(A)$. Thus,

$$f(A)=\{f(x)|x \in A\}.$$

By the definition of a map, the image of an element the domain is always a single element in the codomain. o.k.

Preimage. Let B be a subset of the codomain Y. The elements in X associated with all elements in B constitute a subset in the domain X. We call the subset the preimage of B under f, and write the subset as $f^{-1}(B)$. Thus,

$$f^{-1}(B)=\{x|f(x)\in B\}.$$

The preimage of an element in the codomain is a subset of the domain and may contain any number of elements. o.k.

Injection. A map is injective (one-to-one) if it sends each element in the domain to a distinct element in the codomain. That is, for every two distinct elements a and b in X, $f(a)$ and $f(b)$ are distinct elements in Y.

Surjection. A map is surjective (onto) if it sends at least one element in the domain to every element in the codomain. That is, for every element y in Y, there is at least one element x in X to satisfy $y = f(x)$.

$\Longleftrightarrow Y = f(X)$

Bijection. A map is bijective (one-to-one and onto) if it is both injective and surjective.

Operation

Definition. An *operation* on a set is a map that associates every ordered pair of elements in the set to another element in the set. The set is said to be *closed under the operation.*

Thus, to specify an operation, we need to specify a set S, as well as a rule (i.e., a map) that combines any two elements in the set to give another element in the set. The domain of the map is the Cartesian product $S \times S$, and the codomain of the map is S.

Let a and b be two elements in S. An operation maps the order pair (a,b) to an element c in S. The operation is such a special map that we use a special notation. Instead of $c = f(a,b)$, we write

$$c = a * b.$$

This notation better reflects the nature of an operation: it comes between two elements, rather than before them.

Number Field

Linear algebra involves many sets. We specify each set by describing the properties of its elements. We study elements within each set, and map one set to another. We study families of maps.

Here we go with our first set. A set F is called a number field if the following conditions hold.

Adding two elements in F gives an element in F. To any two elements α and β in F there corresponds an element in F, written as $\alpha + \beta$, called the addition of α and β. The addition obeys the following rules.

1) Addition is commutative: $\alpha + \beta = \beta + \alpha$ for every α and β in F.

2) Addition is associative: $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for every α, β and γ in F.

3) There exists an element in F, called 0 (zero), such that $0 + \alpha = \alpha$ for every α in F.

4) For every α in F, there exists an element γ in F, such that $\alpha + \gamma = 0$. We also write $\gamma = -\alpha$.

This means that $(F,+)$ is a group.

Multiplying two elements in F gives an element in F. To any two elements α and β in F there corresponds an element in F, written as $\alpha\beta$, called the multiplication of α and β. The multiplication obeys the following rules.

5) Multiplication is commutative: $\alpha\beta = \beta\alpha$ for every α and β in F.

6) Multiplication is associative: $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for every α, β and γ in F.

7) There exists an element in F, called 1, such that $1 \cdot \alpha = \alpha$ for every α in F.

8) For every $\alpha \neq 0$ in F, there exists an element γ in F, such that $\alpha\gamma = 1$. We also write $\gamma = 1/\alpha$.

Usually, mathematicians mean something else by "counter example".
You have a statements and a counter example disproves it.

Multiplication is distributive over addition. A final rule involves both operations of addition and multiplication:

9) $\gamma(\alpha+\beta)=\gamma\alpha+\gamma\beta$ for every α, β and γ in F.

Remark. We call each element in the set F a number. Addition and multiplication are two distinct operations. Each operation turns two elements in F into an element in F, namely, $F\times F\to F$. That is, the set F is closed under the two operations. We need to memorize nothing new: the two operations follow the usual arithmetic rules of addition, subtraction, multiplication, and division.

Examples and Counterexamples

The smallest field. The definition of field explicitly mentions two elements 0 and 1. If we define addition by an unusual rule, 1 + 1 = 0, we can confirm that the set $\{0,1\}$ satisfies all the axioms of field.

Counterexample. The set of integers is not a field. Here we assume the usual operations of addition and multiplication. The set of integers violates Axiom 8.

Example. The set of rational numbers is a field. The set of real numbers is a field. The set of complex numbers is a field. Here we follow the usual arithmetic rules of addition, subtraction, multiplication, and division.

The field of complex numbers contains the field of real numbers. The field of real numbers contains the field of rational numbers.

Example. A set consists of all numbers of the form $\alpha+\beta\sqrt{2}$, where α and β are rational numbers. We adopt the usual definition of addition and multiplication for real numbers. This set is a number field. This field is contained in the field of real numbers.

Counterexample. A set consists of all numbers of the form $\alpha\sqrt{2}+\beta\sqrt{3}+\gamma\sqrt{5}$, where α, β and γ are rational numbers. We adopt the usual definition of addition and multiplication for real numbers. The multiplication of two elements in the set does not always give another element in the set. This set is not a number field.

Counterexample. Each element in a set is a piece of gold of some amount. We define the addition of two pieces in the set by melting them together, resulting in a piece in the set. However, we do not have a sensible definition of the multiplication that makes the multiplication of two pieces into another piece. This set is not a field.