Modified Saint-Venant's Principe of the Problem of Curved Bars

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Abstract

The proof of Saint-Venant's Principle for curved bars is discussed and the formulation of Modified Saint-Venant's Principle of the problem is established.The study shows that Saint-Venant's decay of stresses is valid only for the curved bars which are "effectively infinite". It is essential and significant to develop mathematical theories of Modified Saint-Venant's Principle one by one if Elasticity has to be constructed to be rational, logical, rigorous and secure mechanics.

`Keywords` : Saint-Venant's Principe, proof, provability, decay, formulation, "effectively infinite" bars

`AMS` Subject Classifications: 74-02, 74G50

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1 Introduction

Saint-Venant's Principle is essential and fundamental in Elasticity (See Ref.[1] and Ref.[2]). Boussinesq and Love announce statements of Saint-Venant's Principe (See Ref.[3] and Ref.[4]), but Mises and Sternberg argue, by citing counterexamples, that the statements are not clear, suggesting that Saint-Venant's Principle should be proved or given a mathematical formulation (See Ref.[5] and Ref.[6]). Truesdell asserts that if Saint-Venant's Principle of equipollent loads is true, it must be a mathematical consequence of the general equations of Linear Elasticity (See Ref.[7]).

There is no doubt that mathematical proof of Saint-Venant's Principle has become an academic attraction for contributors and much effort has been made for exploring its mysterious implications or deciphering its puzzle. Zanaboni "proved" a theorem trying to concern Saint-Venant's Principle in terms of work and energy (See Refs.[8],[9],[10]). However, Zhao argues that Zanaboni's theorem is false (See Ref.[11]). The work published by Toupin cites more counterexamples to explain that Love's statement is false, and then establishes a formulation of energy decay, which is considered as "a precise mathematical formulation and proof" of Saint-Venant's Principle for the elastic cylinder (See Refs.[12], [13]). Furthermore, Toupin's work seems to set up an example followed by a large number of papers to establish Toupin-type energy decay formulae for branches of continuum mechanics. Since 1965 the concept of energy decay suggested by Toupin has been widely accepted by authors, and various techniques have been developed to construct inequalities of Toupin-type decay of energy which are spread widely in continuum mechanics. Especially, the theorem given by Berdichevskii is considered as a generalization of Toupin's theorem (See Ref.[14]). Horgan and Knowles reviewed the development (See Refs.[15],[16],[17]). However, Zhao points out that Toupin's theory is not a strict mathematical proof, and Toupin's Theorem is not an exact mathematical formulation, of Saint-Venant's Principle. Interestingly and significantly, Saint-Venant's Principle stated by Love is disproved mathematically from Toupin's Theorem, so Toupin's Theorem is mathematically inconsistent with Saint-Venant's Principle (See Ref.[11]).

Zhao disproves mathematically the "general" Saint-Venant's Principle stated by Boussinesq and Love and points out that Special Saint-Venant's Principle or Modified Saint-Venant's Principle can be proved or formulated, (See Ref.[11]).

Saint-Venant's Principle is applied without proof here and there in the literature of Elasticity. It is essential to supplement the literature with mathematical proof or formulation of Special Saint-Venant's Principle or Modified Saint-Venant's Principle of elastic problems one by one unless Elasticity is not to be constructed to be rational, logical, rigorous and secure mechanics.

We suggest the problem of Saint-Venant's Principle for curved bars and establish its formulation of Modified Saint-Venant's Principle.

2 Love's Statement of Saint-Venant's Principle and Its Provability

Love's Statement : "According to this principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces statically equivalent to zero force and zero couple, are of negligible magnitude at distances which are large compared with the linear dimensions of the part."

(See Ref.[4])

Zhao disproves mathematically the “general” Saint-Venant’s Principle stated by Love, but argues by mathematical analysis that Saint-Venant’s decay of strains (then stresses) described by Love’s statement can be proved true by special formulating or adding supplementary conditions to the problems discussed (See Ref.[11]).

3 Saint-Venant’s Principle of the Problem of Curved Bars

3.1 Formulating the Problem

Let us consider a curved bar with a constant narrow rectangular cross section and a circular axis (See Ref. [18]). The end ($\theta = 0$) of the bar is loaded by an equilibrium system of forces, otherwise the bar would be free. Denoting by a and b the inner and outer radii of the boundary and taking the width of the rectangular cross section as unity, the boundary conditions are:

$$r = a \; and \; r = b: \quad \sigma_r = 0, \quad \tau_{r\theta} = 0. \tag{1}$$

$$\theta = \alpha: \quad \sigma_\theta = 0, \quad \tau_{r\theta} = 0. \tag{2}$$

$$\theta = 0: \quad \int_a^b \sigma_\theta dr = 0, \tag{3}$$

$$\int_a^b \sigma_\theta r dr = 0, \tag{4}$$

$$\int_a^b \tau_{r\theta} dr = 0. \tag{5}$$

The conditions of Eq.(1) are satisfied if

$$r = a \; and \; r = b: \quad \phi \; = \; 0, \tag{6}$$

$$\frac{\partial \phi}{\partial r} \; = \; 0. \tag{7}$$

Under the conditions (6) and (7), Eqs. (3), (4) and (5) are satisfied because

$$\int_a^b \sigma_\theta dr = \int_a^b \frac{\partial^2 \phi}{\partial r^2} dr = (\frac{\partial \phi}{\partial r})_a^b = 0, \tag{8}$$

$$\int_a^b \sigma_\theta r dr = \int_a^b \frac{\partial^2 \phi}{\partial r^2} r dr = (\frac{\partial \phi}{\partial r} r)_a^b - \int_a^b \frac{\partial \phi}{\partial r} dr = (\phi)_a^b = 0, \tag{9}$$

$$\int_a^b \tau_{r\theta} dr = -\int_a^b \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \phi}{\partial \theta})dr = -(\frac{1}{r}\frac{\partial \phi}{\partial \theta})_a^b = 0. \quad (10)$$

3.2 Stress Function

Let the stress function be

$$\phi(r,\theta) = \varphi_1(r)\varphi_2(\theta) = \varphi_1(r)e^{-\lambda\theta}. \quad (11)$$

Putting Eq.(11) into

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2})(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2})\ \phi(r,\theta) = 0, \quad (12)$$

we find the solution

$$\phi(r,\theta) = [A\sin(\lambda lnr) + B\cos(\lambda lnr) + Cr^2\sin(\lambda lnr) + Dr^2\cos(\lambda lnr)]e^{-\lambda\theta}. \quad (13)$$

From Eq.(13)

$$\begin{aligned} \frac{\partial \phi}{\partial r} &= \{A\lambda\frac{1}{r}\cos(\lambda lnr) - B\lambda\frac{1}{r}\sin(\lambda lnr) + C[2r\sin(\lambda lnr) + \lambda r\cos(\lambda lnr)] \\ &+ D[2r\cos(\lambda lnr) - \lambda r\sin(\lambda lnr)]\}e^{-\lambda\theta} \end{aligned} \quad (14)$$

The conditions (6) and (7) must be fulfilled by (13) and (14), therefore it is required that

$$A\sin(\lambda lna) + B\cos(\lambda lna) + Ca^2\sin(\lambda lna) + Da^2\cos(\lambda lna) = 0 \quad (15)$$

$$A\sin(\lambda lnb) + B\cos(\lambda lnb) + Cb^2\sin(\lambda lnb) + Db^2\cos(\lambda lnb) = 0 \quad (16)$$

$$\begin{aligned} A\lambda\frac{1}{a}\cos(\lambda lna) &- B\lambda\frac{1}{a}\sin(\lambda lna)\ + C\ [2a\sin(\lambda lna) + \lambda a\cos(\lambda lna)] \\ &+ D[2a\cos(\lambda lna) - \lambda a\sin(\lambda lna)] = 0 \end{aligned} \quad (17)$$

$$\begin{aligned} A\lambda\frac{1}{b}\cos(\lambda lnb) &- B\lambda\frac{1}{b}\sin(\lambda lnb) +\ C\ [2b\sin(\lambda lnb) + \lambda b\cos(\lambda lnb)] \\ &+ D[2b\cos(\lambda lnb) - \lambda b\sin(\lambda lnb)] = 0 \end{aligned} \quad (18)$$

The coefficients *A*-*D* have non-zero solutions from Eqs.(15)-(18) if

$$\lambda = 0 \qquad (19)$$

or

$$a = b. \qquad (20)$$

However, we discard the case of Eq.(19) when Saint-Venant's Principle is discussed because it means no decay at all.

3.3 Stress Components

From Eq.(13) we find the stress components

$$\begin{aligned} \sigma_r &= \{A\frac{\lambda}{r^2}[\cos(\lambda lnr) + \lambda\sin(\lambda lnr)] + B\frac{\lambda}{r^2}[-\sin(\lambda lnr) + \lambda\cos(\lambda lnr)] \\ &+ C[(2+\lambda^2)\sin(\lambda lnr) + \lambda\cos(\lambda lnr)] + D[(2+\lambda^2)\cos(\lambda lnr) \\ &- \lambda\sin(\lambda lnr)]\}\ e^{-\lambda\theta}, \end{aligned} \qquad (21)$$

$$\begin{aligned} \sigma_\theta &= \{-A\frac{\lambda}{r^2}[\cos(\lambda lnr) + \lambda\sin(\lambda lnr)] + B\frac{\lambda}{r^2}[\sin(\lambda lnr) - \lambda\cos(\lambda lnr)] \\ &+ C[(2-\lambda^2)\sin(\lambda lnr) + 3\lambda\cos(\lambda lnr)] + D[(2-\lambda^2)\cos(\lambda lnr) \\ &- 3\lambda\sin(\lambda lnr)]\}\ e^{-\lambda\theta}, \end{aligned} \qquad (22)$$

$$\begin{aligned} \tau_{r\theta} &= \lambda\{A\frac{1}{r^2}[-\sin(\lambda lnr) + \lambda\cos(\lambda lnr)] - B\frac{1}{r^2}[\cos(\lambda lnr) + \lambda\sin(\lambda lnr)] \\ &+ C[\sin(\lambda lnr) + \lambda\cos(\lambda lnr)] + D[\cos(\lambda lnr) - \lambda\sin(\lambda lnr)]\}\ e^{-\lambda\theta}. \end{aligned} \qquad (23)$$

3.4 Saint-Venant's Decay of Stresses and Its Requirement

Condition (2) must be satisfied by Eq.(22) and Eq.(23), and it is required that

$$(\lambda\alpha) \to \infty. \qquad (24)$$

Let

$$\lambda \equiv \gamma r_0/(b-a) = \gamma(b+a)/2(b-a) \qquad (25)$$

where $r_0 = (b+a)/2$ is the radius of the central axis, $(b-a)$ is the depth, of the bar, then $L_\alpha \equiv r_0\alpha/(b-a)$ is the "effective" length of the bar, and Eq.(24) is changed into

$$(\gamma L_\alpha) \to \infty. \qquad (26)$$

Substituting Eq.(25) into Eqs.(21), (22) and (23), we have

$$
\begin{aligned}
\sigma_r &= \{A\frac{\lambda}{r^2}[\cos(\lambda lnr) + \lambda\sin(\lambda lnr)] + B\frac{\lambda}{r^2}[-\sin(\lambda lnr) + \lambda\cos(\lambda lnr)] \\
&+ C[(2+\lambda^2)\sin(\lambda lnr) + \lambda\cos(\lambda lnr)] + D[(2+\lambda^2)\cos(\lambda lnr) \\
&- \lambda\sin(\lambda lnr)]\}\ e^{-\gamma l_\theta}, \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_\theta &= \{-A\frac{\lambda}{r^2}[\cos(\lambda lnr) + \lambda\sin(\lambda lnr)] + B\frac{\lambda}{r^2}[\sin(\lambda lnr) - \lambda\cos(\lambda lnr)] \\
&+ C[(2-\lambda^2)\sin(\lambda lnr) + 3\lambda\cos(\lambda lnr)] + D[(2-\lambda^2)\cos(\lambda lnr) \\
&- 3\lambda\sin(\lambda lnr)]\}\ e^{-\gamma l_\theta}, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\tau_{r\theta} &= \lambda\{A\frac{1}{r^2}[-\sin(\lambda lnr) + \lambda\cos(\lambda lnr)] - B\frac{1}{r^2}[\cos(\lambda lnr) + \lambda\sin(\lambda lnr)] \\
&+ C[\sin(\lambda lnr) + \lambda\cos(\lambda lnr)] + D[\cos(\lambda lnr) - \lambda\sin(\lambda lnr)]\}\ e^{-\gamma l_\theta}, \qquad (29)
\end{aligned}
$$

where $l_\theta \equiv r_0\theta/(b-a)$ is the "effective" distance from the equilibrium system of forces.

From Eq.(20) we know that Eqs.(27), (28) and (29) remain valid if the depth of the bar tends to zero, that is,

$$(b-a) \to 0. \qquad (30)$$

In other words, the bar should be an "effectively infinite" one, that is,

$$L_\alpha \to \infty, \qquad (as\ \ 0 < \gamma < \infty). \qquad (31)$$

And the coefficients A-D in Eqs.(27), (28) and (29) are controlled by Eqs.(15) and (17), or, Eqs.(16) and (18).

From Eqs.(27), (28) and (29) we have

$$\lim_{l_\theta\to\infty}\sigma_r = 0, \quad \lim_{l_\theta\to\infty}\sigma_\theta = 0, \quad \lim_{l_\theta\to\infty}\tau_{r\theta} = 0. \qquad (32)$$

where $l_\theta \to \infty$ is supported by Eq.(30).

Eqs.(27), (28) and (29) and/or (32) are the formulation of Saint-Venant's decay of stresses for the problem of the "effectively infinite" curved bars.

4 Conclusion

Saint-Venant's decay of stresses is valid for the problem of "effectively infinite" curved bars.

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