THE UNIVERSITY OF TEXAS AT AUSTIN

Finite Element Method in Fracture Mechanics

Naoto Sakakibara

5/9/2008

Table of Contents

Summary

The Finite Element Method (FEM) has been one of the most powerful numerical tools for the solution of the crack problem in fracture mechanics. In 1960s, you can find the early application of the finite element method in the papers by Swedlow, Williams and Yang [1965]. Henshell and Shaw [1975] and Barsoum [1976] suggested the quarter point element in order to get accurate solution around crack tip in 1975. On the other hand, Blenzley [1974] developed the enriched elements in 1974. In recent research, Extended Finite Element Method (XFEM), which allows you to calculate the crack propagation without remeshing finite elements, is proposed at the end of 20^{th} century.

Introduction

It is known that the solution of the finite element has some error (5 to 10 %) with the general isoparametric element. In addition to that, it is also reported that the large number of the mesh don't guarantee the solutions in the vicinity of the crack tip. The reason of those inaccurate solutions is caused by the singularity of the stress (strain) field around crack tip as shown below. (Mode I crack tip field from LEFM)

$$\sigma_r = \frac{3C}{\sqrt{r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin^2\left(\frac{\theta}{2}\right)\right]$$

$$\sigma_\theta = \frac{3C}{\sqrt{r}}\cos^3\left(\frac{\theta}{2}\right)$$

$$\sigma_r = \frac{3C}{\sqrt{r}}\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)$$

Due to the lower order of the shape function of the isoparametric elements; those singularity variation is hardly obtained. In this paper, two outstanding techniques will be introduced. Both techniques can achieve the singularity field around the crack tip. One is called Collapsed Quadrilateral Quarter element and the other is enriched elements.

Quarter Point Element

Henshell and Shaw and Barsoum independently found that by moving the mid node of a eight node quadrilateral element to a quarter point position, the desired 1/ variation for strains can be achieved along rays, within the element, the emanate from the node crack tip. [Sanford, 2002] As you can see in the Figure, this can be easily done by moving the node 6 close to the 3 and also locate node 4, 7 and 3 at the same place. Furthermore, as you move the midpoint to the quarter point, the shape function in global coordinate becomes more similar to the $1/\sqrt{\mathrm{r}}$ function. However, when the nodes are collapsed to produce the $1/\sqrt{\mathrm{r}}$ variation, the element reflects only the near field behavior and its size should be restricted to that of the region of validity of the near-field equation. Pu, et al. noted that the 12-node collapsed element gave good results for problems they investigated if r0 were restricted to 1 – 2% of the crack length, which is comparable with the size of the singularity dominated-zone determined by Chona, Irwin, and Sanford [Chona, R., Irein, G., and Sanford, R.J., 1983] In particular, a small number of Quarter Point Element (QEP) surrounding the crack tip results in inadequate modeling of the circumferential displacements, while too small a span angle introduces errors due to excessive element distortion. Therefore, it would seem that employing between 6 and 8 QEPs are reasonable. [I.L.Lim, I.W.Jhonston and S.K.Choi, 1993]

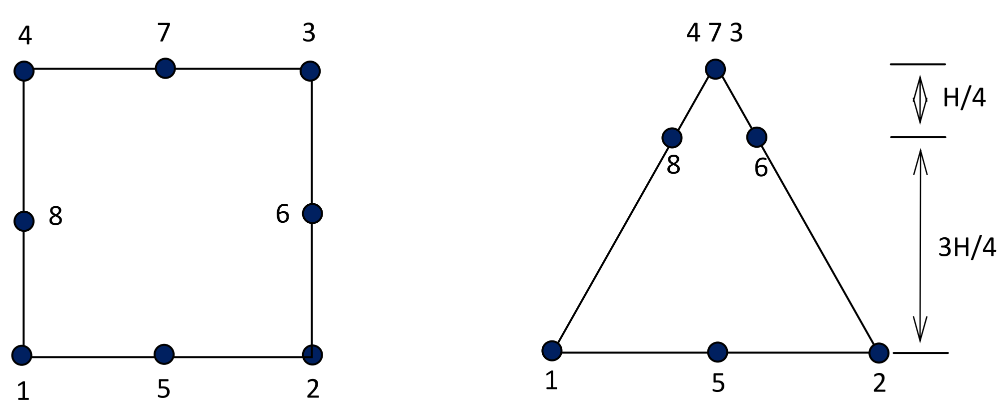

Figure 1 Quadrilateral Isoparametric and Collapsed Quarter Point Element

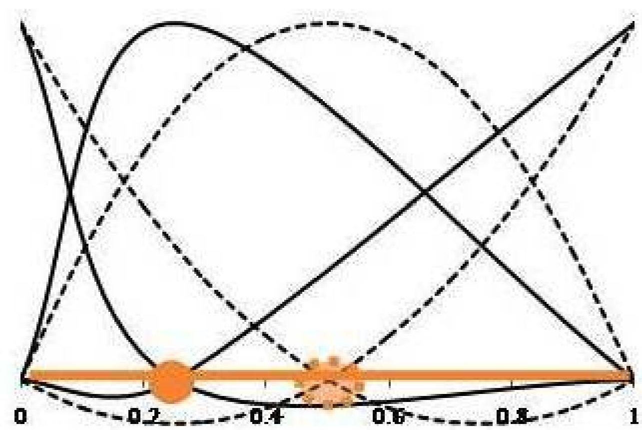

Figure 2 Image of Shape Function

Even though quarter point elements have some difficulty in meshing, there is a strong advantage of capability of using general Finite Element Code. In other words, there is no need to change the formulation of finite element. Both nodes are simply expressed as below.

$$u_i(x) = N_{ij}(x)u_j$$

Transition element

Under special configuration, transitional elements improve the accuracy of stress intensity factor computations. These transitional elements are located in the immediate vicinity of the singular elements with the mid-side nodes so adjusted as to reflect or extrapolate the square root singularity on the stress and strains at the tip of the crack. As shown in the equation, the mid-point of the transitional element can be obtained by this relation. In this formula, the length of the collapsed QPE is defined as 1 and L is the Length from crack tip to the outside of the transitional elements.

$$\beta L = \frac{L + 2\sqrt{L} + 1}{4}$$

According to the M.A.Hussain, it was found that there was improvement in accuracy for a configuration which consisted only singular and transitional elements, when transitional elements are used for a double-edge crack problem.

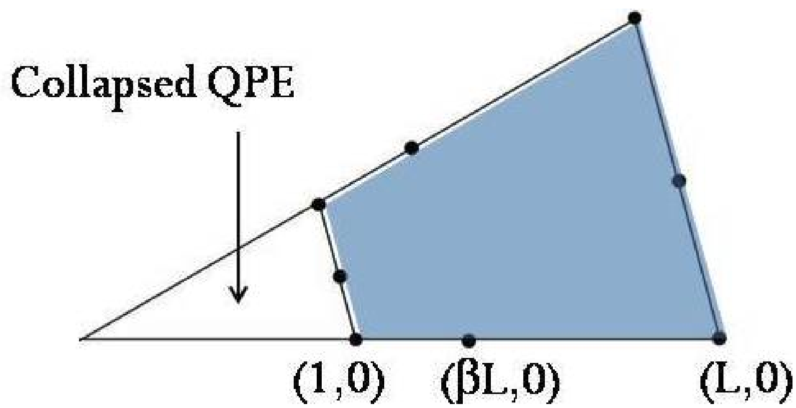

Figure 3 Transition Element

Meshing Rules for QPE and transitional element

Fig.4 is the example of the meshing for the crack tip problem. As you can see in the figure, a plate with certain crack size is given. Quarter point elements are located around the crack tip so that the singularity of the stress & strain field will be satisfied. Transitional elements are deployed right next to the quarter point element. Finally, rest of the region of the plate is meshed by the CPE8 elements.

In general, there is no optimal numbers for the size of the each element. The mesh size should be determined in each problem so that good accuracy is obtained. However, there exist some suggestions in order to define the size of the mesh. For the collapsed quadrilateral QPE, the recommended ratio for the crack length and the distance between crack tip and quarter point is about L-QPE/a=0.05~0.10, where a is distance between crack tip and quarter point. Also, since bigger number of elements in hoop direction will make those elements excessively distorted, number of elements in circumferential direction should be 6~8. Transition elements are expected to be bigger than quarter point elements and it is know that L-Tra/L-QPE ~ 2.5 in Fig.4 gives you a good accuracy under some special problems.

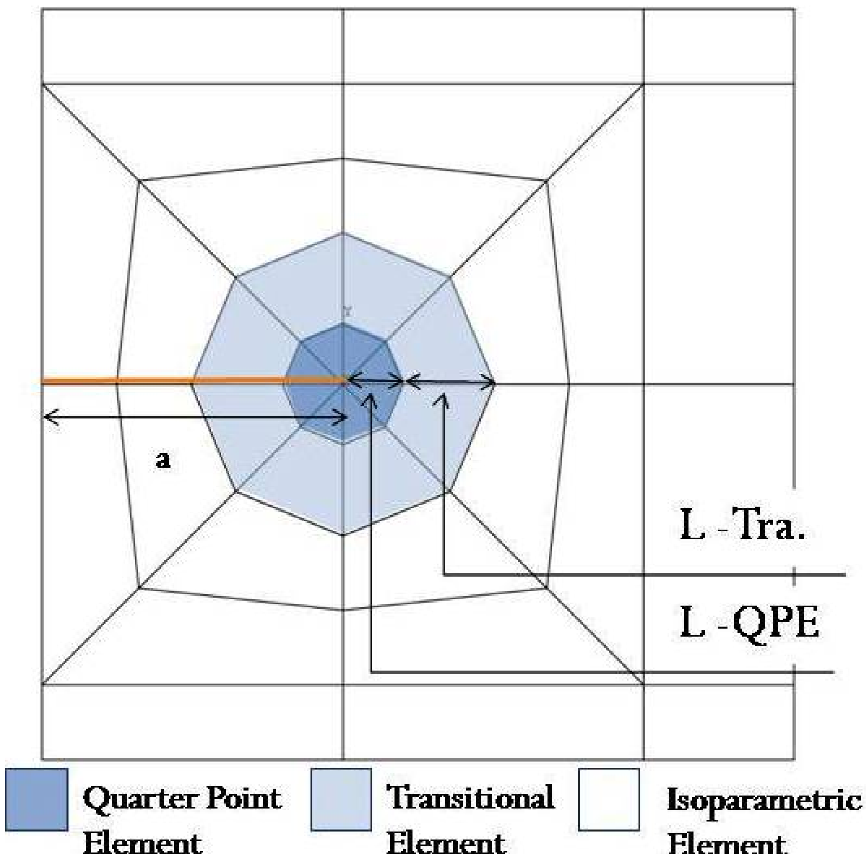

Figure 4   Meshing Configuration

The Fig.5 is the example of the effect of the elements size. In this graph, L-QPE/a appears on the x-axis and the error SIF (Stress Intensity Factor) appears on the y-axis. As you can see in this graph, regardless of method to find the SIF, the error in SIF increases a lot, if you have a small size(LQPE/a is less than 0.05) QPE. The method to find the SIF will be discussed in next section.

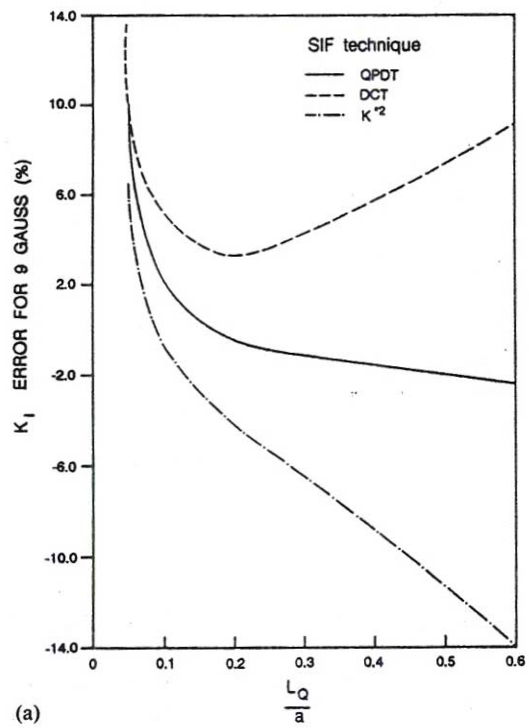

Figure 5 Stress Intensity Factor vs QPE size

Stress Intensity Factor (SIF)

(a) Quarter-point displacement technique

The quarter-point displacement technique (QPDT) was applied in the FEM simulation to evaluate the SIF.

$$K_1 = \frac{2\,G}{\kappa + 1}\frac{\sqrt{2\,\pi}}{L}(v'_B - v'_D)$$

$$K_2 = \frac{2\,G}{\kappa + 1}\frac{\sqrt{2\,\pi}}{L}(u'_B - u'_D)$$

where,

κ = (3 – ν)/ (1+ ν) for plane stress, = 3 - 4 ν for plane strain and axisymmetry,

L = length of QPE along clack face,

u’, v’ = local displacement around crack as showed in Fig.

(b) Displacement correlation technique

Similar to the QPDT method, SIF will be found by the Displacement correlation technique as shown below.

Among these techniques, the DCT is more widely used, although the QPDT is simply to implement.

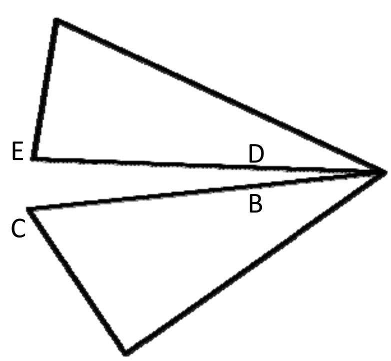

Figure 6 Node # for SIF calculation

Enriched Element

Another class of elements developed to deal with crack problems is known as “enriched” elements. This formulation involves adding the analytic expression of the crack –tip field to the conventional finite element polynomial approximation for the displacement. As you see in the equation below, you add the extra degree of freedom to the approximation of the displacement. In this class of the elements, K_I and K_{II} will are the added degree of freedom and Qs are the functions which represents the singular field around the crack tip. In general, those Q functions are derived from analytical solution.

$$u_i = \sum_k^4 N_k u_{ik} + R(a,b)\left\{K_I\left(Q_{1i} - \sum_k^4 N_k Q_{1\,ik}\right) + K_{II}\left(Q_{2i} - \sum_k^4 N_k\, Q_{2\,ik}\right)\right\}$$

$$Q_{I1} = \frac{1}{G}\sqrt{\frac{\rho}{\pi}}\cos\frac{\theta}{2}\left[\frac{\kappa-1}{2} + \sin^2\frac{\theta}{2}\right]$$

$$Q_{II1} = \frac{1}{G}\sqrt{\frac{\rho}{\pi}}\sin\frac{\theta}{2}\left[\frac{\kappa-1}{2} + \cos^2\frac{\theta}{2}\right]$$

$$Q_{I2} = \frac{1}{G}\sqrt{\frac{\rho}{\pi}}\sin\frac{\theta}{2}\left[\frac{\kappa-1}{2} - \cos^2\frac{\theta}{2}\right]$$

$$Q_{II2} = \frac{1}{G}\sqrt{\frac{\rho}{\pi}}\cos\frac{\theta}{2}\left[\frac{-(\kappa-1)}{2} + \sin^2\frac{\theta}{2}\right]$$

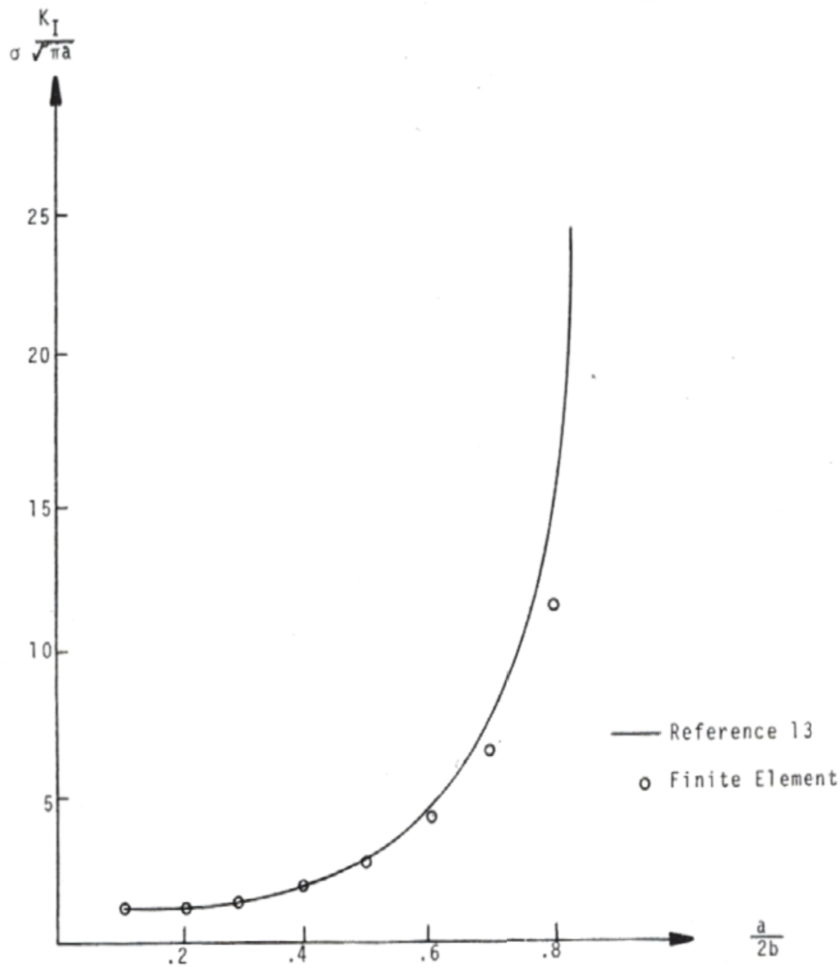

Figure 7 Stress Intensity Factor

This Fig.7 shows the result from Blenzy's paper [Benzly, 1974] with side cracked panel. Solid line is the experimental data of normalized SIF and circles are his Finite Element Analysis data. Basically, he has found that for this type of problem, enriched element method has accurate solution from small crack size to relatively big clack.

It is known that enriched elements can achieve singular field around crack tip very well. However this is true, only if you use the appropriate integration method. For example, Blenzy [Benzly, 1974] used 7 x 7 Gaussian quadrature in order to achieve this higher order functions.

Furthermore, since another DOF is introduce to the expression, stiffness matrix and load vectors are not same as the general FEM code any more.

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{Bmatrix} u \\ K_I \\ K_{II} \end{Bmatrix} = \begin{Bmatrix} F \\ F' \end{Bmatrix}$$

In this expression, K^{11} is the original stiffness matrix and F is load vectors. Other matrices and vectors are introduced as you add extra degree of freedom K_I and K_{II}

NS-F-FEM ver1.0

NS-F-FEM (Naoto Sakakibara –Fracture FEM) was developed by FORTRAN 95 in order to verify those facts. Characteristic information is listed as below.

1. Element – you can choose either general quadrilateral element or quarter point element.
2. Element size – you can specify the size of the element.
3. Geometry – only simple rectangular plate with small crack is available. (As shown in Fig)
4. Material Property – you can input your own material property.
5. SIF (Stress Intensity Factor) – SIF is calculated by quarter point displacement technique.

One of the examples of the analysis is showed in Fig.8 and Fig.9 is the calculation from the ABAQUS with quarter point elements. Applying distributed load on the top edge of the plate made of typical brittle material glass, you can get this deformed configuration as shown below. The bottom edge is fixed.

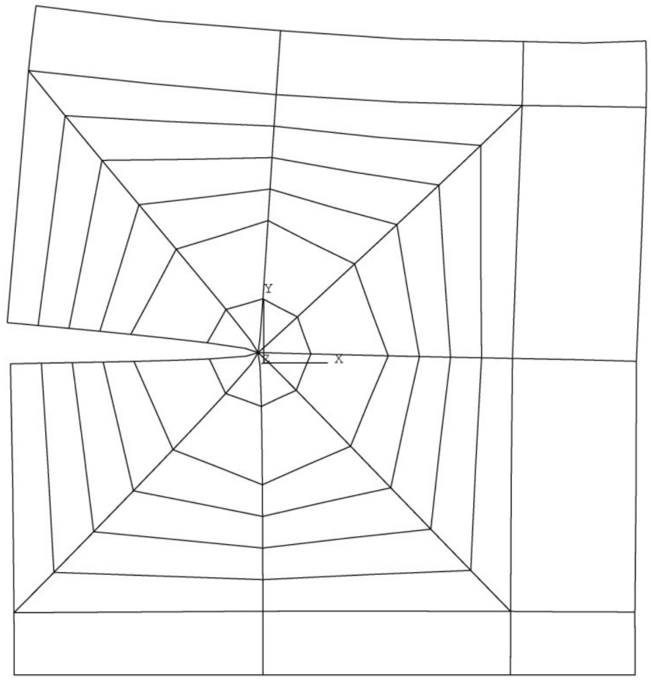

Figure 8 Deformed Configuration of NSFEM

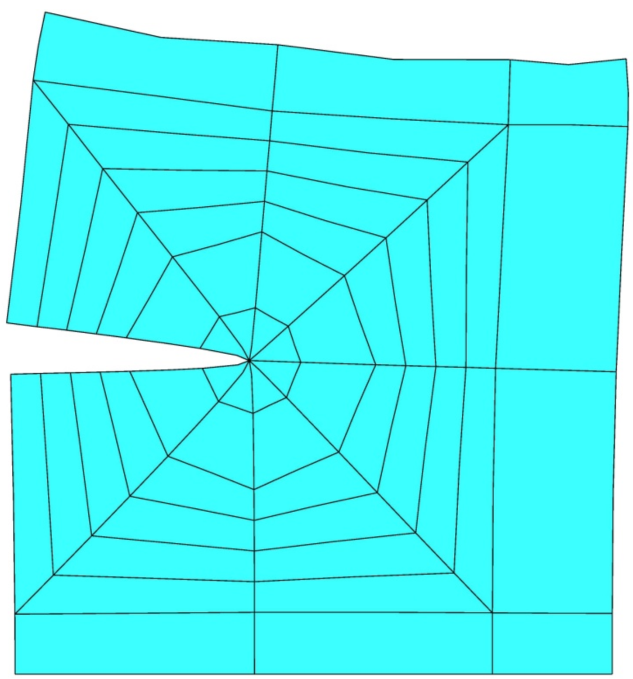

Figure 9 Deformed Configuration of ABAQUS

Introduction to Extended Finite Element Method

At the end of the 20th century, new finite element method called Extended FEM was introduced. This is the new FEM technique mainly for the fracture mechanics. The advantage of this technique is that mesh is independent from the crack geometry, while in the most of the FEM application, mesh should be created along the crack geometry. However, XFEM doesn't need to consider the crack geometry when it is created. This technique first introduced by Belytschko et.al. in their paper. [Nicolas Moes, John Dolbow and Ted Belytschko, 1999] In this section, two main concept of XFEM is explained. Both of them are enriching technique by using special function and adding extra degree of freedom. One of them is adding singular expression and the other is adding discontinuous expression which allows the element to have two different strain and stress field. The equation below is general XFEM expression. First term is represent general FEM approximation of the displacement field and 2nd term applies singular field around crack tip. Finally, 3rd term gives discontinuity to the elements.

$$u(\mathbf{x}) = \sum_{j=1}^{n} N_j(\mathbf{x}) u_j + \sum_{k=1}^{mt} N_k(\mathbf{x}) \left(\sum_{l=1}^{mf} F_l(\mathbf{x}) \mathbf{b}_k \right) + \sum_{h=1}^{n} N_h(\mathbf{x}) H(\xi(\mathbf{x})) \mathbf{a}_h$$

Adding Singularity field

As well as the enriched element explained in this paper before, the basic concept is adding singularity field so that FEM displacement approximation achieve singular field around crack tip. As you can see in the Fig. normally this enrichment is applied at the element including crack tip.

Adding Discontinuity

By applying the discontinuity to the element, we can express two different strain fields in one element. This means that the strain expression in one side of crack is different from the other side of crack. In this Fig.10, region I and II is in a same element but different fields due to the discontinuous function. Also, two dashed line in the picture is assumed two crack edge.

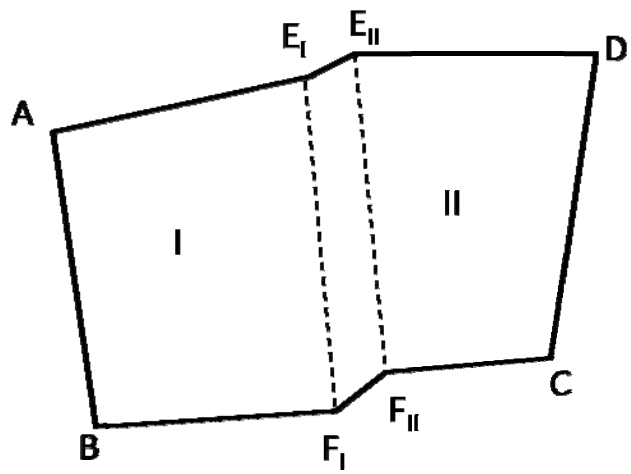

Figure 10 Enriched element with discontinuous function

Enriching Mesh

As shown in previous section, in XFEM elements are enriched by adding singular field and discontinuity. The way to enriching the element is shown below. Singular element should be located at the crack tip and enriched by the discontinuous property element should be deployed at the crack exist like Fig.11

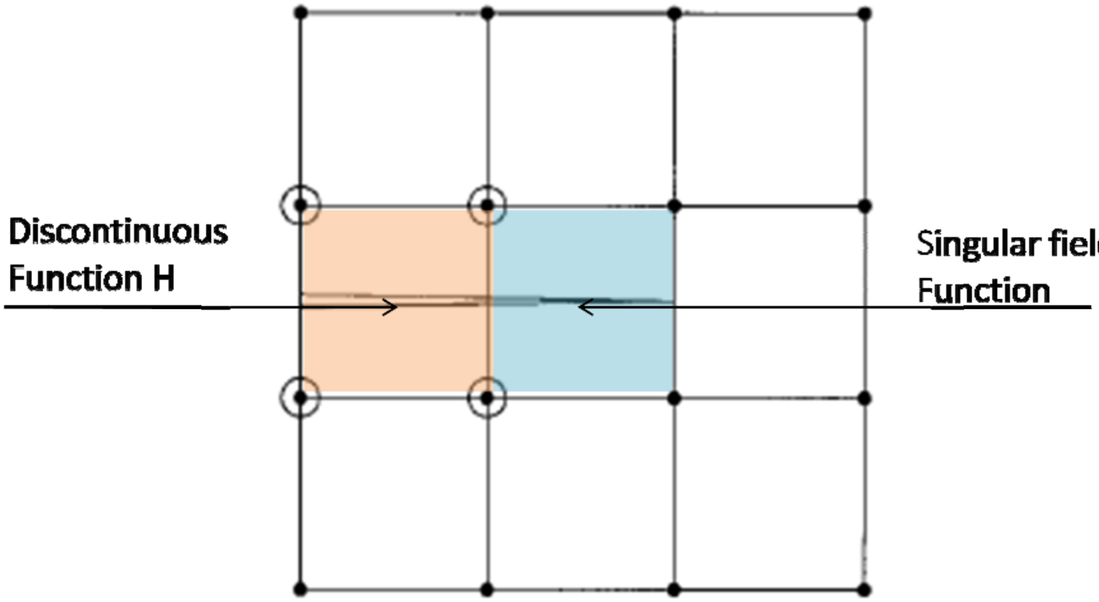

Figure 11 Enriched element around crack tip

Conclusion

In this paper, three FEM techniques for fracture mechanics were introduced. First one is quarter point element. The benefit of this element is you can use general shape function for finite element approximation; therefore this method is relatively easy to use. Second one is enriching the element by derived function in analytical calculation. This is not strongly dependent on mesh size. Both two was introduce about 1970's and nowadays new technique called XFEM is under the research. XFEM enables you to analyze the crack problem without considering the crack geometry. Mesh is independent from the crack geometry. All you need to worry is to enrich appropriate element in right way.

Bibliography

Barsoum, R. (1976). Furthur application of quadratic isoparametric elements to linear fracture mechanics of plate bending and general shells. *Int.J.Num.Meth,Engng* , 11,167-169.

Benzly, S. (1974). Representation of of singularities with isoparametric finite elements. *Int.J.Num.Meth.Engng.* , 8,537-545.

Chona, R., Irein, G., and Sanford, R.J. (1983). The influence of specimen size and shape on the singurarity-dominated zone. *Proceedings, 14th National Symposium on Fracture Mechanics, STP791, Vol.1, American Soc. for Testing and Materials*, (pp. I1-I23). Philadelphia.

Henshell,R.D and Shaw,K.G. (1975). Crack tip finite elements are unnnecessary. *Int.J.Num.Meth.Engng.* , 9,495-507.

I.L.Lim, I.W.Jhonston and S.K.Choi. (1993). Application of singular quadratic distorted isoparametric elements in linear fracture mechanics. *International journal for numerical methods in engineering* , Vol.36, 2473-2499.

I.L.Lim, I.W.Johnston and S.K.Choi. (1992). On stress intensity factor computation from the quater-point element displacements. *Communications in applied numerical methods* , Vol.8, 291-300.

Mohammad, S. (2008). *Extendet finite element.* Blackwell Publishing.

Nicolas Moes, John Dolbow and Ted Belystschko. (1999). A finite element method for crack growth withiout remeshing. *International jounarl for numerical methods in engineering* , 131-150.

Nicolas Moes, John Dolbow and Ted Belytschko. (1999). A finite element method for crack growth without remshing. *Int. J. Nume. Engng* , 46, 131-150.

Sanford, R. (2002). *Principle of Fracture Mechanics.* Upper Saddle River, NJ 07458: Pearson Education, Inc.

Swedlow,J.,L., Williams,M.L. and Yang,W.H. (1965). Elasto-plastic Stresses and strains in cracked plate. . *Proc.1st.Int.Conf. Fracture.,*, (pp. Vol 1, 259-282). Sendai, Japan.