On the Stress Field of a Nonlinear Elastic Solid Torus with a Toroidal Inclusion[*]

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Abstract

In this paper we analyze the stress field of a solid torus made of an incompressible isotropic solid with a toroidal inclusion that is concentric with the solid torus and has a uniform distribution of pure dilatational finite eigenstrains. We use a perturbation analysis and calculate the residual stresses to the first order in the *thinness* ratio (the ratio of the radius of the generating circle and the overall radius of the solid torus). In particular, we show that the stress field inside the inclusion is not uniform. This is in contrast with the corresponding results for infinitely-long and finite circular cylindrical bars and spherical balls with cylindrical and spherical inclusions, respectively. We also show that for a solid torus of any size made of an incompressible linear elastic solid with an inclusion with uniform (infinitesimal) pure dilatational eigenstrains the stress inside the inclusion is not uniform.

Keywords Finite eigenstrains · Geometric mechanics · Nonlinear elasticity · Elastic torus · Inclusion

1 Introduction

Eigenstrains are the anelastic part of the total strain tensor and represent referential rearrangements, changes, distortions, etc. When deformations (more precisely displacement gradients) are large different measures of strain may be considered and an eigenstrain would explicitly depend on the choice of a strain measure. Eigenstrains model many different phenomena, e.g., plasticity [Bilby et al., 1957, Kröner, 1959], thermal strains [Stojanović et al., 1964, Ozakin and Yavari, 2010], swelling [Pence and Tsai, 2005, 2006], and bulk growth [Takamizawa and Matsuda, 1990, Takamizawa, 1991, Rodriguez et al., 1994, Yavari, 2010]. For a detailed discussion of finite eigenstrains see Yavari and Goriely [2013] and Golgoon et al. [2016].

In a seminal paper, Eshelby [1957] showed that for an ellipsoidal inclusion in an infinite linear elastic solid, for uniform eigenstrains the stress inside the inclusion is uniform as well. There have been many investigations in recent years on the validity of this uniformity property for nonlinear elastic solids and inclusions with finite eigenstrains. There are several results in 2D in the case of harmonic solids [Kim and Schiavone, 2007, Kim et al., 2008, Kim and Schiavone, 2008, Ru and Schiavone, 1996, Ru et al., 2005]. In 3D, recently Yavari and Goriely [2013] showed that in the case of cylindrical bars (finite or infinitely-long) and spherical balls with cylindrical and spherical inclusions, respectively, with pure dilatational finite eigenstrains, the stress uniformity property holds for both incompressible isotropic solids and some special classes of compressible isotropic solids. Note that these geometries are simply-connected. Perhaps the simplest example of a non-simply connected body is a hollow cylinder. However, in that case one can only have an annular inclusion. Another simple example of a non-simply connected body is a solid torus. To our best knowledge, finite (or infinitesimal) eigenstrains in a solid torus and their induced residual stresses have not been studied in the literature. In this paper, we investigate this problem in the case of incompressible solids (see Fig. 1).

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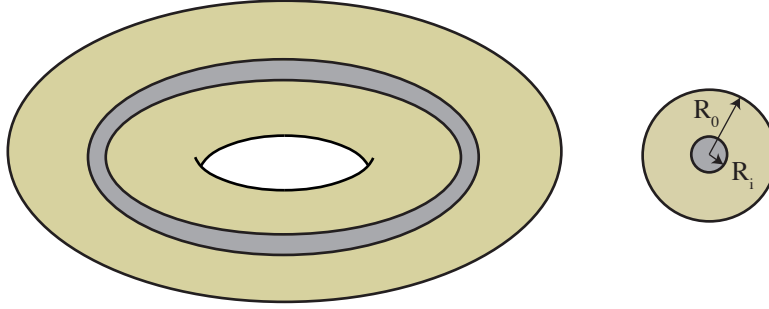[†]Corresponding author, e-mail: arash.yavari@ce.gatech.edu

Figure 1: *A solid torus with a toroidal inclusion that is concentric with it.*

Kydoniefs and Spencer [1965] and Kydoniefs [1966] studied the finite deformation of a torus made of a homogeneous, isotropic, incompressible elastic solid under inflation by uniform internal pressure and under inflation and rotation with a constant angular velocity, respectively. They assumed that the torus in its deformed state is generated by rotating two concentric circles about a line in their plane. Assuming that the radii of the generating circles are small compared to the overall radius of the torus, they obtained approximate solutions for the stress and deformation fields in the torus. Their work was further extended to a solid torus inflated from a torus in its undeformed state by Hill [1980]. He too assumed that the ratio of the radius of the generating circles and the overall radius of the torus (thinness ratio) is small and obtained the solutions to the first order of this small ratio. Under the same assumption, Kydoniefs and Spencer [1967] explored the finite inflation of an elastic toroidal membrane due to uniform internal pressure such that it has a circular cross section in its reference configuration. They obtained the solutions to the second order in the thinness ratio and presented some numerical results for a toroidal membrane made of a Mooney-Rivlin material, describing the dependence of the deformation and the generated stresses on the internal pressure. Krokhmal [2002] studied the displacement boundary-value problem of a linear elastic torus. He reduced the boundary-value problem to an infinite system of linear algebraic equations and developed an analytical technique for solving it.

Toroidal inclusions and inhomogeneities have been observed in the microstructures of both natural and engineered materials [Radi and Sevostianov, 2016]. Onaka et al. [2002] investigated the problem of elastic toroidal inclusions in an infinite linear elastic medium using an averaged Eshelby tensor. They found that the averaged Eshelby tensor of toroidal inclusions on an arbitrary plane is nearly the same as the average of the Eshelby tensors of randomly oriented rod-like inclusions on that plane. Onaka [2003] considered an infinitely extended body having a doughnut-like inclusion with purely dilatational eigenstrains. He observed that near the inclusion there are two points at which all the components of the strain tensor vanish. Note that this is not the case for spherical inclusions with purely dilatational eigenstrains placed in an infinite linear elastic medium, for which strains become null only at infinitely far distances from the inclusion. In another paper by Onaka [2005], the strain field generated by elongated toroidal inclusions were studied and compared with that of doughnut-like and spherical inclusions. It was observed that for an infinitely elongated tubular inclusion, all the strain tensor components in the matrix region surrounded by the inclusion vanish. The reinforcing effects of rigid toroidal inhomogeneities in a linear elastic medium was studied by Argatov and Sevostianov [2011]. They observed that there is no noticeable difference in the reinforcing properties of toroidal and spheroidal inhomogeneities with the same volume and diameter. Kirilyuk [1988] investigated the effects of a toroidal inhomogeneity on the stress concentration in an infinite isotropic medium. They considered two cases: perfect bonding and slipping at the inhomogeneity-matrix interface. As an example, they showed that the difference in the maximum stress could differ up to 40% for the two cases.

In the setting of linearized elasticity, it is known that for a single inclusion with uniform eigenstrain in an infinite domain to have a uniform stress field the inclusion must be an ellipsoid [Liu, 2008, Kang and Milton, 2008]. Earlier, Rodin [1996] had shown that the remarkable property of ellipsoidal inclusions is not shared by polygonal inclusions in 2D or polyhedral inclusions in 3D.

In particular, a toroidal inclusion with uniform eigenstrain in an infinite solid would have a non-uniform stress field. One may now consider a solid torus with an inclusion whose generating circle is concentric with the boundary circle of the solid torus (see Fig. 1). Is the stress field inside such an inclusion with uniform and pure dilatational eigenstrain uniform? We solve this problem for finite dilatational eigenstrains in the case of

a "thin" solid torus made of an incompressible isotropic nonlinear elastic solid. We will show that to the first order in the thinness ratio, stress inside the inclusion is not uniform. We then study the same problem for a solid torus made of an incompressible linear elastic solid with a toroidal inclusion with a uniform infinitesimally small pure dilatational eigenstrain. We show that for any size of the solid torus (not necessarily thin) the stress inside the inclusion is not uniform.

This paper is organized as follows. In section 2 we briefly review some basic concepts of the geometric theory of nonlinear elasticity. In section 3 we formulate the governing equilibrium equations of a solid torus with an axially-symmetric distribution of finite eigenstrains. In section 3.1 we consider a toroidal inclusion that is concentric with the solid torus and calculate the residual stress field using a perturbation analysis. We then present some numerical examples for neo-Hookean solids. Finally, we solve the corresponding problem in linear elasticity in section 3.2. Conclusions are given in section 4.

2 Elements of Geometric Anelasticity

In this section, we tersely review some fundamental elements of the geometric theory of nonlinear elasticity and anelasticity. For more detailed discussions, see [Marsden and Hughes, 1983, Yavari and Goriely, 2012a].

Kinematics. A body $\mathcal{B}$ is assumed to be identified with a Riemannian manifold $(\mathcal{B}, \mathbf{G})$, and a configuration of $\mathcal{B}$ is a smooth embedding $\varphi : \mathcal{B} \to \mathcal{S}$, where $(\mathcal{S}, \mathbf{g})$ is also assumed to be a Riemannian manifold. An affine connection ∇ on a smooth manifold M is a linear map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, where $\mathcal{X}(M)$ indicates the set of all smooth vector fields on M, that has to satisfy some specific properties (see do Carmo [1992] for details). It turns out that there is a unique torsion-free and compatible affine connection associated with any Riemannian manifold, referred to as Riemannian connection (see, for example, [do Carmo, 1992]; [Petersen, 2006]). We denote the Levi-Civita connection associated with the Riemannian manifold $(\mathcal{S}, \mathbf{g})$ by $\nabla^{\mathbf{g}}$. The set of all configurations of $\mathcal{B}$ is denoted by $\mathcal{C}$. A motion is a curve $c : \mathbb{R}^+ \to \varphi_t \in \mathcal{C}$ such that φ_t assigns a spatial point $x = \varphi_t(X) = \varphi(X,t) \in \mathcal{S}$ to every material point $X \in \mathcal{B}$ at any time t. It is assumed that the body is stress-free in its reference configuration, which may have a nontrivial geometry, e.g., in the presence of eigenstrains. The deformation gradient $\mathbf{F}$ is the derivative map of φ defined as

$$\mathbf{F}(X,t) = d\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{S} \,. \tag{2.1}$$

The adjoint of $\mathbf{F}$ is defined as follows

$$\mathbf{F}^{\mathsf{T}}(X,t) : T_{\varphi_t(X)}\mathcal{S} \to T_X\mathcal{B}, \qquad \mathbf{g}\left(\mathbf{F}\mathbf{V}, \mathbf{v}\right) = \mathbf{G}\left(\mathbf{V}, \mathbf{F}^{\mathsf{T}}\mathbf{v}\right), \quad \forall \mathbf{V} \in T_X\mathcal{B}, \, \mathbf{v} \in T_{\varphi_t(X)} \,. \tag{2.2}$$

The Finger deformation tensor is defined as $\mathbf{b}(x,t) = \mathbf{F}(X,t)\mathbf{F}^{\mathsf{T}}(X,t) : T_x\varphi(\mathcal{B}) \to T_x\varphi(\mathcal{B})$. In components, $b^{ab} = F^a{}_A F^b{}_B G^{AB}$. Another measure of strain is the Lagrangian strain tensor that is defined as $\mathbf{E} = \frac{1}{2}\left(\varphi_t^*\mathbf{g} - \mathbf{G}\right)$. The Jacobian of deformation J relates the Riemannian volume elements of the material manifold $dV(X, \mathbf{G})$ and the spatial manifold $dv(\varphi_t(X), \mathbf{g})$ and is written as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} \,, \quad dv = J \, dV \,. \tag{2.3}$$

Constitutive equations. For isotropic solids the energy function W depends only on the principal invariants of $\mathbf{b}$, denoted by I_1, I_2, and I_3. In the case of incompressible solids, $I_3 = 1$, and hence, $W = W(X, I_1, I_2)$. We restrict our attention to isotropic incompressible hyperelastic solids, for which the Cauchy stress has the following representation [Ogden, 1997, Simo and Marsden, 1984]

$$\boldsymbol{\sigma} = \left(-p + 2I_2 W_{I_2}\right)\mathbf{g}^\sharp + 2W_{I_1}\mathbf{b}^\sharp - 2W_{I_2}\mathbf{b}^{-1} \,, \tag{2.4}$$

where p is the Lagrange multiplier associated with the internal incompressibility condition, and $W_{I_1} := \frac{\partial W}{\partial I_1}$, $W_{I_2} := \frac{\partial W}{\partial I_2}$. Note that $\mathbf{b}$ and $\mathbf{C}$ have the same principal invariants, and $\mathbf{b}^\sharp = \varphi_*\left(\mathbf{G}^\sharp\right)$. We assume that the body in the absence of eigenstrains is isotropic. Eigenstrains are modeled by a material metric $\mathbf{G}$ that explicitly depends on the distribution of eigenstrains [Yavari and Goriely, 2013, 2015a, Golgoon and Yavari, 2016]. In other words, stress-free configuration of a body with a distribution of eigenstrains may not be globally realizable in the Euclidean ambient space.
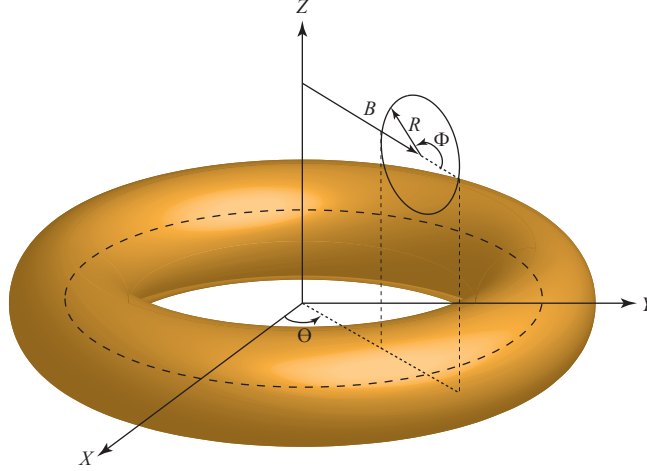
Figure 2: *A solid torus and its toroidal coordinates in the undeformed configuration.*

Equilibrium equations. The localized spatial balance of linear momentum of a body in static equilibrium in the absence of body forces in terms of the Cauchy stress reads $\text{div}\,\boldsymbol{\sigma} = \mathbf{0}$. In components

$$(\text{div}\,\boldsymbol{\sigma})^a = \sigma^{ab}{}_{|b} = \frac{\partial \sigma^{ab}}{\partial x^b} + \sigma^{ac}\gamma^b{}_{cb} + \sigma^{cb}\gamma^a{}_{cb}\,, \tag{2.5}$$

where $\gamma^a{}_{bc}$ denotes the Christoffel symbols of the connection $\nabla^{\mathbf{g}}$ in the local charts $\{x^a\}$, defined as $\nabla^{\mathbf{g}}_{\partial_b}\partial_c = \gamma^a{}_{bc}\partial_a$. Moreover, the Christoffel symbols of the Levi-Civita connection can be directly expressed in terms of the components of the Riemannian metric as

$$\gamma^a{}_{bc} = \frac{1}{2}g^{ak}\left(\frac{\partial g_{kb}}{\partial x^c} + \frac{\partial g_{kc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^k}\right)\,. \tag{2.6}$$

In this paper we model finite eigenstrains in a nonlinear elastic solid by defining a Riemannian material manifold, which has a metric that explicitly depends on the distribution of eigenstrains. This idea has been discussed in detail in our previous works [Yavari and Goriely, 2013, 2015b,a, Golgoon et al., 2016]

3 An Incompressible Isotropic Solid Torus with Axially-Symmetric Finite Eigenstrains

In this section we consider a solid torus generated by rotating a circle with radius R_o about a line in its plane such that the distance from the origin to the center of the circle is B. Let (R, Θ, Φ) and (r, θ, ϕ) be the material and spatial toroidal coordinates as illustrated in Figure 2. In the toroidal coordinates (R, Θ, Φ), the metric of the eigenstrain-free torus is written as

$$\mathbf{G}_o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (B + R\cos\Phi)^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix}\,. \tag{3.1}$$

We assume an axially-symmetric (Θ-independent) eigenstrain (pre-strain) distribution in the torus. Following the construction suggested by Yavari and Goriely [2013] to model eigenstrains, we consider the following material metric[1]

$$\mathbf{G} = e^{\Omega(R,\Phi)}\mathbf{G}_o\,, \tag{3.2}$$

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[1]Similar constructions have been discussed in [Ozakin and Yavari, 2010, Yavari, 2010, Yavari and Goriely, 2012a,b, 2014, 2015b, Sadik and Yavari, 2015, Sozio and Yavari, 2017] to address problems in growth mechanics, thermoelasticity, and the nonlinear mechanics of distributed defects.

where $\Omega(R, \Phi)$ is an arbitrary function describing the inhomogeneous dilatational eigenstrain distribution in the torus. The ambient space is endowed with the Euclidean metric, which in the toroidal coordinates (r, θ, ϕ) has the following representation

$$
\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (b + r\cos\phi)^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix}. \tag{3.3}
$$

Let us consider an axially-symmetric class of deformations of the following form

$$
r = r(R, \Phi), \quad \theta = \Theta, \quad \phi = \phi(R, \Phi). \tag{3.4}
$$

The deformation gradient for this class of deformations reads

$$
\mathbf{F} = \begin{pmatrix} \frac{\partial r}{\partial R} & 0 & \frac{\partial r}{\partial \Phi} \\[2mm] 0 & 1 & 0 \\[2mm] \frac{\partial \phi}{\partial R} & 0 & \frac{\partial \phi}{\partial \Phi} \end{pmatrix}. \tag{3.5}
$$

We assume an incompressible solid, i.e., $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = 1$, which gives us[2]

$$
r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R} = \frac{Re^{\frac{3}{2}\Omega(R,\Phi)}(B + R\cos\Phi)}{r(b + r\cos\phi)}. \tag{3.6}
$$

The Finger deformation tensor reads

$$
\mathbf{b}^\sharp = e^{-\Omega(R,\Phi)} \begin{pmatrix} r_{,R}^2 + \frac{r_{,\Phi}^2}{R^2} & 0 & r_{,R}\phi_{,R} + \frac{r_{,\Phi}\phi_{,\Phi}}{R^2} \\[3mm] 0 & \frac{1}{(B+R\cos\Phi)^2} & 0 \\[3mm] r_{,R}\phi_{,R} + \frac{r_{,\Phi}\phi_{,\Phi}}{R^2} & 0 & \phi_{,R}^2 + \frac{\phi_{,\Phi}^2}{R^2} \end{pmatrix}. \tag{3.7}
$$

The first two principal invariants of $\mathbf{b}$ are ($I_3 = 1$)

$$
I_1 = I + \beta^2, \qquad I_2 = \beta^2 I + \frac{1}{\beta^2}, \tag{3.8}
$$

where

$$
I = e^{-\Omega(R,\Phi)}\left(r_{,R}^2 + r^2\phi_{,R}^2 + \frac{r_{,\Phi}^2 + r^2\phi_{,\Phi}^2}{R^2}\right), \qquad \beta = \frac{Re^{\Omega(R,\Phi)}}{r\left(r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R}\right)}. \tag{3.9}
$$

The inverse of Finger tensor $\mathbf{b}^{-1} = \mathbf{c}$ is written as

$$
\mathbf{b}^{-1} = e^{\Omega(R,\Phi)} \begin{pmatrix} \frac{\phi_{,\Phi}^2 + R^2\phi_{,R}^2}{(r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R})^2} & 0 & -\frac{R^2 r_{,R}\phi_{,R} + r_{,\Phi}\phi_{,\Phi}}{r^2(r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R})^2} \\[3mm] 0 & \frac{(B+R\cos\Phi)^2}{(b+r\cos\phi)^4} & 0 \\[3mm] -\frac{R^2 r_{,R}\phi_{,R} + r_{,\Phi}\phi_{,\Phi}}{r^2(r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R})^2} & 0 & \frac{R^2 r_{,R}^2 + r_{,\Phi}^2}{r^4(r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R})^2} \end{pmatrix}. \tag{3.10}
$$

Following (2.4), the non-zero components of the Cauchy stress read

$$
\sigma^{rr} = -p(R, \Phi) + 2\left(W_{I_1} + \beta^2 W_{I_2}\right)\left(r_{,R}^2 + \frac{r_{,\Phi}^2}{R^2}\right)e^{-\Omega(R,\Phi)} + \frac{2W_{I_2}}{\beta^2}, \tag{3.11a}
$$

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[2]We use Mathematica [Wolfram Research, 2016] for the symbolic computations.

$$\sigma^{r\phi} = 2e^{-\Omega(R,\Phi)} \left(W_{I_1} + \beta^2 W_{I_2} \right) \left(r_{,R}\phi_{,R} + \frac{r_{,\Phi}\phi_{,\Phi}}{R^2} \right) , \tag{3.11b}$$

$$\sigma^{\theta\theta} = -\frac{p(R,\Phi)}{(b + r\cos\phi)^2} + \frac{2e^{-\Omega(R,\Phi)} \left(W_{I_1} + I W_{I_2} \right)}{(B + R\cos\Phi)^2} , \tag{3.11c}$$

$$\sigma^{\phi\phi} = -\frac{p(R,\Phi)}{r^2} + 2 \left(W_{I_1} + \beta^2 W_{I_2} \right) \left(\phi_{,R}{}^2 + \frac{\phi_{,\Phi}{}^2}{R^2} \right) e^{-\Omega(R,\Phi)} + \frac{2W_{I_2}}{\beta^2 r^2} . \tag{3.11d}$$

The physical components of the Cauchy stress are calculated using the relation $\hat{\sigma}^{ab} = \sigma^{ab}\sqrt{g_{aa}g_{bb}}$ [Truesdell, 1953]. Thus

$$\hat{\sigma}^{rr} = \sigma^{rr} , \quad \hat{\sigma}^{r\phi} = r\sigma^{r\phi} , \quad \hat{\sigma}^{\theta\theta} = (b + r\cos\phi)^2 \sigma^{\theta\theta} , \quad \hat{\sigma}^{\phi\phi} = r^2\sigma^{\phi\phi} . \tag{3.12}$$

The non-zero components of the first Piola-Kirchhoff stress tensor, i.e., $P^{aA} = J(F^{-1})^A{}_b\sigma^{ab}$ are written as

$$P^{rR} = e^{-\Omega(R,\Phi)} \left[2r_{,R} \left(W_{I_1} + \beta^2 W_{I_2} \right) + \frac{r\beta\phi_{,\Phi}}{R} \left(\frac{2W_{I_2}}{\beta^2} - p(R,\Phi) \right) \right] , \tag{3.13a}$$

$$P^{r\Phi} = \frac{e^{-\Omega(R,\Phi)}}{R^2} \left[2r_{,\Phi} \left(W_{I_1} + \beta^2 W_{I_2} \right) + rR\beta\phi_{,R} \left(p(R,\Phi) - \frac{2W_{I_2}}{\beta^2} \right) \right] , \tag{3.13b}$$

$$P^{\phi R} = e^{-\Omega(R,\Phi)} \left[2\phi_{,R} \left(W_{I_1} + \beta^2 W_{I_2} \right) + \frac{\beta r_{,\Phi}}{rR} \left(p(R,\Phi) - \frac{2W_{I_2}}{\beta^2} \right) \right] , \tag{3.13c}$$

$$P^{\theta\Theta} = \sigma^{\theta\theta} , \tag{3.13d}$$

$$P^{\phi\Phi} = \frac{e^{-\Omega(R,\Phi)}}{rR^2} \left[2r\phi_{,\Phi} \left(W_{I_1} + \beta^2 W_{I_2} \right) + R\beta r_{,R} \left(\frac{2W_{I_2}}{\beta^2} - p(R,\Phi) \right) \right] . \tag{3.13e}$$

The Christoffel symbol matrices of $\mathbf{g}$ read (cf. 2.6)

$$\boldsymbol{\gamma}^r = [\gamma^r{}_{ab}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(b + r\cos\phi)\cos\phi & 0 \\ 0 & 0 & -r \end{pmatrix} , \quad \boldsymbol{\gamma}^\theta = [\gamma^\theta{}_{ab}] = \begin{pmatrix} 0 & \frac{\cos\phi}{b + r\cos\phi} & 0 \\ \frac{\cos\phi}{b + r\cos\phi} & 0 & -\frac{r\sin\phi}{b + r\cos\phi} \\ 0 & -\frac{r\sin\phi}{b + r\cos\phi} & 0 \end{pmatrix} ,$$

$$\boldsymbol{\gamma}^\phi = [\gamma^\phi{}_{ab}] = \begin{pmatrix} 0 & 0 & \frac{1}{r} \\ 0 & \left(\frac{b}{r} + \cos\phi \right)\sin\phi & 0 \\ \frac{1}{r} & 0 & 0 \end{pmatrix} . \tag{3.14}$$

In the absence of body forces, the non-trivial equilibrium equations are $\sigma^{rb}{}_{|b} = 0$ and $\sigma^{\phi b}{}_{|b} = 0$, which after simplification read (the equilibrium equation in the θ-direction gives $p = p(R,\Phi)$)

$$\frac{\partial\sigma^{rr}}{\partial r} + \frac{\partial\sigma^{r\phi}}{\partial\phi} + \left(\frac{1}{r} + \frac{\cos\phi}{b + r\cos\phi} \right)\sigma^{rr} - \cos\phi(b + r\cos\phi)\sigma^{\theta\theta} - \frac{r\sin\phi}{b + r\cos\phi}\sigma^{r\phi} - r\sigma^{\phi\phi} = 0 , \tag{3.15}$$

$$\frac{\partial\sigma^{r\phi}}{\partial r} + \frac{\partial\sigma^{\phi\phi}}{\partial\phi} + \left(\frac{3}{r} + \frac{\cos\phi}{b + r\cos\phi} \right)\sigma^{r\phi} + \frac{\sin\phi}{r}(b + r\cos\phi)\sigma^{\theta\theta} - \frac{r\sin\phi}{b + r\cos\phi}\sigma^{\phi\phi} = 0 . \tag{3.16}$$

Note that

$$\frac{\partial}{\partial r} = \frac{\phi_{,\Phi}}{\phi_{,\Phi}r_{,R} - \phi_{,R}r_{,\Phi}}\frac{\partial}{\partial R} + \frac{\phi_{,R}}{\phi_{,R}r_{,\Phi} - \phi_{,\Phi}r_{,R}}\frac{\partial}{\partial\Phi} , \qquad \frac{\partial}{\partial\phi} = \frac{r_{,\Phi}}{r_{,\Phi}\phi_{,R} - r_{,R}\phi_{,\Phi}}\frac{\partial}{\partial R} + \frac{r_{,R}}{r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R}}\frac{\partial}{\partial\Phi} . \tag{3.17}$$

Boundary conditions. As we are interested in finding the residual stress field, we assume that the boundary of the torus is traction-free, i.e.,

$$P^{rR} = 0, \quad P^{\phi R} = 0, \qquad R = R_o, \quad -\pi \leq \Phi \leq \pi. \tag{3.18}$$

Finding an exact solution of the PDEs (3.15) and (3.16) does not seem feasible. Therefore, we seek approximate solutions assuming that the radius of the cross section generating the torus is small compared to the radius of revolution [Kydoniefs and Spencer, 1965, Kydoniefs, 1966, Kydoniefs and Spencer, 1967, Kydoniefs, 1967, Hill, 1980]. Hence, we find the solution assuming that R/B and r/b, which are of the same order, are sufficiently small so that the second and higher powers of R/B (and r/b) can be neglected. In doing so, the problem is essentially a perturbation of the problem of finite eigenstrains in an infinitely-long circular cylindrical bar, which was discussed in [Yavari and Goriely, 2013] for the special case of cylindrically-symmetric distribution of eigenstrains.

3.1 A nonlinear solid torus with finite eigenstrains and $r/b \ll 1$ and $R/B \ll 1$

In this section, we restrict our attention to the radially-symmetric dilatational eigenstrain distributions, for which $\Omega = \Omega(R)$. Moreover, we assume that r/b and R/B are sufficiently small and find the solutions to the first order in the thinness ratio $\varepsilon = R_o/B \ll 1$. For the zero-order problem ($\varepsilon \to 0$), the torus becomes a cylinder with the cylindrically-symmetric distribution of purely dilatational eigenstrains, for which, in the cylindrical coordinates, $r = r(R)$, $\phi = \Phi$, $z = \frac{b}{B}Z$, and $p = p(R)$. Therefore, we consider the following asymptotic expansions[3]

$$\begin{aligned} r &= r^{(0)}(R) + r^{(1)}(R, \Phi) + \mathcal{O}(\varepsilon^2), & \phi &= \Phi + \phi^{(1)}(R, \Phi) + \mathcal{O}(\varepsilon^2), \\ p &= p^{(0)}(R) + p^{(1)}(R, \Phi) + \mathcal{O}(\varepsilon^2), & \sigma^{ij} &= \sigma_{(0)}^{ij}(R) + \sigma_{(1)}^{ij}(R, \Phi) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.6) and equating the zero and the first-order terms on both sides one gets

$$r^{(0)} \frac{dr^{(0)}}{dR} = \frac{BR}{b} e^{\frac{3}{2}\Omega(R)}, \tag{3.20}$$

$$\frac{dr^{(0)}}{dR} \phi_{,\Phi}^{(1)} + r_{,R}^{(1)} = \frac{BR}{br^{(0)}} e^{\frac{3}{2}\Omega(R)} \left[\left(\frac{R}{B} - \frac{r^{(0)}}{b} \right) \cos\Phi - \frac{r^{(1)}}{r^{(0)}} \right]. \tag{3.21}$$

Similarly

$$I_1 = I_1^{(0)}(R) + I_1^{(1)}(R, \Phi) + \mathcal{O}(\varepsilon^2), \qquad I_2 = I_2^{(0)}(R) + I_2^{(1)}(R, \Phi) + \mathcal{O}(\varepsilon^2). \tag{3.22}$$

Denoting $\alpha = \frac{dr^{(0)}}{dR}$, one has

$$I_1^{(0)} = \left[\alpha^2 + \frac{r^{(0)2}}{R^2} \right] e^{-\Omega} + \frac{R^2 e^{2\Omega}}{r^{(0)2}\alpha^2}, \qquad I_2^{(0)} = \left[\frac{1}{\alpha^2} + \frac{R^2}{r^{(0)2}} \right] e^{\Omega} + \frac{r^{(0)2}\alpha^2}{R^2 e^{2\Omega}}, \tag{3.23}$$

$$I_1^{(1)} = 2 \left[\frac{r^{(0)2}}{R^2 e^{\Omega}} - \frac{R^2 e^{2\Omega}}{r^{(0)2}\alpha^2} \right] \left(\phi_{,\Phi}^{(1)} + \frac{r^{(1)}}{r^{(0)}} \right) + 2 \left[\frac{\alpha}{e^{\Omega}} - \frac{R^2 e^{2\Omega}}{r^{(0)2}\alpha^3} \right] r_{,R}^{(1)}, \tag{3.24}$$

$$I_2^{(1)} = 2 \left[\frac{r^{(0)2}\alpha^2}{R^2 e^{2\Omega}} - \frac{R^2 e^{\Omega}}{r^{(0)2}} \right] \left(\phi_{,\Phi}^{(1)} + \frac{r^{(1)}}{r^{(0)}} \right) + 2 \left[\frac{r^{(0)2}\alpha}{R^2 e^{2\Omega}} - \frac{e^{\Omega}}{\alpha^3} \right] r_{,R}^{(1)}. \tag{3.25}$$

Assuming that the material is piecewise homogeneous[4] one can write

$$W_{I_1} = W_{I_1}^{(0)} + W_{I_1 I_1}^{(0)} I_1^{(1)} + W_{I_1 I_2}^{(0)} I_2^{(1)} + \mathcal{O}(\varepsilon^2), \qquad W_{I_2} = W_{I_2}^{(0)} + W_{I_2 I_2}^{(0)} I_2^{(1)} + W_{I_1 I_2}^{(0)} I_1^{(1)} + \mathcal{O}(\varepsilon^2), \tag{3.26}$$

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[3]Note that $\sigma_{(0)}^{r\phi} = 0$ and $\hat{\sigma}^{\theta\theta} = \hat{\sigma}_{(0)}^{\theta\theta} + \hat{\sigma}_{(1)}^{\theta\theta} + \mathcal{O}(\varepsilon^2)$.

[4]For the sake of simplicity of calculations, here we do not consider the dependence of W on X, which would be needed in the case of an inhomogeneity. Instead, we model inhomogeneities by assuming different energy functions at different regions of the body.

where $W_{I_1{}^{\alpha_1}I_2{}^{\alpha_2}}^{(0)} := \frac{\partial^{\alpha_1+\alpha_2}W}{\partial^{\alpha_1}I_1\partial^{\alpha_2}I_2}(I_1^{(0)}, I_2^{(0)})$, for $(\alpha_1, \alpha_2 \in \{0,1,2\})$. Expanding (3.11) one obtains the following expressions for the non-zero Cauchy stress components

$$\sigma_{(0)}^{rr} = -p^{(0)} + 2e^{-\Omega}\left[\alpha^2 W_{I_1}^{(0)} + \frac{R^2 e^{2\Omega}}{r^{(0)2}}W_{I_2}^{(0)}\right] + \frac{2r^{(0)2}\alpha^2}{R^2 e^{2\Omega}}W_{I_2}^{(0)}, \tag{3.27a}$$

$$\sigma_{(1)}^{rr} = 4r_{,R}^{(1)}e^{-\Omega}\alpha\left[W_{I_1}^{(0)} + \frac{r^{(0)2}}{R^2 e^{\Omega}}W_{I_2}^{(0)}\right] + 2\left[\frac{r^{(0)2}\alpha^2}{R^2 e^{2\Omega}} + \frac{R^2 e^{\Omega}}{r^{(0)2}}\right]\left(W_{I_2{}^2}^{(0)}I_2^{(1)} + W_{I_1 I_2}^{(0)}I_1^{(1)}\right)$$
$$+ 4\left(\phi_{,\Phi}^{(1)} + \frac{r^{(1)}}{r^{(0)}}\right)\left[\frac{r^{(0)2}\alpha^2}{R^2 e^{2\Omega}} - \frac{R^2 e^{\Omega}}{r^{(0)2}}\right]W_{I_2}^{(0)} + 2\alpha^2 e^{-\Omega}\left(W_{I_1{}^2}^{(0)}I_1^{(1)} + W_{I_1 I_2}^{(0)}I_2^{(1)}\right) - p^{(1)}, \tag{3.27b}$$

$$\sigma_{(1)}^{r\phi} = 2e^{-\Omega}\left[W_{I_1}^{(0)} + \frac{R^2 e^{2\Omega}}{r^{(0)2}\alpha^2}W_{I_2}^{(0)}\right]\left(\alpha\phi_{,R}^{(1)} + \frac{r_{,\Phi}^{(1)}}{R^2}\right), \tag{3.27c}$$

$$\hat{\sigma}_{(0)}^{\theta\theta} = \frac{2b^2 e^{-\Omega}}{B^2}\left[W_{I_1}^{(0)} + e^{-\Omega}\left(\alpha^2 + \frac{r^{(0)2}}{R^2}\right)W_{I_2}^{(0)}\right] - p^{(0)}, \tag{3.27d}$$

$$\hat{\sigma}_{(1)}^{\theta\theta} = \frac{2b^2 e^{-2\Omega}}{B^2}\left(\alpha^2 + \frac{r^{(0)2}}{R^2}\right)\left[W_{I_2{}^2}^{(0)}I_2^{(1)} + W_{I_1 I_2}^{(0)}I_1^{(1)}\right] + \frac{4b^2 e^{-2\Omega}}{B^2}\left[\alpha r_{,R}^{(1)} + \frac{r^{(0)2}}{R^2}\left(\phi_{,\Phi}^{(1)} + \frac{r^{(1)}}{r^{(0)}}\right)\right]W_{I_2}^{(0)} - p^{(1)}$$
$$- \frac{4b^2 e^{-\Omega}\cos\Phi}{B^2}\left(\frac{R}{B} - \frac{r^{(0)}}{b}\right)\left[W_{I_1}^{(0)} + e^{-\Omega}\left(\alpha^2 + \frac{r^{(0)2}}{R^2}\right)W_{I_2}^{(0)}\right] + \frac{2b^2 e^{-\Omega}}{B^2}\left[W_{I_1{}^2}^{(0)}I_1^{(1)} + W_{I_1 I_2}^{(0)}I_2^{(1)}\right], \tag{3.27e}$$

$$\sigma_{(0)}^{\phi\phi} = -\frac{p^{(0)}}{r^{(0)2}} + \frac{2e^{-\Omega}}{R^2}\left[W_{I_1}^{(0)} + \frac{R^2 e^{2\Omega}}{r^{(0)2}\alpha^2}W_{I_2}^{(0)}\right] + \frac{2\alpha^2}{R^2 e^{2\Omega}}W_{I_2}^{(0)}, \tag{3.27f}$$

$$\sigma_{(1)}^{\phi\phi} = \frac{2e^{-\Omega}}{R^2}\left[W_{I_1{}^2}^{(0)}I_1^{(1)} + W_{I_1 I_2}^{(0)}I_2^{(1)}\right] + 2\left(\frac{e^{\Omega}}{r^{(0)2}\alpha^2} + \frac{\alpha^2}{R^2 e^{2\Omega}}\right)\left[W_{I_2{}^2}^{(0)}I_2^{(1)} + W_{I_1 I_2}^{(0)}I_1^{(1)}\right] - \frac{4r^{(1)}e^{\Omega}}{r^{(0)3}\alpha^2}W_{I_2}^{(0)}$$
$$- \frac{1}{r^{(0)2}}\left[p^{(1)} - \frac{2r^{(1)}}{r^{(0)}}p^{(0)}\right] + \frac{4r_{,R}^{(1)}}{\alpha e^{2\Omega}}\left[\frac{\alpha^2}{R^2} - \frac{e^{3\Omega}}{r^{(0)2}\alpha^2}\right]W_{I_2}^{(0)} + \frac{4\phi_{,\Phi}^{(1)}}{R^2 e^{\Omega}}\left[W_{I_1}^{(0)} + \frac{\alpha^2}{e^{\Omega}}W_{I_2}^{(0)}\right]. \tag{3.27g}$$

Using (3.17) and (3.19), the non-trivial zero and first-order equilibrium equations are derived by expanding (3.15) and (3.16) as follows

$$\frac{d\sigma_{(0)}^{rr}}{dr^{(0)}} + \frac{\sigma_{(0)}^{rr}}{r^{(0)}} - r^{(0)}\sigma_{(0)}^{\phi\phi} = 0, \tag{3.28}$$

$$\frac{\sigma_{(1),R}^{rr}}{\alpha} - \frac{r_{,R}^{(1)}}{\alpha^2}\frac{d\sigma_{(0)}^{rr}}{dR} + \sigma_{(1),\Phi}^{r\phi} + \frac{\sigma_{(1)}^{rr}}{r^{(0)}} - \frac{r^{(1)}}{r^{(0)2}}\sigma_{(0)}^{rr} + \frac{\cos\Phi}{b}\left(\sigma_{(0)}^{rr} - \hat{\sigma}_{(0)}^{\theta\theta}\right) - r^{(0)}\sigma_{(1)}^{\phi\phi} - r^{(1)}\sigma_{(0)}^{\phi\phi} = 0, \tag{3.29}$$

$$\frac{\sigma_{(1),R}^{r\phi}}{\alpha} - \frac{r_{,\Phi}^{(1)}}{\alpha}\frac{d\sigma_{(0)}^{\phi\phi}}{dR} + \sigma_{(1),\Phi}^{\phi\phi} + \frac{3}{r^{(0)}}\sigma_{(1)}^{r\phi} + \frac{r^{(0)}\sin\Phi}{b}\left(\frac{\hat{\sigma}_{(0)}^{\theta\theta}}{r^{(0)2}} - \sigma_{(0)}^{\phi\phi}\right) = 0. \tag{3.30}$$

Assuming that $r(0) = 0$, (3.20) has the following solution

$$r^{(0)}(R) = \left(\frac{2B}{b}\int_0^R \zeta e^{\frac{3}{2}\Omega(\zeta)}d\zeta\right)^{\frac{1}{2}}. \tag{3.31}$$

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It follows from (3.28) that $\frac{dp^{(0)}(R)}{dR} = h(R)$, where

$$h(R) = -\frac{2Re^{\Omega(R)}}{kr^{(0)^2}} \left[\left\{ \frac{k^2R^2e^{\frac{3}{2}\Omega(R)}}{r^{(0)^2}} + \frac{r^{(0)^2}}{R^2e^{\frac{3}{2}\Omega(R)}} - 2k \right\} \left(k^2e^{\Omega(R)}W_{I_1}^{(0)}(R) + W_{I_2}^{(0)}(R) \right) - k^3Re^{\Omega(R)}W_{I_1}^{(0)'}(R) \right.$$
$$\left. - k\Omega'(R)R \left(2k^2e^{\Omega(R)}W_{I_1}^{(0)}(R) + \left(1 + \frac{k^2r^{(0)^2}}{R^2} \right) W_{I_2}^{(0)}(R) \right) - kR \left(1 + \frac{k^2r^{(0)^2}}{R^2} \right) W_{I_2}^{(0)'}(R) \right], \quad (3.32)$$

and $k = B/b$. Note that

$$\frac{dW_{I_1}^{(0)}(R)}{dR} = \frac{dI_1^{(0)}}{dR} W_{I_1I_1}^{(0)} + \frac{dI_2^{(0)}}{dR} W_{I_1I_2}^{(0)},$$
$$\frac{dW_{I_2}^{(0)}(R)}{dR} = \frac{dI_2^{(0)}}{dR} W_{I_2I_2}^{(0)} + \frac{dI_1^{(0)}}{dR} W_{I_1I_2}^{(0)}. \quad (3.33)$$

Example: A toroidal inclusion with uniform pure dilatational eigenstrains in a neo-Hookean solid torus. Let us consider the following distribution of eigenstrains

$$\Omega(R) = \begin{cases} \Omega_o, & 0 \le R < R_i \\ 0, & R_i < R \le R_o \end{cases}. \quad (3.34)$$

We assume that the torus is made of an incompressible homogeneous neo-Hookean solid, i.e., $W = \frac{\mu}{2}(I_1 - 3)$, where μ is the shear modulus at the ground state. Therefore, it follows from (3.31) that

$$r^{(0)}(R) = k^{\frac{1}{2}} \begin{cases} e^{\frac{3\Omega_o}{4}} R, & 0 \le R \le R_i \\ \left(R^2 + \gamma_o R_i^2 \right)^{\frac{1}{2}}, & R_i \le R \le R_o \end{cases}, \quad (3.35)$$

where $\gamma_o = e^{\frac{3\Omega_o}{2}} - 1$. Using (3.32), we find the zero-order pressure field as[5]

$$p^{(0)}(R) = \begin{cases} \mu c_i, & 0 \le R < R_i \\ \mu c_o - \frac{k\mu}{2} \left[\frac{\gamma_o}{\gamma_o + \frac{R^2}{R_i^2}} + \ln \left(\frac{\frac{R^2}{R_i^2}}{\gamma_o + \frac{R^2}{R_i^2}} \right) \right], & R_i < R \le R_o \end{cases}, \quad (3.36)$$

where c_o and c_i are constants to be determined after enforcing the boundary conditions (3.18) and the continuity of the traction vector on the inclusion-matrix interface. The continuity of the traction vector on the boundary of the inclusion implies that $\sigma_{(0)}^{rr}$ must be continuous at $R = R_i$. Therefore, c_o and c_i are computed as

$$c_o = k + \frac{k}{2} \ln \left(\frac{\frac{R_o^2}{R_i^2}}{\gamma_o + \frac{R_o^2}{R_i^2}} \right) - \frac{k\gamma_o}{2 \left(\gamma_o + \frac{R_o^2}{R_i^2} \right)}, \quad c_i = c_o + \frac{3k\Omega_o}{4} - \frac{k}{2} \left(1 + e^{-\frac{3\Omega_o}{2}} \right) + ke^{\frac{\Omega_o}{2}}. \quad (3.37)$$

The zero-order stress components are simplified to read

$$\frac{\sigma_{(0)}^{rr}}{\mu} = \begin{cases} ke^{\frac{\Omega_o}{2}} - c_i, & 0 \le R \le R_i \\ k + \frac{k}{2} \ln \left(\frac{\frac{R^2}{R_i^2}}{\gamma_o + \frac{R^2}{R_i^2}} \right) - \frac{k\gamma_o}{2 \left(\gamma_o + \frac{R^2}{R_i^2} \right)} - c_o, & R_i \le R \le R_o \end{cases}, \quad (3.38a)$$

$$\frac{\hat{\sigma}_{(0)}^{\theta\theta}}{\mu} = \begin{cases} \frac{e^{-\Omega_o}}{k^2} - c_i, & 0 \le R < R_i \\ \frac{1}{k^2} - c_o + \frac{k}{2} \left[\frac{\gamma_o}{\gamma_o + \frac{R^2}{R_i^2}} + \ln \left(\frac{\frac{R^2}{R_i^2}}{\gamma_o + \frac{R^2}{R_i^2}} \right) \right], & R_i < R \le R_o \end{cases}, \quad (3.38b)$$

---

[5]Here, using (3.32), we find the zero-order pressure field in the inclusion and the matrix separately. Alternatively, the discontinuous eigenstrain distribution (3.34) may be treated as a step function defined in the entire region, and the pressure field is found from (3.32). In both cases, continuity of the traction vector on the inclusion-matrix interface is needed to find the unknown constants.

$$\frac{R^2 \sigma_{(0)}^{\phi\phi}}{\mu} = \begin{cases} e^{-\Omega_o}\left(1 - \frac{c_i}{k}e^{-\frac{\Omega_o}{2}}\right), & 0 \le R < R_i \\ 1 - \frac{\frac{R^2}{R_i^2}}{k\left(\gamma_o + \frac{R^2}{R_i^2}\right)}\left\{c_o - \frac{k}{2}\left[\frac{\gamma_o}{\gamma_o + \frac{R^2}{R_i^2}} + \ln\left(\frac{\frac{R^2}{R_i^2}}{\gamma_o + \frac{R^2}{R_i^2}}\right)\right]\right\}, & R_i < R \le R_o \end{cases}. \tag{3.38c}$$

Substituting (3.35) into (3.21), one obtains the following relations in the inclusion and the matrix

$$\phi_{,\Phi}^{(1)} + k^{-\frac{1}{2}}e^{-\frac{3\Omega_o}{4}}r_{,R}^{(1)} = \frac{R}{B}\left(1 - k^{\frac{3}{2}}e^{\frac{3\Omega_o}{4}}\right)\cos\Phi - \frac{r^{(1)}}{k^{\frac{1}{2}}e^{\frac{3\Omega_o}{4}}R}, \qquad 0 \le R \le R_i,$$

$$\phi_{,\Phi}^{(1)} + k^{-\frac{1}{2}}\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}r_{,R}^{(1)} = \frac{R}{B}\left(1 - k^{\frac{3}{2}}\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}\right)\cos\Phi - \frac{r^{(1)}}{k^{\frac{1}{2}}R\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}}, \qquad R_i < R \le R_o. \tag{3.39}$$

The expressions (3.27) for the first-order stress components are simplified to read

$$\frac{\sigma_{(1)}^{rr}}{\mu} = \begin{cases} 2k^{\frac{1}{2}}e^{-\frac{\Omega_o}{4}}r_{,R}^{(1)} - \frac{p^{(1)}}{\mu}, & 0 \le R \le R_i \\ \frac{2k^{\frac{1}{2}}r_{,R}^{(1)}}{\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}} - \frac{p^{(1)}}{\mu}, & R_i < R \le R_o \end{cases}, \tag{3.40a}$$

$$\frac{\sigma_{(1)}^{r\phi}}{\mu} = \begin{cases} e^{-\Omega_o}\left(k^{\frac{1}{2}}e^{\frac{3\Omega_o}{4}}\phi_{,R}^{(1)} + \frac{r_{,\Phi}^{(1)}}{R^2}\right), & 0 \le R \le R_i \\ \frac{k^{\frac{1}{2}}\phi_{,R}^{(1)}}{\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}} + \frac{r_{,\Phi}^{(1)}}{R^2}, & R_i < R \le R_o \end{cases}, \tag{3.40b}$$

$$\frac{\hat{\sigma}_{(1)}^{\theta\theta}}{\mu} = \begin{cases} -\frac{2Re^{-\Omega_o}\cos\Phi}{Bk^2}\left(1 - k^{\frac{3}{2}}e^{\frac{3\Omega_o}{4}}\right) - \frac{p^{(1)}}{\mu}, & 0 \le R \le R_i \\ -\frac{2R\cos\Phi}{Bk^2}\left[1 - k^{\frac{3}{2}}\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}\right] - \frac{p^{(1)}}{\mu}, & R_i < R \le R_o \end{cases}, \tag{3.40c}$$

$$\frac{R^2\sigma_{(1)}^{\phi\phi}}{\mu} = \begin{cases} 2k^{-\frac{3}{2}}c_i e^{-\frac{9\Omega_o}{4}}\frac{r^{(1)}}{R} + 2e^{-\Omega_o}\phi_{,\Phi}^{(1)} - k^{-1}e^{-\frac{3\Omega_o}{2}}\frac{p^{(1)}}{\mu}, & 0 \le R \le R_i \\ 2\phi_{,\Phi}^{(1)} - \frac{p^{(1)}}{k\mu\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)} + \frac{2r^{(1)}}{k^{\frac{3}{2}}R\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)^{\frac{3}{2}}}\left(c_o - \frac{k}{2}\left[\frac{\gamma_o}{\gamma_o + \frac{R^2}{R_i^2}} + \ln\left(\frac{\frac{R^2}{R_i^2}}{\gamma_o + \frac{R^2}{R_i^2}}\right)\right]\right), & R_i < R \le R_o \end{cases}. \tag{3.40d}$$

We now seek a solution of the following form (see Appendix B for a proof of this representation)[6]

$$r^{(1)} = \begin{cases} f_i(R)\cos\Phi, & 0 \le R \le R_i \\ f_o(R)\cos\Phi, & R_i \le R \le R_o \end{cases},$$

$$\phi^{(1)} = \begin{cases} g_i(R)\sin\Phi, & 0 \le R \le R_i \\ g_o(R)\sin\Phi, & R_i \le R \le R_o \end{cases}, \tag{3.41}$$

$$\frac{p^{(1)}}{\mu} = \begin{cases} h_i(R)\cos\Phi, & 0 \le R < R_i \\ h_o(R)\cos\Phi, & R_i < R \le R_o \end{cases}.$$

One should also note that this solution is consistent with the symmetry of the problem and the governing equations, i.e., (3.39) and the equations found when (3.38) and (3.40) are substituted into (3.29) and (3.30). Substituting (3.41) into (3.39) gives

$$g_i(R) = \frac{R}{B}\left(1 - k^{\frac{3}{2}}e^{\frac{3\Omega_o}{4}}\right) - k^{-\frac{1}{2}}e^{-\frac{3\Omega_o}{4}}\left(f_i'(R) + \frac{f_i(R)}{R}\right), \qquad 0 \le R \le R_i, \tag{3.42a}$$

$$g_o(R) = \frac{R}{B}\left[1 - k^{\frac{3}{2}}\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}\right] - k^{-\frac{1}{2}}\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}f_o'(R) - \frac{f_o(R)}{k^{\frac{1}{2}}R\left(1 + \gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}}, \qquad R_i \le R \le R_o. \tag{3.42b}$$

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[6]Solutions with a similar form were discussed in [Kydoniefs and Spencer, 1965, Kydoniefs, 1966].

Substituting (3.41) into (3.29) and (3.30) one obtains

$$f_i''(R) + \frac{f_i'(R)}{R} - \frac{3f_i(R)}{2R^2} - \frac{e^{\frac{\Omega_o}{4}} h_i'(R)}{2k^{\frac{1}{2}}} + k^{\frac{1}{2}} e^{\frac{3\Omega_o}{4}} \left(\frac{g_i'(R)}{2} - \frac{g_i(R)}{R} \right) + \frac{1}{2Bk} \left(k^3 e^{\frac{3\Omega_o}{2}} - 1 \right) = 0, \quad (3.43a)$$

$$2R^3(R^2 + \eta)f_o''(R) + (2R^4 - \eta^2)f_o'(R) - R\left(\frac{R^4}{R^2 + \eta} + 2(R^2 + \eta) \right) f_o(R) - k^{-\frac{1}{2}} R^2 (R^2 + \eta)^{\frac{3}{2}} h_o'(R)$$

$$+ k^{\frac{1}{2}} R^4 (R^2 + \eta)^{\frac{1}{2}} g_o'(R) - 2k^{\frac{1}{2}} R(R^2 + \eta)^{\frac{3}{2}} g_o(R) + \frac{R^3}{Bk} \left((k^3 - 1) R^2 - \eta \right) = 0, \quad (3.43b)$$

$$f_i'(R) + \frac{3f_i(R)}{R} - \frac{e^{\frac{\Omega_o}{4}} h_i(R)}{k^{\frac{1}{2}}} - k^{\frac{1}{2}} e^{\frac{3\Omega_o}{4}} \left(R^2 g_i''(R) + 3Rg_i'(R) - 2g_i(R) \right) + \frac{R}{Bk} \left(k^3 e^{\frac{3\Omega_o}{2}} - 1 \right) = 0, \quad (3.44a)$$

$$2R(R^2 + \eta)g_o(R) - R^2(3R^2 + \eta)g_o'(R) - R^3(R^2 + \eta)g_o''(R) - k^{-1} R^3 h_o(R) + k^{-\frac{1}{2}} (R^2 + \eta)^{\frac{3}{2}} f_o'(R)$$

$$+ k^{-\frac{1}{2}} R\left(3(R^2 + \eta)^{\frac{1}{2}} - \frac{\eta^2}{(R^2 + \eta)^{\frac{3}{2}}} \right) f_o(R) + \frac{R(R^2 + \eta)^{\frac{1}{2}}}{Bk^{\frac{3}{2}}} \left((k^3 - 1) R^2 + k^3 \eta \right) = 0, \quad (3.44b)$$

where $\eta = R_i^2 \gamma_o$. Using (3.42), (3.43), and (3.44), one finds the following third-order linear ODEs for $f_i'(R)$ and $f_o'(R)$

$$f_i^{(4)}(R) + \frac{6f_i'''(R)}{R} + 3\left(\frac{f_i''(R)}{R^2} - \frac{f_i'(R)}{R^3} \right) = 0, \qquad 0 \le R \le R_i, \quad (3.45a)$$

$$(R^2 + \eta)^2 \frac{f_o^{(4)}(R)}{R} + 2(R^2 + \eta)\left(3R^2 - \eta \right) \frac{f_o'''(R)}{R^2} + \left(3R^4 + 2\eta^2 \right) \frac{f_o''(R)}{R^3} - 3f_o'(R)$$

$$= \frac{k^{\frac{1}{2}} \eta^2}{BR^2 (R^2 + \eta)^{\frac{1}{2}}} - \frac{k^2 \eta^2 \left(R^2 + 2\eta \right)}{BR^3 (R^2 + \eta)}, \qquad R_i \le R \le R_o. \quad (3.45b)$$

It then follows that (3.45a) has the following solution

$$f_i(R) = c_{i_1} R^2 + \frac{c_{i_2}}{R^2} + c_{i_3} \ln R + c_{i_4}. \tag{3.46}$$

Enforcing $f_i(0) = 0$ implies that $c_{i_2} = c_{i_3} = c_{i_4} = 0$. $f_i(R)$ is now substituted into (3.42a) to obtain $g_i(R)$, from which $h_i(R)$ is calculated using (3.44a) as

$$g_i(R) = \frac{R}{B} \left(1 - k^{\frac{3}{2}} e^{\frac{3\Omega_o}{4}} \right) - 3c_{i_1} k^{-\frac{1}{2}} e^{-\frac{3\Omega_o}{4}} R, \tag{3.47}$$

$$h_i(R) = 8c_{i_1} k^{\frac{1}{2}} e^{-\frac{\Omega_o}{4}} R + \frac{R}{B} \left(2k^{\frac{5}{2}} e^{\frac{5\Omega_o}{4}} - k e^{\frac{\Omega_o}{2}} - k^{-\frac{1}{2}} e^{-\frac{\Omega_o}{4}} \right). \tag{3.48}$$

After some simplifications, the boundary conditions (3.18) give one the following relations

$$f_o(R_o) = \frac{k^{\frac{1}{2}} \left(R_o^2 + \eta \right)^{\frac{3}{2}}}{R_o} g_o'(R_o), \tag{3.49}$$

$$f_o'(R_o) = \frac{\left(R_o^2 + \eta \right)^{\frac{1}{2}}}{2k^{\frac{1}{2}} R_o} h_o(R_o). \tag{3.50}$$

The continuity of the displacement field at the inclusion-matrix interface implies that

$$f_i(R_i) = f_o(R_i), \qquad g_i(R_i) = g_o(R_i). \tag{3.51}$$

The traction vector is defined as $\mathbf{t} = \langle\!\langle \boldsymbol{\sigma}, \boldsymbol{n} \rangle\!\rangle_{\mathbf{g}}$, which in components reads $t^a(x, \boldsymbol{n}) = \sigma^{ac} g_{bc} n^b$. The unit normal vector to the inclusion-matrix interface to the first order in ε reads $\boldsymbol{n} = \hat{\boldsymbol{r}} + n^\phi_{(1)} \hat{\boldsymbol{\phi}}$, where

$$n^\phi_{(1)} = \frac{f_i(R_i) \sin \Phi}{k^{\frac{1}{2}} e^{\frac{3\Omega_o}{4}} R_i} . \tag{3.52}$$

The continuity of the first-order terms of the traction vector on the inclusion-matrix boundary implies that $\hat{\sigma}^{rr}_{(1)}$ and $\hat{\sigma}^{r\phi}_{(1)} + n^\phi_{(1)} \hat{\sigma}^{\phi\phi}_{(0)}$ must be continuous at $R = R_i$, $-\pi \leq \Phi \leq \pi$. These conditions yield

$$h_o(R_i) - h_i(R_i) = 2k^{\frac{1}{2}} e^{-\frac{3\Omega_o}{4}} \left(f'_o(R_i) - e^{\frac{\Omega_o}{2}} f'_i(R_i) \right), \tag{3.53}$$

$$k^{\frac{1}{2}} R_i^2 e^{\frac{9\Omega_o}{4}} \left(e^{\frac{\Omega_o}{2}} g'_i(R_i) - g'_o(R_i) \right) = f_i(R_i) \left(e^{2\Omega_o} - 1 \right). \tag{3.54}$$

We first find the homogeneous solution of (3.45b) using the power series expansion of $f'_o(R)$ centered at $R = 0$, under the assumption that $f'_o(R)$ can be analytically extended to the interval $[0, R_o]$. Note that depending on whether η is positive or negative, or equivalently, Ω_o is positive or negative, (3.45b) has different solutions. We have the following solutions for the homogeneous part of the differential equation, denoted by $f^p_{o,hom}$ and $f^n_{o,hom}$ for the positive and negative pure dilatational eigenstrain Ω_o, respectively, in the interval $[R_i, R_o]$ as follows[7]

$$f^p_{o,hom}(R) = c^p_{o_1} \frac{R}{(R^2 + \eta)^{\frac{1}{2}}} + c^p_{o_2} \left\{ R (R^2 + \eta)^{\frac{1}{2}} + \eta \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}} \right\} + c^p_{o_3} \left\{ R^2 - \frac{\eta R}{(R^2 + \eta)^{\frac{1}{2}}} \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}} \right\} + c^p_{o_4}, \tag{3.55}$$

$$f^n_{o,hom}(R) = c^n_{o_1} \frac{R}{(R^2 + \eta)^{\frac{1}{2}}} + c^n_{o_2} \left\{ R (R^2 + \eta)^{\frac{1}{2}} + \eta \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}} \right\} + c^n_{o_3} \left\{ R^2 - \frac{\eta R}{(R^2 + \eta)^{\frac{1}{2}}} \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}} \right\} + c^n_{o_4}. \tag{3.56}$$

We now use the method of variation of parameters to find the particular solution of (3.45b), which is denoted by $f^p_{o,par}$ and $f^n_{o,par}$ for the positive and negative values of Ω_o, respectively. After some calculations, the general solution of the differential equation for the positive and negative values of the pure dilatational eigenstrain are obtained as[8]

$$\begin{aligned}
f^p_o(R) &= c^p_{o_1} \frac{R}{(R^2 + \eta)^{\frac{1}{2}}} + c^p_{o_2} \left\{ R (R^2 + \eta)^{\frac{1}{2}} + \eta \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}} \right\} + c^p_{o_3} \left\{ R^2 - \frac{\eta R}{(R^2 + \eta)^{\frac{1}{2}}} \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}} \right\} + c^p_{o_4} \\
&\quad + \frac{k^{\frac{1}{2}}}{16B} \left\{ 2\eta^2 J^p_1(R) - \frac{2k^{\frac{3}{2}} \eta^3 R J^p_2(R)}{(R^2 + \eta)^{\frac{1}{2}}} - \frac{\eta \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}}}{(R^2 + \eta)^{\frac{1}{2}}} \left[\frac{k^{\frac{3}{2}} \eta}{R} + \left(k^{\frac{3}{2}} R + (R^2 + \eta)^{\frac{1}{2}} \right) \ln \frac{R^2}{R^2 + \eta} \right] \right. \\
&\quad - 2R (R^2 + \eta)^{\frac{1}{2}} - 8k^{\frac{3}{2}} R^2 - \frac{R\eta}{(R^2 + \eta)^{\frac{1}{2}}} \ln \frac{\eta}{R^4} + \left(k^{\frac{3}{2}} R - (R^2 + \eta)^{\frac{1}{2}} \right) R \ln \frac{R^2}{R^2 + \eta} \\
&\quad \left. + 2k^{\frac{3}{2}} \eta \ln \frac{\eta}{R^2 + \eta} - 2\eta \ln \left(R + (R^2 + \eta)^{\frac{1}{2}} \right) \right\} = f^p_{o,hom}(R) + f^p_{o,par}(R),
\end{aligned} \tag{3.57}$$

---

[7]Note that one can easily verify that (3.55) and (3.56) are indeed the homogeneous solutions of (3.45b) for $R \in [R_i, R_o]$, and therefore, using the power series method is justified.

[8]The dilogarithm function is defined as: $\mathrm{Li}_2(z) = \sum\limits_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \ln(1-\varsigma) \frac{d\varsigma}{\varsigma}$ for $|z| < 1$.

$$f_o^n(R) = c_{o_1}^n \frac{R}{(R^2 + \eta)^{\frac{1}{2}}} + c_{o_2}^n \left\{ R(R^2 + \eta)^{\frac{1}{2}} + \eta \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}} \right\} + c_{o_3}^n \left\{ R^2 - \frac{\eta R}{(R^2 + \eta)^{\frac{1}{2}}} \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}} \right\}$$

$$+ \frac{k^{\frac{1}{2}}}{16B} \left\{ 2\eta^2 J_1^n(R) - \frac{2k^{\frac{3}{2}} \eta^3 R J_2^n(R)}{(R^2 + \eta)^{\frac{1}{2}}} - \frac{\eta \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}}}{(R^2 + \eta)^{\frac{1}{2}}} \left[\frac{k^{\frac{3}{2}} \eta}{R} + \left(k^{\frac{3}{2}} R + (R^2 + \eta)^{\frac{1}{2}} \right) \ln \frac{R^2}{R^2 + \eta} \right] \right.$$

$$- 2R(R^2 + \eta)^{\frac{1}{2}} - 8k^{\frac{3}{2}} R^2 - \frac{R\eta}{(R^2 + \eta)^{\frac{1}{2}}} \ln \frac{-\eta}{R^4} + \left(k^{\frac{3}{2}} R - (R^2 + \eta)^{\frac{1}{2}} \right) R \ln \frac{R^2}{R^2 + \eta}$$

$$\left. + 2k^{\frac{3}{2}} \eta \ln \frac{-\eta}{R^2 + \eta} - 2\eta \ln \left(R + (R^2 + \eta)^{\frac{1}{2}} \right) \right\} + c_{o_4}^n = f_{o,hom}^n(R) + f_{o,par}^n(R), \tag{3.58}$$

where

$$J_1^p(R) = \int \frac{\sinh^{-1} \frac{\zeta}{\eta^{\frac{1}{2}}}}{\zeta(\zeta^2 + \eta)} d\zeta = \frac{1}{2\eta} \left[\text{Li}_2(-u(R)) - \text{Li}_2(u(R)) + \ln \left(\frac{R^2}{R^2 + \eta} \right) \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}} \right], \tag{3.59}$$

$$J_2^p(R) = \int \frac{\sinh^{-1} \frac{\zeta}{\eta^{\frac{1}{2}}}}{\zeta^3(\zeta^2 + \eta)} d\zeta = -\frac{1}{2\eta^2} \left[\frac{(R^2 + \eta)^{\frac{1}{2}}}{R} + \left(\frac{\eta}{R^2} + \ln \frac{R^2}{R^2 + \eta} \right) \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}} \right.$$

$$\left. + \text{Li}_2(-u(R)) - \text{Li}_2(u(R)) \right], \tag{3.60}$$

$$J_1^n(R) = \int \frac{\cosh^{-1} \frac{\zeta}{(-\eta)^{\frac{1}{2}}}}{\zeta(\zeta^2 + \eta)} d\zeta = \frac{1}{2\eta} \left[\text{Li}_2(-u(R)) - \text{Li}_2(u(R)) + \ln \left(\frac{R^2}{R^2 + \eta} \right) \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}} \right], \tag{3.61}$$

$$J_2^n(R) = \int \frac{\cosh^{-1} \frac{\zeta}{(-\eta)^{\frac{1}{2}}}}{\zeta^3(\zeta^2 + \eta)} d\zeta = -\frac{1}{2\eta^2} \left[\frac{(R^2 + \eta)^{\frac{1}{2}}}{R} + \left(\frac{\eta}{R^2} + \ln \frac{R^2}{R^2 + \eta} \right) \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}} \right.$$

$$\left. + \text{Li}_2(-u(R)) - \text{Li}_2(u(R)) \right], \tag{3.62}$$

and $u(R) = \frac{\eta}{\left(R + (R^2 + \eta)^{\frac{1}{2}} \right)^2}.$[9] The function $g_o(R)$ may now be calculated for the positive and negative values of the eigenstrain by substituting (3.57) and (3.58) for f_o, respectively, into (3.42b). We can then find $h_o(R)$ by substituting for f_o and g_o into (3.44b) (see Appendix A for details). Using (3.40) and (3.41), along with the expressions (3.46), (3.47), and (3.48), the first-order physical components of the Cauchy stress are written as

$$\frac{\hat{\sigma}_{(1)}^{rr}}{\mu} = \begin{cases} -\frac{e^{-\frac{\Omega_o}{4}}}{Bk^{\frac{1}{2}}} \left(4Bk c_{i_1} - k^{\frac{3}{2}} e^{\frac{3\Omega_o}{4}} + 2k^3 e^{\frac{3\Omega_o}{2}} - 1 \right) R \cos \Phi, & 0 \leq R \leq R_i \\ \left[\frac{2k^{\frac{1}{2}} R f_o'(R)}{(R^2 + \eta)^{\frac{1}{2}}} - h_o(R) \right] \cos \Phi, & R_i \leq R \leq R_o \end{cases}, \tag{3.63a}$$

$$\frac{\hat{\sigma}_{(1)}^{r\phi}}{\mu} = \begin{cases} -\frac{k^{\frac{1}{2}} e^{-\frac{\Omega_o}{4}}}{B} \left(4Bc_{i_1} + k^2 e^{\frac{3\Omega_o}{2}} - k^{\frac{1}{2}} e^{\frac{3\Omega_o}{4}} \right) R \sin \Phi, & 0 \leq R < R_i \\ k^{\frac{1}{2}} \left[k^{\frac{1}{2}} R g_o'(R) - \frac{(R^2 + \eta)^{\frac{1}{2}} f_o(R)}{R^2} \right] \sin \Phi, & R_i < R \leq R_o \end{cases}, \tag{3.63b}$$

$$\frac{\hat{\sigma}_{(1)}^{\theta\theta}}{\mu} = \begin{cases} -\frac{e^{-\Omega_o}}{Bk^2} \left(8Bk^{\frac{5}{2}} c_{i_1} e^{\frac{3\Omega_o}{4}} + 2k^{\frac{9}{2}} e^{\frac{9\Omega_o}{4}} - 3k^{\frac{3}{2}} e^{\frac{3\Omega_o}{4}} - k^3 e^{\frac{3\Omega_o}{2}} + 2 \right) R \cos \Phi, & 0 \leq R < R_i \\ -\left[\frac{2R}{Bk^2} \left(1 - k^{\frac{3}{2}} \left(1 + \frac{\eta}{R^2} \right)^{\frac{1}{2}} \right) + h_o(R) \right] \cos \Phi, & R_i < R \leq R_o \end{cases}, \tag{3.63c}$$

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[9] Note that $|u(R)| < 1$ for $R \in [R_i, R_o]$, and therefore, $\text{Li}_2(\pm u(R))$ is well-defined.

$$\frac{\hat{\sigma}^{\phi\phi}_{(1)}}{\mu} = \begin{cases} -\frac{e^{-\frac{\Omega_o}{4}}}{Bk^{\frac{1}{2}}}\left(12Bkc_{i_1} - 3k^{\frac{3}{2}}e^{\frac{3\Omega_o}{4}} + 4k^3 e^{\frac{3\Omega_o}{2}} - 1\right)R\cos\Phi\,, & 0 \le R < R_i \\ \frac{1}{R^2}\left(2k^{\frac{1}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}f_o(R) + 2k\left(R^2+\eta\right)g_o(R) - R^2 h_o(R)\right)\cos\Phi\,, & R_i < R \le R_o \end{cases}. \tag{3.63d}$$

Remark 3.1. Note that when $R_i = R_o$, i.e., when the entire solid torus has a uniform pure dilatational eigenstrain, no residual stresses are generated. In this case, we recover the exact solution, for which $r/R = b/B = e^{\frac{\Omega_o}{2}}$. Note that this is indeed the exact solution, as it is stress-free, and thus, the equilibrium equations are trivially satisfied. Also, it satisfies the incompressibility condition (3.6).

Remark 3.2. One can simply check that in the first-order approximation with respect to the thinness ratio the deformed shapes of the outer boundaries of the inclusion and the matrix remain circular with radii equal to those of their corresponding zero-order approximations, but they become eccentric with eccentricity $E = f_o(R_o) - f_i(R_i)$. Note that the inclusion and the matrix outer boundary points rotate with respect to one another after deformation such that their relative rotation is $\Delta(\Phi) = (g_o(R_o) - g_i(R_i))\sin\Phi$ for any pair of points located at an angle Φ in the initial configuration on the outer boundaries of the inclusion and the matrix.

Next, we proceed to numerically calculate the values of the constants k, c_{i_1}, c_{o_1}, c_{o_2}, c_{o_3}, and c_{o_4} using the expressions (3.49), (3.50), (3.51), (3.53), and (3.54) for a neo-Hookean solid torus with a given negative or positive pure dilatational eigentrain.[10]

Numerical results. We now consider some numerical examples and examine the first-order residual stress field (3.63) for inclusions with different values of pure dilatational eigenstrains and various torus geometries. Figures 3 and 4 show the variation of the radial part of the first-order stress components for a torus with $R_o/B = 0.1$, containing inclusions with several values of R_i/R_o, and $\Omega_o = \pm 0.5$. Notice that all the first-order stress components vary linearly with the material radial coordinate in the inclusion. As expected all the stress components undergo a jump at the inclusion-matrix boundary except the radial stress component, which is continuous at the interface. For the positive eigenstrain case (Figure 3), the maximum shear stress in the inclusion and matrix is first increasing, then decreasing as R_i/R_o increases from zero (a torus without eigenstrain) such that the maximum shear stress for a torus with $R_i/R_o = 0.4$ is greater than that of a torus with $R_i/R_o = 0.2$ and $R_i/R_o = 0.6$ in both the inclusion and the matrix. For the case of negative eigenstrain, the maximum shear stress in the torus increases with the increase in R_i/R_o ratio from 0 to 0.8. After that, however, as R_i/R_o increases the maximum shear stress decreases until it becomes zero when $R_i/R_o = 1$ (Figure 4). Note that $R_i/R_o = 1$ corresponds to the entire torus having a uniform pure dilatational eigenstrain distribution, which is stress-free as was discussed earlier in Remark 3.1.

The contour plots of the first-order residual stress components for a torus with $R_o/B = 0.1$ are depicted in Figures 5 to 7. A torus with an inclusion with a negative pure dilatational eigenstrain and $R_i/R_o = 0.4$ is shown in Figure 5. One observes that the shear stress concentrates across the inclusion-matrix interface with its maximum attained at the top and the bottom. For the selected parameters, the first-order circumferential stress component $\hat{\sigma}^{\phi\phi}_{(1)}/\mu$ is negligible in the inclusion, and hence, the circumferential stress component remains uniform to the first order in the inclusion. This is also true for the positive eigenstrain cases $\Omega_o = 0.5$ and $\Omega_o = 0.7$ with $R_i/R_o = 0.4$ and 0.2, respectively (Figures 6 and 7).

Figure 8a illustrates the dependence of b/B on the pure dilatational eigenstrain value Ω_o for different values of R_i/R_o (Note that B and b represent the distance of the center of the inclusion from the origin in the initial and deformed configurations, respectively (see Figure 2).). For positive eigenstrain values, b/B monotonically increases as Ω_o increases, and as expected, the higher the R_i/R_o ratio, the more rapid the increase. For negative eigenstrains, nevertheless, b/B reaches a minimum, which decreases as R_i/R_o increases, and is attained at lower eigestrain values. As was mentioned earlier for both negative and positive eigenstains, as R_i/R_o approaches 1, the b/B curve gets closer to $e^{\frac{\Omega_o}{2}}$ (see Remark 3.1). The variation of the eccentricity ratio E/R_o, where $E = f_o(R_o) - f_i(R_i)$, with respect to Ω_o is shown in Figure 8b for several values of R_i/R_o. As Ω_o increases from 0, the eccentricity decreases until it reaches its minimum, which increases as R_i/R_o increases, and is attained at lower values of Ω_o.[11] For $\Omega_o < 0$, the eccentricity ratio is first increasing, then decreasing as

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[10]Note that the system of equations for the unknown constants is nonlinear in k and linear with respect to the other constants.
[11]Note that positive and negative eccentricity values correspond to the inclusion moving to the left and right relative to the matrix, respectively.

Ω_o decreases starting from zero. The maximum eccentricity corresponds to the lower values of eigenstrains as R_i/R_o increases. Moreover, the maximum eccentricity first increases as R_i/R_o increases, then it decreases. For instance, the maximum eccentricity for a torus with $R_i/R_o = 0.7$ is greater than that of a torus with $R_i/R_o = 0.5$ and $R_i/R_o = 0.9$ when $\Omega_o < 0$. As R_i/R_o approaches 1, the eccentricity ratio tends to zero for any value of the eigenstrain Ω_o.

3.2 A linear elastic solid torus with small eigenstrains

In this section we derive the governing equations of a solid torus made of an incompressible linear elastic material that has a distribution of small eigenstrains. In geometric elasticity, in order to linearize one starts with a reference motion $\overset{\circ}{\varphi}$ and a one-parameter family of motions φ_ϵ such that $\varphi_{\epsilon=0} = \overset{\circ}{\varphi}$ [Marsden and Hughes, 1983, Yavari and Ozakin, 2008]. Let us consider a one-parameter family of motions φ_ϵ such that $\varphi_\epsilon(R, \Theta, \Phi) = (r_\epsilon(R, \Phi), \Theta, \phi_\epsilon(R, \Phi))$. We will linearize about the stress-free configuration $\overset{\circ}{\varphi}(R, \Theta, \Phi) = (R, \Theta, \Phi)$, i.e., $r_{\epsilon=0}(R, \Phi) = R$ and $\phi_{\epsilon=0}(R, \Phi) = \Phi$. The variation field is defined as

$$\delta\varphi(R, \Theta, \Phi) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_\epsilon(R, \Theta, \Phi) = (u(R, \Phi), 0, w(R, \Phi)), \qquad (3.64)$$

where u and w are the non-zero displacement components.

Linearization of the incompressibility constraint. For any motion in the given one-parameter family we have

$$J_\epsilon = \frac{r_\epsilon(b_\epsilon + r_\epsilon \cos\phi_\epsilon)}{e^{\frac{3}{2}\Omega_\epsilon(R,\Phi)}R(B + R\cos\Phi)} \left(\frac{\partial r_\epsilon}{\partial R}\frac{\partial \phi_\epsilon}{\partial \Phi} - \frac{\partial r_\epsilon}{\partial \Phi}\frac{\partial \phi_\epsilon}{\partial R}\right) = 1. \qquad (3.65)$$

Taking derivative with respect to ϵ of both sides and evaluating at $\epsilon = 0$, one obtains

$$u + \frac{R\left[\delta b + u\cos\Phi - wR\sin\Phi\right]}{B + R\cos\Phi} + R(u_{,R} + w_{,\Phi}) = \frac{3}{2}R\delta\Omega. \qquad (3.66)$$

Similarly, from (3.9), one obtains the variation of I and β as

$$\delta I = -2\delta\beta = 2\left(-\delta\Omega + \frac{u}{R} + u_{,R} + w_{,\Phi}\right). \qquad (3.67)$$

Therefore, it follows from (3.67) that $\delta I_1 = \delta I_2 = 0$. To simplify the calculations, we assume that the material is piecewise homogeneous and use (3.11) and (3.67) to find the linearized components of the Cauchy stress tensor as

$$\delta\sigma^{rr} = -\delta p + 2\left(W_{I_1} + W_{I_2}\right)\left(2u_{,R} - \delta\Omega\right), \qquad (3.68a)$$

$$\delta\sigma^{r\phi} = 2\left(W_{I_1} + W_{I_2}\right)\left(w_{,R} + \frac{u_{,\Phi}}{R^2}\right), \qquad (3.68b)$$

$$\delta\sigma^{\theta\theta} = -\frac{\delta p}{(B + R\cos\Phi)^2} + \frac{4\left(W_{I_1} + 2W_{I_2}\right)}{(B + R\cos\Phi)^3}\left[\delta b + u\cos\Phi - wR\sin\Phi\right]$$
$$+ \frac{2}{(B + R\cos\Phi)^2}\left[2W_{I_2}\left(\frac{u}{R} + u_{,R} + w_{,\Phi}\right) - \delta\Omega\left(W_{I_1} + 4W_{I_2}\right)\right], \qquad (3.68c)$$

$$\delta\sigma^{\phi\phi} = -\frac{\delta p}{R^2} + \frac{4\left(W_{I_1} + W_{I_2}\right)}{R^2}\left(\frac{u}{R} + w_{,\Phi} - \frac{\delta\Omega}{2}\right). \qquad (3.68d)$$
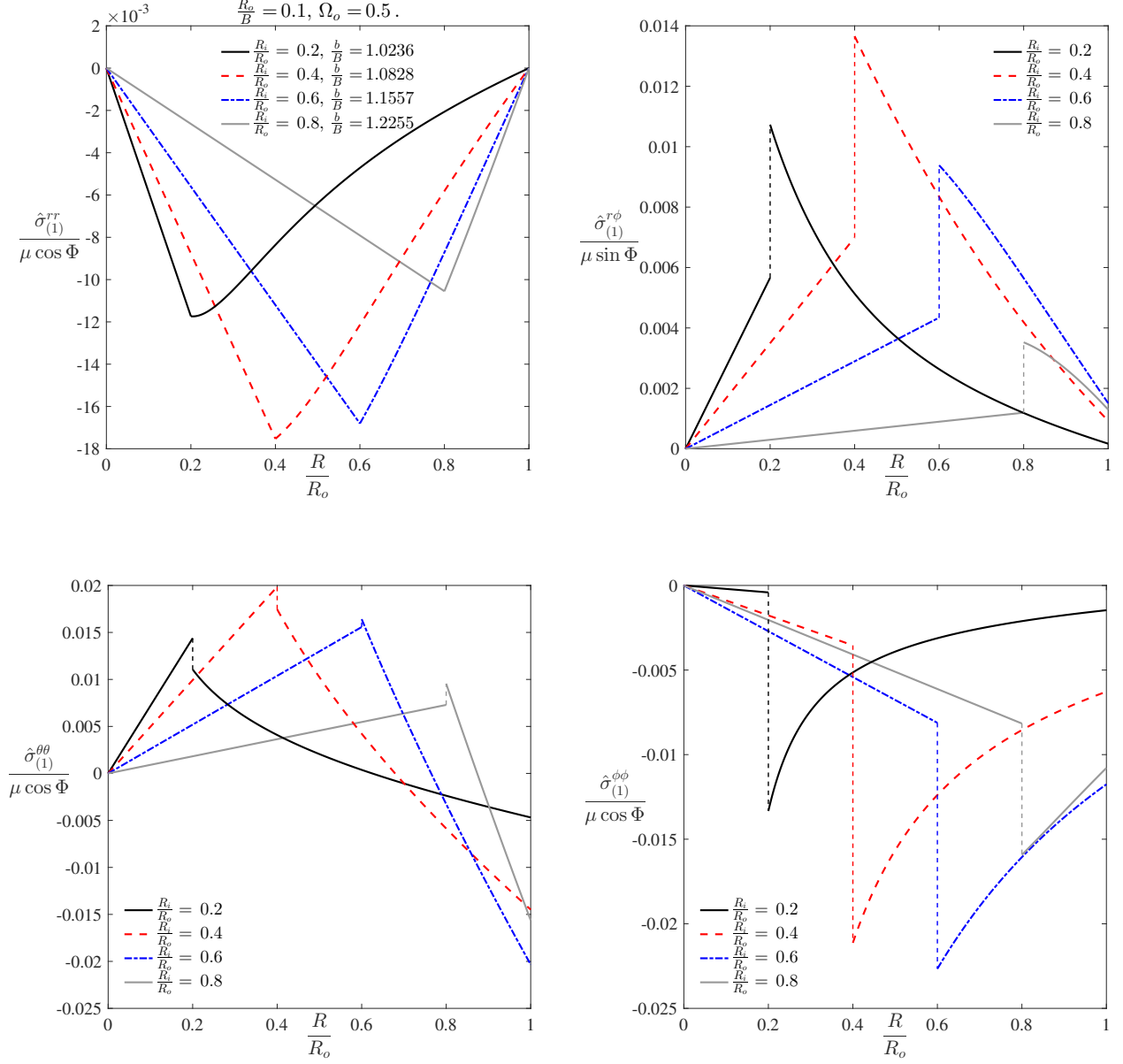
Figure 3: *The radial part of the first-order normalized components of the Cauchy stress tensor for $\frac{R_o}{B} = 0.1$, $\Omega_o = 0.5$, and different values of $\frac{R_i}{R_o}$.*
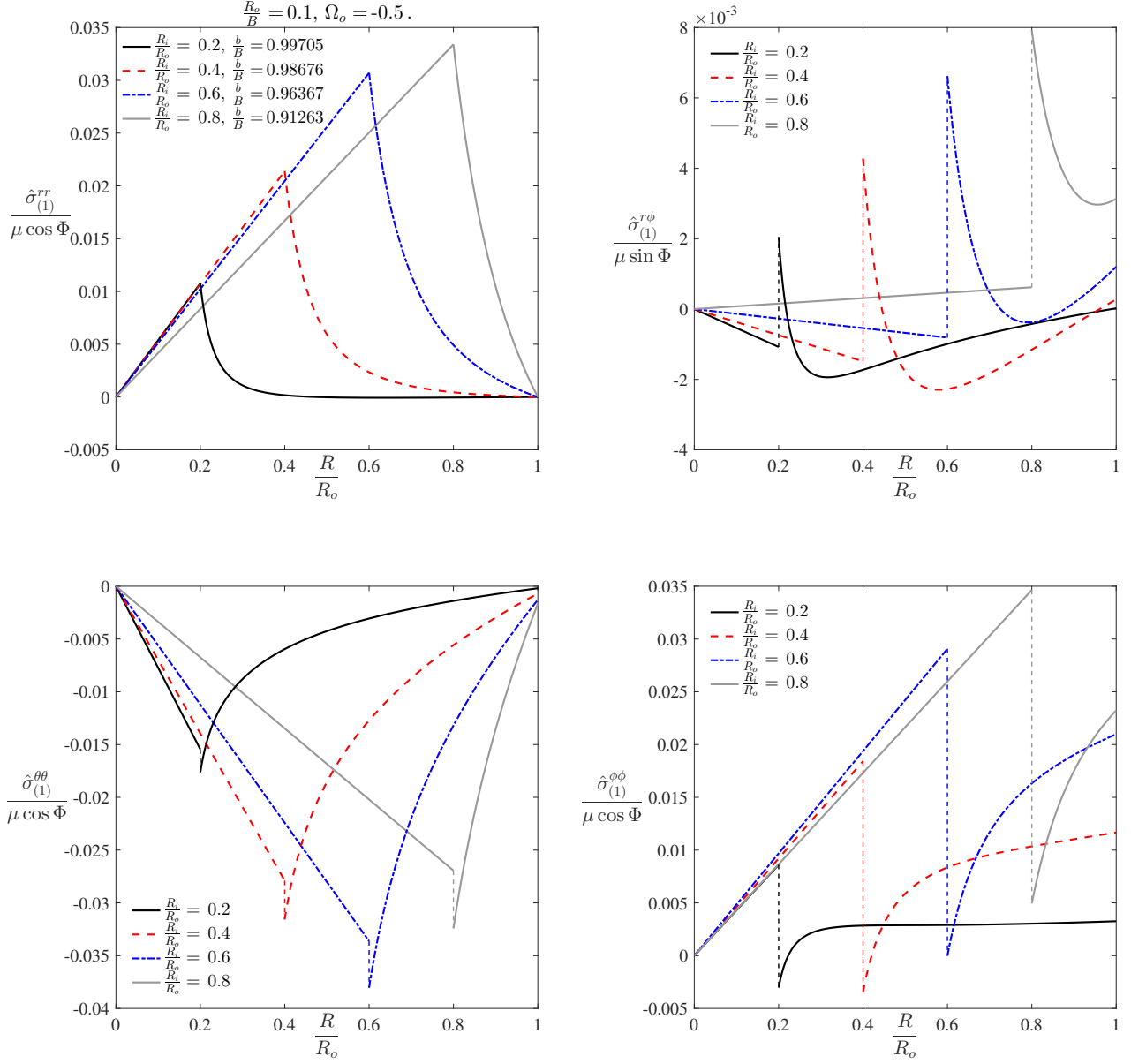
Figure 4: *The radial part of the first-order normalized components of the Cauchy stress tensor for $\frac{R_o}{B} = 0.1$, $\Omega_o = -0.5$, and different values of $\frac{R_i}{R_o}$.*
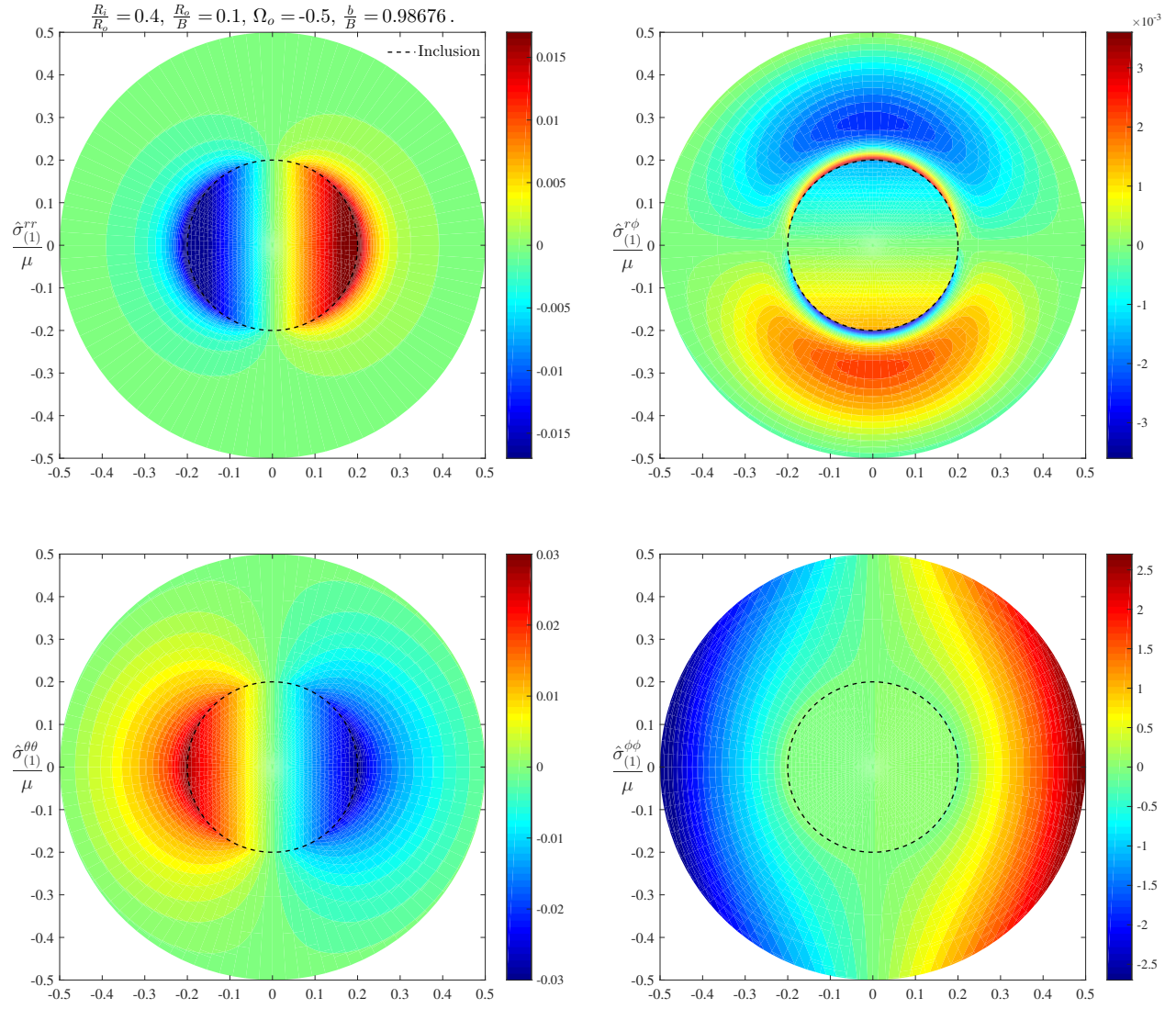
Figure 5: *The first-order physical components of the Cauchy stress in a torus having an inclusion with $\frac{R_i}{R_o} = 0.4$, $\frac{R_o}{B} = 0.1$, and constant pure dilatational eigenstrain distribution $\Omega_o = -0.5$. The ratio of the deformed major radius to the initial major radius of the torus is $\frac{b}{B} = 0.98676$.*

Figure 6: *The first-order physical components of the Cauchy stress in a torus having an inclusion with $\frac{R_i}{R_o} = 0.4$, $\frac{R_o}{B} = 0.1$, and constant pure dilatational eigenstrain distribution $\Omega_o = 0.5$. The ratio of the deformed major radius to the initial major radius of the torus is $\frac{b}{B} = 1.0828$.*
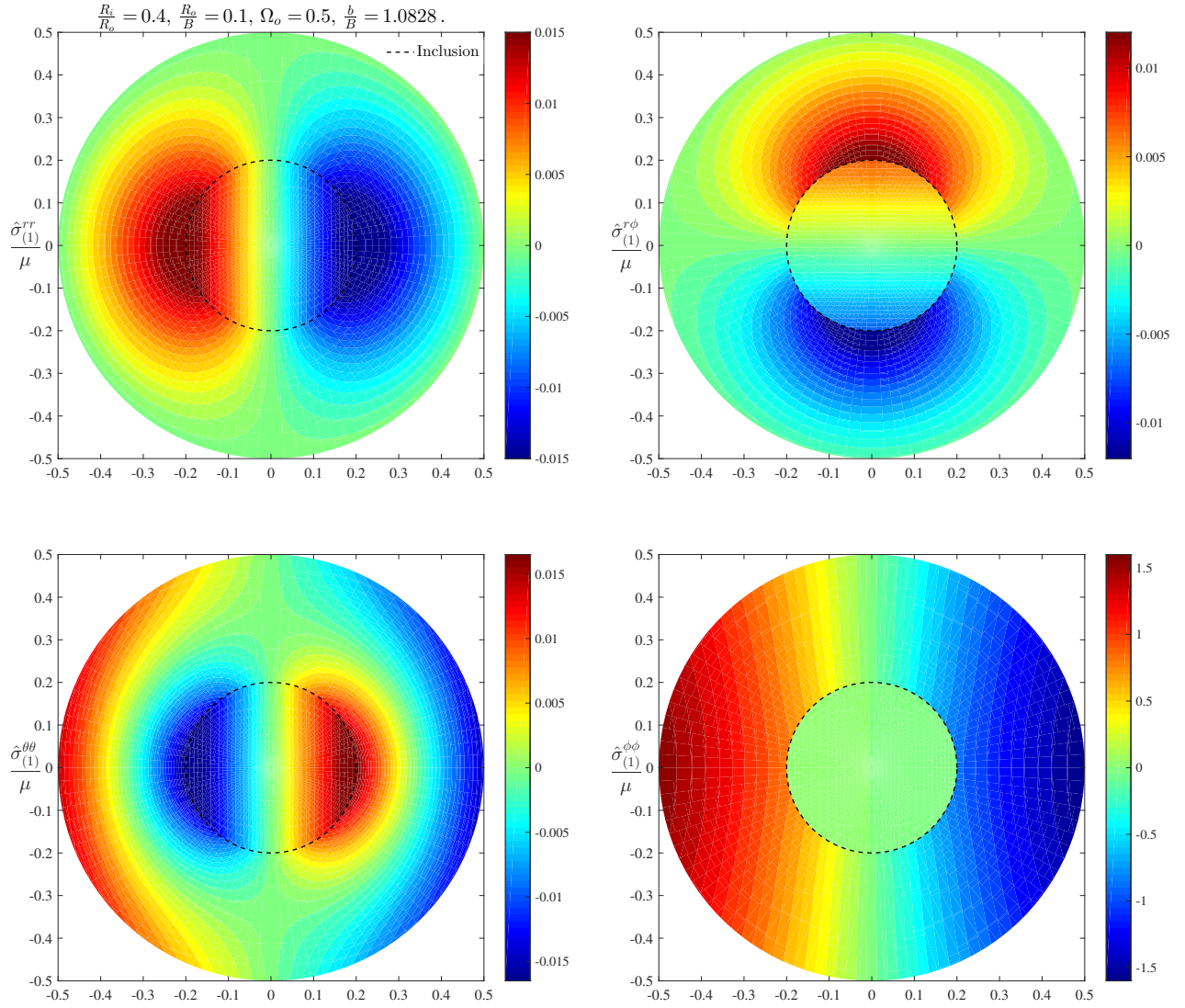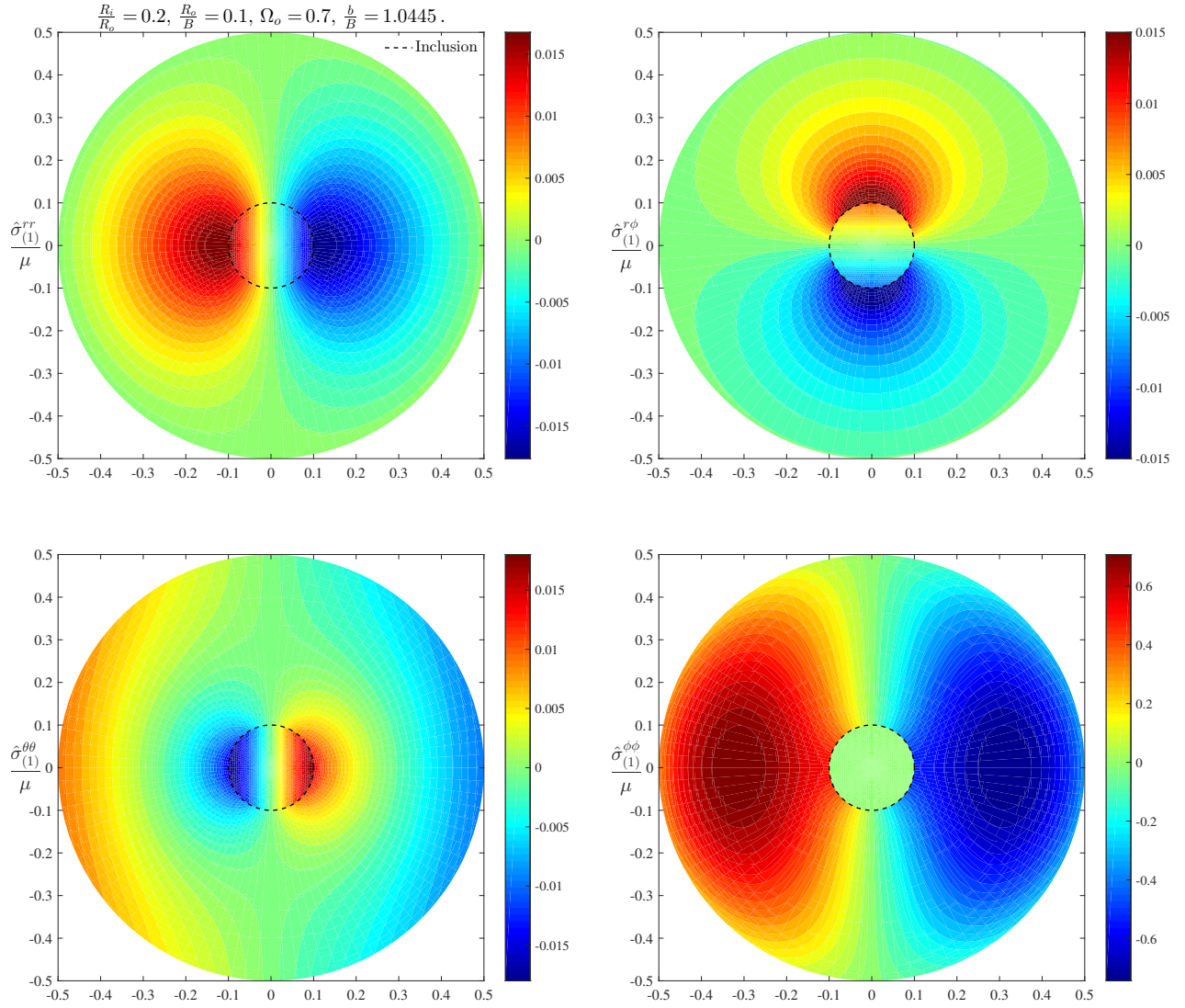
Figure 7: *The first-order physical components of the Cauchy stress in a torus having an inclusion with $\frac{R_i}{R_o} = 0.2$, $\frac{R_o}{B} = 0.1$, and constant pure dilatational eigenstrain distribution $\Omega_o = 0.7$. The ratio of the deformed major radius to the initial major radius of the torus is $\frac{b}{B} = 1.0445$.*
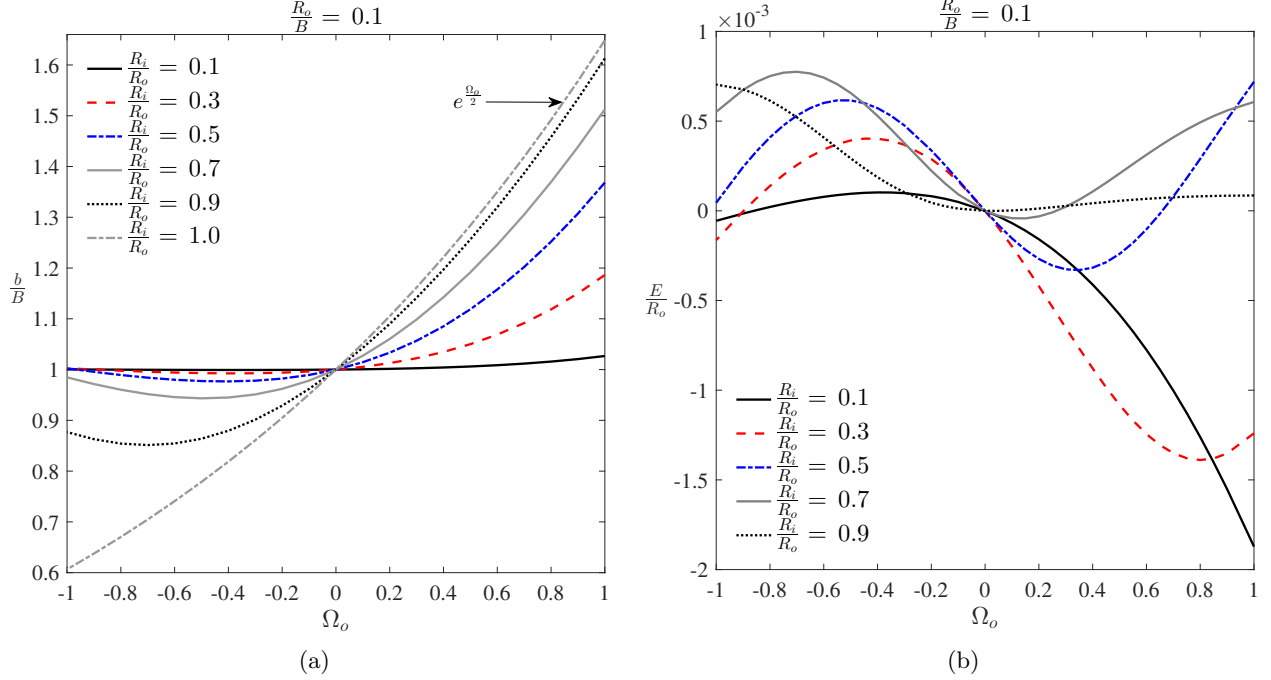
Figure 8: *Variations of $\frac{b}{B}$ and the eccentricity ratio $\frac{E}{R_o}$ as functions of Ω_o for a torus with $\frac{R_o}{B} = 0.1$ and different values of $\frac{R_i}{R_o}$, shown in (a) and (b), respectively.*

Linearization of the equilibrium equations. Using (3.17), linearizing the equilibrium equations (3.15) and (3.16) one obtains

$$\delta\sigma^{rr}_{,R} + \delta\sigma^{r\phi}_{,\Phi} + \left(\frac{1}{R} + \frac{\cos\Phi}{B + R\cos\Phi}\right)\delta\sigma^{rr} - \cos\Phi\,(B + R\cos\Phi)\,\delta\sigma^{\theta\theta} - \frac{R\sin\Phi}{B + R\cos\Phi}\delta\sigma^{r\phi} - R\delta\sigma^{\phi\phi} = 0\,, \quad (3.69)$$

$$\delta\sigma^{r\phi}_{,R} + \delta\sigma^{\phi\phi}_{,\Phi} + \left(\frac{3}{R} + \frac{\cos\Phi}{B + R\cos\Phi}\right)\delta\sigma^{r\phi} + \frac{\sin\Phi}{R}\,(B + R\cos\Phi)\,\delta\sigma^{\theta\theta} - \frac{R\sin\Phi}{B + R\cos\Phi}\delta\sigma^{\phi\phi} = 0\,. \quad (3.70)$$

Substituting the linearized stress components given by (3.68) into the above equations one finds

$$\delta p_{,R} + \frac{4\,(W_{I_1} + W_{I_2})}{R}\left[\frac{u}{R} - u_{,R} - Ru_{,RR} - \frac{u_{,\Phi\Phi}}{2R} - \frac{Rw_{,R\Phi}}{2} + w_{,\Phi} + \frac{R\delta\Omega_{,R}}{2}\right]$$
$$+ \frac{\cos\Phi}{B + R\cos\Phi}\left[-4W_{I_1}u_{,R} + 4W_{I_2}\left(\frac{u}{R} + w_{,\Phi}\right) - 6W_{I_2}\delta\Omega\right] + \frac{2R\,(W_{I_1} + W_{I_2})\sin\Phi}{B + R\cos\Phi}\left(\frac{u_{,\Phi}}{R^2} + w_{,R}\right)$$
$$+ \frac{4\,(W_{I_1} + 2W_{I_2})\cos\Phi}{(B + R\cos\Phi)^2}\,(\delta b + u\cos\Phi - Rw\sin\Phi) = 0\,, \quad (3.71)$$

$$\frac{R\delta p_{,\Phi}}{2} + \frac{R\sin\Phi}{B + R\cos\Phi}\left[2W_{I_1}\,(u + Rw_{,\Phi}) + RW_{I_2}\,(3\delta\Omega - 2u_{,R})\right]$$
$$+ \frac{2R^2\sin\Phi\,(W_{I_1} + 2W_{I_2})}{(B + R\cos\Phi)^2}\,[Rw\sin\Phi - u\cos\Phi - \delta b] - \frac{(W_{I_1} + W_{I_2})\,(3B + 4R\cos\Phi)}{B + R\cos\Phi}\,[u_{,\Phi} + R^2w_{,R}]$$
$$- R\,(W_{I_1} + W_{I_2})\,[u_{,R\Phi} + R^2w_{,RR} + 2w_{,\Phi\Phi} - \delta\Omega_{,\Phi}] = 0\,. \quad (3.72)$$

Linearization of the boundary conditions. Similarly, boundary conditions (3.18) are linearized and are written as

$$2\,(W_{I_1} + W_{I_2})\,(2u_{,R} - \delta\Omega) - \delta p = 0\,, \qquad R = R_o\,, \quad -\pi \le \Phi \le \pi\,, \quad (3.73a)$$

$$w_{,R} + \frac{u_{,\Phi}}{R^2} = 0\,, \qquad R = R_o\,, \quad -\pi \le \Phi \le \pi\,. \tag{3.73b}$$

Example: A toroidal inclusion with uniform pure dilatational eigenstrains in an incompressible linear elastic solid torus. Let us consider the following distribution of eigenstrains in the torus

$$\delta\Omega(R) = \begin{cases} \delta\Omega_o\,, & 0 \le R < R_i \\ 0\,, & R_i < R \le R_o \end{cases}\,. \tag{3.74}$$

Pressure and displacement fields in the torus are written as

$$\begin{aligned}
u(R, \Phi) &= \begin{cases} u_i(R, \Phi), & 0 \le R < R_i \\ u_o(R, \Phi), & R_i < R \le R_o \end{cases}, \\
w(R, \Phi) &= \begin{cases} w_i(R, \Phi), & 0 \le R < R_i \\ w_o(R, \Phi), & R_i < R \le R_o \end{cases}, \\
\delta p(R, \Phi) &= \begin{cases} \delta p_i(R, \Phi), & 0 \le R < R_i \\ \delta p_o(R, \Phi), & R_i < R \le R_o \end{cases}.
\end{aligned} \tag{3.75}$$

Therefore, from (3.66) it follows that

$$u_i + \frac{R\left[\delta b + u_i \cos\Phi - w_i R \sin\Phi\right]}{B + R\cos\Phi} + R(u_{i,R} + w_{i,\Phi}) = \frac{3}{2} R\delta\Omega_o\,, \tag{3.76a}$$

$$u_o\left(B + R\cos\Phi\right) + R\left[\delta b + u_o \cos\Phi - w_o R \sin\Phi\right] + R(B + R\cos\Phi)(u_{o,R} + w_{o,\Phi}) = 0\,. \tag{3.76b}$$

From the continuity of the displacement field at the inclusion-matrix interface, we know that

$$u_i(R_i, \Phi) = u_o(R_i, \Phi)\,, \qquad w_i(R_i, \Phi) = w_o(R_i, \Phi)\,. \tag{3.77}$$

Also, we eliminate the rigid body motion by setting $u_i(0, \Phi) = 0$ and $w(R, 0) = 0$.

We next show that for a torus made of an isotropic incompressible linear elastic solid with a toroidal inclusion having a non-zero uniform pure dilatational eigenstrain distribution, the stress field inside the inclusion cannot be uniform. Let us assume that the stress field inside the inclusion is uniform, i.e., each physical Cauchy stress component is constant. Thus

$$\delta\sigma^{rr} = c_1\,, \quad R\delta\sigma^{r\phi} = c_2\,, \quad (B + R\cos\Phi)^2 \delta\sigma^{\theta\theta} = c_3\,, \quad R^2\delta\sigma^{\phi\phi} = c_4\,, \tag{3.78}$$

where c_1 to c_4 are some constants. After some simplifications, it follows from (3.69) that

$$\frac{c_1 - c_4}{R} + \frac{1}{B + R\cos\Phi}\left(c_1 \cos\Phi - c_3 \cos\Phi - c_2 \sin\Phi\right) = 0\,, \tag{3.79}$$

which implies that $c_1 = c_3 = c_4 = C$ and $c_2 = 0$. Therefore (note that the equilibrium equation (3.70) has already been satisfied)

$$\delta\sigma^{rr} = (B + R\cos\Phi)^2 \delta\sigma^{\theta\theta} = R^2\delta\sigma^{\phi\phi} = C\,, \quad 0 \le R < R_i\,, \tag{3.80}$$

$$\delta\sigma^{r\phi} = 0\,, \qquad\qquad\qquad\qquad\qquad 0 \le R < R_i\,. \tag{3.81}$$

From (3.68) and $\delta\sigma^{rr} - R^2\delta\sigma^{\phi\phi} = 0$, one obtains

$$\frac{u}{R} + w_{,\Phi} = u_{,R}. \tag{3.82}$$

Using the above relation in $\delta\sigma^{rr} - (B + R\cos\Phi)^2 \delta\sigma^{\theta\theta} = 0$, one finds

$$\delta b + u\cos\Phi - Rw\sin\Phi = (B + R\cos\Phi)\frac{2u_{,R}\left(W_{I_1} - W_{I_2}\right) + 3W_{I_2}\delta\Omega_o}{2\left(W_{I_1} + 2W_{I_2}\right)}\,. \tag{3.83}$$

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Similarly, (3.81) implies that

$$w_{,R} + \frac{u_{,\Phi}}{R^2} = 0\,. \tag{3.84}$$

We then use (3.83) and the incompressibility condition (3.76a), along with $u(0, \Phi) = 0$ to conclude that

$$u(R, \Phi) = \frac{\delta\Omega_o}{2}R. \tag{3.85}$$

Substituting the above relation into (3.82) and (3.84), one concludes that $w = 0$. Now going back to (3.83) one finally finds that

$$\delta b = \frac{\delta\Omega_o}{2}B\,. \tag{3.86}$$

This is a contradiction because δb has to depend on the radius of the inclusion R_i. In other words, the above relation is telling us that the change in the overall radius of the solid torus after deformation is independent of the size of the inclusion. In particular, when $R_i \to 0$ we expect $\delta b \to 0$, which is not what (3.86) predicts. This contradiction shows that the stress field inside the inclusion cannot be uniform.

4 Conclusions

In this paper, we studied the residual stress and deformation fields of a solid torus containing a toroidal inclusion with finite eigenstrains that is concentric with the solid torus. We used a perturbation analysis and obtained the stress and displacement fields to the first order in the thinness ratio. We showed that the stress field in the toroidal inclusion is nonuniform, unlike cylindrical and spherical inclusions in infinitely-long and finite circular cylindrical bars and spherical balls, respectively, in which the stress field inside the inclusion is uniform. We presented some numerical results for a neo-Hookean solid torus having an inclusion with a uniform pure dilatational eigenstrain distribution. In particular, we observed that all the first-order stress components in the inclusion have a linear dependence on the referential radial coordinate. Moreover, the maximum shear stress in the torus is first increasing, then decreasing as the relative size of the inclusion increases from zero. We observed shear stress concentration regions across the inclusion-matrix interface for a torus with a negative pure dilatational eigenstrain distribution. Interestingly, the torus exhibits different responses for positive and negative eigenstrain values. It was observed that for the positive eigenstrians, b/B monotonically increases as the eigenstrain Ω_o increases, and the increase is more rapid for inclusions with larger relative sizes. For negative eigesntrains, nonetheless, b/B reaches a minimum, the value of which decreases as the relative size of the inclusion becomes larger. We noticed that in the first-order approximation with respect to the thinness ratio the deformed shapes of the outer boundaries of the matrix and the inclusion are eccentric circles with radii equal to those of their corresponding zero-order approximations. Finally, we proved that the stress field inside a toroidal inclusion with nonzero uniform pure dilatational (infinitesimal) eigenstrains in an isotropic incompressible linear elastic solid torus is always nonuniform for any size of the solid torus.

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Appendix A   Analytical expressions for the functions $g_o(R)$ and $h_o(R)$

$$
\begin{aligned}
g_o^p(R) = {} & -\frac{1}{k^{\frac{1}{2}} R \left(R^2 + \eta\right)^{\frac{1}{2}}} \left[c_{o_1}^p \left(R^2 + \eta\right)^{\frac{1}{2}} + c_{o_2}^p \left(\left(R^2 + \eta\right)^{\frac{1}{2}} \left(3R^2 + 2\eta\right) + R\eta \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}} \right) + R c_{o_4}^p \right. \\
& + c_{o_3}^p \left(R\left(3R^2 + \eta\right) - \eta \left(R^2 + \eta\right)^{\frac{1}{2}} \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}} \right) \right] + \frac{1}{16 B R^3 \left(R^2 + \eta\right)^{\frac{1}{2}}} \left[k^{\frac{3}{2}} R \left(8R^4 + 2R^2\eta + \eta^2\right) \right. \\
& + 2R^2 \left(R^2 + \eta\right)^{\frac{1}{2}} \left(11R^2 + \eta\right) + \eta^2 k^{\frac{3}{2}} \left(R^2 + \eta\right)^{\frac{1}{2}} \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}} - k^{\frac{3}{2}} \eta R^3 \ln(R^2\eta) \\
& + R^2\eta \left(R + k^{\frac{3}{2}} \left(R^2 + \eta\right)^{\frac{1}{2}} \right) \sinh^{-1} \frac{R}{\eta^{\frac{1}{2}}} \ln \frac{R^2}{R^2 + \eta} + R^2 \left(\left(R^2 + \eta\right)^{\frac{1}{2}} - k^{\frac{3}{2}} R \right) \left(3R^2 \ln \frac{R^2}{R^2 + \eta} \right. \\
& \left. + \eta \ln \eta \right) + R^2\eta \left(3k^{\frac{3}{2}} R - 2 \left(R^2 + \eta\right)^{\frac{1}{2}} \right) \ln(R^2 + \eta) + 2R^2\eta^2 \left\{ \frac{R}{\eta} \ln \left(R + \left(R^2 + \eta\right)^{\frac{1}{2}} \right) - R J_1^p(R) \right. \\
& \left. \left. + \eta k^{\frac{3}{2}} \left(R^2 + \eta\right)^{\frac{1}{2}} J_2^p(R) \right\} \right],
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
g_o^n(R) = {} & -\frac{1}{k^{\frac{1}{2}} R \left(R^2 + \eta\right)^{\frac{1}{2}}} \left[c_{o_1}^n \left(R^2 + \eta\right)^{\frac{1}{2}} + c_{o_2}^n \left(\left(R^2 + \eta\right)^{\frac{1}{2}} \left(3R^2 + 2\eta\right) + R\eta \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}} \right) + R c_{o_4}^n \right. \\
& + c_{o_3}^n \left(R\left(3R^2 + \eta\right) - \eta \left(R^2 + \eta\right)^{\frac{1}{2}} \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}} \right) \right] + \frac{1}{16 B R^3 \left(R^2 + \eta\right)^{\frac{1}{2}}} \left[k^{\frac{3}{2}} R \left(8R^4 + 2R^2\eta + \eta^2\right) \right. \\
& + 2R^2 \left(R^2 + \eta\right)^{\frac{1}{2}} \left(11R^2 + \eta\right) + \eta^2 k^{\frac{3}{2}} \left(R^2 + \eta\right)^{\frac{1}{2}} \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}} - k^{\frac{3}{2}} \eta R^3 \ln(-R^2\eta) \\
& + R^2\eta \left(R + k^{\frac{3}{2}} \left(R^2 + \eta\right)^{\frac{1}{2}} \right) \cosh^{-1} \frac{R}{(-\eta)^{\frac{1}{2}}} \ln \frac{R^2}{R^2 + \eta} + R^2 \left(\left(R^2 + \eta\right)^{\frac{1}{2}} - k^{\frac{3}{2}} R \right) \left(3R^2 \ln \frac{R^2}{R^2 + \eta} \right. \\
& \left. + \eta \ln(-\eta) \right) + R^2\eta \left(3k^{\frac{3}{2}} R - 2 \left(R^2 + \eta\right)^{\frac{1}{2}} \right) \ln(R^2 + \eta) + 2R^2\eta^2 \left\{ \frac{R}{\eta} \ln \left(R + \left(R^2 + \eta\right)^{\frac{1}{2}} \right) - R J_1^n(R) \right. \\
& \left. \left. + \eta k^{\frac{3}{2}} \left(R^2 + \eta\right)^{\frac{1}{2}} J_2^n(R) \right\} \right],
\end{aligned}
\tag{A.2}
$$

$$h_o^p(R) = \frac{1}{16Bk^{\frac{1}{2}}R^3\left(R^2+\eta\right)^2}\left[-\left(R^2+\eta\right)^{\frac{1}{2}}\left\{16R^3\left(R^2+\eta\right)^2 + k^3R\left(2R^2+\eta\right)\left(16R^2\left(R^2+\eta\right)-\eta^2\right)\right\}\right.$$

$$+ R^2k^{\frac{3}{2}}\left\{2R\left(4R^4+7R^2\eta+2\eta^2\right)\left(k^{\frac{3}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}-R\right) + k^{\frac{3}{2}}\eta^2\left(3R\left(R^2+\eta\right)^{\frac{1}{2}}\right.\right.$$

$$\left.\left.+ \eta\sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}\right)\right\}\ln\frac{R^2}{R^2+\eta} - 2k^{\frac{3}{2}}R^2\left(R^2+\eta\right)\left(16R^4+14R^2\eta-\eta^2\right) + 2\eta^3k^{\frac{3}{2}}R^2\ln\frac{\eta^{\frac{1}{2}}}{R^2+\eta}$$

$$+ k^3\eta^4\left\{2\eta R^2 J_2^p(R) + \sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}\right\} + 16BkR^2\left\{-c_{o_1}^p\eta^2 + 2c_{o_2}^p\left(R^2+\eta\right)^2\left(4R^2-\eta\right)\right.$$

$$\left.\left.+ c_{o_3}^p R\left(R^2+\eta\right)^{\frac{1}{2}}\left(8R^4+14R^2\eta+7\eta^2\right) + c_{o_3}^p\eta^3\sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}\right\}\right], \tag{A.3}$$

$$h_o^n(R) = \frac{1}{16Bk^{\frac{1}{2}}R^3\left(R^2+\eta\right)^2}\left[-\left(R^2+\eta\right)^{\frac{1}{2}}\left\{16R^3\left(R^2+\eta\right)^2 + k^3R\left(2R^2+\eta\right)\left(16R^2\left(R^2+\eta\right)-\eta^2\right)\right\}\right.$$

$$+ R^2k^{\frac{3}{2}}\left\{2R\left(4R^4+7R^2\eta+2\eta^2\right)\left(k^{\frac{3}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}-R\right) + k^{\frac{3}{2}}\eta^2\left(3R\left(R^2+\eta\right)^{\frac{1}{2}}\right.\right.$$

$$\left.\left.+ \eta\cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}\right)\right\}\ln\frac{R^2}{R^2+\eta} - 2k^{\frac{3}{2}}R^2\left(R^2+\eta\right)\left(16R^4+14R^2\eta-\eta^2\right) + 2\eta^3k^{\frac{3}{2}}R^2\ln\frac{(-\eta)^{\frac{1}{2}}}{R^2+\eta}$$

$$+ k^3\eta^4\left\{2\eta R^2 J_2^n(R) + \cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}\right\} + 16BkR^2\left\{-c_{o_1}^n\eta^2 + 2c_{o_2}^n\left(R^2+\eta\right)^2\left(4R^2-\eta\right)\right.$$

$$\left.\left.+ c_{o_3}^n R\left(R^2+\eta\right)^{\frac{1}{2}}\left(8R^4+14R^2\eta+7\eta^2\right) + c_{o_3}^n\eta^3\cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}\right\}\right]. \tag{A.4}$$

Appendix B Proof of the separability of the first-order deformation and pressure fields in the form given in (3.41)

Let us represent $r^{(1)}$, $\phi^{(1)}$, and $p^{(1)}/\mu$ by appropriate Fourier series expansions, given that they are even, odd, and even 2π-periodic functions,[12] respectively, as

$$r^{(1)} = \frac{r_0^{(1)}(R)}{2} + \sum_{n=1}^{\infty} r_n^{(1)}(R)\cos\left(n\Phi\right),$$

$$\phi^{(1)} = \sum_{n=1}^{\infty} \phi_n^{(1)}(R)\sin\left(n\Phi\right), \tag{B.1}$$

$$\frac{p^{(1)}}{\mu} = \frac{p_0^{(1)}(R)}{2\mu} + \sum_{n=1}^{\infty}\frac{p_n^{(1)}(R)}{\mu}\cos\left(n\Phi\right),$$

where for $n \in \mathbb{N} \cup \{0\}$, $r_n^{(1)}$, $\phi_n^{(1)}$, and $\frac{p_n^{(1)}}{\mu}$ are the real-valued Fourier coefficients given by

$$r_n^{(1)}(R) = \frac{1}{\pi}\int_{-\pi}^{\pi} r^{(1)}(R,\zeta)\cos(n\zeta)d\zeta,$$

$$\phi_n^{(1)}(R) = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi^{(1)}(R,\zeta)\sin(n\zeta)d\zeta, \tag{B.2}$$

$$\frac{p_n^{(1)}}{\mu} = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{p^{(1)}}{\mu}(R,\zeta)\cos(n\zeta)d\zeta.$$

---

[12]This immediately follows from the symmetry of the problem for radially-symmetric eigenstrain distributions.

We now show that all the Fourier coefficients vanish except those with $n = 1$, giving one what one has in (3.41). Substituting (B.1) into (3.39), one obtains the following ODEs in the inclusion and the matrix

$$n\phi_n^{(1)} = -k^{-\frac{1}{2}} e^{-\frac{3\Omega_o}{4}} \left(r_n^{(1)'} + \frac{r_n^{(1)}}{R} \right), \qquad 0 \leq R \leq R_i, \quad n \geq 2,$$

$$n\phi_n^{(1)} = -k^{-\frac{1}{2}} \left(1 + \frac{\eta}{R^2} \right)^{-\frac{1}{2}} \left[\left(1 + \frac{\eta}{R^2} \right) r_n^{(1)'} + \frac{r_n^{(1)}}{R} \right], \qquad R_i \leq R \leq R_o, \quad n \geq 2,$$

$$r_0^{(1)'} + \frac{r_0^{(1)}}{R} = 0, \qquad 0 \leq R \leq R_i,$$

$$\left(1 + \frac{\eta}{R^2} \right) r_0^{(1)'} + \frac{r_0^{(1)}}{R} = 0, \qquad R_i \leq R \leq R_o,$$

(B.3)

where $\eta = R_i^2 \gamma_o$. Similarly, we substitute (B.1) into (3.29) and (3.30) to find the following ODEs for $n \geq 2$ and $n = 0$:

$$r_n^{(1)''} + \frac{r_n^{(1)'}}{R} - \frac{(2+n^2) r_n^{(1)}}{2R^2} - \frac{e^{\frac{\Omega_o}{4}}}{2k^{\frac{1}{2}}} \frac{p_n^{(1)'}}{\mu} + nk^{\frac{1}{2}} e^{\frac{3\Omega_o}{4}} \left(\frac{\phi_n^{(1)'}}{2} - \frac{\phi_n^{(1)}}{R} \right) = 0, \qquad 0 \leq R \leq R_i, \quad \text{(B.4a)}$$

$$2R^3(R^2 + \eta) r_n^{(1)''} + (2R^4 - \eta^2) r_n^{(1)'} - R \left(\frac{R^4}{R^2 + \eta} + \left(1 + n^2 \right) (R^2 + \eta) \right) r_n^{(1)}$$

$$- k^{-\frac{1}{2}} R^2 (R^2 + \eta)^{\frac{3}{2}} \frac{p_n^{(1)'}}{\mu} + nk^{\frac{1}{2}} R^4 (R^2 + \eta)^{\frac{1}{2}} \phi_n^{(1)'} - 2nk^{\frac{1}{2}} R(R^2 + \eta)^{\frac{3}{2}} \phi_n^{(1)} = 0, \qquad R_i \leq R \leq R_o, \quad \text{(B.4b)}$$

and for $n \geq 2$:

$$r_n^{(1)'} + \frac{3r_n^{(1)}}{R} - \frac{e^{\frac{\Omega_o}{4}}}{k^{\frac{1}{2}}} \frac{p_n^{(1)}}{\mu} - \frac{k^{\frac{1}{2}} e^{\frac{3\Omega_o}{4}}}{n} \left(R^2 \phi_n^{(1)''} + 3R\phi_n^{(1)'} - 2n^2 \phi_n^{(1)} \right) = 0, \qquad 0 \leq R \leq R_i, \quad \text{(B.5a)}$$

$$2n^2 R(R^2 + \eta)\phi_n^{(1)} - R^2(3R^2 + \eta)\phi_n^{(1)'} - R^3(R^2 + \eta)\phi_n^{(1)''} - \frac{nR^3}{k} \frac{p_n^{(1)}}{\mu}$$

$$+ nk^{-\frac{1}{2}}(R^2 + \eta)^{\frac{3}{2}} r_n^{(1)'} + nk^{-\frac{1}{2}} R \left(3(R^2 + \eta)^{\frac{1}{2}} - \frac{\eta^2}{(R^2 + \eta)^{\frac{3}{2}}} \right) r_n^{(1)} = 0, \qquad R_i \leq R \leq R_o. \quad \text{(B.5b)}$$

Clearly, $r_n^{(1)} = \phi_n^{(1)} = p_n^{(1)}/\mu = 0$, $n = 0$ and $n \geq 2$ is a solution of the system of linear ordinary differential equations (B.3), (B.4), and (B.5), and hence it is the unique solution satisfying the required boundary conditions (3.18) and the continuity of the displacement and traction fields at the inclusion-matrix interface.