Integral Formulations for 2D Elasticity: 1. Anisotropic Materials

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Abstract

Several sets of boundary integral equations for two dimensional elasticity are derived from Cauchy integral theorem. These equations reveal the relations between displacements and resultant forces, between displacements and tractions, and between the tangential derivatives of displacements and tractions on solid boundary. Special attention is given to the formulation that is based on tractions and the tangential derivatives of displacements on boundary, because its integral kernels have the weakest singularities. The formulation is further extended to include singular points, such as dislocations and line forces, in a finite body, so that the singular stress field can be directly obtained from solving the integral equations on the external boundary without involving the linear superposition technique often used in the literature. Body forces and thermal effect are subsequently included. The general framework of setting up a boundary value problem is discussed and continuity conditions at a non-singular corner are derived. The general procedure in obtaining the elastic field around a circular hole is described, and the stress fields with first and second order singularities are obtained. Some other analytical solutions are also derived by using the formulation.

Keywords: Integral equations, elasticity, Boundary element method

1. Introduction

The mechanical properties of materials greatly depend on their microstructures. In many engineering applications, micro-structures of solids evolve over time. The evolution is often affected by the presence of stress field, such as thin film morphology instability driven by surface diffusion (Yao, 1988; Sorolovitz, 1989). Grain boundary diffusion is inherently coupled with stress field evolution due to mass redistribution inside solid body. Finite element method (FEM), as a general tool for solving stress field, has been used in simulating micro-structure evolution. For example, Bower et al (2004) simulated constitutive response of polycrystals during high temperature plastic deformation and Prevost et al(2000) simulated crack nucleation due to stress dependant surface reaction. However, if one uses FEM as a stress solver in simulating structural evolution, updating mesh can take a big portion of total calculation time. And FEM has to solve the displacements or stresses in the whole field, much of the information is not needed if we are only concerned about the mass transport at the surface.

For repeated calculation of stress field in an evolving body, reducing the calculation cost becomes extremely important. Boundary element method (BEM), due to its fewer degrees of freedom, has always been very attractive in solving stress field. The mesh generation and refinement in BEM is much simpler than in FEM, especially for two dimensional problems, in which we only need to divide a line element into two when it is too long, or eliminate it when it is too short (Liu et al, 2006).

In this paper, a general framework for setting up boundary integral equations is presented. Several sets of singular boundary integral equations with different singularities are derived. They are the relations between displacements (**u**) and resultant forces (**T**), between displacements (**u**) and tractions (**t**), and between the tangential derivatives of displacements

($d\mathbf{u}/ds$) and tractions ($\mathbf{t}$). Displacement and stress fields inside solid are expressed in terms of integrals involving these boundary variables.

The conventional BEM is based on the relation between displacements ($\mathbf{u}$) and tractions ($\mathbf{t}$) along the boundary. However, in calculating the internal stresses inside solid, the integral kernels associated with the displacements have higher order of singularity ($1/r^2$ for 2D), which causes so called *boundary layer effect*, i.e., large numerical error in calculating stresses near boundary (Brebbia, 1978). The boundary layer effect makes the calculation of stresses in structures with high aspect ratio such as thin film a difficult job. In many applications such as stress driven surface morphology evolution due to diffusion, the accurate evaluation of stresses close to boundary is required.

In this paper, most of the attention is given to the integral equations relating tractions ($\mathbf{t}$) and the tangential derivative of displacements ($d\mathbf{u}/ds$). The effort is driven by two factors. One is to eliminate the boundary layer effect mentioned above. The other is to derive a formulation that can directly deal with singular points inside solid, such as dislocations or line forces, without using linear superposition technique. The application of the formulation in dislocation dynamics is discussed.

Using the integral equations of traction and the tangential derivatives of displacements to set up boundary element scheme is not new. Using the fundamental solutions, Ghosh et al. (1986) and Wu et al. (1992) expressed boundary displacements in terms of the boundary integrals of tractions and the tangential derivatives of displacements. Their numerical results showed that the method delivered stresses accurately at internal points which were extremely close to the boundary. Using the derivatives of displacements to increase the accuracy of stress calculation was also used in Okada et al. (1989), in which boundary displacements and tractions were solved by the conventional boundary element method, while internal stresses were calculated from the integrals involving the gradient of displacements and tractions at the boundary.

Previously published boundary integral equations for anisotropic materials (e.g., Rizzo and Shippy, 1970; Benjumea and Sikarskie, 1972; and Wu et al, 1992) were usually derived from fundamental solutions, by using reciprocal theorem or Somigliana's identity. Starting from Cauchy integral theorem, this paper presents a unified approach in deriving integral formulations for 2D elasticity for either anisotropic or isotropic material. Three sets of integral formulations are obtained, and the relations among these formulations are clearly revealed. Beside the boundary integral equations similar to those in Ghosh et al (1986) and Wu et al (1992), several new boundary integral equations are listed here.

The formulations derived in this paper are based on Lekhnitskii-Eshelby-Stroh(LES) representation for anisotropic materials and Cauchy Theorem. The numerical implementation of the formulations presented here will appear in another paper. The complex forms allow the integrals on each boundary element be evaluated analytically and even the shape of each element can be curved. The detailed discussion in numerical implementation of complex variable integral equations will be presented later in another paper. This paper focuses on theoretical aspect of the theory and its applications in obtaining analytical solutions.

2. Lekhnitskii-Eshelby-Stroh (LES) Representation

It has been shown by Eshelby et al. (1953), Stroh (1958) and Lekhnitski (1963), that for a two dimensional problem, i.e. with geometry and external loading invariant in the direction normal to *xy*-plane, the elastic field can be represented in terms of three functions. The displacement u_i, stress σ_{ij}, and the resultant force T_i on an arc are

$$u_i = 2\,\mathrm{Re}[\sum_{\alpha=1}^{3} A_{i\alpha} f_\alpha(z_\alpha)] \qquad T_i = -2\,\mathrm{Re}[\sum_{\alpha=1}^{3} L_{i\alpha} f_\alpha(z_\alpha)] \tag{1}$$

$$\sigma_{2i} = 2\,\mathrm{Re}[\sum_{\alpha=1}^{3} L_{i\alpha} f'_\alpha(z_\alpha)] \qquad \sigma_{1i} = -2\,\mathrm{Re}[\sum_{\alpha=1}^{3} L_{i\alpha} p_\alpha f'_\alpha(z_\alpha)], \tag{2}$$

where function f_α is holomorphic in its argument, $z_\alpha = x + p_\alpha y$, and f'_α is its derivative with respect to the associated argument z_α. Here p_α's are three distinct complex numbers with positive imaginary part, which can be solved as roots of a sixth order polynomial,

$$\left| C_{i1k1} + pC_{i1k2} + pC_{i2k1} + p^2 C_{i2k2} \right| = 0, \tag{3}$$

where C_{ijkl} are the stiffness tensor of the material. Each column of **A** is solved from the eigenvalue problem

$$\sum_{k=1}^{3} (C_{i1k1} + p_\alpha C_{i1k2} + p_\alpha C_{i2k1} + p_\alpha^2 C_{i2k2}) A_{k\alpha} = 0\,. \tag{4}$$

The matrix **L** is given by

$$L_{i\alpha} = \sum_{k=1}^{3} [C_{i2k1} + p_\alpha C_{i2k2}] A_{k\alpha}\,. \tag{5}$$

The notations used in this paper follow those in Suo (1990), who proved that the schemes derived by Lekhnitskii and Eshelby were consistent.

3. Formulation based on u and T

Field solution

Denote $\mathbf{f} \equiv [f_1(z_1), f_2(z_2), f_3(z_3)]^{\mathrm{T}}$, **u** as displacements and **T** as the resultant force vector at the boundary of a solid. From Eq. (1), we have

$$\mathbf{u} = \mathbf{A} \cdot \mathbf{f} + \overline{\mathbf{A}} \cdot \overline{\mathbf{f}} \tag{6a}$$

and $$-\mathbf{T} = \mathbf{L}\cdot\mathbf{f} + \overline{\mathbf{L}}\cdot\bar{\mathbf{f}}\,, \tag{6b}$$

on the boundary. Eliminate $\bar{\mathbf{f}}$ from Eqs. (6a) and (6b), we get

$$\mathbf{f} = \mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\mathbf{u}-\overline{\mathbf{B}}\cdot\mathbf{T}]\,, \tag{7}$$

at the boundary. Here $\mathbf{B} = i\mathbf{A}\mathbf{L}^{-1}$ is a Hermitian matrix.

Denote $\partial\Omega$ as the boundary of a simply connected domain Ω in z-plane ($z = x + iy$, $i = \sqrt{-1}$) and $\partial\Omega_\alpha$ the image of $\partial\Omega$ in z_α-plane. For any internal point z in domain Ω, its image is $z_\alpha = x + p_\alpha y$ in z_α-plane. From Cauchy theorem,

$$f_\alpha(z_\alpha) = \frac{1}{2\pi i}\oint_{\partial\Omega_\alpha}\frac{f_\alpha(\zeta_\alpha)}{\zeta_\alpha - z_\alpha}d\zeta_\alpha\,. \tag{8}$$

Applying Eq. (8) to the three components in Eq. (7), we have $\mathbf{f}$ inside the solid,

$$\mathbf{f} = \frac{1}{2\pi i}\oint_{\partial\Omega}\langle (C+p_\alpha S)/(\varsigma_\alpha - z_\alpha)\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\mathbf{u}-\overline{\mathbf{B}}\cdot\mathbf{T}]\big|_\zeta\, ds \tag{9}$$

where $\langle (C+p_\alpha S)/(\varsigma_\alpha - z_\alpha)\rangle$ is diagonal matrix whose α-th diagonal component is $(C+p_\alpha S)/(\varsigma_\alpha - z_\alpha)$. ζ is a point on $\partial\Omega$, ζ_α is its image on $\partial\Omega_\alpha$, $C=\cos\theta$, $S=\sin\theta$, θ is the tangent angle at ζ (Fig.1) and ds is an infinitesimal arc length at $\partial\Omega$ so that $d\varsigma = (C+iS)ds$ and $d\varsigma_\alpha = (C+p_\alpha S)ds$. The original integrals in Eq. (8) along $\partial\Omega_\alpha$ are converted into integrations along $\partial\Omega$. In this paper, $\langle * \rangle$ represents a diagonal matrix whose α-th diagonal component is indicated inside the bracket.

Inserting Eq. (9) into Eqs. (1) and (2), we obtain displacement and stress field in terms of the boundary integrals of $\mathbf{u}$ and $\mathbf{T}$,

$$\mathbf{u}\big|_z = \mathrm{Re}\{\frac{1}{\pi i}\oint_{\partial\Omega}\mathbf{A}\cdot\langle (C+p_\alpha S)/(\varsigma_\alpha - z_\alpha)\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\mathbf{u}-\overline{\mathbf{B}}\cdot\mathbf{T}]\big|_\zeta\, ds\} \tag{10}$$

$$[\sigma_{21},\sigma_{22},\sigma_{23}]^{\mathrm{T}}\Big|_z = \mathrm{Re}\left\{\frac{1}{\pi i}\int_C \mathbf{L}\cdot\left\langle (C+p_\alpha S)\big/(\zeta_\alpha - z_\alpha)^2\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\mathbf{u}-\overline{\mathbf{B}}\cdot\mathbf{T}]\Big|_\zeta ds\right\} \qquad (11a)$$

$$[\sigma_{11},\sigma_{12},\sigma_{13}]^{\mathrm{T}}\Big|_z = \mathrm{Re}\left\{\frac{-1}{\pi i}\int_C \mathbf{L}\cdot\left\langle (C+p_\alpha S)p_\alpha\big/(\zeta_\alpha - z_\alpha)^2\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\mathbf{u}-\overline{\mathbf{B}}\cdot\mathbf{T}]\Big|_\zeta ds\right\}. \qquad (11b)$$

Although for isotropic materials, **A** and **L** are both singular, Eq. (10-11) still hold for isotropic materials since the matrices above of the form $\mathbf{A}\cdot\langle *\rangle\cdot\mathbf{L}^{-1}$ and $\mathbf{L}\cdot\langle *\rangle\cdot\mathbf{L}^{-1}$ have finite limits.

Boundary integral equations

Eqs. (10-11) clearly show that if resultant forces and displacements at a solid boundary are known, the displacement and stress fields can be directly determined. This section lists the integral equations that can be used to determine unknown boundary displacements and resultant forces.

For a point $\zeta_0 = x_0 + iy_0$ at boundary $\partial\Omega$, its image is $\zeta_\alpha^0 = x_0 + p_\alpha y_0$ at $\partial\Omega_\alpha$. For analytic function $f_\alpha(z_\alpha)$, its boundary value

$$f_\alpha(\zeta_\alpha^0) = \frac{1}{\pi i}\oint_{\partial\Omega_\alpha}\frac{f_\alpha(\zeta_\alpha)}{\zeta_\alpha - \zeta_\alpha^0}d\zeta_\alpha\,. \qquad (12)$$

Since ζ_0 is a point on the boundary, the singular integration at the right hand side of Eq. (12) takes Cauchy Principal Value. If ζ_0 is a corner point, the coefficient at the left hand side is θ_{corner}/π instead of 1, where θ_{corner} is the angle formed by the two sides at the corner. Combining Eqs. (6), (7) and (12), we have

$$\mathbf{T}\Big|_{\varsigma_0} = -2\,\mathrm{Re}\left\{\frac{1}{\pi i}\oint_{\partial\Omega}\mathbf{L}\cdot\left\langle (C+p_\alpha S)\big/(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\mathbf{u}-\overline{\mathbf{B}}\cdot\mathbf{T}]\Big|_\varsigma ds\right\} \qquad (13a)$$

and

$$\mathbf{u}\big|_{\varsigma_0} = 2\,\mathrm{Re}\left\{\frac{1}{\pi i}\oint_{\partial\Omega}\mathbf{A}\cdot\left\langle (C+p_\alpha S)/(\varsigma_\alpha-\varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\mathbf{u}-\overline{\mathbf{B}}\cdot\mathbf{T}]\big|_{\varsigma}\,ds\right\}. \quad (13b)$$

The above two sets of integral equations might be used to construct boundary element scheme for numerical solution.

4. Formulation based on u and t

Displacement and stress fields

In Eq. (9), the resultant forces $\mathbf{T}$ can be replaced by tractions $\mathbf{t}=(t_x,t_y,t_z)^{\mathrm{T}}=d\mathbf{T}/ds$, by integrating by part,

$$\begin{aligned}\mathbf{f}\big|_z &= \frac{1}{2\pi}\oint_{\partial\Omega}\left\langle (C+p_\alpha S)/(\varsigma_\alpha-z_\alpha)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{u}\big|_{\varsigma}\,ds+\\ &\qquad \frac{1}{2\pi i}\oint_{\partial\Omega}\left\langle \ln(\varsigma_\alpha-z_\alpha)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\overline{\mathbf{B}}\cdot\mathbf{t}\big|_{\varsigma}\,ds-\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\overline{\mathbf{B}}\cdot\mathbf{T}\big|_{\varsigma^{(r)}}\end{aligned} \quad (14)$$

where $\varsigma^{(r)}$ is a reference point on the boundary used to determine the single value branch of the logarithmic functions. In Eq. (14), the single-valued branch-cut of $\ln(\varsigma_\alpha-z_\alpha)$ in z_α-plane is line from z_α and passing ς_α^r (Fig. 2a), so that the argument of $(\varsigma_\alpha-z_\alpha)$ is from $\arg(\varsigma_\alpha^{(r)}-z_\alpha)$ to $\arg(\varsigma_\alpha^{(r)}-z_\alpha)+2\pi$. In deriving the last term of Eq. (14), we assume the external loads on the boundary are self-equilibrated ($\oint_{\partial\Omega}\mathbf{t}ds=0$) so that $\mathbf{T}$ is periodic around the boundary.

Putting Eq. (14) into Eq. (1), we have displacement field

$$\begin{aligned}\mathbf{u}\big|_z = \frac{1}{\pi}\mathrm{Re}\Big\{&\oint_{\partial\Omega}\mathbf{A}\cdot\left\langle (C+p_\alpha S)/(\varsigma_\alpha-z_\alpha)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{u}\big|_{\varsigma}\,ds\\ &-i\oint_{\partial\Omega}\mathbf{A}\cdot\left\langle \ln(\varsigma_\alpha-z_\alpha)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\overline{\mathbf{B}}\cdot\mathbf{t}\big|_{\varsigma}\,ds\Big\}\end{aligned}. \quad (15)$$

The term associated with the last term in Eq. (14), involving $\mathbf{T}$ at $\varsigma^{(r)}$, is eliminated because $\mathbf{A}\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\overline{\mathbf{B}}\cdot\mathbf{T}\big|_{\varsigma^r}$ is pure imaginary. The stress field is

$$[\sigma_{21},\sigma_{22},\sigma_{23}]^{\mathrm{T}}=\frac{1}{\pi}\mathrm{Re}\int_{\partial\Omega}\Big[\mathbf{L}\cdot\Big\langle (C+p_\alpha S)/(\zeta_\alpha - z_\alpha)^2\Big\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{u}\big|_{\zeta} + i\mathbf{L}\cdot\big\langle 1/(\varsigma_\alpha - z_\alpha)\big\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\overline{\mathbf{B}}\cdot\mathbf{t}\big|_{\zeta}\Big]ds \tag{16a}$$

$$[\sigma_{11},\sigma_{12},\sigma_{13}]^{\mathrm{T}}=-\frac{1}{\pi}\mathrm{Re}\int_{\partial\Omega}\Big[\mathbf{L}\cdot\Big\langle (C+p_\alpha S)p_\alpha/(\zeta_\alpha - z_\alpha)^2\Big\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{u}\big|_{\zeta} + i\mathbf{L}\cdot\big\langle p_\alpha/(\varsigma_\alpha - z_\alpha)\big\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\overline{\mathbf{B}}\cdot\mathbf{t}\big|_{\zeta}\Big]ds\,. \tag{16b}$$

Boundary integral equations

Consider a boundary point $\zeta_0 = x_0 + iy_0$ at $\partial\Omega$, if we integrate Eq. (13b) by part, or take limit from Eq. (15) by letting z approach to ζ_0 then use Plemelj formulae(Muskhelishvili, 1992), we obtain

$$\mathbf{u}\big|_{\varsigma_0}=\frac{2}{\pi}\mathrm{Re}\Big\{\oint_{\partial\Omega}[\mathbf{A}\cdot\Big\langle (C+p_\alpha S)/(\varsigma_\alpha-\varsigma_\alpha^0)\Big\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{u}\big|_{\zeta} - i\mathbf{A}\cdot\Big\langle \ln(\varsigma_\alpha-\varsigma_\alpha^0)\Big\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\overline{\mathbf{B}}\cdot\mathbf{t}\big|_{\zeta}]ds\Big\}, \tag{17}$$

Since the expression is independent of the reference point, the branch cut for each logarithm function in Eq. (17) can be conveniently chosen to be from ς_α^0 and pointing outward,(Fig. 2b), not intersecting the other part of the boundary.

Eq. (17) is usually used in the conventional boundary element method to solve unknown boundary values from known boundary conditions. Once all the boundary values of **u** and **t** are known, Eqs. (15-16) were used to evaluate internal fields. Same as in Eq. (11), the integral kernels associated with **u** in Eq. (16) have second order singularity, which indicates large numerical error in evaluating the stresses near the boundary.

Integrating Eq. (13a) by part, we have

$$\mathbf{T}\big|_{\varsigma_0} - \mathbf{T}\big|_{\varsigma^{(r)}} = -\frac{1}{\pi}\mathrm{Re}\left\{ \oint_{\partial\Omega} [\mathbf{L}\cdot\left\langle (C+p_\alpha S)/(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle \cdot \mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{u}\big|_\varsigma \right. \\ \left. - i\mathbf{L}\cdot\left\langle \ln(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle \cdot \mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\overline{\mathbf{B}}\cdot\mathbf{t}\big|_\varsigma]ds \right\} . \quad (18a)$$

The branch cut of logarithm function $\ln(\varsigma_\alpha - \varsigma_\alpha^0)$ is a line from ς_α^0 and passing ς_α^r (Fig.2c). The left hand side of Eq. (18a) becomes zero if we let ς_0 and $\varsigma^{(r)}$ be the same point,

$$0 = \mathrm{Re}\left\{ \oint_{\partial\Omega} [\mathbf{L}\cdot\left\langle (C+p_\alpha S)/(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle \cdot \mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{u}\big|_\varsigma \right. \\ \left. - i\mathbf{L}\cdot\left\langle \ln(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle \cdot \mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\overline{\mathbf{B}}\cdot\mathbf{t}\big|_\varsigma]ds \right\} . \quad (18b)$$

Now the branch cut of $\ln(\varsigma_\alpha - \varsigma_\alpha^0)$ is a line from ς_α^0 and pointing outwards, not intersecting other part of the boundary (Fig.2b). The brunch cut sets the difference between the right hand sides of Eqs. (18a) and (18b), though they look exactly the same. Since the left hand side of Eq. (18a) can be expressed as the direct integration of **t** from $\varsigma^{(r)}$ to ς_0, Eqs.(18a) and (18b) both relate **u** and **t** on the boundary, complementing Eq. (17).

5. Formulation based on $d\mathbf{u}/ds$ and t

Displacement and stress fields

In Eq. (14), the integrals containing **u** can be integrated by part, so that

$$\mathbf{f}\big|_z = -\frac{1}{2\pi i}\oint_{\partial\Omega}\left\langle \ln(\varsigma_\alpha - z_\alpha)\right\rangle \cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\left(i\frac{d\mathbf{u}}{ds} - \overline{\mathbf{B}}\cdot\mathbf{t}\right)\Big|_\varsigma ds + \mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot(i\mathbf{u} - \overline{\mathbf{B}}\cdot\mathbf{T})\big|_{\varsigma^{(r)}} \quad (19)$$

The branch cut of $\ln(\varsigma_\alpha - z_\alpha)$ is a line from z_α and passing $\varsigma_\alpha^{(r)}$ (Fig. 2a). In getting the last term, we assume there is no displacement discontinuity at $\varsigma^{(r)}$ and the integration of traction around the boundary is zero. Thus, the displacement field is

$$\mathbf{u}\big|_z = -\frac{1}{\pi}\mathrm{Re}\oint_{\partial\Omega}\mathbf{A}\cdot\left\langle \ln(\varsigma_\alpha - z_\alpha)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\left(\frac{d\mathbf{u}}{ds}+i\overline{\mathbf{B}}\cdot\mathbf{t}\right)\bigg|_\zeta ds + \mathbf{u}\big|_{\varsigma^{(r)}}, \tag{20}$$

The stress field, derived from Eq. (2), is

$$[\sigma_{21},\sigma_{22},\sigma_{23}]^{\mathrm{T}}\big|_z = \frac{1}{\pi}\mathrm{Re}\int_{\partial\Omega}\mathbf{L}\cdot\left\langle 1/(\varsigma_\alpha - z_\alpha)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\left(\frac{d\mathbf{u}}{ds}+i\overline{\mathbf{B}}\cdot\mathbf{t}\right)\bigg|_\zeta ds \tag{21a}$$

$$[\sigma_{11},\sigma_{12},\sigma_{13}]^{\mathrm{T}}\big|_z = -\frac{1}{\pi}\mathrm{Re}\int_{\partial\Omega}\mathbf{L}\cdot\left\langle p_\alpha/(\varsigma_\alpha - z_\alpha)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\left(\frac{d\mathbf{u}}{ds}+i\overline{\mathbf{B}}\cdot\mathbf{t}\right)\bigg|_\zeta ds \tag{21b}$$

Boundary integral equations

Since $f_\alpha(z_\alpha)$ is an analytic function, $f'_\alpha(z_\alpha)$ is also analytic. f_α in Eq. (8) can be replaced by f'_α. Using $df_\alpha = f'_\alpha(\zeta_\alpha)d\zeta_\alpha = \frac{df_\alpha}{ds}ds$, we obtain the integral equations about the derivatives of displacements $d\mathbf{u}/ds$ and tractions $\mathbf{t}$,

$$\frac{1}{\pi i}\oint_{\partial\Omega}\left\langle 1/(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\left[i\frac{d\mathbf{u}}{ds}-\overline{\mathbf{B}}\cdot\mathbf{t}\right]\bigg|_\zeta ds = \left\langle 1/(C_0 + p_\alpha S_0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\left[i\frac{d\mathbf{u}}{ds}-\overline{\mathbf{B}}\cdot\mathbf{t}\right]\bigg|_{\zeta_0} \tag{22}$$

where ς_0 is a point on the boundary $\partial\Omega$, $C_0 = \cos\theta_0$, $S_0 = \sin\theta_0$ and θ_0 is the tangent angle at point ς_0 on $\partial\Omega$ (Fig. 1). Both sides of the equation can be multiplied by matrices $\mathbf{A}$ or $\mathbf{L}$ so that even in the degenerate cases, in which $\mathbf{A}$ and $\mathbf{L}$ are singular, the limit of $\mathbf{A}\cdot\langle *\rangle\cdot\mathbf{L}^{-1}$ or $\mathbf{L}\cdot\langle *\rangle\cdot\mathbf{L}^{-1}$ still exists and the equations still hold even for the case of isotropic materials. Here, $\langle *\rangle$ represents any diagonal matrix listed above.

$d\mathbf{u}/ds$ and $\mathbf{t}$ on the boundary can be explicitly expressed by boundary integrals. First eliminate the matrices at the right hand side of Eq. (22), then take the real part, we have

$$\mathbf{t}\big|_{\zeta_0} = -2\,\mathrm{Re}\{\frac{1}{\pi i}\oint_{\partial\Omega}\mathbf{L}\cdot\left\langle (C_0 + p_\alpha S_0)\big/(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\frac{d\mathbf{u}}{ds} - \overline{\mathbf{B}}\cdot\mathbf{t}]\Big|_{\varsigma} ds\}\,. \quad (23)$$

If take the imaginary part of Eq. (22), we have

$$\frac{d\mathbf{u}}{ds}\bigg|_{\zeta_0} = \mathrm{Im}\{\frac{1}{\pi i}\oint_{\partial\Omega}(\mathbf{B}+\overline{\mathbf{B}})\cdot\mathbf{L}\cdot\left\langle (C_0 + p_\alpha S_0)\big/(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\frac{d\mathbf{u}}{ds} - \overline{\mathbf{B}}\cdot\mathbf{t}]\Big|_{\varsigma} ds\} - \mathrm{Im}(\mathbf{B})\cdot\mathbf{t}\big|_{\varsigma_0} \quad (24)$$

Insert Eq. (23) into the last term of Eq. (24), then use $(\mathbf{B}+\overline{\mathbf{B}}) = 2\,\mathrm{Re}(\mathbf{B})$ and $\mathbf{BL} = i\mathbf{A}$, we have

$$\frac{d\mathbf{u}}{ds}\bigg|_{\zeta_0} = 2\,\mathrm{Re}\{\frac{1}{\pi i}\oint_{\partial\Omega}\mathbf{A}\cdot\left\langle (C_0 + p_\alpha S_0)\big/(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\frac{d\mathbf{u}}{ds} - \overline{\mathbf{B}}\cdot\mathbf{t}]\Big|_{\varsigma} ds\}\,. \quad (25)$$

Eq. (23) and (25) can be directly obtained by using Eq. (6) and the expression of $\mathbf{f}'$. Here we list Eq. (24) because it might provide a different numerical scheme. The equation will also be used in the discussion on problems in half plane.

Because of the weak singularity of the integral kernels of Eq. (20), the displacements at a boundary point ς_0 can be obtained from Eq. (20) by simply replacing z by ς_0,

$$\mathbf{u}\big|_{\varsigma_0} = -\frac{1}{\pi}\mathrm{Re}\oint_{\partial\Omega}\mathbf{A}\cdot\left\langle \ln(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot(\frac{d\mathbf{u}}{ds} + i\overline{\mathbf{B}}\cdot\mathbf{t})\Big|_{\zeta} ds + \mathbf{u}\big|_{\varsigma^{(r)}}\,. \quad (26a)$$

Let $\varsigma^{(r)} = \varsigma_0$, we have

$$\oint_{\partial\Omega}\mathbf{A}\cdot\left\langle \ln(\varsigma_\alpha - \varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot(\frac{d\mathbf{u}}{ds} + i\overline{\mathbf{B}}\cdot\mathbf{t})\Big|_{\zeta} ds = 0 \quad (26b)$$

Eq. (26b) can also be obtained directly since f'_α is analytic and $\oint_{\partial\Omega_\alpha}\ln(\zeta_\alpha - \zeta_\alpha^0)f'_\alpha(\zeta_\alpha)d\zeta_\alpha = 0$.

The branch cut of the logarithm function is a line from point ς_0, going outward and not intersecting the boundary at other places (Fig. 2c). Matrix $\mathbf{A}$ in Eq. (26a) should be replaced by $\mathbf{L}$ in evaluating the resultant forces on the boundary. Even though $\mathbf{A}$ and $\mathbf{L}$ are singular for degenerate cases, the limit of $\mathbf{A}\cdot\left\langle \ln(\varsigma_\alpha-\varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}$ exists, so Eq. (26) holds for any materials. Indeed, if we take the limit of Eq. (26b) for isotropic materials, then pick the real part, we obtain the integral equations used in Ghosh et al(1986) for constructing a boundary element scheme.

Another interesting fact is that $\ln(\varsigma_\alpha-\varsigma_\alpha^0)$ in Eq. (26) can be replaced by any analytic function of $\varsigma_\alpha-\varsigma_\alpha^0$, so

$$\oint_{\partial\Omega}\mathbf{A}\cdot\left\langle g_\alpha(\varsigma_\alpha-\varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\left(\frac{d\mathbf{u}}{ds}+i\overline{\mathbf{B}}\cdot\mathbf{t}\right)\Big|_\varsigma ds=0. \tag{27a}$$

Let $g_\alpha=(\varsigma_\alpha-\varsigma_\alpha^0)^n$. When $n=0$, we get the conditions of displacement continuity and force balance. For $n>2$, we can convert $d\mathbf{u}/ds$ into $\mathbf{u}$ by integration by part, so

$$n\oint_{\partial\Omega}\mathbf{A}\cdot\left\langle(\varsigma_\alpha-\varsigma_\alpha^0)^{n-1}\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{u}\big|_\varsigma ds=i\oint_{\partial\Omega}\mathbf{A}\cdot\left\langle(\varsigma_\alpha-\varsigma_\alpha^0)^{n}\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\overline{\mathbf{B}}\cdot\mathbf{t})\big|_\varsigma ds \tag{27b}$$

Matrix **A** is put in front so that the equations above still hold for degenerate cases.

6. Field around an internal singular point

Now let's assume that there is a singular stress field around an internal point $z_0=x_0+iy_0$ and stresses vanish at infinity. We can construct a contour to circumvent the singular point as shown in Fig. 3. The stress field inside the contour is expressed by Eq. (21). Apply Cauchy theorem Eq. (8) to $\mathbf{f}'$ and use Eq. (7), we can get the integral expression of $\mathbf{f}'$ in terms of boundary values of $d\mathbf{u}/ds$ and tractions **t**, similar to the left hand side of Eq. (22). When the outside part of the contour, Γ_{out}, approaches to infinity, because of vanishing $d\mathbf{u}/ds$ and tractions **t** at infinity, the integrations on Γ_{out} vanish. The integrations on Γ^- and Γ^+ cancel each other if displacements across Γ^- and Γ^+ are continuous or just differ by a constant vector. Thus, the stress field can be fully expressed by the integrations on Γ^0. Since z is at the outside of Γ^0, $\left|(\varsigma_\alpha - z_\alpha^0)/(z_\alpha - z_\alpha^0)\right| < 1$ and the kernel $1/(\varsigma_\alpha - z_\alpha)$ in the integral expression of f'_α can be expanded around the singular point z_α^0, then

$$f'_\alpha(z_\alpha) = \sum_{n=1} \frac{a_\alpha^{(n)}}{(z_\alpha - z_\alpha^0)^n}. \tag{28}$$

where $a_\alpha^{(n)}$ is the α−th component of a vector $\mathbf{a}^{(n)}$,

$$\mathbf{a}^{(n)} = \frac{1}{2\pi i}\int_{\Gamma_0(CCW)} \left\langle (\varsigma_\alpha - z_\alpha^0)^{n-1} \right\rangle \cdot \mathbf{L}^{-1} \cdot (\mathbf{B} + \overline{\mathbf{B}})^{-1} \cdot (i\frac{d\mathbf{u}}{ds} - \overline{\mathbf{B}} \cdot \mathbf{t})\Big|_{\zeta} ds \ . \tag{29}$$

The above integration is along Γ_0, from the upper bank of the cut going counterclockwise to the lower bank.

Stress field of first order singularity — Dislocation and concentrated force

When stress has first order singularity around a point $z_0 = x_0 + iy_0$, $d\mathbf{u}/ds$ and tractions $\mathbf{t}$ have $1/r$ singularity too, and for any $n \geq 2$, $\mathbf{a}^{(n)}$ vanishes as Γ_0 becomes infinitely small. Assume $\oint_{\Gamma_0(CCW)} \frac{\partial \mathbf{u}}{\partial s} ds = \mathbf{b}$ (dislocation) and $\oint_{\Gamma_0} \mathbf{t} ds = \mathbf{F}$ (concentrated force), we have

$$\mathbf{a}^{(1)} = \frac{1}{2\pi}\mathbf{L}^{-1} \cdot (\mathbf{B} + \overline{\mathbf{B}})^{-1} \cdot (\mathbf{b} + i\overline{\mathbf{B}} \cdot \mathbf{F}). \tag{30}$$

Here we can see that the two conditions for displacement and traction can be given independently. Letting $\mathbf{F} = 0$, we get the solution around a dislocation. Letting $\mathbf{b} = 0$, we get the solution around a concentrated force. The result is the same as in Stroh (1958) and Suo(1990).

Stress field of second order singularity

Now assume that around a small circle of radius r around z_0, $\oint_{\Gamma_0(CCW)} \frac{\partial \mathbf{u}}{\partial s} ds = 0$ and $\oint_{\Gamma_0} \mathbf{t} ds = 0$ so that $\mathbf{a}^{(1)}$ vanishes. Assume $\mathbf{t}$ and $d\mathbf{u}/ds$ have second order singularity around point z_0. Let $re^{i\vartheta} = \varsigma - z_0$, ς is a point on the circle. Since $\mathbf{t}$ and $\mathbf{u}$ are periodic function of ϑ both of them can be expressed by the Fourior series of ϑ. Let's only consider the in-plane case. Using Eq. (29), we have

$$\mathbf{a}^{(2)} = \frac{1}{2\pi} \mathbf{L}^{-1} \cdot (\mathbf{B} + \overline{\mathbf{B}})^{-1} \cdot (\mathbf{u}_s + i\overline{\mathbf{B}} \cdot \mathbf{t}_c) + \frac{1}{2\pi} \langle p_\alpha \rangle \cdot \mathbf{L}^{-1} \cdot (\mathbf{B} + \overline{\mathbf{B}})^{-1} \cdot (-\mathbf{u}_c + i\overline{\mathbf{B}} \cdot \mathbf{t}_S) \quad (34)$$

where

$$\mathbf{u}_S = \int_0^{2\pi} \mathbf{u} r \sin \vartheta d\vartheta, \quad \mathbf{u}_C = \int_0^{2\pi} \mathbf{u} r \cos \vartheta d\vartheta \quad (35a)$$

$$\mathbf{t}_S = \int_0^{2\pi} \mathbf{t} r^2 \sin \vartheta d\vartheta, \quad \mathbf{t}_C = \int_0^{2\pi} \mathbf{t} r^2 \cos \vartheta d\vartheta \,. \quad (35b)$$

Since $\mathbf{u}$ has singularity of $1/r$ and $\mathbf{t}$ has $1/r^2$ singularity, the above 4 pairs of constants are the first order Fourier transformations of the angular functions of displacements and traction. The relations among these constants can be established by inserting Eqs.(34), (28) into Eq. (35),

$$\frac{\mathbf{u}_s - i\mathbf{u}_c}{2} = \mathbf{A} \cdot \left\langle \frac{-1}{p_\alpha + i} \right\rangle \cdot \mathbf{L}^{-1} \cdot (\mathbf{B} + \overline{\mathbf{B}})^{-1} \cdot (\mathbf{u}_s + i\overline{\mathbf{B}} \cdot \mathbf{t}_c) + \mathbf{A} \cdot \left\langle \frac{p_\alpha}{p_\alpha + i} \right\rangle \cdot \mathbf{L}^{-1} \cdot (\mathbf{B} + \overline{\mathbf{B}})^{-1} \cdot (\mathbf{u}_c - i\overline{\mathbf{B}} \cdot \mathbf{t}_S)$$
$$\frac{\mathbf{t}_c + i\mathbf{t}_s}{2} = \mathbf{L} \cdot \left\langle \frac{-1}{p_\alpha + i} \right\rangle \cdot \mathbf{L}^{-1} \cdot (\mathbf{B} + \overline{\mathbf{B}})^{-1} \cdot (\mathbf{u}_s + i\overline{\mathbf{B}} \cdot \mathbf{t}_c) + \mathbf{L} \cdot \left\langle \frac{p_\alpha}{p_\alpha + i} \right\rangle \cdot \mathbf{L}^{-1} \cdot (\mathbf{B} + \overline{\mathbf{B}})^{-1} \cdot (\mathbf{u}_c - i\overline{\mathbf{B}} \cdot \mathbf{t}_S) .$$

(36)

The following equations have been used in deriving above results,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos \vartheta}{\cos \vartheta + p_\alpha \sin \vartheta} d\vartheta = \frac{i}{p_\alpha + i}, \text{ and } \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin \vartheta}{\cos \vartheta + p_\alpha \sin \vartheta} d\vartheta = \frac{1}{p_\alpha + i} \quad \text{for } \mathrm{Im}(p_\alpha) > 0.$$

From Eq. (36), it is obvious that

$$\mathbf{u}_c + i\mathbf{u}_s = \mathbf{B}\cdot(\mathbf{t}_c + i\mathbf{t}_s)\,. \tag{37}$$

Following the procedure, we can derive similar equations for higher order solutions. So the procedure can be used to solve asymptotic solution near a cylindrical hole when either tractions or displacements are known at the surface of the hole.

A cylindrical hole subject to internal pressure

As an example, let's consider a cylindrical hole of radius r_0 in an infinite body subjects to internal pressure of *p*. The center of the hole is z_0. We seek the asymptotic solution. The remotes stress vanishes at infinity, so the stress field will not have constant term. Since the total resultant force on the hole is zero and there is no displacement jump across a cut from z_0, the stress field will not have the first order singular terms. The first non-zero term of stress would be the second order term and can be expressed by $\mathbf{f}'$ in Eq. (28) if the coefficients in Eq. (29) are known. Using Eq. (35), we get $\mathbf{t}_c + i\mathbf{t}_s = \pi p r_0^2 \begin{pmatrix} 1 \\ i \end{pmatrix}$. Insert it into Eq. (37), we get

$$\mathbf{u}_c + i\mathbf{u}_s = \pi p r_0^2 \mathbf{B}\cdot\begin{pmatrix} 1 \\ i \end{pmatrix}. \tag{38}$$

Taking the real and imaginary parts, we obtain all the coefficients needed in Eq. (34) to calculate the coefficients $\mathbf{a}^{(2)}$. Higher order terms can be proved to be zero.

7. Applications of singular field solutions

Application to discrete dislocation dynamics in a finite body

One motivation for studying integral formulation is to develop a fast and convenient stress solver for simulating dislocation motion in a solid with finite boundary. So far most of dislocation dynamics simulations assumed periodic boundary conditions (Kubin, et al, 1992; Fang, et al., 1993), and only very few had considered the effect of finite boundary, (van der Giesson et al., 1995; Fivel., et al, 1997) which is important when simulating dislocation motion near contact surface in tribology, or studying the size effect in micro-plasticity. van der Giessen et al. (1995) solved the stress field in a finite body using a linear superposition technique: adopt the analytical solution for dislocation in infinite body, calculate the corresponding tractions on the solid boundary, apply reverse tractions as external load then solve the non-singular reduced problem using finite element method. If the integral formulation based on $\mathbf{t}$ and $d\mathbf{u}/ds$ is used, such linear superposition technique is unnecessary. By cutting small holes around all dislocations (Fig. 3), the integrations on these circles with respect to a fixed point ζ_0 on the external boundary can be analytically obtained. Eqs. (23) and (24) on the external boundary become,

$$\begin{aligned}\left.\begin{pmatrix}\mathbf{t}\\ d\mathbf{u}/ds\end{pmatrix}\right|_{\zeta_0} &= \frac{2}{\pi}\mathrm{Re}\{\oint_{\partial\Omega}\begin{pmatrix}-\mathbf{L}\\ \mathbf{A}\end{pmatrix}\cdot\left\langle (C_0+p_\alpha S_0)/(\varsigma_\alpha-\varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\left.[\frac{d\mathbf{u}}{ds}+i\overline{\mathbf{B}}\cdot\mathbf{t}]\right|_{\varsigma}ds\}\\ &+\frac{2}{\pi}\sum_{k=1}^{n}\mathrm{Re}\{\begin{pmatrix}\mathbf{L}\\ -\mathbf{A}\end{pmatrix}\cdot\left\langle (C_0+p_\alpha S_0)/(z_\alpha^{(k)}-\varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{b}^{(k)}\}\end{aligned} \tag{39}$$

and Eq.(26b) becomes

$$\begin{aligned}0 &= \oint_{\partial\Omega}\mathbf{A}\cdot\left\langle \ln(\varsigma_\alpha-\varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\left.(\frac{d\mathbf{u}}{ds}+i\overline{\mathbf{B}}\cdot\mathbf{t})\right|_{\varsigma}ds\\ &-\sum_{\mathrm{k}=1}^{\mathrm{n}}\mathbf{A}\cdot\left\langle \ln(z_\alpha^{(k)}-\varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{b}\end{aligned} \tag{40}$$

where $z^{(k)}$ is the position and $\mathbf{b}^{(k)}$ the Burger's vector of the kth dislocation, and n is the total number of dislocations.

From the above boundary integral equations, $\mathbf{t}$ and $d\mathbf{u}/ds$ (or $\mathbf{u}$) can be directly solved. Subsequently, the stress field can be calculated. It has the expression of Eq.(21) plus the contributions from the dislocations, using the solution from Eqs. (28) and (30). The displacement field inside the body has the same expression as Eq. (40), replacing ς_α^0 by z_α.

In dislocation dynamics simulation, if the boundary does not move, the stiffness matrix is fixed. The dislocation motion only changes the generalized 'force' vector. We only need to solve the inverse of the stiffness matrix once. However, when many dislocations nucleate or annihilate at the surface, the external surface profile changes due to large plastic deformation so we need to update the stiffness matrix. This can be done easily when BEM is used. Eqs. (39) and (40) can also be written in incremental form.

With body forces and temperature gradient

When discrete concentrated forces $\mathbf{F}^{(k)}$ are present at $z^{(k)}$ (k=1,..,n), a set of integral equations similar to Equ. (39) and (40) can be obtained, except that the vector $\mathbf{b}^{(k)}$ in Eqs. (39) and (40) should be replaced by $i\overline{\mathbf{B}}\cdot\mathbf{F}^{(k)}$.

When there is continuously distributed body force with force density $\mathbf{f}_{Body}$, the boundary integral equations become

$$\begin{aligned}\left.\begin{pmatrix}\mathbf{t}\\ d\mathbf{u}/ds\end{pmatrix}\right|_{\zeta_0} &= \frac{2}{\pi}\mathrm{Re}\{\oint_{\partial\Omega}\begin{pmatrix}-\mathbf{L}\\ \mathbf{A}\end{pmatrix}\cdot\left\langle (C_0+p_\alpha S_0)/(\varsigma_\alpha-\varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\left.[\frac{d\mathbf{u}}{ds}+i\overline{\mathbf{B}}\cdot\mathbf{t}]\right|_{\varsigma} ds\}\\ &+\frac{2}{\pi}\mathrm{Re}\{\int_{\Omega}\begin{pmatrix}\mathbf{L}\\ -\mathbf{A}\end{pmatrix}\cdot\left\langle (C_0+p_\alpha S_0)/(z_\alpha-\varsigma_\alpha^0)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot i\overline{\mathbf{B}}\cdot\mathbf{f}_{body}dA\}\end{aligned}\tag{41}$$

Replace $\mathbf{b}$ in Eq. (40) by $i\overline{\mathbf{B}}\cdot\mathbf{f}_{body}$, and the summation by integration, we have a set of boundary integral equations

$$0 = \mathrm{Re} \oint_{\partial\Omega} \mathbf{A} \cdot \left\langle \ln(\varsigma_\alpha - \varsigma_\alpha^0 \right\rangle \cdot \mathbf{L}^{-1} \cdot (\mathbf{B} + \overline{\mathbf{B}})^{-1} \cdot (\frac{d\mathbf{u}}{ds} + i\overline{\mathbf{B}} \cdot \mathbf{t})\Big|_{\zeta} ds$$

$$- \mathrm{Re}\{\int_{\Omega} i\mathbf{A} \cdot \left\langle \ln(z_\alpha - \varsigma_\alpha^0 \right\rangle \cdot \mathbf{L}^{-1} \cdot (\mathbf{B} + \overline{\mathbf{B}})^{-1} \cdot \overline{\mathbf{B}} \cdot \mathbf{f}_{body} dA\} . \tag{42}$$

The branch cut of the logarithmic function is shown in Fig. 2c. It is well known that temperature gradient is mathematically equivalent to a body force distribution $f_i^{body} = C_{ijkl}\alpha_{kl}T_{,j}$, where α_{ij} is thermal expansion tensor which gives the thermal strain $\varepsilon_{kl}^T = \alpha_{kl}\Delta T$. T is the temperature distribution. Similar replacements can also be done for the strain due to phase transformation.

8. Discussion on Boundary Value Problem and Multiply-Connected Domain

Boundary value problem

For each of above integral formulations, we have more than two boundary integral equations. Some integral equations are given in complex form. Discussion on the complex variable boundary element method can be found in Hromada (1984) for some simple potential problems. The detailed discussion on the numerical implementation and the possibility of devising new numerical schemes will be discussed in the sequel paper.

From the derivation, we can see that boundary integral equations in different formulation are not exactly equivalent. For example, **u** and **T** are single valued so we can obtain Eq. (14), the integral equation about **u** and **t**, from Eq. (9), the equations based on **u** and **T**. However, to get Eq. (9) from (14), we need one extra condition,

$$\oint_{\partial\Omega} \mathbf{t} ds = 0 , \tag{43a}$$

so that **T** is single valued along the boundary. Similarly, Eq. (19), the integral equation about **t** and $d\mathbf{u}/ds$, is equivalent to Eq. (14) only when

$$\oint \frac{d\mathbf{u}}{ds} ds = 0 . \tag{43b}$$

To solve elasticity problem, we need boundary conditions. The conventional boundary element method is based on Eq. (17). From elasticity theory, for each direction at any point on a solid boundary, if either displacement or traction is given, stress field is uniquely determined. To avoid rigid body motion and rotation, we need at least one point be fully constrained and another extra displacement boundary condition at another point, such as the tangential derivative of the normal displacement be given or one displacement component is given. These conditions are needed for all the formulations.

In a general mixed boundary value problem, on some segments of the boundary, displacements are given, and on some, tractions are given. For numerical implementation based on the interpolation of $d\mathbf{u}/ds$ on each boundary element, instead of $\mathbf{u}$, prescribing displacements along a boundary segment is equivalent to giving $d\mathbf{u}/ds$ along the segment and the displacements at one chosen point on the segment. The known part of $d\mathbf{u}/ds$ and $\mathbf{t}$ along the boundary can be directly applied to the boundary integral equations, Eq. (26b) or any one set from Eq. (23) to (25). The given displacements at the chosen point of each displacement-given segment can be included using Eq. (26a). All these, together with Eqs. (43a) and (43b), can be used to solve unknown part of $d\mathbf{u}/ds$ and $\mathbf{t}$ along the boundary.

Mathematically, if there is a singular point on the boundary, we also need to specify the singularity conditions at the point, such as the range of singularity to ensure the uniqueness of the solution.

Continuity condition at a non-singular corner point

Same as boundary traction, the tangential derivatives of displacements are usually not continuous at a corner point due to the abrupt change of orientation along the boundary. Assume the corner is a non-singular point so that the stress state is well defined and the derivative of analytic function f_α is same along all the directions,

$$\left\{\frac{ds}{d\xi_\alpha}\left\{\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\frac{d\mathbf{u}}{ds}-\overline{\mathbf{B}}\cdot\mathbf{t}]\right\}_\alpha\right\}_- = \left\{\frac{ds}{d\xi_\alpha}\left\{\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\frac{d\mathbf{u}}{ds}-\overline{\mathbf{B}}\cdot\mathbf{t}]\right\}_\alpha\right\}_+ .\text{(44a)}$$

Re-arrange the above equation, we have

$$(\mathbf{B}+\overline{\mathbf{B}})\cdot\mathbf{L}\cdot\left\langle (C_+ + p_\alpha S_+)/(C_- + p_\alpha S_-)\right\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\,d\mathbf{u}/ds-\overline{\mathbf{B}}\cdot\mathbf{t}]_- = [i\,d\mathbf{u}/ds-\overline{\mathbf{B}}\cdot\mathbf{t}]_+ \text{(44b)}$$

where $C_\pm$ and $S_\pm$ are the cosine and sine of the tangent angles at the two side of the corner along the boundary. The positive tangential direction is counter-clock-wise. Continuity condition of **t** at a corner was often discussed in boundary element method. The equation above provides rigorous continuity conditions at a non-singular corner point.

Multiply connected domain

All the above equations are derived for single simply-connected domain. Now let's consider a multiply-connected domain in Fig. 4. If there is no displacement jump or body forces on Γ_+ and Γ_-, the integrations involving **u**, **t**, and $d\mathbf{u}/ds$ on Γ_+ and Γ_- cancel each other, so the integral equations about **u**, **t**, and $d\mathbf{u}/ds$ also hold for multiply connected. If there is a constant displacement jump $\Delta\mathbf{u}_0$ across Γ_+ and Γ_- and there is a net resultant force $\Delta\mathbf{T}_0$ on the inner boundary Γ_{in}, then $\mathbf{u}_+ - \mathbf{u}_- = \Delta\mathbf{u}_0$ and $\mathbf{T}_+ - \mathbf{T}_- = \Delta\mathbf{T}_0$, the formulation based on **u** and **T** gives

$$\mathbf{f}\big|_z = \frac{1}{2\pi i}\oint_{\Gamma_{\rm in},\Gamma_{\rm out}} \langle (C + p_\alpha S)/(\varsigma_\alpha - z_\alpha)\rangle \cdot \mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot[i\mathbf{u}-\overline{\mathbf{B}}\cdot\mathbf{T}]\big|_\zeta \, ds \\ + \frac{1}{2\pi i}\langle \ln[(b_\alpha - z_\alpha)/(a_\alpha - z_\alpha)]\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot(i\Delta\mathbf{u}_0 - \overline{\mathbf{B}}\cdot\Delta\mathbf{T}_0) \quad , \quad (45)$$

where a and b are the intersect points of the horizontal cut and the inner and external boundaries. Corresponding boundary integral equations should be changed accordingly. Similarly, under the same conditions, for formulations based on $\mathbf{t}$ and $\mathbf{u}$, Eq. (14) needs one extra term at the right hand side, which is

$$\frac{1}{2\pi}\langle \ln[(b_\alpha - z_\alpha)/(a_\alpha - z_\alpha)]\rangle\cdot\mathbf{L}^{-1}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\Delta\mathbf{u}_0 \,.$$

For formulation based on $\mathbf{t}$ and $d\mathbf{u}/ds$, integral equations (23-26) do not change. We can modify Eqs. (43a) and (43b) to express the extra conditions.

9. Other Analytical Examples

In this section, some solutions of half plane problems, crack and interface problems are derived by using the formulation based on $d\mathbf{u}/ds$ and $\mathbf{t}$.

***Half plane problem*s**

a) Upper half plane without dislocation

The integral equations can be greatly simplified for half plane problem, in which the boundary of the solid is x-axis and infinity. Assume the solid occupies the upper half plane and stress vanishes at infinity. Along x-axis, $\zeta_\alpha = \mathrm{Re}(\zeta)$, and $\mathbf{t} = -(\sigma_{yx}, \sigma_{yy}, \sigma_{yz})^{\mathrm{T}}$. Eq. (23) becomes

$$-\mathrm{Re}(\mathbf{B})\cdot\mathbf{t}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\xi - x}[\mathrm{Im}(\mathbf{B})\cdot\mathbf{t} + \frac{\partial\mathbf{u}}{\partial\xi}]d\xi \,, \quad (46a)$$

and Eq. (24) becomes

$$\frac{d\mathbf{u}}{dx}(x) = -\operatorname{Im}(\mathbf{B})\cdot\mathbf{t}(x) + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Re}(\mathbf{B})\cdot\mathbf{t}(\xi)}{\xi - x}d\xi \tag{46b}$$

It is obvious that Eq. (46a) and Eq. (46b) are equivalent. Eq. (46b) has the same form as the one derived in Yu (1999). It explicitly gives the solution for problems with given traction.

If there is a concentrated line force **F** exerted at point $(x_0,0)$ on the boundary. Eq. (46b) gives

$$\frac{d\mathbf{u}}{dx}(x) = -\operatorname{Im}(\mathbf{B})\cdot\mathbf{F}\delta(x_0) + \frac{\operatorname{Re}(\mathbf{B})\cdot\mathbf{F}}{\pi(x_0 - x)}. \tag{47}$$

where δ is the Dirac Delta function. Put Eq. (47) into (21), use residue theorem, we get the stress field at z

$$[\sigma_{21,}\sigma_{22},\sigma_{23}]^{\mathrm{T}}\Big|_z = \operatorname{Re}\{\frac{1}{\pi i}\mathbf{L}\cdot\left\langle\frac{1}{z_\alpha - x_0}\right\rangle\cdot\mathbf{L}^{-1}\cdot\mathbf{F}\}$$

$$[\sigma_{11},\sigma_{12},\sigma_{13}]^{\mathrm{T}}\Big|_z = \operatorname{Re}\{\frac{1}{\pi i}\mathbf{L}\cdot\left\langle\frac{p_\alpha}{z_\alpha - x_0}\right\rangle\cdot\mathbf{L}^{-1}\cdot\mathbf{F}\}. \tag{48}$$

Thus if all the tractions along the x-axis are known, the whole stress field can be obtained by integrating Eq. (48) along x-axis, replacing **F** by ***t**dx*.

Now let's solve the half plane problem in which the boundary displacements are given. Along x-axis, Eq. (22) can also be re-written as

$$\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{1}{\xi - x}[i\overline{\mathbf{B}}^{-1}\cdot\frac{d\mathbf{u}}{ds} - \mathbf{t}]\Big|_\xi d\xi = (i\overline{\mathbf{B}}^{-1}\cdot\frac{d\mathbf{u}}{ds} - \mathbf{t})\Big|_x \tag{49}$$

Equating the real parts at the two sides of (49) gives

$$\mathbf{t}\big|_x = -\operatorname{Im}(\overline{\mathbf{B}}^{-1})\cdot\frac{d\mathbf{u}}{dx}\Big|_x - \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{\xi - x}\operatorname{Re}(\overline{\mathbf{B}}^{-1})\cdot\frac{d\mathbf{u}}{d\xi}\Big|_\xi d\xi, \tag{50a}$$

And equating the imaginary parts gives

$$\mathrm{Re}(\overline{\mathbf{B}}^{-1})\cdot\left.\frac{d\mathbf{u}}{ds}\right|_x=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{\xi-x}\left.[\mathrm{Im}(\overline{\mathbf{B}}^{-1})\cdot\frac{d\mathbf{u}}{d\xi}+\mathbf{t}]\right|_\xi d\xi\,. \tag{50b}$$

Eq.(50a) gives the solution for problems with given boundary displacements.

Now consider the following problem. Assume the surface is rigidly held for $|x|>a$, and for $|x|<a$, the surface is punched by a rigid indentor so that the displacements $\mathbf{u}=\mathbf{\Delta}$ for $|x|<a$, $y=0$. For the given boundary condition,

$$\frac{d\mathbf{u}}{dx}=\mathbf{\Delta}[\delta(x+a)-\delta(x-a)]\,. \tag{51}$$

Insert it into Eq. (50a), we have

$$\left.\mathbf{t}\right|_x=-[\delta(x+a)-\delta(x-a)]\,\mathrm{Im}(\overline{\mathbf{B}}^{-1})\cdot\mathbf{\Delta}+\frac{1}{\pi}\frac{2a}{a^2-x^2}\mathrm{Re}(\mathbf{B}^{-1})\cdot\mathbf{\Delta} \tag{52}$$

Insert Eqs. (51-52) into Eq. (19), pick the infinity as the reference point, we have

$$\mathbf{f}=\frac{1}{2\pi i}\left\langle\ln\frac{z_\alpha-a}{z_\alpha+a}\right\rangle\cdot\mathbf{A}^{-1}\cdot\mathbf{\Delta}\,. \tag{53}$$

In deriving the above equation, $\overline{\mathbf{B}}^{-1}=\overline{\mathbf{B}^{-1}}$ is used.

b)**With the presence of dislocation**

Assume there are dislocations located at z_0 in upper half plane, the integral equations corresponding to Eq. (46b) is

$$\left.\frac{d\mathbf{u}}{dx}\right|_x=-\mathrm{Im}(\mathbf{B})\cdot\mathbf{t}(x)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}(\mathbf{B})\cdot\mathbf{t}(\xi)}{\xi-x}d\xi-\frac{1}{\pi}(\mathbf{B}+\overline{\mathbf{B}})\cdot\mathrm{Im}\{\mathbf{L}\cdot\left\langle\frac{1}{z_\alpha^{(0)}-x}\right\rangle\cdot\mathbf{L}^{-1}\}\cdot(\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot\mathbf{b}\,, \tag{54}$$

according to Eqs. (22) and (30). If there are more dislocations, terms similar to the last one of above equation can be added. Insert Eq. (54) into the boundary integral expression for **f**, we have

$$
\begin{aligned}
\mathbf{f} = & \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\langle \ln(\xi - z_\alpha)\right\rangle \cdot \mathbf{L}^{-1}\cdot \mathbf{t} d\xi \\
& + \frac{1}{2\pi}[\left\langle \ln(z_\alpha^{(0)} - z_\alpha)\right\rangle \cdot \mathbf{L}^{-1} + \mathbf{L}^{-1}\cdot \overline{\mathbf{L}}\cdot \left\langle \ln(\bar{z}_\alpha^{(0)} - z_\alpha)\right\rangle \cdot \overline{\mathbf{L}}^{-1}]\cdot (\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot \mathbf{b}
\end{aligned} \quad . \tag{55}
$$

A constant vector term which is determined by the choice of reference point is ignored in Eq. (55), which gives the solution for traction-given boundary value problems. Similarly, if displacement is given along the boundary, we can calculate the tractions. Multiply $\overline{\mathbf{B}}^{-1}$ to the two sides of Eq. (22), then take the imaginary part, we obtain the traction along the boundary,

$$
\begin{aligned}
\mathbf{t}\big|_x = & \operatorname{Im}(\mathbf{B}^{-1})\cdot \left.\frac{d\mathbf{u}}{dx}\right|_x - \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{\xi - x}[\operatorname{Re}(\mathbf{B}^{-1})\cdot \frac{d\mathbf{u}}{d\xi}]\Big|_\xi d\xi \\
& - \frac{1}{\pi}\operatorname{Im}\{(\mathbf{B}^{-1} + \overline{\mathbf{B}}^{-1})\cdot \mathbf{A}\cdot \left\langle \frac{1}{z_\alpha^{(0)} - x}\right\rangle \cdot \mathbf{L}^{-1}\}\cdot (\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot \mathbf{b}
\end{aligned} \tag{56}
$$

And finally, we can get field solution using the boundary integral expression for field and residue theorem,

$$
\begin{aligned}
\mathbf{f} = & -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left\langle \ln(z_\alpha - \xi)\right\rangle \cdot \mathbf{A}^{-1}\cdot \frac{d\mathbf{u}}{d\xi} d\xi \\
& + \frac{i}{2\pi}[\left\langle \ln(z_\alpha^{(0)} - z_\alpha)\right\rangle \cdot \mathbf{L}^{-1} + \mathbf{A}^{-1}\cdot \overline{\mathbf{A}}\cdot \left\langle \ln(\bar{z}_\alpha^{(0)} - z_\alpha\right\rangle \cdot \overline{\mathbf{L}}^{-1}]\cdot (\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot \mathbf{b}
\end{aligned} \quad . \tag{57}
$$

Crack and interface problems

For completeness, the following gives the basic boundary integral equations for lower half plane with the presence of dislocations at $z^{(0)}$ with $\operatorname{Im}(z^{(0)}) < 0$,

$$
\frac{d\mathbf{u}}{dx}(x) = \operatorname{Im}(\mathbf{B})\cdot \mathbf{t}(x) + \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Re}(\mathbf{B})\cdot \mathbf{t}(\xi)}{\xi - x}d\xi - \frac{1}{\pi}(\mathbf{B}+\overline{\mathbf{B}})\cdot \operatorname{Im}\{\mathbf{L}\cdot \left\langle \frac{1}{z_\alpha^{(0)} - x}\right\rangle \cdot \mathbf{L}^{-1}\}\cdot (\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot \mathbf{b} \tag{58a}
$$

$$
-\mathbf{t}\big|_x = \operatorname{Im}(\mathbf{B}^{-1})\cdot \left.\frac{d\mathbf{u}}{dx}\right|_x + \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\operatorname{Re}(\mathbf{B}^{-1})}{\xi - x}\cdot \frac{d\mathbf{u}}{d\xi}]\Big|_\xi d\xi - \frac{1}{\pi}\operatorname{Im}\{(\mathbf{B}^{-1} + \overline{\mathbf{B}}^{-1})\cdot \mathbf{A}\cdot \left\langle \frac{1}{z_\alpha^{(0)} - x}\right\rangle \cdot \mathbf{L}^{-1}\}\cdot (\mathbf{B}+\overline{\mathbf{B}})^{-1}\cdot \mathbf{b} \tag{58b}
$$

Combine the equations for upper and lower half planes, we obtain the integral equations for crack or contact problems,

$$\frac{d\Delta\mathbf{u}}{dx}(x) = \mathrm{Im}(\mathbf{H})\cdot\mathbf{t}(x) - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}(\mathbf{H})\cdot\mathbf{t}(\xi)}{\xi - x}d\xi + \text{dislocation terms} \quad (59),$$

where $\Delta\mathbf{u} = \mathbf{u}^{+} - \mathbf{u}^{-}$, $\mathbf{t} = \mathbf{t}^{-} = (\sigma_{yx}, \sigma_{yy}, \sigma_{yz})^{\mathrm{T}}$ and $\mathbf{H} = \mathbf{B}^{(1)} + \overline{\mathbf{B}}^{(2)}$, where superscript (1) indicates the material in upper half plane, and (2) for the material in lower half plane. In deriving the above equation, we used traction continuity condition at the interface, $\mathbf{t}^{+} = -\mathbf{t}^{-} = -(\sigma_{yx}, \sigma_{yy}, \sigma_{yz})^{\mathrm{T}}$. Eq. (59) is very useful in solving crack and contact problems. Using the continuity or jump condition at the bonded or contact interface, we can solve the integral equations analytically. The general solution of the above integral equation is given in the literature (Mushkhelishvili,1992).

Anti-plane problem

For materials with *xy*-plane as mirror plane, the in-plane and anti-plane deformation can be decoupled (Stroh, 1956 and Suo,1990). Let's consider the anti-plane deformation. The corresponding eigen-value in Eq. (3) is

$$p_3 = (s_{45} + i(s_{44}s_{55} - s_{45}^2)^{1/2})/s_{55} \quad (60)$$

Only one holomorphic function $f(z_3)$ is needed to represent deformation,

$$u_3 = 2\,\mathrm{Re}(Af(z_3)), \qquad T_3 - 2\,\mathrm{Re}(Lf(z_3)) \quad (61a)$$

$$\sigma_{23} = 2\,\mathrm{Re}(Lf(z_3)) \qquad \sigma_{23} = -2\,\mathrm{Re}(Lp_3 f(z_3)), \quad (61b)$$

For the anti-plane problems, matrices and vectors in Section 2 reduce to scalars, with $L = -1$, $A = iB$, and $B = (s_{44}s_{55} - s_{45}^2)^{1/2}$ which is a real number. u_3 and BT_3 are the real and imaginary parts of an analytic function. All the integral equations can be simplified by replacing the matrixes by the above three scalars. For example, Eq. (15) becomes

$$u_3\big|_z = \frac{1}{2\pi}\mathrm{Re}\left\{\oint_{\partial\Omega} -i\frac{C+p_3S}{\varsigma_3 - z_3}u_3\big|_\varsigma ds - \oint_{\partial\Omega}\ln(\varsigma_3 - z_3)Bt_3\big|_\varsigma ds\right\} \tag{62}$$

and Eq. (25) becomes

$$\left.\frac{du_3}{ds}\right|_{\varsigma_0} = \mathrm{Re}\{\frac{1}{\pi i}\oint_{\partial\Omega}\frac{C_0+p_3S_0}{\varsigma_3-\varsigma_3^0}(\frac{du_3}{ds}+iB\cdot t_3)\Big|_\varsigma ds\} \tag{63}$$

For the upper half plane,

$$B\sigma_{23}(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\xi - x}\frac{du_3(\xi)}{d\xi}d\xi\,, \tag{64a}$$

$$\frac{du_3}{dx}(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{B\cdot\sigma_{23}(\xi)}{\xi - x}d\xi \tag{64b}$$

For crack or interface problem,

$$\frac{d\Delta u_3}{dx}(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H\cdot\sigma_{23}(\xi)}{\xi - x}d\xi\,, \tag{65}$$

where H is the summation of B in the upper and lower half planes. The stress field around a screw dislocation can be easily obtained.

10. Conclusion

Starting from LES representation for anisotropic materials and Cauchy integral theorem, this paper presents the unified approach in deriving boundary integral equations for 2D elasticity. The relations among the different integral equations are clearly shown. These equations provide more flexibility in devising numerical schemes in the boundary element method. Among different formulations, formulation based on $d\mathbf{u}/ds$ and $\mathbf{t}$ is given special attention. The advantage of the new formulation in numerical implementation is obvious. It gives a straight forward approach for calculating stress field within a finite solid body which has dislocations inside. If linear interpolating $d\mathbf{u}/ds$ on each element instead of quadratically interpolating $\mathbf{u}$, the boundary element method based on the formulation has fewer degrees of freedom (about *half* for traction given problem). The kernels in the integral equations for $d\mathbf{u}/ds$ and $\mathbf{t}$ have the weakest singularities among all three formulations, so that the boundary effect which occurred in conventional boundary element method can be eliminated (Ghosh et al, 1986; Wu et al, 1992). The feature is very important if the stress state near boundary is of great interest, such as in simulation of surface morphology evolution due to surface diffusion of reaction.

Through several examples, we show that the formulation is very useful in deriving analytical solutions. Stress field around a singular point is derived for field with first and second order stress singularities. The scheme presented here can be used to obtain the asymptotic solutions around a circular hole. The solution for a pressured circular hole is obtained. Formulations can be simplified for half plane problem, which can be further used in fracture mechanics and contact problems. The stress fields in a half plane for problems of prescribed boundary traction or displacements are derived, with or without the presence of dislocation.

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Figure Caption:

Fig. 1 A A boundary in z-plane. θ is the tangent angle at ζ, $d\zeta = e^{i\theta} ds$ and θ_0 is the tangent angle at ζ_0.

Fig. 2 The branch cuts of logarithmic functions.

Fig. 3 The integration contour for field with a dislocation.

Fig. 4 The integration contour for a multiply connected domain.

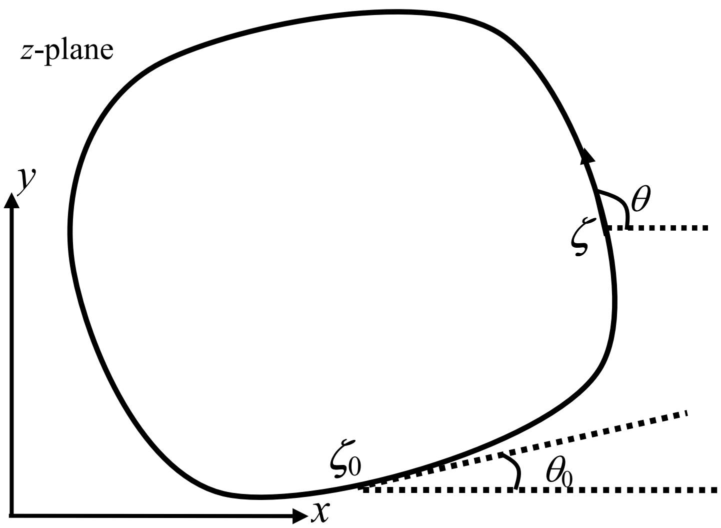
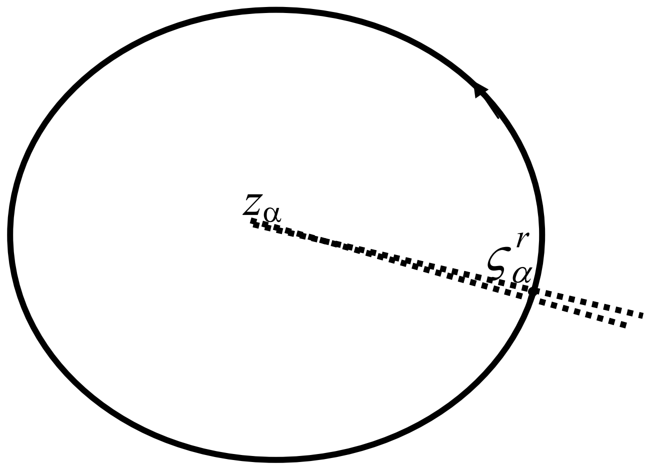
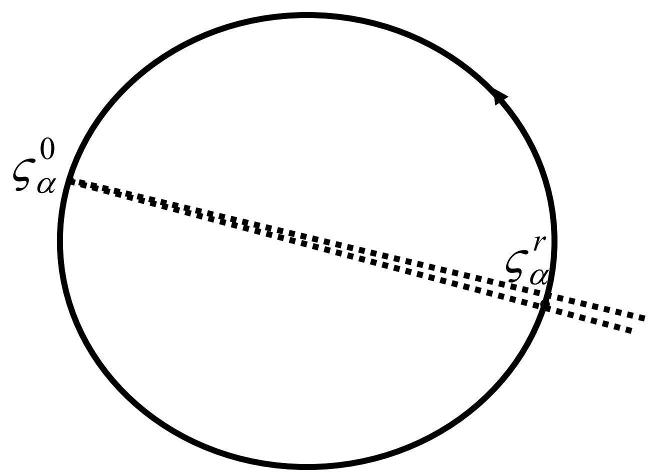


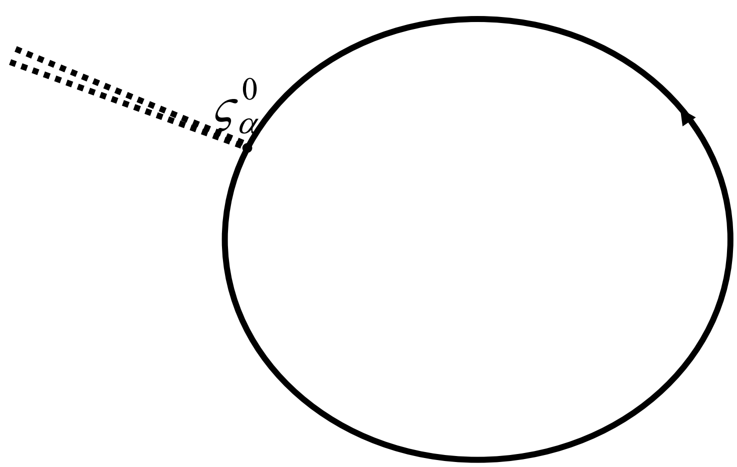
Fig. 1 A boundary in z-plane.


z_α
ς_α^r

(a)


ς_α^0
ς_α^r

(b)


ς_α^0

(c)

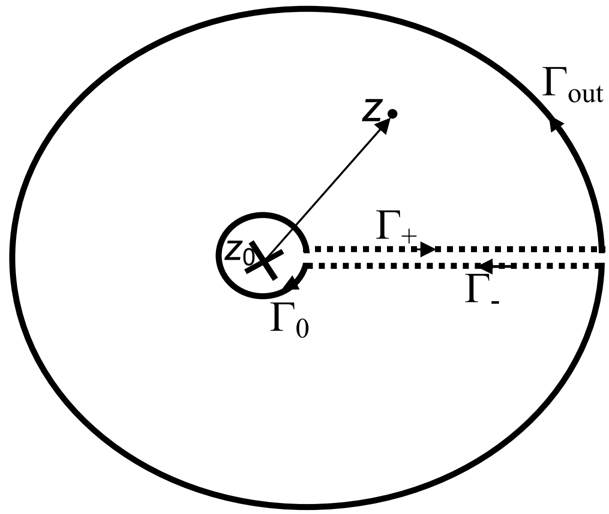


Fig. 3 The integration contour for field with a dislocation.

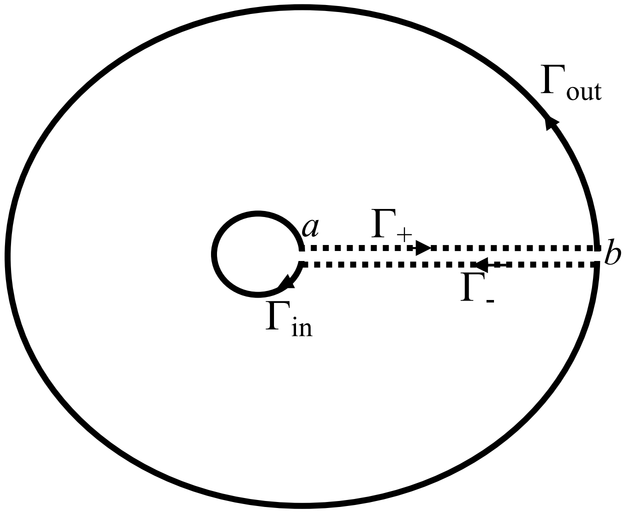


Fig. 4 The integration contour for a multiply connected domain.