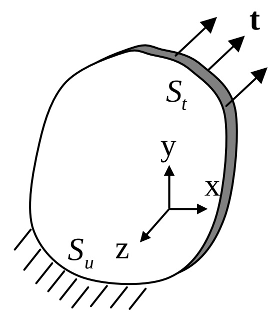
Plane Elasticity Problems

Main Reference: *Theory of Elasticity*, by S.P. Timoshenko and J.N. Goodier, McGraw-Hill, New York. Chapters 2,3,4.

A thin sheet subject to loads in its plane. A thin sheet of an isotropic material lies in the plane (x, y), and is subject to load in the plane of the sheet. The top and the bottom surfaces of the sheet are traction-free. The edge of the sheet is subject to two kinds of the boundary conditions: at each material particle on the edge either the displacement or the traction is prescribed. In the latter case, we write

$$\sigma_{xx} n_x + \tau_{xy} n_y = t_x$$

$$\tau_{xy} n_x + \sigma_{yy} n_y = t_y$$



where t_x and t_y are components of the traction vector prescribed on the edge of the sheet, and n_x and n_y are the components of the unit vector normal to the edge of the sheet. The above equations provide two conditions at each point on the edge of the sheet.

Semi-inverse method. We next go into the interior of the sheet. We have already obtained a full set of governing equations of linear elasticity. However, no general approach exists to solve partial differential equations analytically. Numerical methods are now readily available to solve any elasticity problem that you can pose. In this introductory course, to gain insight into solid mechanics, we will make reasonable guesses of solutions, and see if they satisfy all the governing equations. This trial-and-error approach has a name: it is called the semi-inverse method.

It seems reasonable to guess that the stress field in the sheet only has nonzero components in its plane: $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$, and that the components out of the plane vanish: $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$. Furthermore, we guess that the in-plane components of stress may vary with x and y, but are independent of z. That is, the stress field in the sheet is described by three functions of two variables:

$$\sigma_{xx}(x, y), \ \ \sigma_{yy}(x, y), \ \ \tau_{xy}(x, y) .$$

Will these guesses satisfy the governing equations of elasticity? Let us go through these equations one by one.

Equilibrium equations. Using the guessed stress field, we reduce the three equilibrium equations to two equations:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 .$$

These two equations by themselves are insufficient to determine the three components of stress.

Stress-strain relations. Given the guessed stress field, the 6 components of the strain field are

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E}, \quad \varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu \frac{\sigma_{xx}}{E}, \quad \gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}), \ \ \gamma_{xz} = \gamma_{yz} = 0 .$$

Strain-displacement relations. Recall the 6 strain-displacement relations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \ \ \varepsilon_{yy} = \frac{\partial v}{\partial y}, \ \ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}, \ \ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \ \ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} .$$

It seems reasonable to assume that the in-plane displacements u and v vary only with x and y, but not with z. From these guesses, together with the conditions that $\gamma_{xz} = \gamma_{yz} = 0$, we find that

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0 .$$

Thus, w is independent of x and y, and can only be a function of z. If we insist that ε_{zz} be independent of z, and from $\varepsilon_{zz} = \partial w / \partial z$, then ε_{zz} must be a constant, $\varepsilon_{zz} = c$, and $w = cz + b$. On the other hand, we also have $\varepsilon_{zz} = -\nu(\sigma_{xx} + \sigma_{yy})/E$, which may not be a constant. This inconsistency shows that our guesses are incorrect.

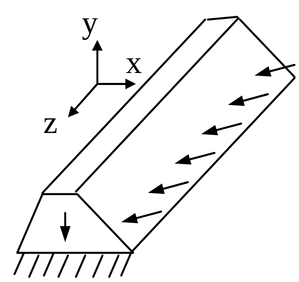
Plane stress problem. Instead of abandoning these guesses, we will just call our guesses the plane-stress approximation. If you neglect the inconsistency between $\varepsilon_{zz} = c$ and $\varepsilon_{zz} = -\nu(\sigma_{xx} + \sigma_{yy})/E$, at least the following set of equations look self-consistent:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E}, \quad \varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu \frac{\sigma_{xx}}{E}, \quad \gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

These are 8 equations for 8 functions. We will focus on these 8 equations.



Plane strain problem. Consider an infinitely long cylinder with the axis in the z-direction, and a cross section in the (x, y) plane. The cylinder is constrained from deforming in the z-direction, and the loading is invariant along the z-direction. Consequently, the displacement field takes the form:

$$u(x, y), \ v(x, y), \ w = 0 .$$

From the strain-displacement relations, we find that only the three in-plane strains are nonzero:

$$\varepsilon_{xx}(x,y) = \frac{\partial u}{\partial x}, \varepsilon_{yy}(x,y) = \frac{\partial v}{\partial y}, \gamma_{xy}(x,y) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} .$$

The three out-of-plane strains vanish: $\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$.

We assume the material is isotropic and linearly elastic. Because $\gamma_{xz} = \gamma_{yz} = 0$, the stress-strain relations imply that $\tau_{xz} = \tau_{yz} = 0$. From $\varepsilon_{zz} = 0$ and $\varepsilon_{zz} = (\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy})$, we obtain that

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}).$$

Thus,

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}) = \frac{1-\nu^2}{E}\left(\sigma_{xx} - \frac{\nu}{1-\nu}\sigma_{yy}\right)$$

$$\varepsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}) = \frac{1-\nu^2}{E}\left(\sigma_{yy} - \frac{\nu}{1-\nu}\sigma_{xx}\right)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

These three stress-strain relations look similar to those under the plane stress conditions, provided we make the following substitutions:

$$\bar{E} = \frac{E}{1-\nu^2}, \quad \bar{\nu} = \frac{\nu}{1-\nu}.$$

The quantity $\bar{E}$ is called the plane strain modulus.

A theorem in calculus. Given two functions $f(x,y)$ and $g(x,y)$, there exists a function $P(x,y)$, such that

$$f = \frac{\partial P(x,y)}{\partial y}, \quad g = \frac{\partial P(x,y)}{\partial x},$$

if and only if the two functions satisfy the condition

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}.$$

Proof. In calculus we know that the order of partial differentiation can be changed, such as

$$\frac{\partial}{\partial x}\left(\frac{\partial A(x,y)}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial A(x,y)}{\partial x}\right).$$

Consequently, the conditions

$$f = \frac{\partial P(x,y)}{\partial y}, \quad g = \frac{\partial P(x,y)}{\partial x}$$

imply

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}.$$

We next prove the converse is also true. For a given function $f(x,y)$, we can always find a function $A(x,y)$, such that

$$f = \frac{\partial A(x,y)}{\partial y}.$$

Indeed, we can obtain $A(x,y)$ by integration:

$$A(x,y) = \int f(x,y)dy + h(x),$$

where $h(x)$ is an arbitrary function. From the prescribed condition

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y},$$

we obtain that

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial A(x,y)}{\partial y}\right).$$

Integrating with respect to y, we obtain that

$$g = \frac{\partial A(x,y)}{\partial x} + k(x),$$

where $k(x)$ is another arbitrary function. Let

$$P = A(x,y) + \int k(x)dx,$$

so that

$$f = \frac{\partial P(x,y)}{\partial y}, \quad g = \frac{\partial P(x,y)}{\partial x}.$$

The Airy stress function. We now apply the above theorem to the equilibrium equations. From $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$, we deduce that there exists a function $A(x,y)$, such that

$$\sigma_{xx} = \frac{\partial A}{\partial y}, \quad \tau_{xy} = -\frac{\partial A}{\partial x}.$$

From $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$, we deduce that that there exists a function $B(x,y)$, such that

$$\sigma_{yy} = \frac{\partial B}{\partial x}, \quad \tau_{xy} = -\frac{\partial B}{\partial y}.$$

Note that is the above the shear stress τ_{xy} have been expressed in two ways, so that

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}.$$

From this equation we deduce that that there exists a function $\phi(x,y)$, such that

$$A = \frac{\partial \phi}{\partial y}, \quad B = \frac{\partial \phi}{\partial x}.$$

The function $\phi(x,y)$ is known as the Airy (1862) stress function. The three components of the stress field can now be represented by the stress function:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \; \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \; \tau_{xy} = -\frac{\partial^2 \phi}{\partial y \partial x}.$$

Recall that we have started with the two equilibrium equations governing the three stresses.

Using the stress-strain relations, we can also express the three components of strain field in terms of the Airy stress function:

$$\varepsilon_{xx} = \frac{1}{E}\left(\frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2}\right), \quad \varepsilon_{yy} = \frac{1}{E}\left(\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2}\right), \; \gamma_{xy} = -\frac{2(1+\nu)}{E}\frac{\partial^2 \phi}{\partial x \partial y}.$$

Compatibility equation. Recall the strain-displacement relations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \; \varepsilon_{yy} = \frac{\partial v}{\partial y}, \; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

Eliminate the two displacements in the three strain displacement relations, and we obtain that

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.$$

This equation is known as the compatibility equation.

Biharmonic equation. Inserting the expressions of the strains in terms of $\phi(x,y)$ into the compatibility equation, and we obtain that

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0.$$

This equations can also be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = 0.$$

It is called the biharmonic equation.

Thus, once $\phi(x,y)$ is solved from the above PDE, we can calculate stresses and strains. After the strains are obtained, the displacement field can be obtained by integrating the strain-displacement relations.

Dependence on elastic constants. For a plane problem with traction-prescribe boundary conditions, both the governing equation and the boundary conditions can be expressed in terms of ϕ. All these equations are independent of elastic constants. Consequently, the stress field in such a boundary value problem is independent of the elastic constants. Once we go over specific examples, we will find that the above statement is only correct for boundary value problems in simply connected regions. For multiply connected regions, the above equations in terms of ϕ do not guarantee that the displacement field is continuous. When we insist that displacement field be continuous, elastic constants may enter the stress field.

Saint-Venant's principle. When a load is applied in a region small compared to the overall size of a body, and the load has a vanishing resultant force and resultant moment, then the stress field decays rapidly away from the region where the load is applied.

While Saint-Venant's principle cannot be proved in such a loose form, we can certainly give a few examples to illustrate the idea.

- We have used this principle in discussing the laminate, where we have neglected the edge effects.
- For a spherical cavity in a block of material under internal pressure, the stress field in the block is

$$\sigma_{rr} = -p\left(\frac{a}{r}\right)^3, \ \sigma_{\theta\theta} = \frac{1}{2}p\left(\frac{a}{r}\right)^3 .$$

Another example that supports Saint-Venant's principle is give below.

A half space subject to periodic traction on the surface. An elastic body occupies a half space, $x > 0$. On the surface of the body, $x = 0$, the traction is prescribed as

$$\sigma_{xx}(0,y,z) = \sigma_0 \cos ky, \quad \tau_{xy}(0,y,z) = \tau_{xz}(0,y,z) = 0 .$$

The applied load is sinusoidal, with amplitude σ_0, and wave number k. The wave number k relates to the wavelength as $k = 2\pi / \lambda$.

The applied load is localized in a region of length λ. Within a period, the load has a vanishing resultant force and a vanishing resultant moment. We expect the field of stress in the body decays over the length scale λ. To ascertain this expectation, we wish to determine the stress field inside the body.

The body clearly deforms under the plane strain conditions. Recall that

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \ \sigma_{yy} = \frac{\partial^2 \phi}{\partial y^2}, \ \tau_{xy} = -\frac{\partial^2 \phi}{\partial y \partial x} .$$

Inspecting the boundary conditions, we guess that the Airy function takes the form

$$\phi(x,y) = f(x)\cos ky .$$

The biharmonic equation becomes

$$\frac{d^4 f}{dx^4} - 2k^2 \frac{d^2 f}{dx^2} + k^4 f = 0 .$$

This is a homogenous ODE with constant coefficients. A solution is of the form

$$f(x) = e^{\alpha x} .$$

Insert this form into the ODE, and we obtain that

$$\left(\alpha^2 - k^2\right)^2 = 0 .$$

The algebraic equation has double roots of $\alpha = -k$, and double roots of $\alpha = +k$. Consequently, the general solution is of the form

$$f(x) = Ae^{kx} + Be^{-kx} + Cxe^{kx} + Dxe^{-kx} ,$$

where *A*, *B*, *C* and *D* are constants of integration.

We expect that the stress field vanishes as $x \to +\infty$, so that the stress function should be of the form

$$f(x) = Be^{-kx} + Dxe^{-kx} .$$

We next determine the constants *B* and *D* by using the boundary conditions.

The stress fields are

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = -\left(Be^{-kx} + Dxe^{-kx}\right)k^2 \cos ky$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \left(-Be^{-kx} + \frac{D}{k}e^{-kx} - Dxe^{-kx}\right)k^2 \sin ky$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = \left(Be^{-kx} - 2\frac{D}{k}e^{-kx} + Dxe^{-kx}\right)k^2 \cos ky$$

Recall the boundary conditions

$$\sigma_{xx}(0,y) = \sigma_0 \cos ky, \quad \tau_{xy}(0,y) = 0 .$$

We find that

$$B = -\sigma_0 / k^2, \quad D = -\sigma_0 / k .$$

The stress field inside the material is

$$\sigma_{xx} = \sigma_0 (1 + kx) e^{-kx} \cos ky$$

$$\tau_{xy} = -\sigma_0 kxe^{-kx} \sin ky$$

$$\sigma_{yy} = \sigma_0 (1 - kx) e^{-kx} \cos ky$$

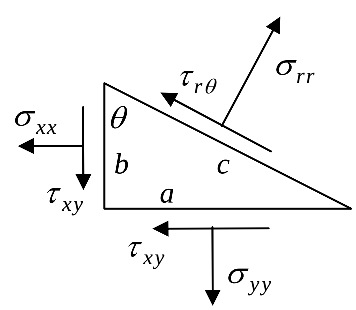
The stress field decays exponentially. This decay once again provides an example of Saint-Venant's principle.

Change of the components of a state of stress due to change of coordinates. A material particle is in a state of plane stress. If we represent the material particle by a square in the (x,y) coordinates, the components of stress are $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$. If we represent the same material particle under the same state of stress by a square in the (r,θ) coordinates, the components of stress are $\sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta}$. The two sets of the components of the same state of stress are related as

$$\sigma_{rr} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{r\theta} = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$



These relations are derived in your course on the strength of materials. They are a consequence of force balance. For example, consider the force balance of the triangle in the radial direction. We obtain that

$$\sigma_{rr} c = \sigma_{xx} b \cos\theta + \sigma_{yy} a \sin\theta + \tau_{xy} b \sin\theta + \tau_{xy} a \cos\theta .$$

Note that $a = c\sin\theta$ and $b = c\cos\theta$. Recall the identities of trigonometry:

$$\cos\theta\cos\theta = \frac{1-\cos 2\theta}{2}, \quad \sin\theta\sin\theta = \frac{1+\cos 2\theta}{2}, \quad 2\sin\theta\cos\theta = \sin 2\theta .$$

Governing equations in polar coordinates. The Airy stress function is a function of the polar coordinates, $\phi(r,\theta)$. The stresses are expressed in terms of the Airy stress function:

$$\sigma_{rr} = \frac{\partial^2\phi}{r^2\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial\phi}{r\partial\theta}\right)$$

The biharmonic equation is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{\partial\phi}{r\partial r} + \frac{\partial^2\phi}{r^2\partial\theta^2}\right) = 0 .$$

The stress-strain relations in polar coordinates are similar to those in the rectangular coordinates:

$$\varepsilon_{rr} = \frac{\sigma_{rr}}{E} - \nu\frac{\sigma_{\theta\theta}}{E}, \quad \varepsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \nu\frac{\sigma_{rr}}{E}, \quad \gamma_{r\theta} = \frac{2(1+\nu)}{E}\tau_{r\theta} .$$

The strain-displacement relations are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{\partial u_\theta}{r\partial\theta}, \quad \gamma_{r\theta} = \frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} .$$

Stress field symmetric about an axis. Consider a very special case: the stress function be $\phi(r)$. The biharmonic equation becomes

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr}\right) = 0 .$$

Each term in this equation has the same dimension in the independent variable *r*. Such an ODE is known as an equi-dimensional equation. A solution to an equi-dimensional equation is of the form

$$\phi = r^m .$$

Inserting into the biharmonic equation, we obtain that

$$m^2(m-2)^2 .$$

The fourth order algebraic equation has a double root of 0 and a double root of 2. Consequently, the general solution to the ODE is

$$\phi(r) = A\log r + Br^2\log r + Cr^2 + D .$$

where *A*, *B*, *C* and *D* are constants of integration.

The components of the stress field are

$$\sigma_{rr} = \frac{\partial^2\phi}{r^2\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = \frac{A}{r^2} + B(1 + 2\log r) + 2C ,$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2\log r) + 2C ,$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial\phi}{r\partial\theta}\right) = 0 .$$

The stress field is a linear in *A*, *B* and *C*.

Lame problem. The contributions due to A and C are familiar: they are the same as the Lame problem:

$$\sigma_{rr} = C + \frac{A}{r^2},$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = C - \frac{A}{r^2}.$$

The constants A and C are determined by using the boundary conditions. For example, for a hole of radius a in an infinite sheet subject to a remote biaxial stress S, the stress field in the sheet is

$$\sigma_{rr} = S\left[1 - \left(\frac{a}{r}\right)^2\right], \quad \sigma_{\theta\theta} = S\left[1 + \left(\frac{a}{r}\right)^2\right].$$

The stress concentration factor of this hole is 2. We may compare this problem with that of a spherical cavity in an infinite elastic solid under remote tension:

$$\sigma_{rr} = S\left[1 - \left(\frac{a}{r}\right)^3\right], \quad \sigma_{\theta\theta} = S\left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right].$$

A cut-and-weld operation. How about the contributions due to B? Let us study the stress field (Timoshenko and Goodier, pp. 77-79). The stress function is

$$\phi(r) = Br^2 \log r.$$

The corresponding stresses are

$$\sigma_{rr} = B(1 + 2\log r), \quad \sigma_{\theta\theta} = B(3 + 2\log r), \quad \tau_{r\theta} = 0.$$

The strains are

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) = \frac{B}{E}[(1 - 3\nu) + 2(1 - \nu)\log r]$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) = \frac{B}{E}[(3 - \nu) + 2(1 - \nu)\log r]$$

$$\gamma_{r\theta} = 0$$

So far everything is straightforward. The next idea requires some imagination, and surprised me when I first learned about the idea. Even though the field of stress is axisymmetric, the filed of displacement need not be axisymmetric. If the field of displacement is allowed to be non-axisymmetric, the field has two components, $u_r(r,\theta)$ and $u_\theta(r,\theta)$. Imagine a ring, with a wedge of angle α cut off. The ring with the missing wedge was then welded together. In the welded ring, the stress field is axisymmetric, but the displacement field is not axisymmetric.

To obtain the displacements, recall the strain-displacement relations

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{\partial u_\theta}{r\partial\theta}, \quad \gamma_{r\theta} = \frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}.$$

Integrating ε_{rr}, we obtain that

$$u_r(r,\theta) = \frac{B}{E}[2(1 - \nu)r \log r - (1 + \nu)r] + f(\theta),$$

where $f(\theta)$ is a function still undetermined. Integrating $\varepsilon_{\theta\theta}$, we obtain that

$$u_\theta(r,\theta) = \frac{4Br\theta}{E} - \int_0^\theta f(q)dq + g(r),$$

where $g(r)$ is another function still undetermined. Inserting the two displacements into the expression

$$\gamma_{r\theta} = \frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = 0,$$

and we obtain that

$$\frac{df(\theta)}{d\theta} + \int_0^\theta f(q)dq = g(r) - r\frac{dg(r)}{dr}.$$

In the equation, the left side is a function of θ, and the right side is a function of r. Consequently, the both sides must equal a constant independent of r and θ, namely,

$$\frac{df(\theta)}{d\theta} + \int_0^\theta f(q)dq = G$$

$$g(r) - r\frac{dg(r)}{dr} = G$$

where G is a constant. The solutions to the two ODEs take the forms:

$$f(\theta) = H\sin\theta + K\cos\theta$$

$$g(r) = Fr + G$$

Substituting these forms into the ODEs, we obtain that $G = H$.

The two displacements are

$$u_r = \frac{B}{E}\left[2(1-\nu)r\log r - (1+\nu)r\right] + H\sin\theta + K\cos\theta$$

$$u_\theta = \frac{4Br\theta}{E} + Fr + H\cos\theta - K\sin\theta$$

Inspecting the above equations, we observe that F represents a rigid-body rotation, and H and K represent a rigid-body translation. The rigid-body rotation and translation will not generate any stress.

Now we can give an interpretation of B. Imagine a ring, with a wedge of angle α cut off. The ring with the missing wedge was then welded together. This operation requires that after a rotation of a circle, the displacement is

$$u_\theta(2\pi) - u_\theta(0) = \alpha r$$

This condition gives

$$B = \frac{\alpha E}{8\pi}.$$

This cut-and-weld operation clearly introduces a stress field in the ring. The field of stress is axisymmetric, but the field of displacement is non-axisymmetric.

An alternative approach. We can also solve the cut-and weld problem directly in terms of displacement field. Upon expecting the geometry of the operation, we can readily convince ourselves that the hoop displacement must be

$$u_\theta(r,\theta) = \frac{\alpha r\theta}{2\pi}.$$

The inspection should also convince us that the radial displacement must be independent of θ, namely,

$$u_r(r,\theta) = f(r).$$

We can then insert the above expressions into the general equations of linear elasticity and derive the ODE for $f(r)$, which can be solved once appropriate boundary conditions are prescribed.

Separation of variables. We next consider the stress function as a function of two variables, $\phi(r,\theta)$. One can obtain many solutions by using the procedure of separation of variable, assuming that

$$\phi(r,\theta) = f(r)g(\theta).$$

Formulas for stresses and displacements can be found on p. 205, *Deformation of Elastic Solids*, by A.K. Mal and S.J. Singh.

A circular hole in an infinite sheet under remote shear. Remote from the hole, the sheet is in a state of pure shear:

$$\tau_{xy} = S, \quad \sigma_{xx} = \sigma_{yy} = 0 .$$

The remote stresses in the polar coordinates are

$$\sigma_{rr} = S\sin 2\theta, \quad \sigma_{\theta\theta} = -S\sin 2\theta, \quad \tau_{r\theta} = S\cos 2\theta .$$

Recall that

$$\sigma_{rr} = \frac{\partial^2\phi}{r^2\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial\phi}{r\partial\theta}\right).$$

We guess that the stress function must be in the form

$$\phi(r,\theta) = f(r)\sin 2\theta .$$

The biharmonic equation becomes

$$\left(\frac{d^2}{dr^2} + \frac{d}{rdr} - \frac{4}{r^2}\right)\left(\frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{r\partial r} - \frac{4f}{r^2}\right) = 0 .$$

A solution to this equi-dimensional ODE takes the form $f(r) = r^m$. Inserting this form into the ODE, we obtain that

$$\left((m-2)^2 - 4\right)\left(m^2 - 4\right) = 0 .$$

The algebraic equation has four roots: 2, -2, 0, 4. Consequently, the stress function is

$$\phi(r,\theta) = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right)\sin 2\theta .$$

The stress components inside the sheet are

$$\sigma_{rr} = \frac{\partial^2\phi}{r^2\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\sin 2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial\phi}{r\partial\theta}\right) = \left(-2A - 6Br^2 + \frac{6C}{r^4} + \frac{2D}{r^2}\right)\cos 2\theta .$$

To determine the constants A, B, C, D, we invoke the boundary conditions:

- Remote from the hole, namely, $r \to \infty$, $\sigma_{rr} = S\sin 2\theta, \tau_{r\theta} = S\cos 2\theta$, giving $A = -S/2, B = 0$.
- On the surface of the hole, namely, $r = a$, $\sigma_{rr} = 0, \tau_{r\theta} = 0$, giving $D = Sa^2$ and $C = -Sa^4/2$.

The stress field inside the sheet is

$$\sigma_{rr} = S\left[1 + 3\left(\frac{a}{r}\right)^4 - 4\left(\frac{a}{r}\right)^2\right]\sin 2\theta$$

$$\sigma_{\theta\theta} = -S\left[1 + 3\left(\frac{a}{r}\right)^4\right]\sin 2\theta$$

$$\tau_{r\theta} = S\left[1 - 3\left(\frac{a}{r}\right)^4 + 2\left(\frac{a}{r}\right)^2\right]\cos 2\theta$$

A hole in an infinite sheet subject to a remote uniaxial stress. Use this as an example to illustrate linear superposition. A state of uniaxial stress is a linear superposition of a state of pure shear and a state of biaxial tension. The latter is the Lame problem. When the sheet is subject to remote biaxial tension of magnitude *S*, the stress field in the sheet is given by

$$\sigma_{rr} = S\left[1 - \left(\frac{a}{r}\right)^2\right], \quad \sigma_{\theta\theta} = S\left[1 + \left(\frac{a}{r}\right)^2\right].$$

Illustrate the superposition in figures. Show that under uniaxial tensile stress, the stress around the hole has a concentration factor of 3. Under uniaxial compression, material may split in the loading direction.

A line force acting on the surface of a half space. An elastic material occupies the half space $x > 0$, and is subject to a line force on its surface in the direction of *x*. Let P be the force per unit length. Recall that the field equations have no length scale. The boundary conditions in this problem also have no length scale. This consideration, along with linearity, requires that the stress field in the half space take the form

$$\sigma_{ij}(r,\theta) = \frac{P}{r} g_{ij}(\theta),$$

where $g_{ij}(\theta)$ are dimensionless functions of θ. Recall that

$$\sigma_{rr} = \frac{\partial^2 \phi}{r^2 \partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial \phi}{r \partial \theta}\right)$$

We guess that the stress function takes the form

$$\phi(r,\theta) = rPf(\theta),$$

where $f(\theta)$ is a dimensionless function of θ. A quick note: a homework problem will show that this guess is incomplete for a line force in an infinite block. This guess, however, suffices for the present problem.

Inserting this guess of the stress function into the biharmonic equation, we obtain an ODE for $f(\theta)$:

$$f + 2\frac{d^2 f}{d\theta^2} + \frac{d^4 f}{d\theta^4} = 0 .$$

The general solution to the ODE is

$$f(\theta) = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta ,$$

so that

$$\phi(r,\theta) = rP(A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta).$$

Observe that $r\sin\theta = y$ and $r\cos\theta = x$ do not contribute to any stress, so we drop these two terms. By the symmetry of the problem, we look for stress field symmetric about $\theta = 0$, so that we will drop the term $\theta\cos\theta$. Consequently, the stress function takes the form

$$\phi(r,\theta) = rPC\theta\sin\theta .$$

We can calculate the components of the stress field:

$$\sigma_{rr} = \frac{2CP\cos\theta}{r}, \; \sigma_{\theta\theta} = \tau_{r\theta} = 0 .$$

This field satisfies the traction boundary conditions, $\sigma_{\theta\theta} = \tau_{r\theta} = 0$ at $\theta = 0$ and $\theta = \pi$. To determine *C*, we require that the resultant force acting on a cylindrical surface of radius *r* balance the line force *P*. On each element $rd\theta$ of the surface, the radial stress provides a vertical component of force $\sigma_{rr}\cos\theta rd\theta$. The force balance of the half cylinder requires that

$$P + \int_{-\pi/2}^{\pi/2} \sigma_{rr}\cos\theta rd\theta = 0 .$$

Integrating, we obtain that $C = -1/\pi$.

The stress components in the *x*-*y* coordinates are

$$\sigma_{xx} = -\frac{2P}{\pi x}\cos^4\theta, \; \sigma_{yy} = -\frac{2P}{\pi x}\sin^2\theta\cos^2\theta, \; \tau_{xy} = -\frac{2P}{\pi x}\sin\theta\cos^3\theta$$

The displacement field is

$$u_r = -\frac{2P}{\pi E}\cos\theta\log r - \frac{(1-\nu)P}{\pi E}\theta\sin\theta$$

$$u_\theta = -\frac{2\nu P}{\pi E}\sin\theta + \frac{2P}{\pi E}\sin\theta\log r - \frac{(1-\nu)P}{\pi E}\theta\cos\theta - \frac{(1-\nu)P}{\pi E}\sin\theta$$

The solution of a line force on the surface of a half space can used to construct solutions to other problems by linear superposition. We illustrate this idea using the following example.

An application. S. Ho, C. Hillman, F.F. Lange and Z. Suo, Surface cracking in layers under biaxial, residual compressive stress, *J. Am. Ceram. Soc.* **78**, 2353-2359 (1995). In treating laminates, we have so far ignored the edge effect. This approximation has troubled us because we know that the edge often initiates failure. Here is a phenomenon discovered in the lab of Fred Lange at UCSB. A thin layer of material 1 was sandwiched in two thick blocks of material 2. Material 1 has a smaller coefficient of thermal expansion than material 2, so that, upon cooling, material 1 develops a biaxial compression in the plane of the laminate. The two blocks are nearly stress-free. Of course, these statements are only valid at a distance larger than the thickness of the thin layer. It was observed in experiment that the thin layer cracked, as shown in Fig. 1.

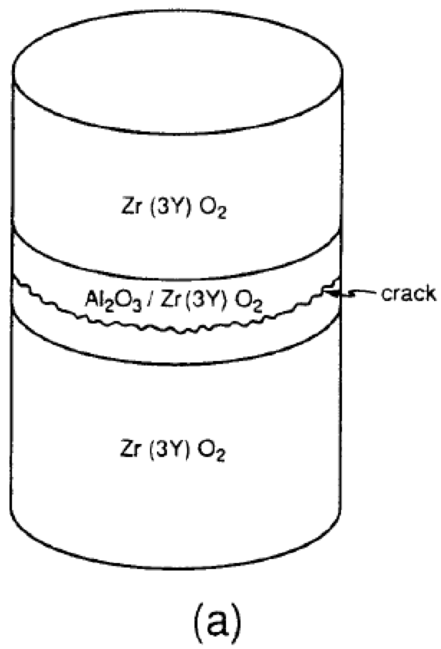


Fig. 1. (a) A thin layer of $Al_2O_3/Zr(Y)O_2$ is bonded between two blocks of $Zr(Y)O_2$. A crack runs parallel to the interfaces, in the $Al_2O_3/Zr(Y)O_2$ layer. (b) An optical micrograph of a crack running in the $Al_2O_3/Zr(Y)O_2$ layer. (c) SEM micrograph of fracture surface showing sequential positions of the crack front (partial dashed lines) extending from the surface near the center of the $Al_2O_3/Zr(Y)O_2$ layer.

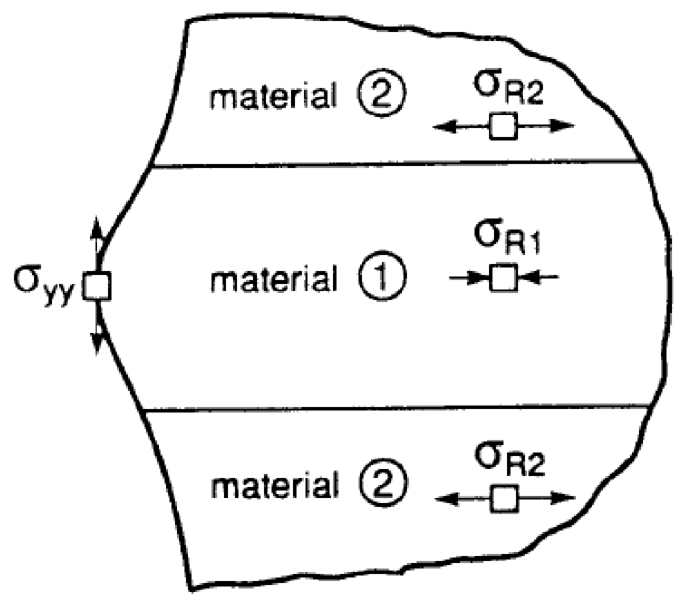


Fig. 2. Far away from the edge, the stress is biaxial in the plane of the laminate, compressive in $Al_2O_3/Zr(Y)O_2$, and tensile in $Zr(Y)O_2$. At the edge, there is a tensile stress normal to the interfaces in $Al_2O_3/Zr(Y)O_2$.

It is clear from Fig. 2 that a tensile stress σ_{yy} can develop near the edge. We would like to know its magnitude, and how fast it decays as we go into the layer.

We analyze this problem by a linier superposition, as illustrated in Fig. 3. Let σ_M be the magnitude of the biaxial stress in the thin layer far from the edge. In Problem A, we apply a compressive traction of magnitude σ_M on the edge of the thin layer, so that the thin layer is under the uniform biaxial stress, and the two thick blocks are stress-free. In problem B, we remove thermal expansion misfit, but apply a tensile traction on the edge of the thin layer. The original problem is the superposition of Problem A and Problem B. In particular, the stress field σ_{yy} in the original problem is the same as the stress σ_{yy} in Problem B.

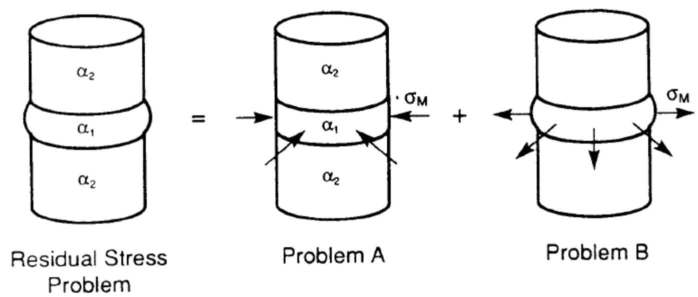


Fig. 3. The residual stress problem is a superposition of the following two problems: (problem A) a band of pressure of magnitude σ_M is applied in addition to the thermal mismatch; (problem B) a band of tensile traction of magnitude σ_M is applied, and there is no thermal mismatch.

Because the thickness of the thin layer is small compared to its lateral dimensions, the edge of the layer in Problem B is approximately under the plane strain conditions. With reference to the inset in Fig. 4, let us calculate the stress distribution $\sigma_{yy}(x,0)$. Recall that when a half space is subject to a line force P, the stress is given by

$$\sigma_{yy} = -\frac{2P}{\pi x}\sin^2\theta\cos^2\theta .$$

We now consider a line force acting at $y=\eta$. On an element of the edge, $d\eta$, the tensile traction applied the line force $P=-\sigma_M d\eta$. Summing up over all elements, we obtain the stress field in the layer:

$$\sigma_{yy}(x,0) = \int_{-t/2}^{t/2}\frac{2\sigma_M d\eta}{\pi x}\sin^2\theta\cos^2\theta .$$

We next evaluate this integral. Note that $\eta = x\tan\theta$, and let $\tan\beta = t/2x$. Consequently, $d\eta = \frac{x}{\cos^2\theta}d\theta$, and the integral becomes that

$$\sigma_{yy}(x,0) = \frac{2\sigma_M}{\pi}\int_{-\beta}^{\beta}\sin^2\theta d\theta = \frac{2\sigma_M}{\pi}\int_{-\beta}^{\beta}\frac{1-\cos 2\theta}{2}d\theta$$

Integrating, we obtain that

$$\sigma_{yy}(x,0) = \frac{2\sigma_M}{\pi}\left(\beta - \frac{1}{2}\sin 2\beta\right).$$

Consider two limiting cases. At the edge of the layer, $x/t \to 0$ and $\beta = \pi/2$, so that

$$\sigma_{yy}(0,0) = \sigma_M .$$

This stress is tensile, and causes the cracking of the thin layer.

Far from the edge, $t/x \to 0$, recall the Taylor expansion:

$$\sin 2\beta = 2\beta - \frac{(2\beta)^3}{6} + \ldots$$

We obtain that

$$\sigma_{yy}(x,0) \to \frac{\sigma_M}{6\pi}\left(\frac{t}{x}\right)^3 .$$

Thus, this component of stress decays as x^{-3}. We may regard this decay as an example of Saint-Venant's principle.

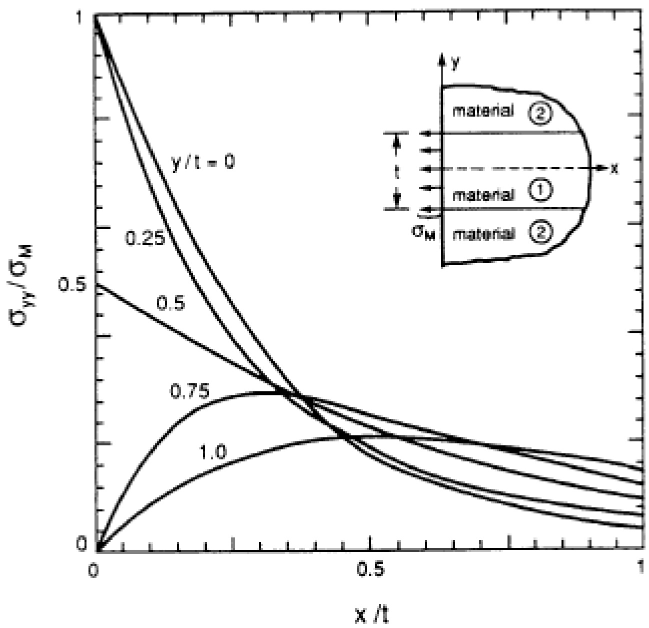


Fig. 4. Distribution of the stress component $\sigma_{yy}(x, y)$ near the edge. The elastic mismatch in this system is assumed to be zero.