

## **Formula for stress incrementation in a Kelvin-Voigt body**

The mechanical response is modeled as a single-element Kelvin-Voigt body in the UMAT. Starting from the constitutive equation for the Kelvin-Voigt body in threedimensional notation ([1] page 147), a scheme for stress incrementation suitable for programming is derived in the following. The strain terms and the stress terms are uncoupled and treated separately. The strain-related term *StrainTerm* is given by

$$StrainTerm = \lambda \left( tr(\underline{\underline{\varepsilon}}) + \theta_\lambda tr(\dot{\underline{\underline{\varepsilon}}}) \right) \delta_{ij} + 2\mu \left( \varepsilon_{ij} + \theta_\mu \dot{\varepsilon}_{ij} \right)$$

Incrementation of the stress is done by applying the backward difference method. Quantities denoting rates are replaced by the ratio of a finite change divided by the respective time increment; all other quantities are augmented by their respective increments. The replacement rules are given by substituting

$$\begin{aligned} \varepsilon_{ij} &\rightarrow \varepsilon_{ij} + \Delta\varepsilon_{ij}, & tr(\underline{\underline{\varepsilon}}) &\rightarrow tr(\underline{\underline{\varepsilon}}) + \Delta tr(\underline{\underline{\varepsilon}}), \\ \dot{\varepsilon}_{ij} &\rightarrow \frac{\Delta\varepsilon_{ij}}{\Delta t}, & tr(\dot{\underline{\underline{\varepsilon}}}) &\rightarrow \frac{\Delta tr(\underline{\underline{\varepsilon}})}{\Delta t} \end{aligned}$$

Upon multiplication with the time increment and lumping like terms the intermediate result is

$$\begin{aligned} StrainTerm \cdot \Delta t = & \\ & \lambda \left( \Delta t \cdot tr(\underline{\underline{\varepsilon}}) + \Delta tr(\underline{\underline{\varepsilon}}) \cdot (\Delta t + \theta_\lambda) \right) \delta_{ij} + \\ & 2\mu \left( \Delta t \cdot \varepsilon_{ij} + \Delta\varepsilon_{ij} \cdot (\Delta t + \theta_\mu) \right) \end{aligned}$$

The same procedure needs to be performed for the stress-related terms of the constitutive equation. It is, by analogy, labeled *StressTerm* and defined by

$$StressTerm = \sigma_{ij}$$

Upon incrementation

$$\sigma_{ij} \rightarrow \sigma_{ij} + \Delta\sigma_{ij}$$

the current value of *StressTerm* is

$$StressTerm = \sigma_{ij} + \Delta\sigma_{ij}$$

Upon multiplication by the time increment one arrives at

$$StressTerm \cdot \Delta t = \Delta t (\sigma_{ij} + \Delta\sigma_{ij})$$

From this point on it is convenient to separate the discussion of normal and shear stresses. The convention adopted in ABAQUS is that the stress and strain components are arranged in vector form with six components, with the three normal components listed first. The presence

of the Kronecker symbol ensures decoupling of the normal and shear components. Hence, in the two following subsections on normal and shear stress incrementation the vectors are represented by three components only to economize space.

## Normal stress incrementation

For the normal components of the stress and the strain, the indices i and j are identical, thus

$$\delta_{ij} = 1.$$

The symbol ‘trace of’ represents

$$tr(\underline{\underline{\varepsilon}}) = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

If the x-component is under consideration

$$\varepsilon_{ij} = \varepsilon_{xx},$$

$$\sigma_{ij} = \sigma_{xx}.$$

The increment of a sum equals the sum of increments

$$\Delta tr(\underline{\underline{\varepsilon}}) = \Delta \varepsilon_{xx} + \Delta \varepsilon_{yy} + \Delta \varepsilon_{zz}$$

The equality

$$StressTerm \cdot \Delta t = StrainTerm \cdot \Delta t$$

can now be solved for the stress increment in the x-direction giving, upon grouping like terms,

$$\begin{aligned} \Delta \sigma_{xx} = & (\lambda + 2\mu) \cdot \varepsilon_{xx} + \lambda \cdot (\varepsilon_{yy} + \varepsilon_{zz}) + \\ & \Delta \varepsilon_{xx} \cdot \left( \lambda \cdot \left( 1 + \frac{\theta_\lambda}{\Delta t} \right) + 2\mu \cdot \left( 1 + \frac{\theta_\mu}{\Delta t} \right) \right) + \\ & (\Delta \varepsilon_{yy} + \Delta \varepsilon_{zz}) \cdot \lambda \cdot \left( 1 + \frac{\theta_\lambda}{\Delta t} \right) - \sigma_{xx} \end{aligned}$$

For brevity let

$$\begin{aligned} A &\rightarrow \lambda \cdot \left( 1 + \frac{\theta_\lambda}{\Delta t} \right) + 2\mu \cdot \left( 1 + \frac{\theta_\mu}{\Delta t} \right), & B &\rightarrow \lambda \cdot \left( 1 + \frac{\theta_\lambda}{\Delta t} \right) \\ C &\rightarrow \lambda + 2\mu, & D &\rightarrow \lambda \\ F &\rightarrow 2\mu \cdot \left( 1 + \frac{\theta_\mu}{\Delta t} \right), & G &\rightarrow 2\mu \end{aligned}$$

Hence

$$\begin{aligned} \Delta \sigma_{xx} = & C \cdot \varepsilon_{xx} + D \cdot (\varepsilon_{yy} + \varepsilon_{zz}) + \\ & \Delta \varepsilon_{xx} \cdot A + (\Delta \varepsilon_{yy} + \Delta \varepsilon_{zz}) \cdot B - \sigma_{xx} \end{aligned}$$

The equation for the stress increment is now rewritten in tensor notation for all normal stress components with the vector definitions

$$\begin{aligned} StressIncr &= \begin{pmatrix} \Delta\sigma_{xx} \\ \Delta\sigma_{yy} \\ \Delta\sigma_{zz} \end{pmatrix}, & StrainIncr &= \begin{pmatrix} \Delta\epsilon_{xx} \\ \Delta\epsilon_{yy} \\ \Delta\epsilon_{zz} \end{pmatrix}, \\ Stress &= \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix}, & Strain &= \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{pmatrix} \end{aligned}.$$

The ABAQUS manual suggests the term *DDSDDE* for the Jacobian of the constitutive equation defining the relation between the stress increment and the strain increment. It is given by

$$DDSDDE = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}$$

The stress increment and the current strain are linked by a tensor that, by analogy, may be called

$$DDSDE = \begin{pmatrix} C & D & D \\ D & C & D \\ D & D & C \end{pmatrix}.$$

The stress increment further depends on the stress before incrementation. They are linked by

$$DDSDS = -\underline{\underline{1}}$$

where  $\underline{\underline{1}}$  stands for the identity matrix. With the above tensor definitions the stress incrementation is cast into the final form of a sum of matrix-vector products

$$\begin{aligned} StressIncr &= \\ &DDSDDE.StrainIncr + \\ &DDSDE.Strain + \\ &DDSDS.Stress \end{aligned}.$$

This is the result for the normal stress incrementation. The code furnishes separable expressions, allowing all contributions to the stress increment be studied individually.

### Shear stress incrementation

Unequal indices *i* and *j* identify the components of the shear stresses, and hence

$$\delta_{ij} = 0.$$

The stress increment is given by

$$\Delta\sigma_{ij} = 2\mu \left( \varepsilon_{ij} + \Delta\varepsilon_{ij} \cdot \left( 1 + \frac{\theta_\mu}{\Delta t} \right) \right) - \sigma_{ij}.$$

The ABAQUS convention demands that shear strains be given in terms of  $\gamma_{ij}$ ; the conversion

$$\varepsilon_{ij} = \frac{\gamma_{ij}}{2}, \quad \Delta\varepsilon_{ij} = \frac{\Delta\gamma_{ij}}{2}$$

yields

$$\Delta\sigma_{ij} = \mu \left( \gamma_{ij} + \Delta\gamma_{ij} \cdot \left( 1 + \frac{\theta_\mu}{\Delta t} \right) \right) - \sigma_{ij}.$$

Again, the quantity linking the stress increment to the strain increment is termed *DDSDDE*

$$DDSDDE = \frac{F}{2} \underline{\underline{1}}.$$

The contribution of the current strain is computed by

$$DDSDE = \frac{G}{2} \underline{\underline{1}}$$

and the one of the current stress tensor by

$$DDSDS = -\underline{\underline{1}}.$$

With the definitions

$$\begin{aligned} StressIncr &= \begin{pmatrix} \Delta\sigma_{ij} \\ \Delta\sigma_{ik} \\ \Delta\sigma_{jk} \end{pmatrix}, & StrainIncr &= \begin{pmatrix} \Delta\gamma_{ij} \\ \Delta\gamma_{ik} \\ \Delta\gamma_{jk} \end{pmatrix}, \\ Stress &= \begin{pmatrix} \sigma_{ij} \\ \sigma_{ik} \\ \sigma_{jk} \end{pmatrix}, & Strain &= \begin{pmatrix} \gamma_{ij} \\ \gamma_{ik} \\ \gamma_{jk} \end{pmatrix}, \end{aligned}$$

the formula for stress incrementation is formally identical to the one for the normal stress incrementation, namely

$$\begin{aligned} StressIncr &= \\ &DDSDDE.StrainIncr + \\ &DDSDE.Strain + \\ &DDSDS.Stress \end{aligned}.$$

[1] J. Lemaître, J.-L. Chaboche: Mécanique des matériaux solides, chapitre 3.2.1 "Solide de Kelvin-Voigt"