6. Applications of Linear Elastic Fracture Mechanics-Practical Issues

References:
J. W. Hutchinson, *Notes on Nonlinear Fracture Mechanics* (http://imechanica.org/node/755);
Alan Zehnder, *Lecture Notes on Fracture Mechanics* (http://hdl.handle.net/1813/3075).

Design rules at multiscales. Length scale matters for the purpose of design. For example, a large structure component may have features like holes and corners, which lead to stress concentration under loading. At this length scale (~ 10^{-2} m), cracks or other defects are not visible (or allowable). Thus, the design rule would base on the geometrical features, loading conditions, and macroscopic material properties. Focusing onto the material at the locations of stress concentration, we see microstructures (e.g., grain boundaries, particle inclusions, fibers) along with inevitable defects at the microscale (~10^{-5} m). Typically, cracks of similar lengths are present and thus a fracture mechanics based design rule may be established based on the energy release rate or stress intensity factors along with fracture toughness of the material. The features of the microstructure would affect both the material toughness and fracture driving force. At this scale, the material toughness has to be determined from experimental measurements. More fundamental understanding of the fracture process may be achieved through atomistic modeling (~ 10^{-9} m), which would potentially be able to predict the fracture toughness based on atomistic structures.

An example practice may be drawn from the design of microelectronic devices at the chip and package levels. The package structures are large, with features like corners, chip-package interface, and solder balls. Design of the package structures must pay close attention to the stresses around the solder balls, especially those near the corner of the chip-package interface. At the chip level, complex material structures are built as the transistors and circuits at very small scales (10^{-6} - 10^{-9} m). Here, a reliability based design requires that defects of similar sizes do not grow under specific test and service conditions.

Case study: matrix cracking in a composite (Lu et al., Acta Metall. Mater. 39, pp. 1883-1890, 1991). Thermal expansion misfit can cause matrix cracking in fiber-reinforced intermetallic composites (e.g., SiC fibers in $MoSi_2$). Consider a long fiber in an infinite matrix. The stress field induced by the differential thermal expansion can be obtained by solving a 2D plane strain problem (Lame solution), assuming no cracks and perfect bonging at the fiber-matrix interface. The stress field in the fiber is uniform:

$$\sigma_r^f = \sigma_\theta^f = -\frac{1}{2}\sigma_T \text{ and } \sigma_z^f = -\sigma_T$$

where $\sigma_T = \frac{E}{1-\nu}\int_{T_0}^{T}(\alpha_m - \alpha_f)dT$. Here, for simplicity, the fiber and matrix are assumed to have identical elastic moduli. The stress in the matrix decays away from the fiber:

$$\sigma_\theta^m = \frac{1}{2}\sigma_T\left(\frac{a}{r}\right)^2,\ \sigma_r^m = -\frac{1}{2}\sigma_T\left(\frac{a}{r}\right)^2,\text{ and } \sigma_z^m = 0.$$

Apparently, the maximum hoop stress in the matrix is $\frac{1}{2}\sigma_T$, independent of the fiber size. However, experiments have shown that the fiber size is important and there exists a critical size, below which matrix cracking does not occur. Taking the approach of fracture mechanics, we can calculate the stress intensity factor at the tip of a R-crack by the method of linear superposition:

$$K(c) = \int_a^{a+c} \sqrt{\frac{c/2+\xi}{c/2-\xi}} \frac{\sigma_\theta^m(r)dr}{\sqrt{\pi c/2}}$$

where c is the crack length and $\xi = r - a - c/2$. The kernel of the integration is the analytical solution for a finite crack in an infinite plate, subjected to concentrated forces on the crack faces (see *The Stress Analysis of Cracks Handbook* by Tada et al.) . Again, the assumption of identical elastic properties for the fiber and matrix is taken to simplify the solution. The integral gives that

$$K(c) = \sigma_T \sqrt{a} \left(\frac{\pi}{8}\frac{c}{a}\right)^{1/2} \left(1 + \frac{c}{2a}\right)^{-3/2}$$

For a fixed crack size, the stress intensity factor depends on the fiber size. The maximum stress intensity factor is reached when $a \approx 2c$, and $K_{\max} = 0.24\sigma_T\sqrt{a}$. Therefore, a critical fiber size is predicted by comparing $K_{\max}$ to the fracture toughness of the matrix material:

$$a_c = \left(\frac{K_{Ic}}{0.24\sigma_T}\right)^2.$$

Note that this prediction of the critical fiber size requires no information about the defect size in the material.

Resistance curve (R-curve). In a perfectly brittle material, the condition for initiation of crack growth and for continuation of quasi-static is: $K = K_c$, with the fracture toughness K_c being a constant. For intermediate and low strength materials, however, the applied stress intensity factor has to increase after crack initiation. In this case, fracture of the material is characterized by a resistance curve (or R-curve), $K_R(\Delta a)$, where Δa is the amount of crack growth. Typically, K_R increases and reaches a steady state, $K_R = K_{SS}$, for $\Delta a > L_{SS}$. The value of K_{SS} can be many times of the initiation toughness K_c. The transition length, L_{SS}, is much longer in low/intermediate strength metals than that in a glass or high strength, low toughness metals.

For a material characterized by a R-curve, the condition for continued crack growth (quasi-static) at the current crack length ($a = a_0 + \Delta a$) is: $K(a) = K_R(\Delta a)$. If this condition is satisfied, the condition for the stability of crack growth is

$$\left(\frac{\partial K}{\partial a}\right)_L < \frac{dK_R}{da}$$

Where the partial derivative of K with respect to a is to be taken under prescribed loading conditions (e.g., displacement control or load control). When $\left(\frac{\partial K}{\partial a}\right)_L \geq \frac{dK_R}{da}$, the crack growth (if occurs) becomes unstable. A comparison between the material R-curve and the loading curve, $K(a)$, can thus be used to predict the stability of crack growth.

Take the double cantilever beam as an example. The loading curve is given by $K(a) = Pa\left(\frac{12}{b^3}\right)^{1/2}$ or equivalently, $K(a) = \frac{E\Delta}{4a^2}\sqrt{3b^3}$. For a perfectly brittle material (i.e., $K_R \equiv K_c$ and $\frac{dK_R}{da} = 0$), the crack growth is always unstable under a prescribed load (*P*) and always stable under a prescribed displacement (Δ). For a material characterized by a R-curve, however, the crack growth is stable if the prescribed load is lower than a critical value, $P < P_c$, where P_c gives a loading curve tangential to the R-curve and depends on the initial crack length.

Fatigue and fatigue crack growth. Under cyclic loading (e.g., thermal cycles, vibrations), a component may fail after a number of cycles. Two practical approaches may be used to analyze the fatigue behavior. One is based on the relationship between the amplitude of applied cyclic stress and the number of cycles to failure (*S-N curve*), which can be measured by a standard fatigue test. A few observations from the fatigue tests are:

- The frequency of the applied cyclic load as well as the shape of the loading cycle has little effect on the number of cycles to failure.
- The relationship between the stress magnitude σ_a and the number of cycles to failure N_f can be approximately expressed by a power law (Basquin's law): $\sigma_a N_f^n = C$, where *n* and *C* are two constants to be determined for a specific material by the fatigue test.
- Some materials like steel have a fatigue limit ($\sim 0.5\sigma_y$), below which the number of cycles to failure is essentially infinity. Other materials such as aluminum do not have such a limit, in which case the fatigue strength may be taken as the stress amplitude corresponding to a specific number of cycles to failure (e.g., $N_f = 10^7$).
- Mean stress effect. Goodman's rule gives a simple relation for the effect of the mean stress on the stress amplitude of cyclic loading that leads to fatigue failure after N_f cycles: $\sigma_a(N_f, \sigma_m) = \sigma_a(N_f, 0)\left(1 - \frac{\sigma_m}{\sigma_{UTS}}\right)$. The mean stress effect on Al is better described by Gerber's parabola: $\sigma_a(N_f, \sigma_m) = \sigma_a(N_f, 0)\left(1 - \left(\frac{\sigma_m}{\sigma_{UTS}}\right)^2\right)$.
- Miner's rule for cumulative damage. For a loading history with periods of cyclic loads with different amplitudes and/or different mean stresses, the fatigue damage accumulated over the cycles may be estimated by simply adding up the percentages

of the loading cycles over the numbers of cycles to failure under each loading condition: $D = \sum_i \frac{N_i}{N_{fi}}$. The fatigue failure is then expected when $D \geq 1$.

The second approach is based on the concepts of fracture mechanics. Assuming a sample with a crack of length *a*. For simplicity, consider 2D, mode I, plane strain, and small scale yielding conditions. The fatigue behavior is then characterized by the growth of the crack under cyclic loading. When the stress intensity factor at the crack tip cycles in the range, $0 \leq K \leq \Delta K$, the rate of crack growth per cycle, da/dN, may be measured as a function of ΔK. At values of ΔK approaching K_{Ic}, the growth rate is high. At sufficiently low values of ΔK, the crack growth is exceeding low and a threshold value K_{th} can be identified, below which the crack essentially does not grow. For intermediate values of ΔK between K_{th} and K_{Ic}, the data often can be fit into a power-law relation (Paris's Law):

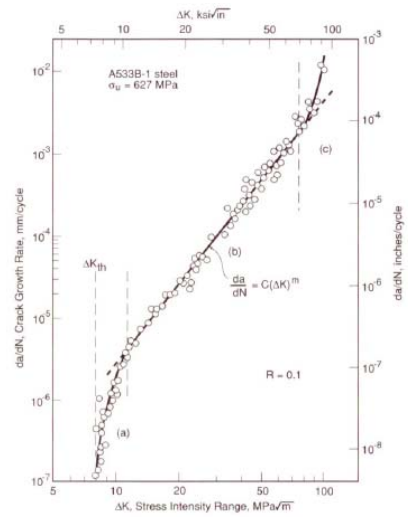


Figure 11.3 Fatigue crack growth rates over a wide range of stress intensities for a ductile pressure vessel steel. Three regions of behavior are indicated: (a) slow growth near the threshold ΔK_{th}, (b) intermediate region following a power equation, and (c) unstable rapid growth. (Plotted from the original data for the study of [Paris 72].)

$$\frac{da}{dN} = A(\Delta K)^m$$

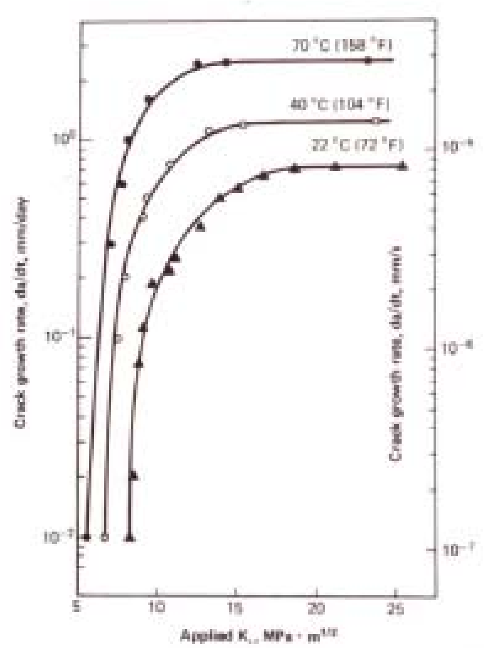
where *m* is typically between 2 and 4.

The fatigue crack growth data and its relationship with ΔK can be used to estimate the remaining fatigue life of a cracked component. Suppose that inspection methods have detected a crack of size a_0 in the component. A critical crack size a_c is specified or estimated by setting $K(a_c) = K_{Ic}$. The number of cycles to failure of the component is then predicted as:

$$N_f = \int_{a_0}^{a_c} \frac{da}{(da/dN)} = \int_{a_0}^{a_c} \frac{da}{A(\Delta K)^m}$$

Usually ΔK is a function of the crack length under a cyclic stress or displacement loading. For example, for an edge crack in a large plate under a remote stress cycling between 0 and $\sigma_{\max}$, $\Delta K = 1.12\sigma_{\max}\sqrt{\pi a}$.

Environmentally assisted cracking (stress corrosion, subcritical cracking, static fatigue). Under a static loading, a crack may grow slowly in a corrosive environment (e.g., moisture). This can happen when $K < K_{Ic}$, thus called subcritical cracking. The mechanism of subcritical cracking may be understood as a result of two effects: (1) the applied stress, although not sufficient to cause cracking by itself, opens up the crack faces, allowing transport of water molecules or other chemical species into the cracked region; (2) the environmental species reduces the bond strength at the crack tip by chemical or physical reaction processes, thus assisting crack growth. Both the transport and reaction processes are time dependent, controlling the rate of crack growth. It was also found that the surface energy of a material in air can be significantly smaller that that in vacuum (Obreimoff, 1930).



The subcritical crack growth rate can be measured as function of the applied stress intensity factor *K*. For values of *K* close to the toughness K_{Ic}, the crack growth is increasingly fast, approaching the regime of fast fracture. At sufficiently low values of K, the crack growth is exceeding low and a threshold value K_{th} can be identified, below which the crack essentially does not grow. Typical data for environmentally assisted cracking show two distinct regions. In the first region with *K* slightly above K_{th}, the crack growth rate is very sensitive to the value of *K*. Here, the crack opening is relatively small, and the transport of the environmental species controls the crack growth rate. A power law is often used to described the subcritical cracking in this region:

$$\frac{da}{dt} = BK^n$$

where the exponent n is typically around 10.

In the second region, as *K* increases, the crack growth rate saturates at a level sensitive to the environmental conditions (e.g., temperature, moisture). Here, the crack opening is large enough for the transport of the environmental species. The process that limits the crack growth rate is predominantly the reaction at the crack tip.