Lecture 8 Rayleigh Instability

Rayleigh (1878) examined a common experience: a thin jet of liquid is unstable and breaks into droplets. When a jet is thin enough, the effect of gravity is negligible compared to surface energy. The jet changes its shape to reduce the total surface energy. Liquid flow sets the time.

Similar instability in solids. Phenomena similar to the Rayleigh instability occur in solid state; see Rodel and Glaeser (1990) for an experimental demonstration and for a literature survey. For example, at a high temperature, a penny-shaped pore in a solid first blunts its edge, from which finger-like tunnels emerge, and the tunnels then break into small cavities (Lange and Clarke 1982).

Why does a cylinder evolve into a row of droplets? Srolovitz and Safran (1986) gave a simple geometric argument. Assume that the surface energy density is isotropic, and the free energy of the system is the surface area times the surface tension. One has to show is that the cylinder has a larger surface area than the row of spheres. Consider a long cylinder of radius *R*, and a row of droplets, each of radius *b*. Imagine that the cylinder evolves to the droplets by first perturb the surface with a wavelength λ. The volume per wavelength of the cylinder equals the volume of each droplet, so that the droplet radius is given by

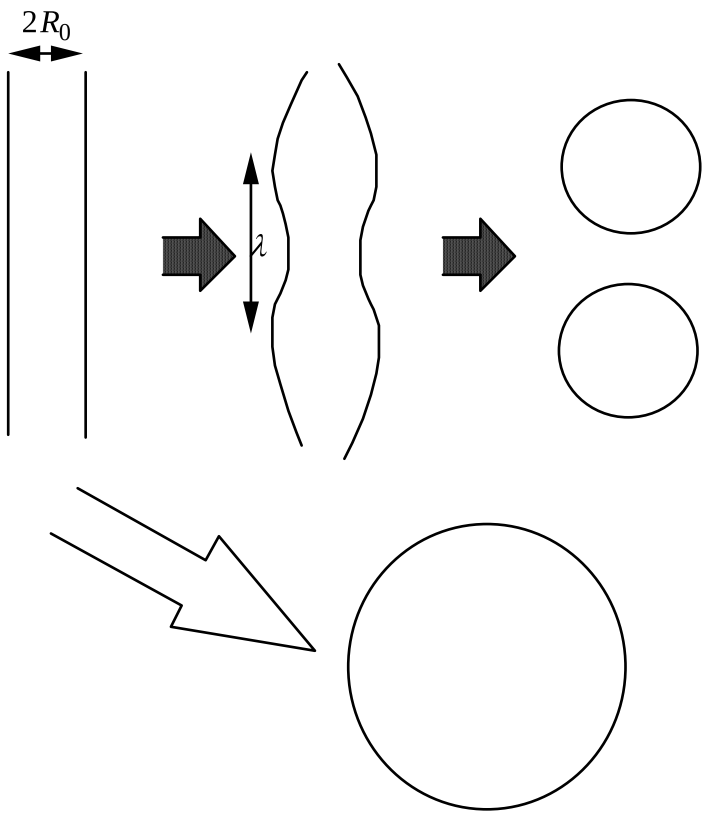
$$b = \left(3R^2\lambda / 4\right)^{1/3}.$$

The free energy per wavelength of the cylinder is $2\pi R\lambda\gamma$. The free energy per droplet is $4\pi b^2\gamma$.

The free energy of cylinder is larger than the row of droplets if

$$\lambda > 9R / 2.$$

From this geometric (energetic) consideration, one expects that a fiber can evolve to a row of spheres of large enough radii.



But wait. Doesn't a single big sphere have the smallest surface area? Why does the cylinder evolve to many small droplets, rather than a single large sphere? The energetic consideration does not answer this question. The answer has to do with kinetics. It takes a short time for the cylinder to evolve to a row of spheres. When the cylinder evolves into droplets, mass transport stops, preventing the system from reaching the minimal energy configuration, a single large sphere. Here we have assumed a certain kind of mass transport mechanism, such as fluid flow or solid diffusion. If, instead, the cylinder is sealed in a small bag, it will evolve to a single large sphere via vapor transport.

Why do atoms diffuse from the troughs to crests? I always thought that a smooth surface has a low free energy. For the cylinder to evolve to droplets, it must first become wavy.

Why do atoms diffuse from the troughs to crests? The answer has to do with a simple geometric fact: the cylinder lives in three dimensions. At each point on the surface there are two principal curvatures: one corresponds to the circle in the section normal to the cylinder axis, and the other corresponds the wave in the section along the cylinder axis. We will call the two curvatures K_1 and K_2. At a trough, the circle is small. At a crest, the circle is big. Consequently, the difference in K_1 drives atoms to diffuse from the trough to the crest, amplifying the wave amplitude. On the other hand, K_2 is positive at a crest, and negative at a trough, so that the difference in K_2 drives atoms to diffuse from the crest to the trough, decreasing the wave amplitude. The difference in K_2 is small when the wavelength is very long.

How fast will the cylinder evolve? To answer this question requires us to study the mass transport process. Perturb the radius of the cylinder in the form:

$$r(r,t) = R[1+\varepsilon(t)\cos(kz)]$$

We will carry out a linear perturbation analysis. We retain terms linear in the perturbation amplitude ε.

The surface has two principal curvatures. One curvature is the inverse of the radius of a cross section:

$$K_1 = \frac{1}{r} = \frac{1}{R}[1-\varepsilon\cos(kz)].$$

For the cylinder of the wavy surface, the curvature increases at troughs, and decreases at crests. The other curvature is in the cross-section of the cylinder along the axial direction:

$$K_2 = -\frac{\partial^2 r}{\partial z^2} = \varepsilon R k^2 \cos(kz).$$

This curvature is negative at troughs, and positive at crests. As discussed before, the two principal curvatures have the opposite trends.

The surface velocity is

$$v_n = \frac{\partial r}{\partial t} = R\frac{\partial \varepsilon}{\partial t}\cos(kz)$$

In the governing equation $v_n = B\nabla^2(K_1 + K_2)$, for the small perturbation, the surface Laplacian is $\nabla^2 = \partial^2 / \partial z^2$, so that

$$\frac{d\varepsilon}{dt} = \frac{\varepsilon}{\tau},$$

with

$$\tau = \frac{R^4 / B}{(kR)^2 - (kR)^4} .$$

This is an ordinary differential equation for the wave amplitude $\varepsilon(t)$. For the initial condition $\varepsilon = \varepsilon(0)$ at $t = 0$, the solution to the ordinary differential equation is

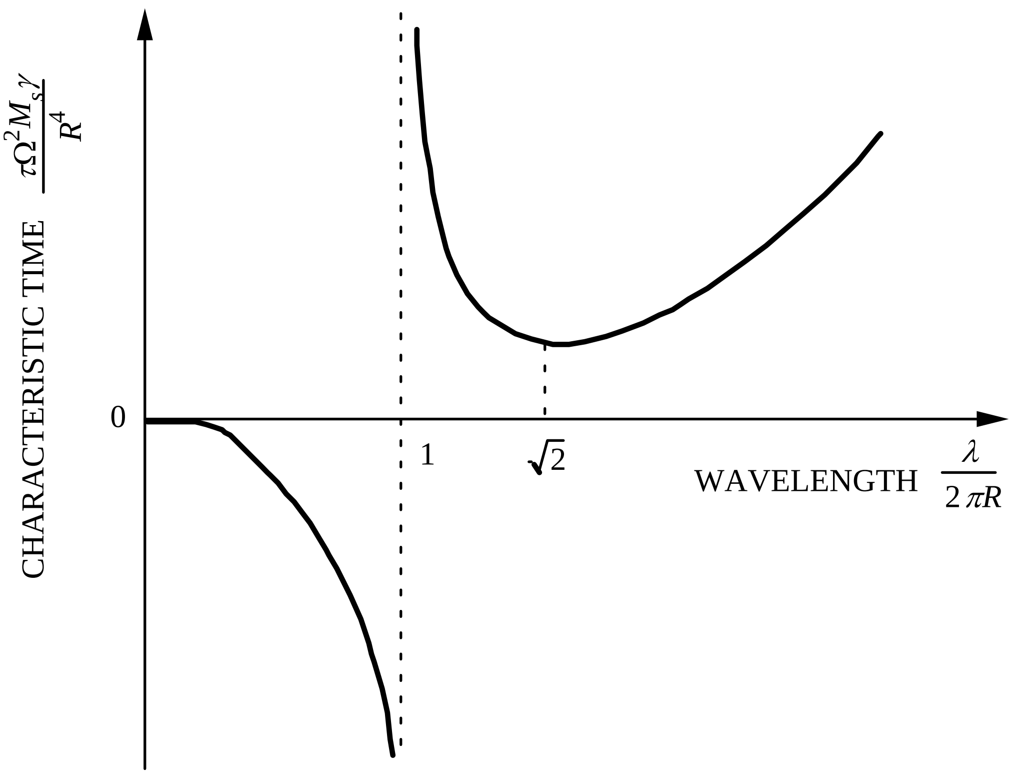
$$\varepsilon(t) = \varepsilon(0)\exp(t / \tau).$$

This shows how fast the wave amplitude will change.

Wavelength selection. The characteristic time τ varies with the wavelength. When $\lambda < 2\pi R$, the perturbation increases the free energy, $\tau < 0$, and the perturbation diminishes with the time. When $\lambda > 2\pi R$, the perturbation decreases the free energy, $\tau > 0$, and the perturbation amplifies with the time. A very long wavelength, mass transport over a long distance takes a long time. Consequently, τ minimizes at an intermediate wavelength, given by

$$\lambda_m = 2\sqrt{2}\pi R .$$

If perturbations of all wavelengths have an identical initial amplitude, $\varepsilon(0)$, the perturbation of wavelength λ_m grows most rapidly at $t = 0$. It is often assumed that this wavelength would win the race even at later times, and the fiber eventually breaks at this wavelength.



The time scale. Corresponding to the fastest growing wavelength, the characteristic time is

$$\tau_m = 4R^4 / B .$$

This is the time scale for the Raleigh instability to occur. We could have obtained this scaling from dimensional considerations. Recall that $B = \gamma_s \Omega D_s \delta_s / kT$. The Rayleigh instability occurs over a long time if the radius is large or the temperature is low.

The above conclusion is made on the basis of the linear stability analysis, where high order terms of ε have been ignored. A complete simulation of the surface evolution is necessary to take into account of the actual initial imperfection and large shape change (Nichols 1976). Yu and Suo (1999) developed a finite element method to evolve axisymmetric surfaces.

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