# Waves

*Yet another play of two actors: inertia and elasticity*

**References.**

J.D. Achenbach, Wave propagation in elastic solids. North-Holland, Amsterdam (1973). Also see Achenbach's speech upon receiving the Timoshenko Medal (http://imechanica.org/node/185).

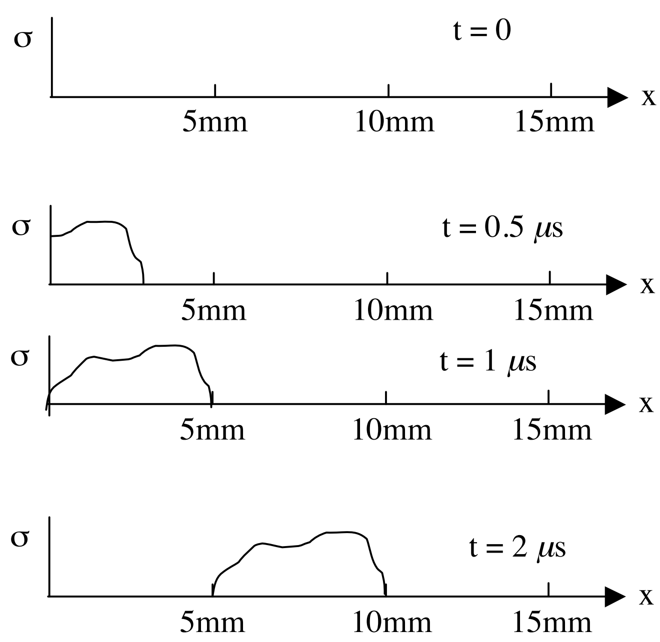
D. Royer and E. Dieulesaint, Elastic Waves in Solids, I and II. Springer, Berlin (2000).

**Light and sound**. A wave is a disturbance in motion. What is being disturbed? In what medium does the disturbance move? A comparison between light and sound is instructive.

- *What is being disturbed?* Light is a traveling disturbance of electric field and magnetic field. Sound is a traveling disturbance of stress field and displacement field.
- *Media*. Light can propagate in vacuum, as well as in certain materials. Sound must propagate in elastic media. Spring of air, liquid, solid. When a noisy watch is suspended in a glass jar with a thread. In the beginning, you can see the watch and hear its noise. When the air is sucked out of the jar, you can still see the watch, but cannot hear it.
- *Speeds*. The light wave speed is about $3\times 10^8$ m/s in vacuum. The sound speed is about 340 m/s in air, 1500 m/s in water, and 5000 m/s in steel.
- *Frequencies and wavelengths*. $f = c/\lambda$. Visible electromagnetic waves. Red: $\lambda = 0.7\mu\text{m}$, $f = 4.3\times 10^{14}$ Hz. Violet: $\lambda = 0.4\mu\text{m}$, $f = 7.5\times 10^{14}$ Hz. Audible sound waves. 20 Hz to 20 kHz. In air, they correspond to 17 m and 17 mm. In water, they correspond to 75 m and 7.5 cm.

**Ultrasound.** Ultrasound has frequencies too high to be detected by the human ear. Ultrasound can be generated using piezoelectric materials, which convert an electric field to a stress field. High frequencies correspond to short wavelengths.

- Ultrasound imaging: see baby inside mother.
- Non-destructive evaluation (NDE): detect flaws inside materials.
- Surface Acoustic Wave (SAW) devices for wireless applications.



**Longitudinal wave in a rod.** The speed of sound in steel is ~ 5 km/s. Imagine that a hammer hits one end of a steel rod. The hit starts at time zero, and lasts for 1 $\mu$s. A frame-by-frame "movie" shows the stress profiles at several times.

- Before time zero, these is no stress in a steel rod.
- At $t$ = 0, a hammer starts to hit the end of the rod.
- At $t$ = 0.5 $\mu$s, 2.5 mm of rod is under compression, but the rest of the rod is stress-free. The delay is caused by the inertia of the matter.
- At $t$ = 1 $\mu$s, 5 mm of rod is under compression, but the rest of the rod is stress-free.
- At $t$ = 2 $\mu$s, the same 5 mm wave packet travels for 5 mm. The rod behind and ahead of the packet is stress-free.

**Equations that govern an elastic rod moving along its axial direction**. So far we have appealed to our daily experience about waves. What do the equations say? Recall the equations that govern an elastic rod moving along it axial direction. Let the axial direction of the rod coincide with the $x$-direction. Name every material particle in the rod by the coordinate $x$ of the material particle when the rod is in the undeformed state. Relative to its place in the undeformed state, the material particle $x$ at time $t$ has displacement $u(x,t)$. The velocity of the material particle $x$ at time $t$ is

$$v = \frac{\partial u(x,t)}{\partial t}.$$

The material particle is under strain

$$\varepsilon = \frac{\partial u(x,t)}{\partial x}.$$

The rod is linearly elastic, so that the stress relates to the strain by Hooke's law:

$$\sigma = E\varepsilon,$$

where $E$ is Young's modulus, which is constant for all material particles and at all time. The balance of momentum requires that

$$E\frac{\partial \sigma(x,t)}{\partial x} = \rho \frac{\partial^2 u(x,t)}{\partial t^2},$$

where $\rho$ is the density, which is constant for all material particles and at all time.

**The D'Alambert solution**. A combination of the above equations gives

$$E\frac{\partial^2 u(x,t)}{\partial x^2} = \rho \frac{\partial^2 u(x,t)}{\partial t^2}.$$

This partial differential equation governs the function $u(x,t)$, and is known as the equation of motion. We can determine the field $u(x,t)$ by solving this PDE in conjunction with initial conditions and boundary conditions. For the time being, let us leave aside the initial and boundary conditions, and just look at the equation of motion by itself.

In the above, we have guessed that a wave can travel in the rod at a constant speed and maintaining an invariant waveform. We now write this guess in a mathematical form:

$$u(x,t) = f(x - ct),$$

where $c$ is the wave speed. At $t$ = 0, the displacement field is $u(x,0)=f(x)$. At time $t$, the displacement field is $u(x,t)=f(x-ct)$, which has the same shape as that at time $t$ = 0, but moves to the right by a distance $ct$. That is, the function $u(x,t)=f(x-ct)$ represents a fixed waveform traveling at speed $c$ to the right.

The displacement varies with both the material particle and time, $u(x,t)$. The expression $u(x,t)=f(x-ct)$, however, says that the displacement field in the rod is a function of a single variable, $\xi = x-ct$. What is the waveform $f(\xi)$ and the wave speed $c$? To find out, let us insert our guess into the equation of motion. Using the chain rule in differential calculus, we obtain that

$$\frac{\partial u(x,t)}{\partial x} = \frac{df(\xi)}{d\xi}\frac{\partial \xi(x,t)}{\partial x} = \frac{df(\xi)}{d\xi}$$

and

$$\frac{\partial u(x,t)}{\partial t} = \frac{df(\xi)}{d\xi}\frac{\partial \xi(x,t)}{\partial t} = -c\frac{df(\xi)}{d\xi}$$

Insert into the equation of motion, and we obtain that

$$E\frac{d^2 f}{d\xi^2} = \rho c^2 \frac{d^2 f}{d\xi^2}$$

Consequently, the equation of motion is satisfied by *any* function $f(\xi)$ provided that the wave speed is given by

$$c = (E/\rho)^{1/2}.$$

For example, for steel, $E = 210\mathrm{GPa}$ and $\rho = 7800\,\mathrm{kg/m^3}$. The above formula gives the speed of wave propagating in a rod of steel: $c \approx 5\,\mathrm{km/s}$. The solution has the two features of non-dispersive waves: the waves travel at a specific speed, and any arbitrary waveform remains unchanged as the waves travels.

We must differentiate two quantities: the velocity of a material particle and the wave speed. The velocity of the material particle $x$ at time $t$ is $v = \partial u(x,t)/\partial t$, so that

$$v(x,t) = -c\frac{df(\xi)}{d\xi}.$$

The stress is given by $\sigma = E\partial u(x,t)/\partial x$, so that

$$\sigma(x,t) = E\frac{df(\xi)}{d\xi}.$$

Note that for every material particle and at any time, the stress relates to the velocity as

$$\sigma = -Rv,$$

where

$$R = \sqrt{E\rho}.$$

The quantity $R$ is called the *acoustic impedance*.

Similarly, $g(x+ct)$ represents a fixed profile of displacement traveling at speed $c$ to the left. Let $\xi = x-ct$ and $\eta = x+ct$. Any functions $f(\xi)$ and $g(\eta)$ satisfy the equation of motion. Because the PDE is linear, it is also satisfied by the linear combination:

$$u(x,t) = f(x-ct) + g(x+ct).$$

The velocity and the stress are

$$v(x,t) = -c\frac{df(\xi)}{d\xi} + c\frac{dg(\eta)}{d\eta},$$

$$\sigma(x,t) = E\frac{df(\xi)}{d\xi} + E\frac{dg(\eta)}{d\eta}.$$

Note that $\sigma = -Rv$ for a wave propagates in the positive $x$ direction, and that $\sigma = +Rv$ for a wave propagates in the negative $x$ direction.

The field $u(x,t)$ is governed by

- A PDE (i.e., the equation of motion)
- Initial conditions (i.e., the displacement field and velocity field in the rod at time zero.)
- Boundary conditions (i.e., the displacement and the stress at the two ends of the rod)

In reaching the D'Alambert solution, we have used the PDE, but not the initial conditions and the boundary conditions. We next illustrate how the initial conditions and boundary conditions come into play.

**An initial-value problem**. Pull a long steel rod with two grips. Hold the forces constant before time zero. Release the grips at time zero. We'd like to find out the waves generated in the rod afterward.

Before the grips are released. The rod has a static stress field, $s(x)$. When the grips are released, at time zero, the stress field is still the same as the static field:

$$E\left[F(x) + G(x)\right] = s(x).$$

At time zero, there is no velocity, so that

$$c\left[-F(x) + G(x)\right] = 0.$$

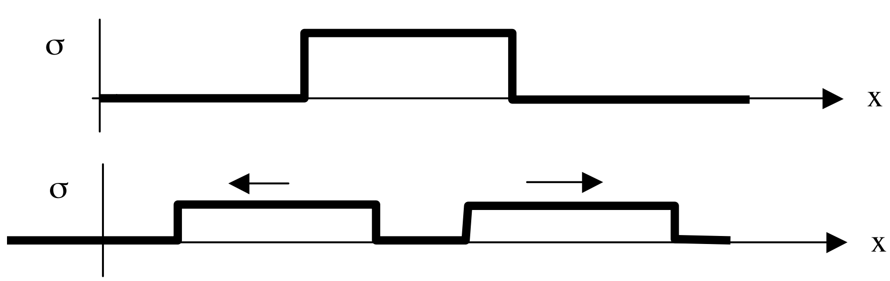
A combination of the two equations gives that

$$F(x) = G(x) = \frac{1}{2E}s(x).$$

The stress field afterward is

$$\sigma(x,t) = E\left[F(x-ct) + G(x+ct)\right]$$
$$= \frac{1}{2}\left[s(x-ct) + s(x+ct)\right]$$

After the grips are released, the static stress profile splits into two waves, one traveling to the right, and the other to the left. This solution is correct before either wave hits the end of the rod.




**Reflection from a free end.** We next illustrate how boundary conditions come into play. An incident wave in the rod hits the free end of the rod, and reflects back into the rod.

Let's say the rod lies on the axis, $x < 0$, and the free end is at $x = 0$. The incident wave travels from the left to the right. The boundary condition: Stress vanishes at all time at $x = 0$. An incident compressive wave, after reflection, becomes a tensile wave. Dynamic fracture.

How do we obtain the solution by doing algebra? We know the shape of the incident wave,

$$\sigma^I = s(x - ct).$$

That is, we know the function $s(\xi)$. The reflected wave travels from the right to the left. It must be of the form

$$\sigma^R = h(x + ct).$$

Its shape $h(\eta)$ is to be determined. The stress in the rod is the sum of the incident and the reflected wave:

$$\sigma(x,t) = s(x - ct) + h(x + ct).$$

How to determine the function $h(\eta)$? The boundary condition: at the free end $x = 0$, the stress is zero at all time, $\sigma(0,t) = 0$. Put this boundary condition to the general expression for the stress, and we have

$$0 = s(-ct) + h(ct).$$

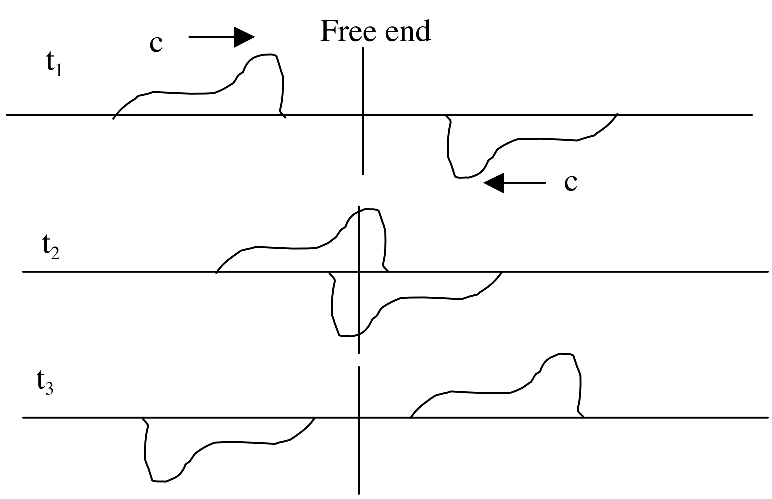
Denote $ct$ by $Q$. We have

$$h(Q) = -s(-Q).$$

That is, we have expressed the function $h$ in terms of the known function $s$. The independent variable can be anything. The stress in the rod is given by

$$\sigma(x,t) = s(x - ct) - s(-x - ct).$$

The first term is the incident wave, running in the positive $x$-direction. The second term is the reflected wave, running in the negative $x$-direction. At the free end, $x = 0$, the stress indeed vanishes at all time. If an incident wave induces a compressive stress in the rod, the reflected wave induces a tensile stress in the rod.




**Exercise**. Analyze the reflection from a fixed end.

**Standing waves.** A normal mode is a *standing wave*. For example, consider a rod fixed at one end and free to move on the other end. From the two boundary conditions, the standing wave of the longest wavelength has $\lambda = 4L$. Only a quarter of this wavelength is inside the rod. The frequency of the fundamental mode is given by $f = c/\lambda$, which recovers the result we obtained before:

$$\omega_1 = \frac{\pi}{2}\sqrt{\frac{E}{\rho}}.$$

We can understand other normal modes in the similar way.

**An alternative form of the D'Alambert solution** is

$$v(x,t) = -F\left(\frac{x}{c} - t\right) - G\left(-\frac{x}{c} - t\right),$$

$$\sigma(x,t) = RF\left(\frac{x}{c} - t\right) - RG\left(-\frac{x}{c} - t\right).$$

**Reflection and transmission at a joint of two rods of different materials.** Two rods are joined at $x = 0$. The two rods have the same cross-sectional area. The rod on the left-hand side has properties

$$E_1, \rho_1, c_1 = \left(E_1/\rho_1\right)^{1/2}, R_1 = \sqrt{E_1\rho_1}.$$

The rod on the right-hand side has properties

$$E_2, \rho_2,\ c_2 = \left(E_2/\rho_2\right)^{1/2}, R_2 = \sqrt{E_2\rho_2}.$$

An incident wave comes from rod 1 towards the joint. Upon hitting the joint, the wave is partly reflected back to rod 1, and partly transmitted into rod 2.

Let the incident wave be

$$v^I(x,t) = -F\left(\frac{x}{c_1} - t\right).$$

The function *f* is the incident waveform, and is known. We need to solve for the wave reflected into rod 1, and the wave transmitted into rod 2.

We expect that the reflected wave takes the form

$$v^R(x,t) = -aF\left(-\frac{x}{c_1} - t\right).$$

We interpret various pieces of this guess as follows:

- Both the velocity and the stress are continuous at all time across the joint $x = 0$, so that the reflected wave in rod 1 should take a waveform similar to the waveform of the incident wave, $F(\cdot)$.
- The amplitude of the reflected wave can be different from that of the incident wave, so we multiply a dimensionless number $a$.
- The reflected wave runs in the negative $x$-direction, so we place the negative sign in front of $x$.
- The reflected wave runs in rod 1 at the wave speed $c_1$.

Everything about the reflected wave is known, except for the dimensionless number $a$.

In rod 1, the net field is the superposition of the incident wave and the reflected wave:

$$v_1(x,t) = -F\left(\frac{x}{c_1} - t\right) - aF\left(-\frac{x}{c_1} - t\right),$$

$$\sigma_1(x,t) = R_1 F\left(\frac{x}{c_1} - t\right) - R_1 aF\left(-\frac{x}{c_1} - t\right).$$

Following a similar line of reasons, we expect that the transmitted wave in rod takes the form

$$v_2(x,t) = -bF\left(\frac{x}{c_2} - t\right),$$

$$\sigma_2(x,t) = R_2 bF\left(\frac{x}{c_2} - t\right).$$

Everything about the transmitted wave is known, except for the dimensionless number *b*, which is the ratio of the amplitude of the transmitted wave over the amplitude of the incident wave.

To determine the two numbers, *a* and *b*, we invoke the boundary conditions: at the joint $x = 0$ and for all time, the velocity of the material particle in rod 1 equals that in rod 2, and the stress in rod 1 equals that in rod 2. Thus,

$$1 + a = b$$

$$R_1(1 - a) = R_2 b$$

This set of linear algebraic equations is solved to give the two dimensionless numbers:

$$a = \frac{R_1 - R_2}{R_1 + R_2}, \quad b = \frac{2R_1}{R_1 + R_2}.$$

Note several special cases:

- When the two rods have the same impedance, $R_1 = R_2$, the two numbers become $a = 0$ and $b = 1$. When the incident wave from rod 1 hits the joint, the wave does not reflect, but fully transmits into rod 2. The two rods are said to have matched impedance.
- When rod 2 has much lower impedance than rod 1, $R_2 / R_1 << 1$, the two numbers become $a = 1$ and $b = 2$. The force transmitted to rod 2 is vanishingly small, so that all the energy of the incident wave in rod 1 will be reflected back into rod 1. The force of the reflected wave has the sign opposite from that of the incident wave.
- When rod 2 has much higher impedance than rod 1, $R_2 / R_1 >> 1$, the two numbers become $a = -1$ and $b = 0$. The displacement transmitted to rod 2 is vanishingly small, so that all the energy of the incident wave in rod 1 will be reflected back into rod 1. The reflected wave keeps the sign of the force as that of the incident wave.

**Dynamics of bending**. We next consider bending of a beam. Let *x* be the coordinate of a cross section when the beam is in the reference configuration, i.e., when the beam is not bent. In the current state, the deflection of the beam is $w(x,t)$, the slope is

$$\theta = \frac{\partial w(x,t)}{\partial x},$$

and the curvature is

$$K = \frac{\partial \theta(x,t)}{\partial x}.$$

The deflection, slope, and curvature specify the deformation geometry.

The material model of the beam is specified by a relation between the curvature and the bending moment *M*, namely,

$$M = EIK ,$$

where *E* is Young's modulus, and *I* is the second moment of the cross section. The quantity $EI$ is the bending stiffness.

Next consider an element of the beam between *x* and $x + dx$. The balance of moment of the element dictates that

$$S = -\frac{\partial M(x,t)}{\partial x},$$

where $S(x,t)$ is the shear force acting on the cross section *x* at time *t*. Applying Newton's second law in the direction of deflection, we obtain that

$$\frac{\partial S(x,t)}{\partial x} = \rho A \frac{\partial^2 w(x,t)}{\partial t^2},$$

where $\rho$ is the mass density, and *A* the area of the cross section.

A combination of the above equations gives

$$EI \frac{\partial^4 w(x,t)}{\partial x^4} = -\rho A \frac{\partial^2 w(x,t)}{\partial t^2}.$$

This is the equation of motion for the deflection $w(x,t)$. This PDE, together with initial and boundary conditions, determines $w(x,t)$. Once $w(x,t)$ is determined, we can calculate other quantities such as the bending moment and shear force. For example, we have studied the vibration of the beam in a homework problem.

**Bending wave.** We now study a special kind of dynamic behavior of a beam: a wave traveling in the beam with a constant speed and an invariant waveform:

$$w(x,t) = f(\xi), \quad \xi = x - ct .$$

Both the wave speed *c* and the waveform *f* are unknown.

Inserting this guess into the PDE, we obtain that

$$EI \frac{d^4 f(\xi)}{d\xi^4} = -\rho A c^2 \frac{d^2 f(\xi)}{d\xi^2} .$$

This ODE has the general solution

$$f(\xi) = a_1 \sin\left(\sqrt{\frac{\rho A}{EI}} c\xi\right) + a_2 \cos\left(\sqrt{\frac{\rho A}{EI}} c\xi\right) + a_3 \xi + a_4 ,$$

where $a_1, a_2, a_3, a_4$ are constants.

We will drop the rigid body motion $a_3 \xi + a_4$, and focus one of the sinusoidal motion:

$$f(\xi) = a \sin\left(\sqrt{\frac{\rho A}{EI}} c\xi\right).$$

To interpret this solution, we write

$$w(x,t) = a \sin(kx - \omega t),$$

where *k* is the wave number, and $\omega$ the frequency. They relate to the wave speed *c* by

$$k = c\sqrt{\frac{\rho A}{EI}}, \quad \omega = c^2 \sqrt{\frac{\rho A}{EI}} .$$

Thus, for a bending wave to travel at a constant speed and invariant waveform, the speed *c* can be arbitrary, and the waveform must be sinusoidal. Once a speed *c* is given, so are the wave number and the frequency.

Observe that the wave speed increases with the wave number, and becomes infinite for very short wavelengths. This is clearly an artifact of our model. The equation of motion is based on the classical theory of beams. The theory breaks down when the wavelength approaches the dimension of the cross section.

**Dispersive wave.** For a non-sinusoidal wave to travel in the beam, the wave may be a sum of many sinusoidal waves. Each sinusoidal wave has its own wave number and frequency, and travels at its own velocity. Consequently, as the non-sinusoidal wave travels, the waveform will change. A wave whose speed varies with its wave number is known as a dispersive wave. By contrast, the longitudinal wave in a rod is nondispersive, so that an arbitrary waveform can propagate in the rod without change.

Let us look at the sinusoidal wave again:

$$w(x,t) = a\sin(kx - \omega t).$$

The wave speed is given by $c = \omega / k$. For a nondispersive wave, the wave speed is constant, independent of the wave number, so that the frequency $\omega$ must be linear in the wave number *k*. For a dispersive wave, however, the wave speed varies with the wave number, so that the frequency is a *nonlinear function* of the wave number, $\omega(k)$. This function is known as the **dispersion relation**. For the bending wave, for example, the dispersion relation can be obtained by eliminating *c* from the $k - c$ and $\omega - c$ relations, so that

$$\omega(k) = k^2\sqrt{\frac{EI}{\rho A}}.$$

This is the dispersion relation for the bending wave.

**Phase velocity.** For a dispersive wave, it is useful to distinguish two kinds of velocities: phase velocity and group velocity. The speed of the sinusoidal wave

$$c_p(k) = \frac{\omega(k)}{k}$$

is known as the phase velocity. For example, the phase velocity of the pure sinusoidal wave in the beam is

$$c_p(k) = \sqrt{\frac{EI}{\rho A}}k.$$

**Group velocity.** The group velocity is defined as

$$c_g(k) = \frac{d\omega(k)}{dk}.$$

For the wave in the beam, the group velocity is

$$c_g(k) = 2\sqrt{\frac{EI}{\rho A}}k.$$

The definition of the group velocity is motivated as follows. Consider two sinusoidal waves with the same amplitude but slightly different wave numbers:

$$a\sin(k_1x - \omega_1 t), \quad a\sin(k_2 x - \omega_2 t).$$

The overall response is the superposition of the two waves:

$$w(x,t) = a\sin(k_1 x - \omega_1 t) + a\sin(k_2 x - \omega_2 t)$$
$$= 2a\cos\left(\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right)\sin\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)$$

The combined wave is a wave of the average wave number $(k_1 + k_2)/2$, modulated by a wave of a smaller wave number $(k_1 - k_2)/2$ (i.e., a longer wave length). The former wave is called the carrier, and the latter the group. The group propagates at the velocity

$$\frac{\omega_1 - \omega_2}{k_1 - k_2} \approx \frac{d\omega(k)}{dk}.$$

Sketch a carrier wave modulated by a group wave.

**Plane waves in a 3D homogeneous, elastic solid.** Consider a wave traveling in a body. At a given time, the wave is localized in a region in the body. When size of the region affected by the wave is small compared to size of the body in all three directions, we may as well regard the body as an infinite body. A material particle in the body is unaffected before the wave arrives. Such a wave is known as a bulk wave.

We now consider a particular kind of bulk wave: a plane wave, characterized by two directions:

1. **The direction of propagation**. The wave propagates in a fixed direction at all time.
2. **The direction of displacement**. All material particles move in the same direction at all time.

Because the governing equations have no length scale, a plane wave in an infinite body is non-dispersive. At a given time, the amplitude of the displacement is the same for all material particles in any plane normal to the direction of propagation.

**Longitudinal wave in an isotropic, homogeneous, elastic solid**. An isotropic elastic solid supports plane waves of two types, longitudinal waves and transverse waves. For a longitudinal wave, the direction of displacement coincides with the direction of propagation. For a transverse wave, the direction of displacement is normal to the direction of propagation. We consider the longitudinal wave in this lecture, and leave the transverse wave as a homework problem.

*Deformation geometry*. Because all directions are equivalent in an isotropic material, we only need to consider plane waves propagating in one direction, denoted by $x$. For a longitudinal wave, every material particle moves in the same direction as the direction of wave propagation. Thus, at time $t$ all the material particles in the plane of coordinate $x$ have the same displacement $u(x,t)$. Consequently, the only nonzero strain component is

$$\varepsilon_x = \frac{\partial u(x,t)}{\partial x}.$$

That is, the solid is in a state of *uniaxial strain*.

*Material model*. Because of Poisson's effect, this strain induces normal stresses in all three directions. Symmetry dictates that $\sigma_y = \sigma_z$. Writing Hooke's law in the $y$-direction, we obtain that

$$\varepsilon_y = 0 = \frac{1}{E}\left(\sigma_y - \nu\sigma_z - \nu\sigma_x\right),$$

so that

$$\sigma_y = \sigma_z = \frac{\nu}{1-\nu}\sigma_x.$$

Writing Hooke's law in the $x$-direction, we obtain that

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right),$$

or

$$\sigma_x = \frac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)}\varepsilon_x .$$

The coefficient is the stiffness under the uniaxial strain conditions.

*Newton's second law* reduces to

$$\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} .$$

Putting the three ingredients together, we obtain the equation of motion:

$$\frac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)}\frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} .$$

Except for the coefficient on the left-hand side, this equation of motion is identical to that for the rod. Thus, the same solution procedure applies. The longitudinal wave speed is

$$c_l = \sqrt{\frac{E\left(1-\nu\right)}{\rho\left(1+\nu\right)\left(1-2\nu\right)}} .$$

Take a representative value of Poisson's ratio, $\nu = 0.3$, and we obtain that

$$c_l = 1.2\sqrt{E/\rho} .$$

Recall that the wave speed in a rod is $\sqrt{E/\rho}$. Let us try to understand when the longitudinal wave in a rod is slower than the longitudinal wave in a 3D solid. For the longitudinal wave in a rod, we assume that the rod is in a state of uniaxial stress. This assumption is reasonable when the length scale of the wave is large compared to the cross section of the rod. Under this condition, the rod can deform freely in the transverse direction. The stress-strain relation is

$$\sigma_x = E\varepsilon_x$$

For the longitudinal wave in the 3D solid, the transverse deformation is prohibited. This constraint effectively stiffen the stress-strain relation, which becomes:

$$\sigma_x = \frac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)}\varepsilon_x .$$

Consequently, the longitudinal wave in a rod is slower than the longitudinal wave in a three-dimensional solid.

**Transverse wave in an isotropic elastic solid**. For a transverse wave, all material particles move in a direction normal to the direction of the wave propagation. Thus, the displacement field takes the form $v\left(x,t\right)$. Following the same procedure, we find that the transverse wave speed is

$$c_t = \sqrt{\frac{E}{2\rho\left(1+\nu\right)}} .$$

Take a representative value of Poisson's ratio, $\nu = 0.3$, and we obtain that

$$c_t = 0.6\sqrt{E/\rho} .$$

Thus, the longitudinal wave is about twice as fast as the transverse wave.

**The general form of a plane wave in an isotropic elastic solid.** For a longitudinal wave, the time-dependent field of displacement is

$$\mathbf{u}(\mathbf{x},t) = \mathbf{s}f(\xi),$$

where

$$\xi = \frac{\mathbf{s}\cdot\mathbf{x}}{c_l} - t,$$

and

$$c_l = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}}.$$

We interpret various aspects of this form as follows:

- **s** is the unit vector pointing in the direction of propagation.
- In the three dimensional space, the distance measured along the direction of propagation **s** is given by the inner product $\mathbf{s}\cdot\mathbf{x}$. Thus, the variable $\xi = \frac{\mathbf{s}\cdot\mathbf{x}}{c_l} - t$ for the plane wave in three dimensions generalizes $\xi = \frac{x}{c} - t$ for the wave in one dimension.
- The plane wave is nondispersive, so that the waveform can be an arbitrary function $f(\cdot)$. This waveform is a scalar, representing the magnitude of the field of the displacement vector. At a fixed time, the magnitude of the displacement field varies for particles in the direction **s**, but is constant for particles in any plane normal to **s**.
- For the longitudinal wave, the direction of the field of displacement vector in the entire body coincides with the direction of propagation, so we multiply $\mathbf{s}$ in the front.
- For an isotropic material, the speed of longitudinal wave, $c_l$, is the same for any direction of propagation.

For a transverse wave, the field of displacement is

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}g\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c_t} - t\right), \quad c_t = \sqrt{\frac{E}{2\rho(1+\nu)}}$$

where **s** is the unit vector pointing in the direction of propagation, **a** is any unit vector normal to **s**, $c_t$ is the speed of the transverse wave, and $g$ is the waveform of displacement.

The general form of a plane wave propagates in direction **s** in an infinite isotropic material is the superposition of three waves:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{s}f\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c_l} - t\right) + \mathbf{a}g\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c_t} - t\right) + \mathbf{b}h\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c_t} - t\right),$$

where **a** and **b** are unit vectors normal to **s**. Vectors **a** and **b** are also normal to each other. In time, the field of displacement will split into two lumps: the longitudinal wave runs faster than the transverse wave.

**Exercise**. A transverse wave propagates in an isotropic material. At any given material particle, it is observed that the magnitude of the displacement is time-independent, but the direction of the displacement rotates at frequency $\omega$ . Write a possible displacement field.

**Exercise.** In an isotropic material, a wave can radiate from a source with spherical symmetry. Consider wave of the form

$$u(r,t) = \frac{1}{r} f\left(\frac{r}{c} - t\right).$$

The field has the spherical symmetry. Motivate the $1/r$ dependence using energy consideration. Show that this displacement field satisfies the equation of motion. Confirm that the speed of the spherical wave is the same as that of longitudinal plane wave.

**Anisotropic, homogeneous, elastic solid.** Recall the three ingredients of solid mechanics. As a material model, we assumes that the stress is linear in strain:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}.$$

The solid is homogeneous and purely elastic, so that the stiffness tensor remains constant for all material particles and at all time. The stiffness tensor is defined by the energy density as a quadratic form of the strain tensor:

$$W = \frac{1}{2} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}.$$

For a generally anisotropic solid, the stiffness tensor has symmetries of two kinds:

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{klij}.$$

The stiffness tensor is also positive-definite. That is

$$C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > 0.$$

for any non-zero strain tensor.

The strain relates to the displacement by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The momentum balance requires that

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}.$$

A combination of the three ingredients leads to the equation of motion

$$C_{ijkl}\frac{\partial^2 u_k}{\partial x_l \partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}.$$

This is a set of three PDEs for the displacement field $\mathbf{u}(\mathbf{x},t)$.

**Plane waves.** For a plane wave in the body, the displacement field takes the form

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a} f(\xi), \quad \xi = \frac{\mathbf{x}\cdot\mathbf{s}}{c} - t,$$

where $\mathbf{s}$ is the unit vector in the direction of propagation, $\mathbf{a}$ the unit vector in the direction of displacement, $c$ the wave speed, and $f$ the waveform. In an anisotropic material, for a given direction of propagation $\mathbf{s}$, the direction of displacement $\mathbf{a}$ may neither coincide with the direction of propagation nor be perpendicular to the direction of propagation. The direction of displacement, $\mathbf{a}$, is determined as a part of the solution to the equation of motion.

Insert the expression of the plane wave into the equation of motion, and we obtain that

$$C_{ijkl}s_l s_j a_k = \rho c^2 a_i.$$

The second-rank tensor, $C_{ijkl}s_l s_j$, is known as the acoustic tensor, which depends on both the stiffness tensor and the direction of propagation. Given the direction of propagation $\mathbf{s}$, the above equation defines an eigenvalue problem of the tensor $C_{ijkl}s_l s_j$, with $\rho c^2$ being the eigenvalue, and $\mathbf{a}$ the eigenvector.

The acoustic tensor $C_{ijkl}s_l s_j$ is symmetric and positive-definite. To confirm that the acoustic tensor is symmetric, namely, $C_{ijkl}s_l s_j = C_{kjil}s_l s_j$, recall a property of the stiffness tensor: $C_{ijkl} = C_{klij}$. To confirm that the acoustic tensor is positive-definite, we need to show that $C_{ijkl}s_l s_j a_i a_k > 0$ for any non-zero vectors **s** and **a**. The nine products $a_i s_j$ are components of a second rank tensor. This tensor in general is not symmetric. Write the symmetric part of the tensor as $\beta_{ij} = \left(a_i s_j + a_j s_i\right)/2$. Recall the properties of the stiffness tensor, $C_{ijkl} = C_{ijlk}$ and $C_{ijkl} = C_{jikl}$. We conclude $C_{ijkl}s_l s_j a_i a_k = C_{ijkl}\beta_{ij}\beta_{kl}$. Further recall a property of the stiffness tensor: $C_{ijkl}\beta_{ij}\beta_{kl} > 0$ for any non-zero symmetric tensor $\beta_{ij}$.

According to linear algebra, a symmetric, positive-definite, second-rank tensor has three real and positive eigenvalues. Associated with each eigenvalue is an eigenvector, $\mathbf{a}$, determined up to a scalar; without loss of generality, we will set each eigenvector to be a unit vector. The three eigenvectors associated with the three eigenvalues are orthogonal to one another. The three eigenvectors, however, in general are neither along nor normal to the direction of preparation **s**. That is, for an anisotropic material and an arbitrary direction of propagation, each plane wave is neither longitudinal wave nor transverse wave.

Thus, associated with a direction of propagation **s**, there exist plane waves of three types. Their speeds, $c', c'', c'''$, and their directions of displacement, $\mathbf{a}', \mathbf{a}'', \mathbf{a}'''$, are the eigenvalues and the eigenvectors of the acoustic tensor $C_{ijkl}s_l s_j$. By liner superposition, the displacement field of a plane wave propagates in direction **s** takes the general form

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}' f\left(\frac{\mathbf{x}\cdot\mathbf{s}}{c'} - t\right) + \mathbf{a}'' g\left(\frac{\mathbf{x}\cdot\mathbf{s}}{c''} - t\right) + \mathbf{a}''' h\left(\frac{\mathbf{x}\cdot\mathbf{s}}{c'''} - t\right).$$

Each of the three functions, $f, g, h$, is a function of a single variable. Each function represents the waveform of a plane wave. These functions are undetermined by the equation of motion, but by initial conditions.

**Exercise**. For a cubic crystal, longitudinal waves can propagate in the directions of symmetry, such as an edge of the cube, a diagonal of a face, and a diagonal of the cube. Calculate the speeds of the longitudinal waves in these three directions for copper.

**Exercise**. Determine the acoustic tensor of an isotropic material. Solve the eigenvalue problem of this acoustic tensor. Relate your results to the longitudinal and transverse waves.

**Stress**. Consider a plane wave in a body:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a} f(\xi), \quad \xi = \frac{\mathbf{x}\cdot\mathbf{s}}{c} - t.$$

where **a** is the unit vector in the direction of displacement, **s** the unit vector in the direction of propagation, *c* the wave speed, and $f(\xi)$ the waveform.

The velocity of the material particle **x** at time *t* is

$$\mathbf{v} = \frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}.$$

A direct calculation gives that

$$\mathbf{v}(\mathbf{x},t) = -\mathbf{a}\frac{df(\xi)}{d\xi}.$$

At all time, every material particle moves in the direction in parallel with **a**.

The stress field is given by

$$\sigma_{ij} = C_{ijpq}\frac{\partial u_p(\mathbf{x},t)}{\partial x_q}.$$

A direct calculation gives that

$$\sigma_{ij} = \frac{C_{ijpq}s_q a_p}{c}\frac{\partial f(\xi)}{\partial \xi}.$$

In the material consider a plane normal to a unit vector **n**. The traction on the plane is given by

$$t_i = \sigma_{ij}n_j.$$

A direct calculation gives that

$$t_i = \frac{C_{ijpq}n_j a_p s_q}{c}\frac{df(\xi)}{d\xi}.$$

Denote the vector

$$b_i = \frac{C_{ijpq}n_j a_p s_q}{c}.$$

We can write the traction in the form

$$\mathbf{t} = \mathbf{b}\frac{df(\xi)}{d\xi}.$$

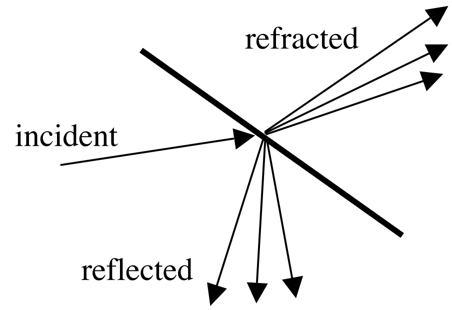
**Slowness**. The vector $\mathbf{s}/c$ is known as the slowness vector. Because **s** is a unit vector, the amplitude of the slowness vector is $1/c$. For a given unit vector **s**, we find the speeds $c', c'', c'''$ of the three plane waves propagating in the direction **s**. Start from an origin, draw the three slowness vectors, $\mathbf{s}/c', \mathbf{s}/c'', \mathbf{s}/c'''$. Repeat this procedure for unit vector **s** in every direction. The tips of the slowness vectors form three surfaces, known as the **slowness diagram**. For example, for an isotropic solid, the slowness diagram consists of two spheres, radii $1/c_l$ and $1/c_t$. The latter is degenerated from two spheres with the same radius. For an anisotropic solid, the surfaces need not be spheres.

**Exercise**. Plot the slowness diagram for single crystal copper. You can choose to plot the slowness diagram in several cross sections.

**Reflection and transmission.** Two materials, each filling a half space, are bonded on a planar interface normal to unit vector **n**. To avoid any discussion involving material symmetry, we assume that both materials are generally anisotropic. A plane wave propagates in one material and incident upon the interface. The incident wave takes the form

$$\mathbf{u}^I = \mathbf{a}f\left(\frac{\mathbf{x}\cdot\mathbf{s}}{c} - t\right),$$

where **a** is the unit vector in the direction of displacement, **s** the unit vector in the direction of propagation, $c$ the wave speed, and $f(\xi)$ the waveform. The incident wave generates three reflected waves and three transmitted waves. We expect that all six waves are plane waves, and will determine their speeds, directions of propagation, and waveforms.




**Exercise**. When the direction of the incident wave is normal to the interface, the incident wave may induce reflected waves all three types, and transmitted waves of all three types. All the six waves propagate in the direction normal to the interface, **n**. Given the waveform of the incident wave, determine the waveforms of the all six reflected and transmitted waves.

**Speeds and directions of propagation of the reflected and transmitted waves**. We next consider the general case when the direction of the incident wave, **s**, differs from the normal direction of the interface, **n**. When the two vectors **s** and **n** are not in the same direction, they form a plane, known as the **plane of incidence**. A given incident wave may cause reflected waves of three types, and transmitted waves of three types. For each of the six waves, we need to determine its direction of propagation, direction of displacement, the wave speed and the waveform, such that the displacements and the tractions in the two materials are continuous across the interface at all time. We place the origin of **x** on the interface.

We make a guess: each of the six waves takes a similar waveform as that of the incident wave. That is, each one of the six waves takes the form

$$\mathbf{u}' = A'\mathbf{a}'f\left(\frac{\mathbf{s}'\cdot\mathbf{x}}{c'} - t\right).$$

For this wave, $\mathbf{a}'$ is the unit vector in the direction of displacement, $\mathbf{s}'$ the unit vector in the direction of propagation, and $c'$ the wave speed. This wave has the same waveform $f(\xi)$, except for the amplitude of the displacement; the scalar $A'$ is to be determined. We verify this guess by finding the six waves that satisfy the boundary conditions: the continuity of displacement and traction at the interface.

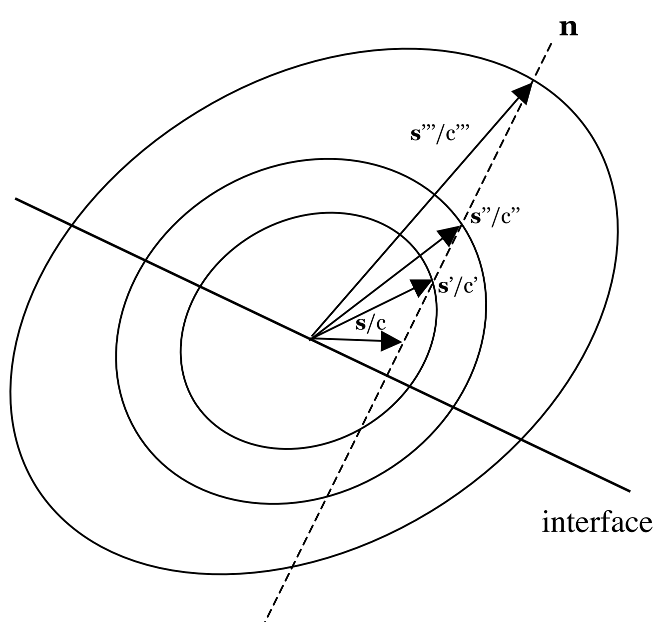
To satisfy these boundary conditions, for all **x** on the interface we make the two arguments of the waveform equal:

$$\frac{\mathbf{s}'\cdot\mathbf{x}}{c'} = \frac{\mathbf{s}\cdot\mathbf{x}}{c}.$$

Rewrite this equation as

$$\left(\frac{\mathbf{s}'}{c'} - \frac{\mathbf{s}}{c}\right)\cdot\mathbf{x} = 0.$$

This equation holds for all **x** on the interface. Thus, the vector $\frac{\mathbf{s}'}{c'} - \frac{\mathbf{s}}{c}$ is perpendicular to the interface, namely, is in the direction of **n**. Consequently, the three vectors, **n**, **s**, and $\mathbf{s}'$ are in the same plane. This conclusion holds



for all six reflected and transmitted waves, so that all the six reflected and transmitted waves propagate in directions that lie in in the plane of incidence.

The directions of propagation of the reflected and transmitted waves, as well as their speeds, may be determined by using the slowness diagram. The figure illustrates the procedure to determine the slowness vectors of the three transmitted waves, using the following information:

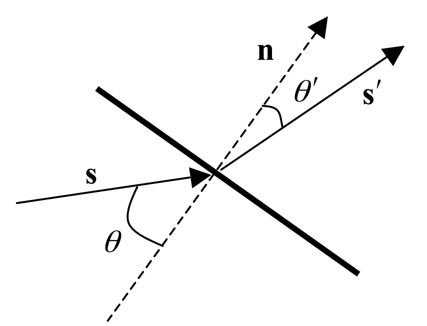
- the slowness diagram of the crystal of the transmitted waves,
- the normal direction of the interface, **n**, and
- the slowness vector of the incident wave, $\mathbf{s}/c$.

This procedure determines the slowness vectors of the three transmitted waves, $\mathbf{s}'/c', \mathbf{s}''/c'', \mathbf{s}'''/c'''$.

From this geometric construction, it is evident that when the speed $c$ of the incident wave is small, i.e., $1/c$ is large, there may be fewer or no refracted waves. The condition of no refracted wave is known as **total reflection**. Also evident from the geometric construction that the condition of the total reflection also depends on the direction of propagation of the incident wave.

Because the vector $\frac{\mathbf{s}'}{c'} - \frac{\mathbf{s}}{c}$ is normal to the interface, the projection of this vector on the interface must vanish, namely,

$$\frac{\sin\theta'}{c'} = \frac{\sin\theta}{c}.$$




where $\theta$ and $\theta'$ are the angles from the normal vector of the interface to $\mathbf{s}$ and $\mathbf{s}'$. This equation is known as **Snell's law**, which determines the directions of the reflected and refracted waves.

The arguments we have used are quite general, so that Snell's law is applicable to many waves, e.g., elastic waves and electromagnetic waves.

**Amplitudes of waves reflected from a free surface.** A solid fills a half space, with no traction acting on its surface. A known wave

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}f\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c} - t\right).$$

is incident upon the surface of the half space. Here $\mathbf{a}$ is the unit vector in the direction of displacement, $\mathbf{s}$ is the unit vector in the direction of propagation, $c$ is the wave speed, and $f(\cdot)$ is the waveform.

Waves of three types reflect back into the solid. For one of the reflected waves, let $c'$ be the speed, $\mathbf{s}'$ be the direction of propagation, and $\mathbf{a}'$ the direction of displacement. The waveform of the reflected must be similar to that of the incident wave, $f(\cdot)$. Thus, the displacement field associated with this reflected wave is

$$A'\mathbf{a}'f\left(\frac{\mathbf{s}'\cdot\mathbf{x}}{c'} - t\right),$$

where $A'$ is the ratio of the amplitude of the reflected wave over the amplitude of the incident wave. Similarly, let $(c'', \mathbf{s}'', \mathbf{a}'', A'')$ and $(c''', \mathbf{s}''', \mathbf{a}''', A''')$ be the corresponding quantities for the other two reflected waves. The net displacement field in the solid is the linear superposition of the four waves:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}f\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c} - t\right) + A'\mathbf{a}'f\left(\frac{\mathbf{s}'\cdot\mathbf{x}}{c'} - t\right) + A''\mathbf{a}''f\left(\frac{\mathbf{s}''\cdot\mathbf{x}}{c''} - t\right) + A'''\mathbf{a}'''f\left(\frac{\mathbf{s}'''\cdot\mathbf{x}}{c'''} - t\right)$$

Everything else about the reflected waves has been determined except for the amplitudes $A', A'', A'''$.

To determine $A', A'', A'''$, we use the traction free boundary conditions. Let $\mathbf{n}$ be the unit vector normal to the surface of the half space. The traction vector on the surface is

$$\mathbf{t} = \mathbf{b}f\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c} - t\right) + A'\mathbf{b}'f\left(\frac{\mathbf{s}'\cdot\mathbf{x}}{c'} - t\right) + A''\mathbf{b}''f\left(\frac{\mathbf{s}''\cdot\mathbf{x}}{c''} - t\right) + A'''\mathbf{b}'''f\left(\frac{\mathbf{s}'''\cdot\mathbf{x}}{c'''} - t\right).$$

As described before, the vector $\mathbf{b}$ is calculated from

$$b_i = \frac{C_{ijpq}n_j a_p s_q}{c}.$$

We can similarly calculate the corresponding vectors for the reflected waves. When $\mathbf{x}$ is on the surface of the solid, all the arguments of the function are equal. The traction-free condition $\mathbf{t} = \mathbf{o}$ gives that

$$\mathbf{b} + A'\mathbf{b}' + A''\mathbf{b}'' + A'''\mathbf{b}''' = \mathbf{o}.$$

This set of three linear algebraic equations determines the amplitude $A', A'', A'''$.

**Exercise.** A wave incident upon the interface between two materials generates three reflected waves and three refracted waves. The amplitudes of the six waves are determined by the continuity of the displacement vector and the traction vector. Establish a set of six equations that determine the six amplitudes. Use the impedance tensors in the final result.

**Reflection from a free surface of an isotropic material**. If the incident wave is a transverse wave with the displacement vector perpendicular to the plane of incidence, the wave is entirely reflected as a wave of the same kind, and the amplitude is 1.

If the incident wave is a transverse wave with the displacement vector in the plane of incidence, the reflected consists of a transverse wave of the same kind and a longitudinal wave. The amplitudes of the two reflected waves are

$$A_t = \frac{\sin 2\theta_l \sin 2\theta - (c_l / c_t)^2 \cos^2 2\theta}{\sin 2\theta_l \tan 2\theta + (c_l / c_t)^2 \cos^2 2\theta}, \quad A_l = \frac{2(c_l / c_t)\sin 2\theta \cos 2\theta}{\sin 2\theta_l \sin 2\theta + (c_l / c_t)^2 \cos^2 2\theta}.$$

If the incident wave is a longitudinal wave, the reflected wave consists of a longitudinal wave and a transverse wave with the displacement vector in the plane of incidence. The amplitudes of the two reflected waves are

$$A_l = \frac{\sin 2\theta_t \sin 2\theta - (c_l / c_t)^2 \cos^2 2\theta_t}{\sin 2\theta_t \sin 2\theta + (c_l / c_t)^2 \cos^2 2\theta_t}, \quad A_t = -\frac{2(c_l / c_t)\sin 2\theta \cos 2\theta_t}{\sin 2\theta_t \sin 2\theta + (c_l / c_t)^2 \cos^2 2\theta_t}.$$

**Surface waves**. Many features of surface waves can be described with little mathematics.

- Opening sentences of Rayleigh (1885). "It is proposed to investigate the behavior of waves upon the plane free surface of an infinite homogeneous isotropic elastic solid, their character being such that the disturbance is confined to a superficial region, of thickness comparable with the wavelength."
- Closing sentences of Rayleigh (1885). "It is not impossible that the surface waves here investigated play an important part in earthquakes, and in the collision of elastic solids.

Diverging in two dimensions only, they must acquire at a great distance from the source a continually increasing preponderance."

- *Direction of propagation*. Surface wave can propagate with a curved wave front. We will focus on surface waves propagating in a fix direction, with a straight wave front. Such waves cause plane strain conditions: The displacement vector is in the plane defined by the vector normal to the surface and the direction of propagation.
- *Waveform*. The half space lacks length scale. The surface wave is non-dispersive: a wave of any arbitrary waveform propagates without changing the waveform.
- *Wave speed*. For an isotropic material, the surface wave speed is the same for any direction of propagation. A dimensional consideration indicates that the surface wave (i.e., the Rayleigh wave) speed should take the form $v_R \sim \sqrt{\mu / \rho}$. The dimensionless pre-factor depends on Poisson's ratio.
- *Motion of material particles*. For a plane wave in the bulk of an elastic solid, the direction of displacement is fixed for all material particles at all time. For a surface wave, each material particle moves along a curve.
- Because the acoustic wave speed is much lower than the light wave speed, acoustic waves have been used in analog signal processing for wireless communication.

**References on surface waves**.


- Lord Rayleigh, On waves propagated along the pane surface of an elastic solid. The Proceedings of London Mathematical Society 17, 4-11 (1885).
- R.M. White and F.W. Voltmer, Direct piezoelectric coupling to surface elastic waves. Applied Physics Letters 7, 314-317 (1965)
- D.M. Barnett, Bulk, surface, and interfacial waves in anisotropic linear elastic solids. International Journal of Solids and structures 37, 45-54 (2000).
- C.K. Campbell, Surface Acoustic Wave Devices. Academic Press (1998).


**Stroh representation.** (Stroh, A.N., 1962. Steady state problems in anisotropic elasticity. *Math. Phys*. 41, 77-103.) An anisotropic elastic solid fills a half space. Let **n** be the unit vector normal to the surface of the solid. A source of stress is moving through the body at a constant speed $v$ in the direction **m**, which is a unit vector normal to **n**. The field in the solid is independent of the coordinate normal to both **m** and **n**. Let $(x,y)$ be a coordinate system moving at the source speed, relating to the fixed coordinates as

$$x = \mathbf{m} \cdot \mathbf{x} - vt, \qquad y = \mathbf{n} \cdot \mathbf{x}.$$

To an observer moving at speed $v$, the steady state field is time-independent. That is, the displacements are functions of $(x,y)$.

Consider a specific steady-state displacement field,

$$\mathbf{u}(\mathbf{x},t) = \mathbf{A} f(x + py),$$

where **A** is a vector, $p$ a number, and $f(\cdot)$ a function of one variable. The displacement field will satisfy the equation of motion provided

$$C_{ijkl}(m_l + pn_l)(m_j + pn_j)A_k = \rho v^2 A_i.$$

This is a set of linear algebraic equations for **A**. To represent a nontrivial displacement field, **A** cannot be zero. Consequently, the determinant of the above equations must vanish, which leads to a polynomial equation of degree six in $p$. Denote the six roots by $p_\alpha$, labeling $\alpha = \pm 1, \pm 2, \pm 3$ so that, if complex roots occur, $p_{+\alpha}, p_{-\alpha}$ are complex conjugates, and giving the positive $\alpha$ to the root with positive imaginary part. The labeling for real roots will be specified later.

Following Stroh, we will only consider the case that the $p_\alpha$ are all distinct; equal roots may be regarded as the limiting case of distinct roots. For each $p_\alpha$, we can determine a column $\mathbf{A}_\alpha$ up to a scaling factor. Make $\mathbf{A}_\alpha$ real when $p_\alpha$ is real, and $\mathbf{A}_\alpha$, $\mathbf{A}_{-\alpha}$ complex conjugates when $p_\alpha$ is complex.

For any six arbitrary functions $f_\alpha(\cdot)$, the linear combination

$$\mathbf{u}(\mathbf{x},t) = \sum_{\alpha=\pm1}^{\pm3} \mathbf{A}_\alpha f_\alpha(x + p_\alpha y)$$

satisfies the equation of motion. Summation over a Greek suffix will always be indicated explicitly. Any steady state solution can be represented in this form; the six Stroh functions $f_\alpha$ are to be determined by boundary conditions. Make $f_\alpha$ real when $p_\alpha$ is real, and $f_\alpha, f_{-\alpha}$ complex conjugate when $p_\alpha$ is complex, so that the displacements will always be real-valued.

We write the above in terms of components:

$$u_i(\mathbf{x},t) = \sum_{\alpha=\pm1}^{\pm3} A_{i\alpha} f_\alpha(z_\alpha),$$

where $A_{i\alpha}$ are the components of the vector $\mathbf{A}_\alpha$, and $z_\alpha = x + p_\alpha y$. The stress is given by $\sigma_{ijkl} = C_{ijkl} u_{k,l}$, and can be express in terms the six functions:

$$\sigma_{ij}(\mathbf{x},t) = \sum_{\alpha=\pm1}^{\pm3} C_{ijkl}(m_l + p_\alpha n_l) A_{k\alpha} f_\alpha(z_\alpha).$$

We use ( $'$ ) to indicate the differentiation of any one-variable function. The traction on the plane normal to the unit vector **n** is given by $t_i = \sigma_{ij} n_j$. Thus,

$$\mathbf{t}(\mathbf{x},t) = \sum_{\alpha=\pm1}^{\pm3} L_{i\alpha} f'_\alpha(z_\alpha),$$

with

$$L_{i\alpha} = C_{ijkl}(m_l + p_\alpha m_l) n_j A_{k\alpha}.$$

The above equations hold for any speed $v$ whether greater or less than the sonic speeds of the solid. When $v = 0$, all the $p_\alpha$ are complex. When $v$ is sufficiently large, all the $p_\alpha$ are real. There are three critical speeds, $V_3 \le V_2 \le V_1$. When $v$ passes $V_\alpha$, a pair of roots $p_{\pm\alpha}$ change from complex to real. If then $v < V_\alpha$, the roots $p_{\pm\alpha}$ are complex, and the functions $f_{\pm\alpha}$ are complex analytic functions. If $v > V_\alpha$, the two roots $p_{\pm\alpha}$ are real, and the equations, $x + p_{\pm\alpha} y = \text{constant}$, are the characteristic lines, along which the real functions $f_{\pm\alpha}$ have constant values.

**Rayleigh wave.** A wave can propagate undiminished along the surface of a half space. However, the wave is localized near the surface, and the amplitude decays beneath the surface. This wave is known as the Rayleigh wave. There is no intrinsic length scale in this problem, so that the Rayleigh wave is non-dispersive.

Say we seek a surface wave with the velocity $v$ below the transverse wave, so that all the eigenvalues are complex. Let the complex potential be $f_1(z_1), f_2(z_2), f_3(z_3)$. The displacement field is

$$u_i = \sum_{\alpha=1}^{3} A_{i\alpha} f_\alpha\left(z_\alpha\right) + \sum_{\alpha=1}^{3} \bar{A}_{i\alpha} \bar{f}_\alpha\left(\bar{z}_\alpha\right).$$

The traction vector is

$$t_i = \sum_{\alpha=1}^{3} L_{i\alpha} f'_\alpha\left(z_\alpha\right) + \sum_{\alpha=1}^{3} \bar{L}_{i\alpha} \bar{f}'_\alpha\left(\bar{z}_\alpha\right).$$

On the surface of the half space, $y = 0$, the traction vanishes, so that

$$\mathbf{Lf}(x) + \bar{\mathbf{L}}\bar{\mathbf{f}}(x) = 0.$$

The first term is analytic in the lower half plane, and the second term is analytic in the upper half plane. Consequently,

$$\mathbf{Lf}(z) = 0$$

for any *z*. This is an eigenvalue problem. To have a nonvanishing field, we must require that

$$\det \mathbf{L} = 0.$$

This equation determines the velocity of the surface wave. The surface wave is also nondispersive.

The complex potentials are given by

$$\mathbf{f}(z) = \mathbf{h} g(z),$$

where $\mathbf{h}$ is an eigenvector of $\mathbf{L}$ associated with the Rayleigh wave, and $g$ is an arbitrary function.

**Rayleigh wave on the surface of an isotropic solid**. For a half space, the anti-plane and the in-plane deformation decouple. We will only consider the in-plane deformation. We have

$$p_{\pm 1}^2 = v^2 / c_l^2 - 1, \qquad p_{\pm 2}^2 = v^2 / c_t^2 - 1.$$

The longitudinal and shear wave speeds are given by

$$c_l = \left[\frac{2(1-\nu)\mu}{(1-2\nu)\rho}\right]^{1/2}, \; c_t = \left(\frac{\mu}{\rho}\right)^{1/2},$$

It is evident that $c_l$ and $c_s$ are also the critical speeds. When the crack speed $v$ surpasses $c_l$ or $c_s$, a pair of roots change from complex to real.

The related matrices are all 2 by 2, as given by

$$\mathbf{A} = \begin{bmatrix} 1 & -p_2 \\ p_1 & 1 \end{bmatrix}, \quad \mathbf{L} = \mu \begin{bmatrix} 2p_1 & 1-p_2^2 \\ -\left(1-p_2^2\right) & 2p_2 \end{bmatrix},$$

For a subsonic crack, $v < c_t$, all roots are imaginary numbers:

$$p_1 = i\sqrt{1 - v^2/c_l^2}, \qquad p_2 = i\sqrt{1 - v^2/c_t^2}.$$

The speed of the surface wave is determined by $\det \mathbf{L} = 0$, namely,

$$4\sqrt{\left(1 - v^2/c_l^2\right)\left(1 - v^2/c_t^2\right)} = \left(2 - v^2/c_t^2\right)^2.$$

This is an algebraic equation for $v$, and must be solved numerically. When $\nu = 1/3$, $v_R / c_t = 0.932$.

**Waves in laminates.** Let us now look at waves propagating in a direction parallel to the interfaces of a laminate of different materials. An example is the Love wave, propagating in a laminate of a layer bonded to a half space. The thickness of each layer provides a length scale, so that the waves are dispersive. Each material has its own set of complex functions. We look for solutions of the form

$$f_\alpha(z_\alpha) = a_\alpha \exp(ikz_\alpha) + b_\alpha \exp(-ikz_\alpha).$$

Such a function ensures that the $x$ dependence is sinusoidal, with the wave number $k$. The wave velocity $v$ and the constants $a_\alpha$ and $b_\alpha$ are determined by requiring continuity of traction and displacements, which leads to an eigenvalue problem. The algebra is messy but straightforward.