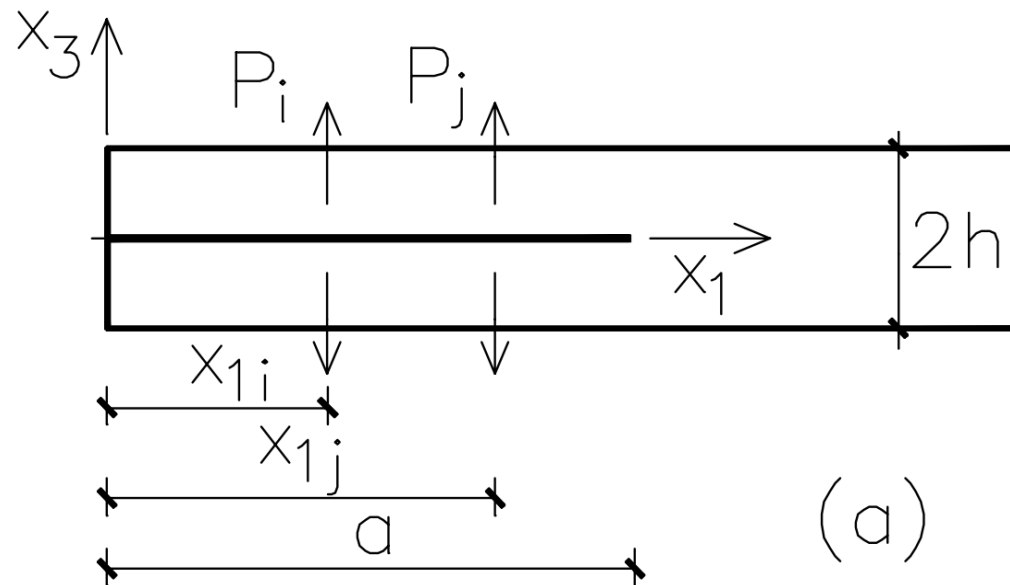


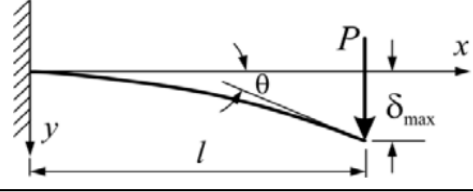
The scope of this note is to assess the localized compliance of a double cantilever beam specimen determined by stress intensity factors and Castigliano's theorem. It will be compared to the compliance obtained from simple beam theory and thus, the stress intensity factors will also be from simple beam theory.

The local compliance is denoted  $\lambda_{ij}$  which translates to the deflection of  $P_i$  at  $x_{1i}$  due to a force couple,  $P_j$ , acting at  $x_{1j}$ . Castigliano's theorem states:

$$\lambda_{ji} = \frac{2}{E} \cdot \int_0^a \frac{K_I(x_{1i}, P_i) \cdot K_I(x_{1j}, P_j)}{P_i \cdot P_j} da$$



Consider a single cantilever beam with a concentrated load,  $P$ , at distance  $l$  from the base

1. Cantilever Beam – Concentrated load $P$ at the free end		
	$\theta = \frac{Pl^2}{2EI}$	$y = \frac{Px^2}{6EI}(3l - x)$
2. Cantilever Beam – Concentrated load $P$ at any point		

The deflection at distance  $x$  from base is:

$$y(x, P, E, I, l) := \frac{P \cdot x^2}{6 \cdot E \cdot I} \cdot (3l - x) \quad \text{for } x \leq l$$

With  $I = (h^3)/12$  for a beam of unit depth and the height  $h$  with  $\delta = 2y$  being the normal opening displacement of the double cantilever we get:

$$\delta(x, P, E, h, l) := 2y\left(x, P, E, \frac{h^3}{12}, l\right) \text{ factor} \rightarrow \frac{4 \cdot P \cdot x^2 \cdot (3l - x)}{E \cdot h^3} \quad \text{for } x \leq l$$

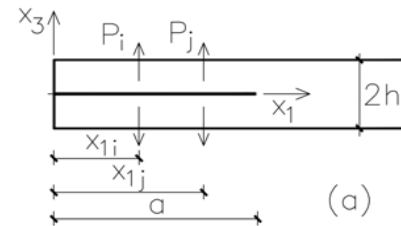
We now switch to a coordinate system where  $x_1$  originates at the free end of the beams and is positive toward the crack tip. Thus, we need to replace  $x$  and  $l$  appropriately with respect to the new reference:

$l$  was the distance from the load,  $P_i$ , to the crack tip:

$$l(x_{1i}, a) := a - x_{1i}$$

And  $x$  the distance from the crack tip to the point where we want to know the opening displacement:

$$x(x_{1j}, a) := a - x_{1j}$$



The displacement at  $x_{1j}$  due to a force couple,  $P_i$ , acting at  $x_{1i}$  is then:

$$\delta_{ji}(x_{1i}, x_{1j}, a, P, E, h) := \delta(a - x_{1j}, P, E, h, a - x_{1i}) \text{ factor} \rightarrow \frac{4 \cdot P \cdot (a - x_{1j})^2 \cdot (2 \cdot a - 3 \cdot x_{1i} + x_{1j})}{E \cdot h^3} \quad \text{for } x_{1i} \leq x_{1j}$$

And the the compliance at  $x_{1j}$  for a force couple acting at  $x_{1i}$  is thus:

$$\lambda_{ji}(x_{1i}, x_{1j}, a, E, h) := \frac{\delta_{ji}(x_{1i}, x_{1j}, a, P, E, h)}{P} \rightarrow \frac{4 \cdot (a - x_{1j})^2 \cdot (2 \cdot a - 3 \cdot x_{1i} + x_{1j})}{E \cdot h^3} \quad \text{for } x_{1i} \leq x_{1j}$$

From Bettis's theorem we get:

$$\lambda_{ij} = \lambda_{ji}$$

### Compliance from stress intensity factors

To determine the local compliance using stress intensity factors we use:

$$K_I(x_1, a, h, P) := \frac{\sqrt{12}P}{3} \cdot \frac{(a - x_1)}{h^2}$$

,where  $a-x_1$  is the distance from the force couple  $P$  to the crack tip.

For  $P_i$  and  $P_j$  respectively we get:

$$K_I(x_{1i}, a, h, P_i) := \frac{\sqrt{12}P_i}{3} \cdot \frac{(a - x_{1i})}{h^2} \quad K_I(x_{1j}, a, h, P_j) := \frac{\sqrt{12}P_j}{3} \cdot \frac{(a - x_{1j})}{h^2}$$

According to the Castigliano theorem, the localized compliance can be determined by:

$$\lambda_{ij}(x_{1i}, x_{1j}, a, h, E) := \frac{2}{E} \cdot \int_0^a \frac{K_I(x_{1i}, a, h, P_i) \cdot K_I(x_{1j}, a, h, P_j)}{P_i \cdot P_j} da \text{ factor} \rightarrow \frac{4 \cdot a \cdot (2 \cdot a^2 - 3 \cdot a \cdot x_{1i} - 3 \cdot a \cdot x_{1j} + 6 \cdot x_{1i} \cdot x_{1j})}{E \cdot h^3}$$

but this is different from the local compliance obtained directly from simple beam theory. However, we get the same result if we change the limits of the integral from 0 to  $a$  in to from  $x_{1j}$  to  $a$ !

$$\lambda_{ij}(x_{1i}, x_{1j}, a, h, E) := \frac{2}{E} \cdot \int_{x_{1j}}^a \frac{K_I(x_{1i}, a, h, P_i) \cdot K_I(x_{1j}, a, h, P_j)}{P_i \cdot P_j} da \text{ factor} \rightarrow \frac{4 \cdot (a - x_{1j})^2 \cdot (2 \cdot a - 3 \cdot x_{1i} + x_{1j})}{E \cdot h^3}$$