

1. Introduction

Fracture Mechanics is an interdisciplinary field. For a brief review of its history and development, read: *F. Erdogan, Fracture Mechanics, Int. J. Solids and Structures 37, 171-183, 2000*. For a review on its application in advanced device technologies, read: *R. F. Cook and Z. Suo, Mechanisms active during fracture under constraint, MRS Bulletin 27 (1), 45-51, 2002*.

Engineering design based on strength of materials assumes flawless materials and structures. The approach of fracture mechanics allows flaw-tolerant design, based on toughness. The former applies mostly for ductile materials (e.g., metals and alloys), while the latter for brittle or quasi-brittle materials (e.g., ceramics and composites).

The fundamental science underlying fracture is rich, spanning from physics and chemistry at the atomic scale to micromechanics of materials and to continuum mechanics of structures (e.g., airplanes and computer chips). This course will take a top-down approach, focusing on the continuum mechanics aspects, with brief introductions to the micromechanics and physical/chemical processes down the scale.

By entering this class, you are assumed to have taken at least one graduate course in Solid Mechanics, presumably Elasticity (Solid I). We will start our discussion with a brief review of Elasticity.

No textbook is required. Lecture notes and reading materials will be handed out in class.

Use of iMechanica. A collection of online resources are posted at <http://imechanica.org/node/2555>, including lecture notes from other universities (Harvard, Cornell, etc.) and links to online books and papers. Both the lecture notes and homework sets of this class will be posted in iMechanica. The first homework is posted at <http://imechanica.org/node/2556>, asking each student to post a comment as self introduction. To do this, you need to register an account with your email address, which is otherwise free. Future homework may also involve posting at iMechanica.

There will be no final exam. Each student is expected to select a Fracture Mechanics related topic and finish a term paper by the end of the semester. Each student will also give an oral presentation of the term paper in the last week of the class session (April 28/30, 2008). Both the presentations and term papers shall be posted on iMechanica, open for comments from others.

2. Elements of Linear Elasticity

A classic textbook is *Theory of Elasticity*, by S.P. Timoshenko and J.N. Goodier, McGraw-Hill, New York.

Material particle. A solid is made of atoms, each atom is made of electrons, protons and neutrons, and each proton or neutron is made of... This kind of description of matter is too detailed, and is not used in the theory of elasticity. Instead, we develop a **continuum theory**, in which a material particle contains many atoms, and represents their average behavior.

Displacement field. Any configuration of the material particles can be used as a reference configuration. The displacement is the vector by which a material particle moves *relative* to its position in the reference configuration. If all the material particles in the body move by the same displacement vector, the body as a whole moves by a rigid-body translation. If the material particles in the body move *relative* to one another, the body deforms. For example, in a bending beam, material particles on one face of the beam move apart from one another (tension), and material particles on the other face of the beam move toward one another (compression). In a vibrating plate, the displacement of each material particle is a function of time.

We label each material particle by its coordinates (x, y, z) in the reference configuration. At time t , the material particle (x, y, z) has the displacement $u(x, y, z, t)$ in the x -direction, $v(x, y, z, t)$ in the y -direction, and $w(x, y, z, t)$ in the z -direction. A function of the spatial coordinates is known as a **field**. *The displacement field is a time-dependent vector field.* It is sometimes convenient to write the coordinates of a material particle in the reference configuration as (x_1, x_2, x_3) , and the displacement vector as $u_1(x_1, x_2, x_3, t)$, $u_2(x_1, x_2, x_3, t)$, $u_3(x_1, x_2, x_3, t)$, or x_i and u_i in the index notation with $i = 1, 2, 3$.

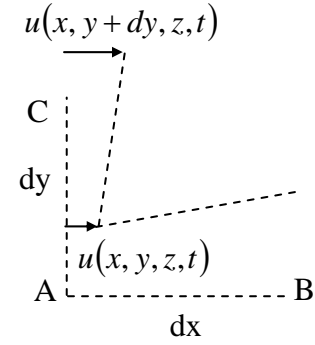
Strain field. Given a displacement field, we can calculate the strain field. Consider two material particles in the reference configuration: particle A at (x, y, z) and particle B at $(x + dx, y, z)$. In the reference configuration, the two particles are distance dx apart. At a given time t , the two particles move to new locations. The x -component of the displacement of particle A is $u(x, y, z, t)$, and that of particle B is $u(x + dx, y, z, t)$. Consequently, the distance between the two particles elongates by $u(x + dx, y, z, t) - u(x, y, z, t)$. The axial strain in the x -direction is

$$\varepsilon_x = \frac{u(x + dx, y, z, t) - u(x, y, z, t)}{dx} = \frac{\partial u}{\partial x}.$$

This is a strain of material particles in the vicinity of (x, y, z) at time t . The function $\varepsilon_x(x, y, z, t)$ is a component of the strain field of the body.

The **shear strain** is defined as follows. Consider two lines of material particles. In the reference configuration, the two lines are perpendicular to each other. The deformation changes

the included angle by some amount. This change in the angle defines the shear strain, γ . We now translate this definition into a strain-displacement relation. Consider three material particles A, B, and C. In the reference configuration, their coordinates are $A(x, y, z)$, $B(x + dx, y, z)$, and $C(x, y + dy, z)$. In the deformed configuration, in the x -direction, particle A moves by $u(x, y, z, t)$ and particle C by $u(x, y + dy, z, t)$. Consequently, the deformation rotates line AC about axis z by an angle (clockwise)



$$\theta_1 = \frac{u(x, y + dy, z, t) - u(x, y, z, t)}{dy} = \frac{\partial u}{\partial y}.$$

Similarly, the deformation rotates line AB about axis z by an angle (counterclockwise)

$$\theta_2 = \frac{v(x + dx, y, z, t) - v(x, y, z, t)}{dx} = \frac{\partial v}{\partial x}.$$

Consequently, the shear strain in the xy plane is the net change in the included angle ($\theta_1 + \theta_2$):

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

For a body in the three-dimensional (3D) space, the strain state of a material particle is described by a total of six components.

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \epsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{aligned}$$

Another definition of the shear strain relates the definition here by $\epsilon_{xy} = \gamma_{xy} / 2$. With this new definition, we can write the six strain-displacement relations neatly in the index notation as

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

The state of stress at a material particle. Imagine a three-dimensional body. The body may not be in equilibrium (e.g., the body may be vibrating). The material property is unspecified (e.g., the material can be solid or fluid). Imagine a material particle inside the body. What *state of stress* does the material particle suffer from?

To talk about internal forces, we must expose them by drawing a **free-body diagram**. Represent the material particle by a small cube (Fig. 1), with its edges parallel to the coordinate

axes. Cut the cube out from the body to expose all the forces on its 6 faces. Define stress as force per unit area. On each face of the cube, there are three **stress components**, one normal to the face (normal stress), and the other two tangential to the face (shear stresses). Now the cube has six faces, so there are a total of 18 stress components. A few points below get us organized.

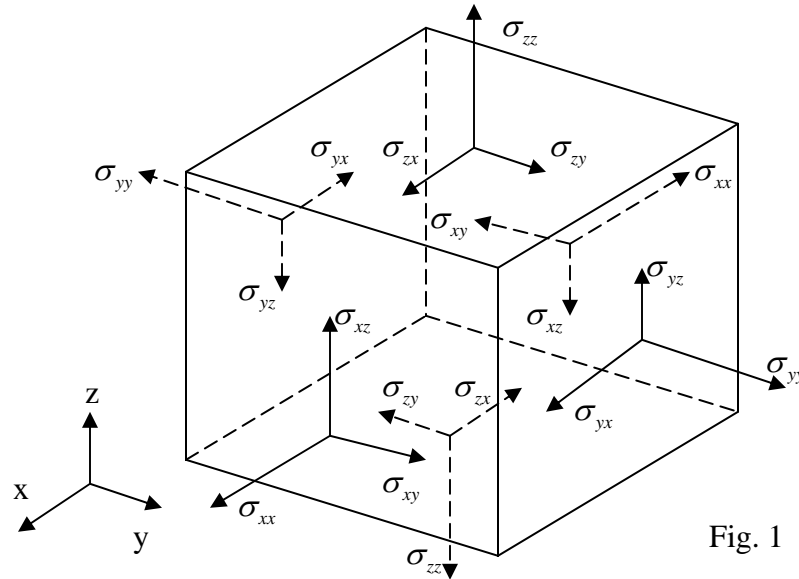


Fig. 1

- Index notation of stress: σ_{ij} . The first subscript signifies the direction of the stress component (the associated force). The second subscript signifies the direction of the vector normal to the face. Here it is more convenient to write the coordinates as (x_1, x_2, x_3) , which is equivalent to (x, y, z) . The indices i and j take values from 1 to 3.
- Sign convention. On a face whose normal is in the positive direction of a coordinate axis, the stress component is positive when it points to the positive direction of the axis. On a face whose normal is in the negative direction of a coordinate axis, the stress component is positive when it points to the negative direction of the axis.
- Equilibrium of the cube. As the size of the cube shrinks, the forces that scale with the volume (gravity, inertia effect) are negligible. Consequently, the forces acting on the cube faces must be in static equilibrium. Normal stress components form pairs ($6/2 = 3$ independent). Shear stress components form quadruples ($12/4 = 3$ independent). Consequently, only 6 independent stress components are needed to describe the state of stress of a material particle.
- Write the six stress components in a 3 by 3 symmetric matrix:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}.$$

Hooke's law. For an isotropic, homogeneous solid, only two independent constants are needed to describe its elastic property: Young's modulus E and Poisson's ratio ν . In addition, a thermal expansion coefficient α characterizes strains due to temperature change. When

temperature changes by ΔT , thermal expansion causes a strain $\alpha\Delta T$ in all three directions. The combination of multi-axial stresses and a temperature change causes normal strains

$$\begin{aligned}\varepsilon_x &= \frac{1}{E} [\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha\Delta T \\ \varepsilon_y &= \frac{1}{E} [\sigma_y - \nu(\sigma_z + \sigma_x)] + \alpha\Delta T \\ \varepsilon_z &= \frac{1}{E} [\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha\Delta T\end{aligned}$$

The shear strains are

$$\varepsilon_{xy} = \frac{1+\nu}{E} \sigma_{xy}, \varepsilon_{yz} = \frac{1+\nu}{E} \sigma_{yz}, \varepsilon_{zx} = \frac{1+\nu}{E} \sigma_{zx}$$

Recall that $\varepsilon_{xy} = \gamma_{xy}/2$ and the shear modulus $G = \frac{E}{2(1+\nu)}$.

Using the index notation, the six stress-strain relation may be written as

$$\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \nu \sigma_{kk} \delta_{ij} + \alpha\Delta T \delta_{ij}.$$

The symbol δ_{ij} stands for 0 when $i \neq j$ and for 1 when $i = j$. We adopt the convention that a repeated index implies a summation over 1, 2 and 3. Thus, $\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$.

Alternatively, the stress components can be obtained from the strain components

$$\sigma_{ij} = \frac{E}{1+\nu} \left(\varepsilon_{ij} + \frac{\nu}{1-2\nu} \varepsilon_{kk} \delta_{ij} \right) - \frac{E}{1-2\nu} \alpha\Delta T \delta_{ij}$$

Isotropy. A material is isotropic when response in one direction is the same as in any other direction. Metals and ceramics in polycrystalline form are isotropic at macro-scale, even though their constituents—grains of single crystals—are anisotropic. Woods, single crystals, uniaxially fiber reinforced composites are *anisotropic* materials.

Traction vector. Imagine a plane inside a material. The plane has the unit normal vector \mathbf{n} , with three components n_1, n_2, n_3 . The force per area on the plane is called the **traction**. The traction is a vector, with three components:

$$\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}.$$

For an arbitrary plane, you can calculate the traction vector from

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Or, in the index notation

$$t_i = \sigma_{ij} n_j$$

Thus, the six components σ_{ij} are indeed sufficient to characterize the state of stress. The traction-stress relation is the consequence of the equilibrium of a **tetrahedron** formed by the particular plane and the three coordinate planes (Fig. 2).

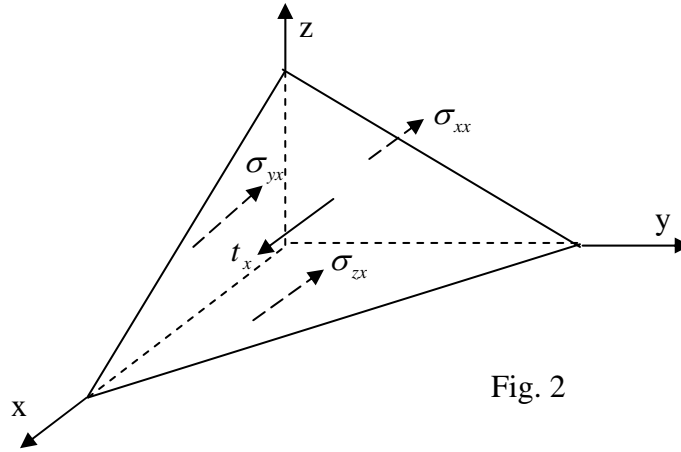


Fig. 2

Stress field. Imagine the three-dimensional body again. At time t , the material particle (x, y, z) is under a state of stress $\sigma_{ij}(x, y, z, t)$. Denote the distributed external force per unit volume by $\mathbf{b}(x, y, z, t)$. An example is the gravitational force, $b_z = -\rho g$. The stress and the displacement are time-dependent fields. Each material particle has the acceleration vector $\partial^2 u_i / \partial t^2$. Cut a small differential element, of edges dx, dy and dz (Fig. 3). Let ρ be the density. The mass of the differential element is $\rho dx dy dz$. Apply Newton's second law in the x -direction, and we obtain that

$$\begin{aligned} & dydz[\sigma_{xx}(x+dx, y, z, t) - \sigma_{xx}(x, y, z, t)] \\ & + dxdz[\sigma_{yx}(x, y+dy, z, t) - \sigma_{yx}(x, y, z, t)] \\ & + dxdy[\sigma_{zx}(x, y, z+dz, t) - \sigma_{zx}(x, y, z, t)] + b_x dxdydz = \rho dxdydz \frac{\partial^2 u_x}{\partial t^2} \end{aligned}$$

Divide both sides of the above equation by $dxdydz$, and we obtain that

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x = \rho \frac{\partial^2 u_x}{\partial t^2}.$$

This is the momentum balance equation in the x -direction. Similarly, the momentum balance equations in the y - and z -direction are

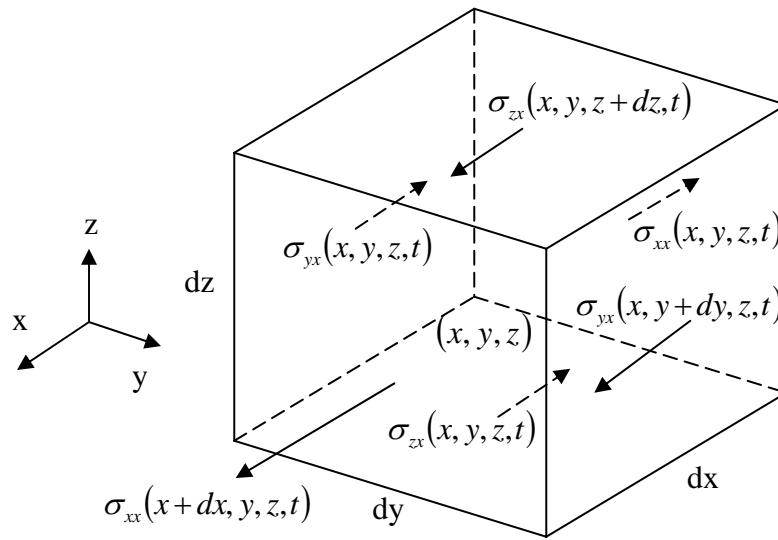


Fig. 3

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y = \rho \frac{\partial^2 u_y}{\partial t^2}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = \rho \frac{\partial^2 u_z}{\partial t^2}$$

Using the index notation and the summation convention, we write the three equations of momentum balance as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = \rho \frac{\partial^2 u_i}{\partial t^2}.$$

When the body is in equilibrium (for static problems), we drop the acceleration terms from the above equations.

Summary of Elasticity Concepts

Players: Fields

- **Stress field:** stress state varies from particle to particle (positional property).
Stress field is represented by 6 functions, $\sigma_{xx}(x, y, z; t)$, $\sigma_{xy}(x, y, z; t)$...
- **Displacement field** is represented by 3 functions, $u(x, y, z; t)$, $v(x, y, z; t)$, $w(x, y, z; t)$.
- **Strain field** is represented by 6 functions.

Rules: 3 elements of solid mechanics

- Momentum balance (Equilibrium for static problems)
- Deformation geometry (Kinematics, strain-displacement relations)
- Material law (Hooke's law for linear elasticity)

Complete equations of elasticity

- 15 Partial differential equations.
- 15 variables (functions)

Boundary conditions

- Prescribe displacement.
- Prescribe traction.

Initial conditions

For dynamic problems (e.g., vibration and wave propagation), one also need prescribe initial displacement and velocity fields.

Solving boundary value problems.

- **Idealization, analytical solutions.**
S.P. Timoshenko and J.N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York.
- **Handbook solutions.**
R.E. Peterson, *Stress Concentration Factors*, John Wiley, New York, 1974. 2nd edition by W.D. Pilkey, 1997.
- **Brute force, numerical methods.**
Finite Element Methods.

Basic Equations of Elasticity

Equilibrium Equations

In index notation

$$\sigma_{ij,j} + b_j = 0$$

Or explicitly

$$\begin{aligned}\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x &= 0 \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y &= 0 \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z &= 0\end{aligned}$$

Strain-Displacement Relation

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

$$\begin{aligned}\varepsilon_{xx} &= \frac{\partial u_x}{\partial x}, & \varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right), \\ \varepsilon_{yy} &= \frac{\partial u_y}{\partial y}, & \varepsilon_{zx} &= \frac{1}{2} \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right), \\ \varepsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \varepsilon_{xy} &= \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)\end{aligned}$$

Stress-Strain Relation

$$\varepsilon_{ij} = \frac{1+\nu}{E} \left(\sigma_{ij} - \frac{\nu}{1+\nu} \sigma_{kk} \delta_{ij} \right)$$

Alternatively

$$\sigma_{ij} = \frac{E}{1+\nu} \left(\varepsilon_{ij} + \frac{\nu}{1-2\nu} \varepsilon_{kk} \delta_{ij} \right)$$

Compatibility Equation

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik}$$

Surface Traction

$$t_i = \sigma_{ij} n_j$$

3D Elasticity: Equations in other coordinates

Cylindrical Coordinates (r, θ, z)

u, v, w are the displacement components in the radial, circumferential and axial directions, respectively. Inertia terms are neglected.

$$\begin{aligned}\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + b_r &= 0 \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + b_\theta &= 0 \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + b_z &= 0\end{aligned}$$

$$\begin{aligned}\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \varepsilon_{\theta z} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right), \\ \varepsilon_{\theta\theta} &= \frac{1}{r} \left(\frac{\partial u_\theta}{\partial \theta} + u_r \right), & \varepsilon_{zr} &= \frac{1}{2} \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right), \\ \varepsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \varepsilon_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)\end{aligned}$$

Spherical Coordinates (r, θ, ϕ)

θ is measured from the positive z -axis to a radius; ϕ is measured round the z -axis in a right-handed sense. u, v, w are the displacements components in the r, θ, ϕ directions, respectively. Inertia terms are neglected.

$$\begin{aligned}\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r} (2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} - \sigma_{r\theta} \cot \theta) + b_r &= 0 \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r} [(\sigma_{\theta\theta} - \sigma_{\phi\phi}) \cot \theta + 3\sigma_{r\theta}] + b_\theta &= 0 \\ \frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r} (3\sigma_{r\phi} + 2\sigma_{\theta\phi} \cot \theta) + b_\phi &= 0\end{aligned}$$

$$\begin{aligned}\varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \varepsilon_{\theta\phi} &= \frac{1}{2r \sin \theta} \left(\frac{\partial u_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial \theta} \sin \theta - u_\phi \cos \theta \right), \\ \varepsilon_{\theta\theta} &= \frac{1}{r} \left(\frac{\partial u_\theta}{\partial \theta} + u_r \right), & \varepsilon_{r\phi} &= \frac{1}{2} \left(\frac{\partial u_\phi}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r} \right), \\ \varepsilon_{\phi\phi} &= \frac{1}{r \sin \theta} \left(\frac{\partial u_\phi}{\partial \phi} + u_r \sin \theta + u_\theta \cos \theta \right), & \varepsilon_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)\end{aligned}$$

Example: Lamé Problem. As an example of boundary value problems, consider a spherical cavity in a large body, remote hydrostatic tension. The symmetry of the problem makes spherical coordinate system convenient.

List nonzero quantities.

- the radial displacement u ,
- the radial stress σ_r , two equal hoop stresses $\sigma_\theta = \sigma_\phi$,
- the radial strain ε_r , two equal hoop strains $\varepsilon_\theta = \varepsilon_\phi$.
- They are all functions of r .

List equations. Use the basic equation sheet. Simplify to the special symmetry.

- Equilibrium equation:
$$\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0.$$
- Deformation geometry:
$$\varepsilon_r = \frac{du}{dr}, \varepsilon_\theta = \frac{u}{r}.$$
- Material law (linear elasticity, Hooke's law):

$$\varepsilon_r = \frac{1}{E}(\sigma_r - 2\nu\sigma_\theta)$$

$$\varepsilon_\theta = \frac{1}{E}[(1-\nu)\sigma_\theta - \nu\sigma_r]$$

Reduce to a single ODE. The above are a set of 5 equations for 5 functions of r . You can follow a number of approaches to solve them. I'll follow the approach below. I want to obtain a single equation for the radial stress, σ_r . From the equilibrium equation, I express σ_θ in terms of σ_r :

$$\sigma_\theta = \sigma_r + \frac{r}{2} \frac{d\sigma_r}{dr}.$$

Then I use the material law to express both strains in terms of σ_r :

$$\varepsilon_r = \frac{1}{E} \left[(1-2\nu)\sigma_r - \nu r \frac{d\sigma_r}{dr} \right]$$

$$\varepsilon_\theta = \frac{1}{E} \left[(1-2\nu)\sigma_r + (1-\nu) \frac{r}{2} \frac{d\sigma_r}{dr} \right]$$

I can eliminate u from the two equations of deformation geometry, and resulting equation is in terms of the two strains,

$$\varepsilon_r = d(r\varepsilon_\theta)/dr.$$

Express this equation in terms of the radial stress, and I have

$$\frac{d^2 \sigma_r}{dr^2} + \frac{4}{r} \frac{d\sigma_r}{dr} = 0.$$

Solve the ODE. This is an **equidimensional** equation. The solution is of form $\sigma_r = r^m$. Substitute $\sigma_r = r^m$ into the ODE, and we find two roots: $m = 0$ and $m = -3$. Consequently, the full solution is

$$\sigma_r = A + \frac{B}{r^3},$$

where A and B are constants to be determined by the boundary conditions.

From the equilibrium equation, the hoop stress is given by

$$\sigma_\theta = A - \frac{B}{2r^3}.$$

The boundary conditions are

- Prescribed remote stress: $\sigma_r = S$ as $r = \infty$.
- Traction-free at the cavity surface: $\sigma_r = 0$ as $r = a$

Upon determining the two constants A and B , we obtain the stress distribution

$$\sigma_r = S \left[1 - \left(\frac{a}{r} \right)^3 \right], \quad \sigma_\theta = S \left[1 + \frac{1}{2} \left(\frac{a}{r} \right)^3 \right].$$

Plot the stresses as a function of r .

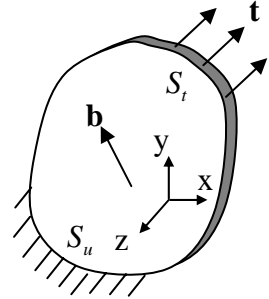
Stress concentration factor. Note that the hoop stress is nonzero near the cavity surface, where the hoop stress reaches the maximum. The stress concentration factor is the ratio of the maximum stress over the applied stress. In this case, the stress concentration factor is $3/2$.

Plane Elasticity Problems

Plane stress problem. A thin sheet of an isotropic material is subject to loads in the plane of the sheet. The sheet lies in the plane (x, y) . The top and the bottom surfaces of the sheet are traction-free. The edge of the sheet may have two kinds of the boundary conditions: displacement prescribed, or traction prescribed. In the latter case, we write

$$\begin{aligned}\sigma_{xx}n_x + \tau_{xy}n_y &= t_x \\ \tau_{xy}n_x + \sigma_{yy}n_y &= t_y\end{aligned}$$

where t_x and t_y are components of the traction vector prescribed on the edge of the sheet, and n_x and n_y are the components of the unit vector normal to the edge of the sheet. The above two equations provide two conditions for the components of the stress tensor along the edge.



Semi-inverse method. We next go into the interior of the sheet. We already have obtained a full set of governing equations for 3D linear elasticity problems. No general approach exists to solve partial differential equations analytically. However, numerical methods are now readily available to solve any elasticity problem that you can pose. Here, to gain some insight into solid mechanics, we will make reasonable guesses of solutions, and see if they satisfy all the governing equations as well as boundary conditions. This *trial-and-error* approach is sometimes called the semi-inverse method.

It seems reasonable to guess that the stress field in the sheet only has nonzero components in its plane: $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$, and the components out of the plane vanish: $\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0$ (satisfying the traction-free boundary conditions on the faces). Furthermore, we guess that the in-plane stress components may vary with x and y , but are independent of z (a reasonable approximation for thin sheet). That is, the stress field in the sheet is described by three functions:

$$\sigma_{xx}(x, y), \quad \sigma_{yy}(x, y), \quad \sigma_{xy}(x, y).$$

Will these guesses satisfy the governing equations of elasticity? Let us go through these equations one by one.

Equilibrium equations. Using the guessed stress field, we reduce the three equilibrium equations to two equations (zero body force):

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0.$$

These two equations by themselves are insufficient to determine the three functions.

Stress-strain relations. Given the guessed stress field, the 6 components of the strain field are

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E}, \quad \varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu \frac{\sigma_{xx}}{E}, \quad \gamma_{xy} = \frac{2(1+\nu)}{E} \sigma_{xy}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}), \quad \gamma_{xz} = \gamma_{yz} = 0.$$

It seems reasonable to assume that the in-plane displacements u and v vary only with x and y , but not with z (again, a good approximation for thin sheets). From these guesses, together with the conditions that $\gamma_{xz} = \gamma_{yz} = 0$, we find that

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial y} = 0.$$

Thus, w is independent of x and y , and can only be a function of z . If we insist that ε_{zz} be independent of z (because the stresses are independent of z), and from $\varepsilon_{zz} = \partial w / \partial z$, then ε_{zz} must be a constant (independent of x and y), $\varepsilon_{zz} = c$, and $w = cz + b$. On the other hand, we also have $\varepsilon_{zz} = -\nu(\sigma_{xx} + \sigma_{yy})/E$, which may not be a constant. This inconsistency shows that our guesses are incorrect.

Summary of equations for plane stress problems. Instead of abandoning these guesses, we will just call our guesses the plane-stress approximation. If we ignore the inconsistency between $\varepsilon_{zz} = c$ and $\varepsilon_{zz} = -\nu(\sigma_{xx} + \sigma_{yy})/E$, at least the following set of equations look self-consistent:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E}, \quad \varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu \frac{\sigma_{xx}}{E}, \quad \gamma_{xy} = \frac{2(1+\nu)}{E} \sigma_{xy}$$

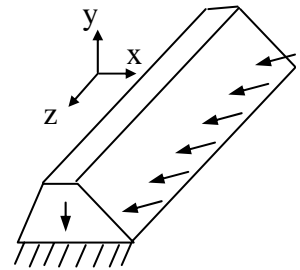
$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

These are 8 equations for 8 functions (thus complete). We will focus on these 8 equations.

Plane strain problem. An infinitely long cylinder with the axis in the z -direction, and a cross section in the (x, y) plane. The loading is invariant along the z -direction. Consequently, the displacement field takes the form:

$$u(x, y), \quad v(x, y), \quad w = 0.$$

From the strain-displacement relations, we find that only the three in-plane strains are nonzero: $\varepsilon_{xx}(x, y), \varepsilon_{yy}(x, y), \gamma_{xy}(x, y)$. The three out-of-plane strains vanish: $\varepsilon_{zz} = \gamma_{xz} = \gamma_{yz} = 0$.



Because $\gamma_{xz} = \gamma_{yz} = 0$, the stress-strain relations imply that $\tau_{xz} = \tau_{yz} = 0$ (for isotropic materials). From $\varepsilon_{zz} = 0$ and $\varepsilon_{zz} = (\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy})$, we obtain that

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}).$$

Thus,

$$\begin{aligned}\varepsilon_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}) = \frac{1-\nu^2}{E}\left(\sigma_{xx} - \frac{\nu}{1-\nu}\sigma_{yy}\right) \\ \varepsilon_{yy} &= \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}) = \frac{1-\nu^2}{E}\left(\sigma_{yy} - \frac{\nu}{1-\nu}\sigma_{xx}\right) \\ \gamma_{xy} &= \frac{2(1+\nu)}{E}\sigma_{xy}\end{aligned}$$

These three stress-strain relations look similar to those under the plane stress conditions, provided we make the following substitutions:

$$\bar{E} = \frac{E}{1-\nu^2}, \quad \bar{\nu} = \frac{\nu}{1-\nu}.$$

The quantity \bar{E} is called the plane strain modulus.

The equilibrium equations for plane strain problems are identical to those for plane stress problems. The strain-displacement equations are the same too, except for that $\varepsilon_{zz} = 0$ under the plane strain condition without any inconsistency. Therefore, any solution to plane stress problems can be easily transferred to corresponding plane strain problems with the substitutions of the plane strain modulus and Poisson's ratio.

The Airy stress function. Many plane elasticity problems can be solved by using the Airy stress function. From the equilibrium equation, $\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0$, we deduce that there exists a function $A(x, y)$, such that

$$\sigma_{xx} = \frac{\partial A}{\partial y}, \quad \sigma_{xy} = -\frac{\partial A}{\partial x}.$$

From $\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0$, we deduce that there exists a function $B(x, y)$, such that

$$\sigma_{yy} = \frac{\partial B}{\partial x}, \quad \sigma_{xy} = -\frac{\partial B}{\partial y}.$$

Finally, from $\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y}$, we deduce that there exists a function $\phi(x, y)$, such that

$$A = \frac{\partial\phi}{\partial y}, \quad B = \frac{\partial\phi}{\partial x}.$$

The function $\phi(x, y)$ is known as the Airy (1862) stress function. The three components of the stress field can now be represented by the stress function:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}.$$

Using the 2D stress-strain relations, we can also express the three components of strain field in terms of the Airy stress function:

$$\varepsilon_{xx} = \frac{1}{E} \left(\frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2} \right), \quad \varepsilon_{yy} = \frac{1}{E} \left(\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2} \right), \quad \gamma_{xy} = -\frac{2(1+\nu)}{E} \frac{\partial^2 \phi}{\partial x \partial y}.$$

Recall that, for plane strain problems, E and ν are substituted by \bar{E} and $\bar{\nu}$.

Compatibility equation. The three strain components are related to two displacement components (a redundancy)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

To make sure there exist two displacement fields, $u(x, y)$ and $v(x, y)$, the strain components must satisfy a relation (to remove the redundancy)

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.$$

This equation is known as the compatibility equation.

Biharmonic equation. Inserting the expressions of the strains in terms of Airy stress function $\phi(x, y)$ into the compatibility equation, we obtain that

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0.$$

This equations can also be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = 0.$$

It is called the biharmonic equation.

Thus, a procedure to solve a plane elasticity problem (with no body force) is to solve the biharmonic equation for $\phi(x, y)$, and then calculate stresses and strains. After the strains are obtained, the displacement field can be obtained by integrating the strain-displacement relations (compatibility satisfied already).

Dependence on elastic constants. For a plane elasticity problem with traction-prescribe

boundary conditions, both the governing equations and the boundary conditions can be expressed in terms of ϕ . All these equations are independent of elastic constants. Consequently, the stress field in such a boundary value problem is independent of the elastic constants (but the strain and displacement are not). However, this is only correct for boundary value problems in simply connected regions. For multiply connected regions, the above equations in terms of ϕ do not guarantee that the displacement field is continuous. When we insist that displacement field be continuous, elastic constants may enter the solutions for the stress field.

Plane elasticity equations in polar coordinates. The Airy stress function is a function of the polar coordinates, $\phi(r, \theta)$. The stresses are expressed in terms of the Airy stress function:

$$\sigma_{rr} = \frac{\partial^2 \phi}{r^2 \partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{\partial \phi}{r \partial \theta} \right)$$

The biharmonic equation is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{r^2 \partial \theta^2} \right) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial \phi}{r \partial r} + \frac{\partial^2 \phi}{r^2 \partial \theta^2} \right) = 0.$$

The stress-strain relations in polar coordinates are similar to those in the rectangular coordinates:

$$\varepsilon_{rr} = \frac{\sigma_{rr}}{E} - \nu \frac{\sigma_{\theta\theta}}{E}, \quad \varepsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \nu \frac{\sigma_{rr}}{E}, \quad \gamma_{r\theta} = \frac{2(1+\nu)}{E} \tau_{r\theta}$$

The strain-displacement relations are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{\partial u_\theta}{r \partial \theta}, \quad \gamma_{r\theta} = \frac{\partial u_r}{r \partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}.$$

Stress field symmetric about an axis. Let the stress function be $\phi(r)$. The biharmonic equation becomes

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \left(\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} \right) = 0.$$

Each term in this equation has the same dimension in the independent variable r . Such an ODE is known as an equi-dimensional equation. A solution to an equi-dimensional equation is of the form

$$\phi = r^m.$$

Inserting into the biharmonic equation, we obtain that

$$m^2(m-2)^2 = 0.$$

The fourth order algebraic equation has a double root of 0 and a double root of 2. Consequently, the general solution to the ODE is

$$\phi(r) = A \log r + B r^2 \log r + C r^2 + D.$$

where A , B , C and D are constants of integration.

The components of the stress field are

$$\begin{aligned}\sigma_{rr} &= \frac{\partial^2 \phi}{r^2 \partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{A}{r^2} + B(1 + 2 \log r) + 2C, \\ \sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} = -\frac{A}{r^2} + B(3 + 2 \log r) + 2C, \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r} \left(\frac{\partial \phi}{r \partial \theta} \right) = 0.\end{aligned}$$

The stress field is linear in A , B and C .

A circular hole in an infinite sheet under remote stresses. First consider an equibiaxial stress far away from the hole. This problem is axisymmetric with respect to the center of the hole (Lame problem in 2D). For a hole of radius a in an infinite sheet, the boundary conditions are:

- Remote from the hole, namely, $r \rightarrow \infty$, $\sigma_{rr} = \sigma_{\theta\theta} = S$, giving $C = S/2$, $B = 0$.
- On the surface of the hole, namely, $r = a$, $\sigma_{rr} = 0$, giving $A = -Sa^2$.

Thus, the stress field inside the sheet is

$$\sigma_{rr} = S \left[1 - \left(\frac{a}{r} \right)^2 \right], \quad \sigma_{\theta\theta} = S \left[1 + \left(\frac{a}{r} \right)^2 \right].$$

The stress concentration factor of this hole is 2. We may compare this problem with that of a spherical cavity in an infinite elastic solid under remote tension (stress concentration factor 3/2):

$$\sigma_{rr} = S \left[1 - \left(\frac{a}{r} \right)^3 \right], \quad \sigma_{\theta\theta} = S \left[1 + \frac{1}{2} \left(\frac{a}{r} \right)^3 \right].$$

Next, consider **a circular hole in an infinite sheet under remote shear**. Remote from the hole, the sheet is in a state of pure shear:

$$\sigma_{xy} = S, \quad \sigma_{xx} = \sigma_{yy} = 0.$$

The remote stresses in the polar coordinates are

$$\sigma_{rr} = S \sin 2\theta, \quad \sigma_{\theta\theta} = -S \sin 2\theta, \quad \sigma_{r\theta} = S \cos 2\theta.$$

Recall that

$$\sigma_{rr} = \frac{\partial^2 \phi}{r^2 \partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r} \left(\frac{\partial \phi}{r \partial \theta} \right).$$

Noting the θ dependence of the stress components, we guess that the stress function must be in the form

$$\phi(r, \theta) = f(r) \sin 2\theta.$$

The biharmonic equation then becomes

$$\left(\frac{d^2}{dr^2} + \frac{d}{rdr} - \frac{4}{r^2} \right) \left(\frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{r\partial r} - \frac{4f}{r^2} \right) = 0.$$

A solution to this equi-dimensional ODE takes the form $f(r) = r^m$. Inserting this form into the ODE, we obtain that

$$((m-2)^2 - 4)(m^2 - 4) = 0.$$

The algebraic equation has four roots: 2, -2, 0, 4. Consequently, the stress function is

$$\phi(r, \theta) = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D \right) \sin 2\theta.$$

The stress components inside the sheet are

$$\begin{aligned} \sigma_{rr} &= \frac{\partial^2 \phi}{r^2 \partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = - \left(2A + \frac{6C}{r^4} + \frac{4D}{r^2} \right) \sin 2\theta \\ \sigma_{\theta\theta} &= \frac{\partial^2 \phi}{\partial r^2} = \left(2A + 12Br^2 + \frac{6C}{r^4} \right) \sin 2\theta \\ \sigma_{r\theta} &= - \frac{\partial}{\partial r} \left(\frac{\partial \phi}{r \partial \theta} \right) = \left(-2A - 6Br^2 + \frac{6C}{r^4} + \frac{2D}{r^2} \right) \cos 2\theta. \end{aligned}$$

To determine the constants A, B, C, D, we invoke the boundary conditions:

- Remote from the hole, namely, $r \rightarrow \infty$, $\sigma_{rr} = S \sin 2\theta$, $\sigma_{r\theta} = S \cos 2\theta$, giving $A = -S/2$, $B = 0$.
- On the surface of the hole, namely, $r = a$, $\sigma_{rr} = 0$, $\sigma_{r\theta} = 0$, giving $D = Sa^2$ and $C = -Sa^4/2$.

Thus, the stress field inside the sheet is

$$\begin{aligned} \sigma_{rr} &= S \left[1 + 3 \left(\frac{a}{r} \right)^4 - 4 \left(\frac{a}{r} \right)^2 \right] \sin 2\theta \\ \sigma_{\theta\theta} &= -S \left[1 + 3 \left(\frac{a}{r} \right)^4 \right] \sin 2\theta \\ \sigma_{r\theta} &= S \left[1 - 3 \left(\frac{a}{r} \right)^4 + 2 \left(\frac{a}{r} \right)^2 \right] \cos 2\theta \end{aligned}$$

Note that the hoop stress at the surface of the hole is $\sigma_{\theta\theta}(r=a) = -4S \sin 2\theta$.

A circular hole in an infinite sheet subject to a remote uniaxial stress. Use this as an example to illustrate linear superposition. A state of uniaxial stress is a linear superposition of a state of pure shear and a state of biaxial tension. When the sheet is subject to remote tension of magnitude S , the stress field in the sheet is given by

$$\sigma_{rr} = \frac{S}{2} \left[1 - \left(\frac{a}{r} \right)^2 \right] - \frac{S}{2} \left[1 + 3 \left(\frac{a}{r} \right)^4 - 4 \left(\frac{a}{r} \right)^2 \right] \cos 2\theta.$$

$$\sigma_{\theta\theta} = \frac{S}{2} \left[1 + \left(\frac{a}{r} \right)^2 \right] + \frac{S}{2} \left[1 + 3 \left(\frac{a}{r} \right)^4 \right] \cos 2\theta$$

$$\sigma_{r\theta} = \frac{S}{2} \left[1 - 3 \left(\frac{a}{r} \right)^4 + 2 \left(\frac{a}{r} \right)^2 \right] \sin 2\theta$$

Illustrate the superposition in figures. Show that under uniaxial tensile stress, the stress around the hole has a concentration factor of 3. Under uniaxial compression, material may split in the loading direction.

Plane stress or plane strain? For a circular hole in an infinite sheet under remote stress, how should the solution depend on the sheet thickness? Apparently, the solution for the in-plane stresses are identical for plane stress and plane strain, while the strain and displacement are different. Whether it is plane stress or plane strain depends on the thickness of the sheet in comparison to the hole size (two relevant length scales). If the thickness is small compared to the hole radius, the plane stress condition applies. In the other limit, when the thickness is much larger than the hole radius, the plane strain condition applies. However, for a sheet of finite thickness, the stress state is always plane stress at the surface, no matter how thick it is. Therefore, for finite thickness, the stress/strain near the hole cannot be assumed to be plane stress or plane strain; rather, it is a three-dimensional problem. How would this affect the stress concentration as well as the deformation near the hole?

Stress concentration of an elliptic hole in a large plate. The stress field around an elliptic hole in a large plate under remote stress can be solved by using the complex variable method (Inglis, 1913; Kolosov, 1914). The detailed solution can be found in *Theory of Elasticity* by S.P. Timoshenko and J.N. Goodier, Chapter 6. In particular, for an elliptic hole with semi-axis a in the x direction and b in the y direction, and the plate is subject to a remote uniaxial stress S in the y direction, it is found that the maximum stress, occurring at the point of the hole boundary intersected by the x axis ($x = \pm a$ and $y = 0$), is

$$\sigma_{yy} = S \left(1 + 2 \frac{a}{b} \right)$$

The factor of stress concentration therefore depends on the aspect ratio of the elliptic hole. When $a = b$, it reduces to the solution for a circular hole with a stress concentration factor 3. The stress increases without limit as the hole becomes more and more slender.

The stress at the hole boundary where it is intersected by the y axis (i.e., $y = \pm b$ and $x = 0$) is $\sigma_{xx} = -S$, independent of the aspect ratio.