Curved Beams

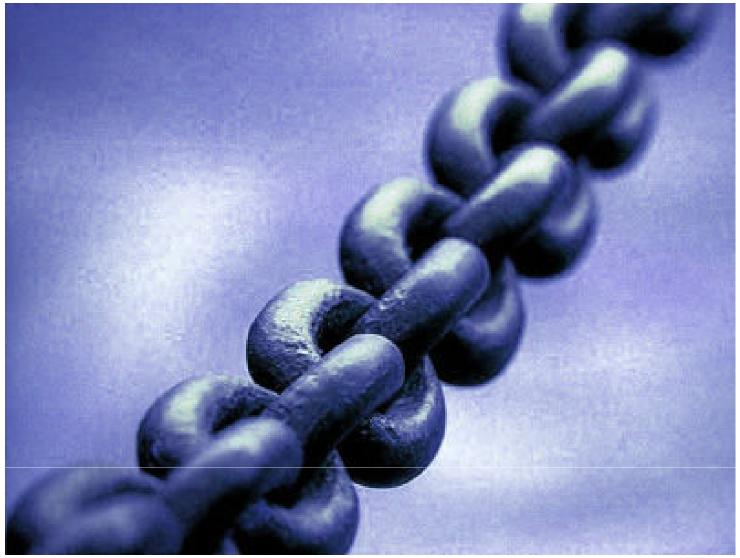
A Theoretical and FEM Investigation

Nathan Thielen
Christian Wylonis
ES240

What are Curved Beams?

- A beam is curved if the line formed by the centroids of all the cross sections is not straight.

http://www.photogen.com/free-photos/data/media/10/179_7908.jpg



http://www.timyoung.net/contrast/images/chain02.jpg

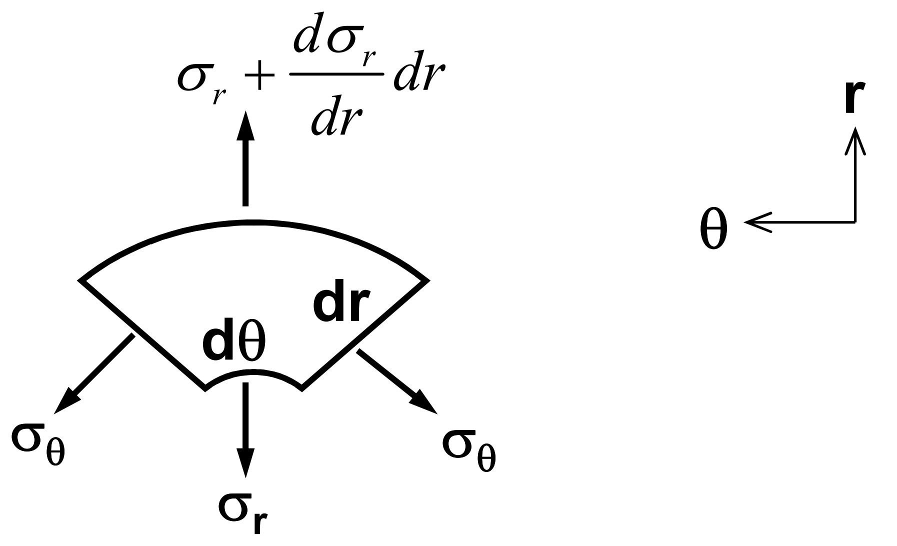
- Curved beams loaded in bending appear as arches, chains, hooks, an d many other connectors.

Assumptions

- Constant Cross Section
- Circular Arc Formed by the Centroid Curve
- Isotropic
- Homogeneous
- Elastic

Force Balance

Assume Shear Stresses are Zero (No Shear Stresses on the Surfaces)



Force Balance:

$$\frac{\partial \sigma_r}{\partial r} + \frac{(\sigma_r - \sigma_\theta)}{r} = 0$$

Stress Compatibility

(Stress Boundary Conditions)

2-D Strain Compatibility (Plane Strain):

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

2-D Stress Compatibility (Hooke's Law and Force Balance):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\sigma_x + \sigma_y\right) = \nabla^2\left(\sigma_x + \sigma_y\right) = 0$$

Converting to Polar Coordinates:

$$\nabla^2\left(\sigma_r + \sigma_\theta\right) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\left(\sigma_r + \sigma_\theta\right) = 0$$

General Solution

Compatibility:

$$\nabla^2\left(\sigma_r+\sigma_\theta\right)=\frac{\partial^2\left(\sigma_r+\sigma_\theta\right)}{\partial r^2}+\frac{1}{r}\frac{\partial\left(\sigma_r+\sigma_\theta\right)}{\partial r}=0$$

(Euler's Differential Equation)

Solve:

$$\left(\sigma_r+\sigma_\theta\right)=k_1\ln\left(\frac{r}{a}\right)+k_2\Rightarrow-\sigma_\theta=\sigma_r-k_1\ln\left(\frac{r}{a}\right)-k_2$$

Force Balance:

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\left(2\sigma_r-k_1\ln\left(\frac{r}{a}\right)-k_2\right)=0$$

(First Order Linear ODE)

Solve:

$$\sigma_r\left(r\right)=C_1+C_2\ln\left(\frac{r}{a}\right)+\frac{C_3}{r^2}\qquad\qquad\sigma_\theta\left(r\right)=C_1+C_2\left(1+\ln\left(\frac{r}{a}\right)\right)-\frac{C_3}{r^2}$$

Boundary Conditions

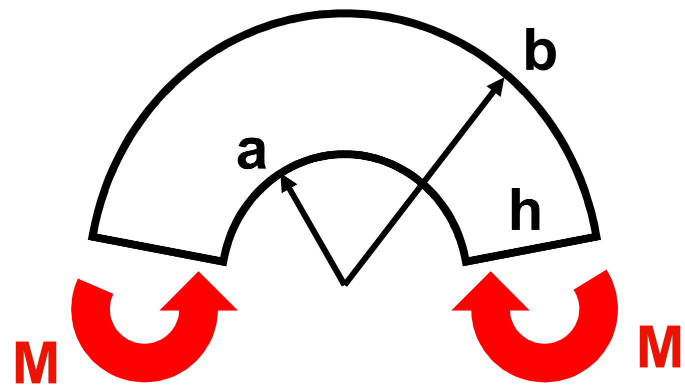
$$\sigma_r(r) = C_1 + C_2 \ln\left(\frac{r}{a}\right) + \frac{C_3}{r^2} \qquad \sigma_\theta(r) = C_1 + C_2\left(1 + \ln\left(\frac{r}{a}\right)\right) - \frac{C_3}{r^2}$$

Boundary Conditions:

$$(i)\ \sigma_r(r=a) = \sigma_r(r=b) = 0$$

$$(ii)\ \int_a^b t\sigma_\theta dr = 0$$

$$(iii)\ \int_a^b rt\sigma_\theta dr = M$$



σ_θ is not zero at r=a or r=b, therefore the distributed normal stresses (σ_θ) cause the moment (M).

Applying Boundary Conditions

$$(i)\sigma_r(r=a)=C_1+\frac{C_3}{a^2}=0$$

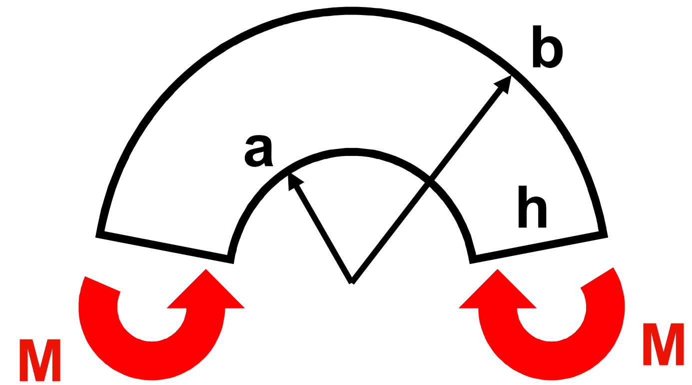
$$(ii)\int_a^b t\sigma_\theta dr=\int_a^b t\left[C_1+C_2\left(1+\ln\left(\frac{r}{a}\right)\right)-\frac{C_3}{r^2}\right]dr=0$$

$$(iii)\int_a^b rt\sigma_\theta dr=\int_a^b rt\left[C_1+C_2\left(1+\ln\left(\frac{r}{a}\right)\right)-\frac{C_3}{r^2}\right]dr=M$$

Solve:

$$C_1=\frac{M}{a^2t\ln\left(\frac{b}{a}\right)}\qquad C_2=\frac{(a+b)(a-b)M}{a^2b^2t\left[\ln\left(\frac{b}{a}\right)\right]^2}\qquad C_3=-\frac{M}{t\ln\left(\frac{b}{a}\right)}$$

Curved Beam Stress Results



$$\sigma_r(r)=\left(\frac{4M}{tb^2N}\right)\left[\left(1-\frac{a^2}{b^2}\right)\ln\left(\frac{r}{a}\right)-\left(1-\frac{a^2}{r^2}\right)\ln\left(\frac{b}{a}\right)\right]$$

$$\sigma_\theta(r)=\left(\frac{4M}{tb^2N}\right)\left[\left(1-\frac{a^2}{b^2}\right)\left(1+\ln\left(\frac{r}{a}\right)\right)-\left(1+\frac{a^2}{r^2}\right)\ln\left(\frac{b}{a}\right)\right]$$

Where:

$$N=\left(1-\frac{a^2}{b^2}\right)^2-4\left(\frac{a^2}{b^2}\right)\ln\left(\frac{b}{a}\right)=const.$$

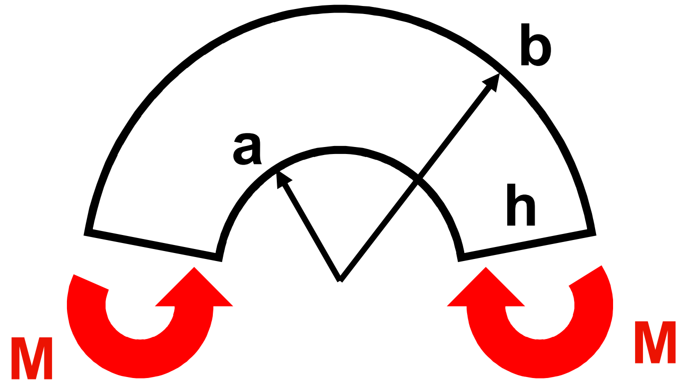
Simple Example

$$h = 0.1mm$$

$$M = 10N \cdot mm$$

$$a = 1mm$$

$$b = 3mm$$



$$\sigma_r\left(r = 2mm\right) = -36MPa$$

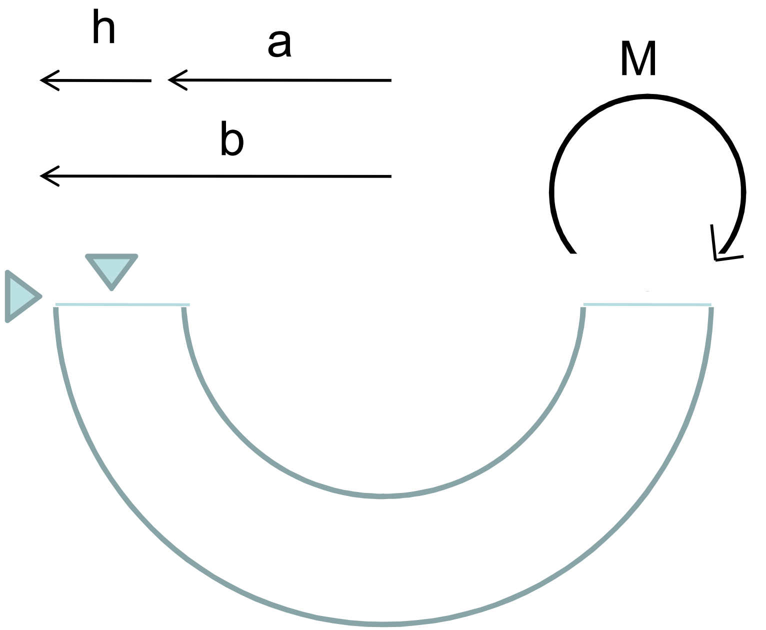
$$\boxed{\sigma_\theta\left(r = 2mm\right) = 23MPa}$$

The neutral axis is NOT the centroid axis (i.e. the center plane is not a neutral axis). This is the PRIMARY difference between a curved beam and a straight beam.

Curved Beams FEM

Models compared to theoretical results

The model



Constants: $h = 0.4$ m
$M = 40$ Nm
$t = 1$ m

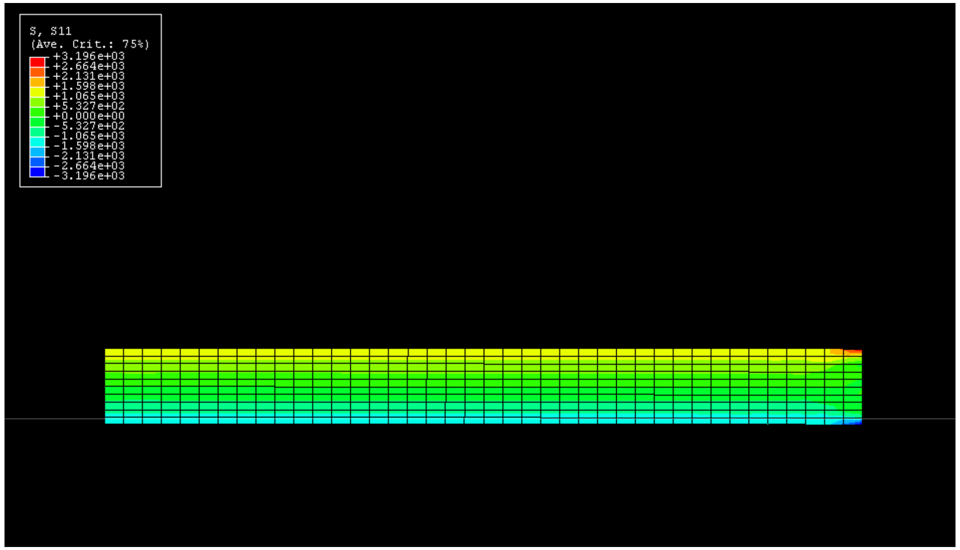
Case 1: Straight beam
$a = \infty$
$b = \infty$

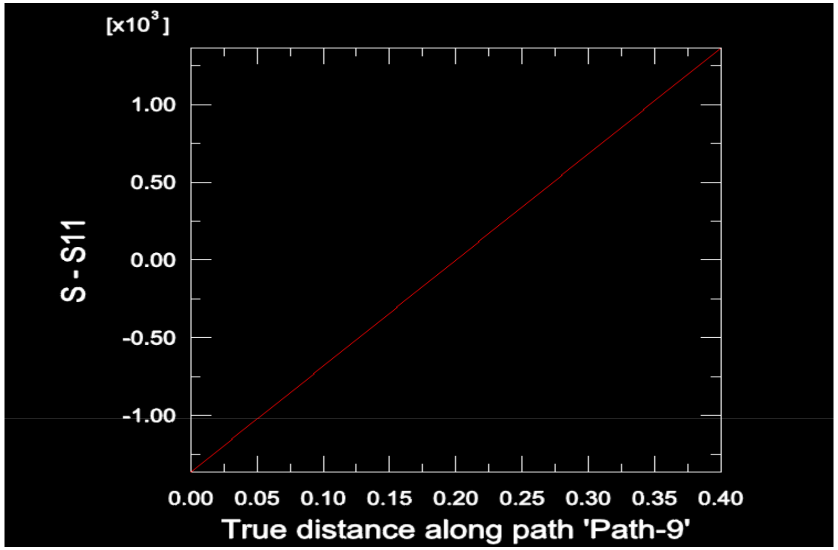
Case 2: $a = 0.6$ m
$b = 1.0$ m

Case 3: $a = 0.4$ m
$b = 0.8$ m

Case 4: $a = 0.2$ m
$b = 0.6$ m

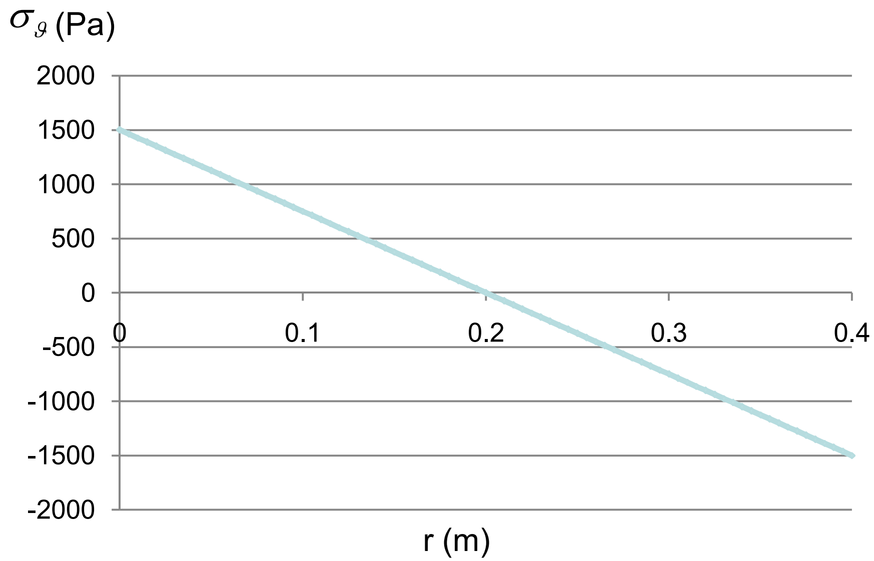
Straight Beam



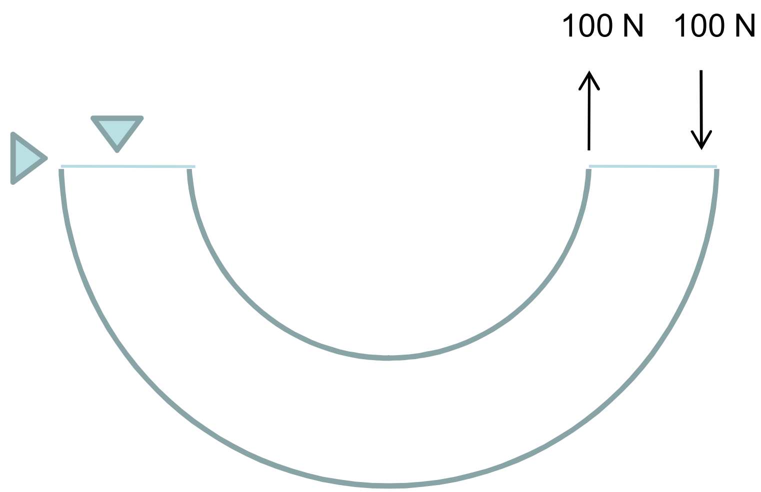


$$\sigma_\vartheta = \sigma_x = \frac{-M \times y}{I} = \frac{-100 \times y}{\frac{1}{12} \times 0.4^3}$$

- Neutral axis = 0.2 m (0.2 m)
- σ_ϑ along the bottom = -1368 Pa (-1500 Pa)
- σ_ϑ along the top = 1382 Pa (1500 Pa)



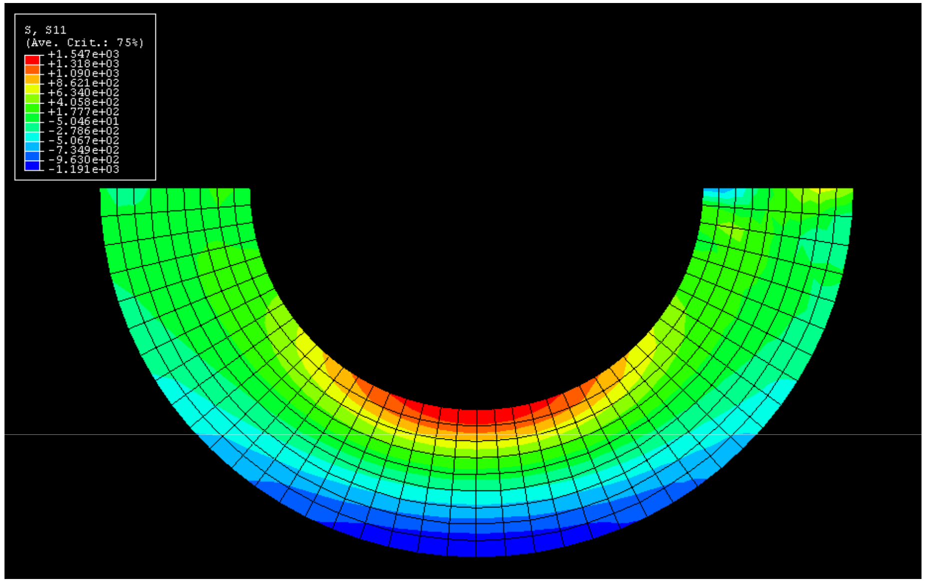
Things to keep in mind

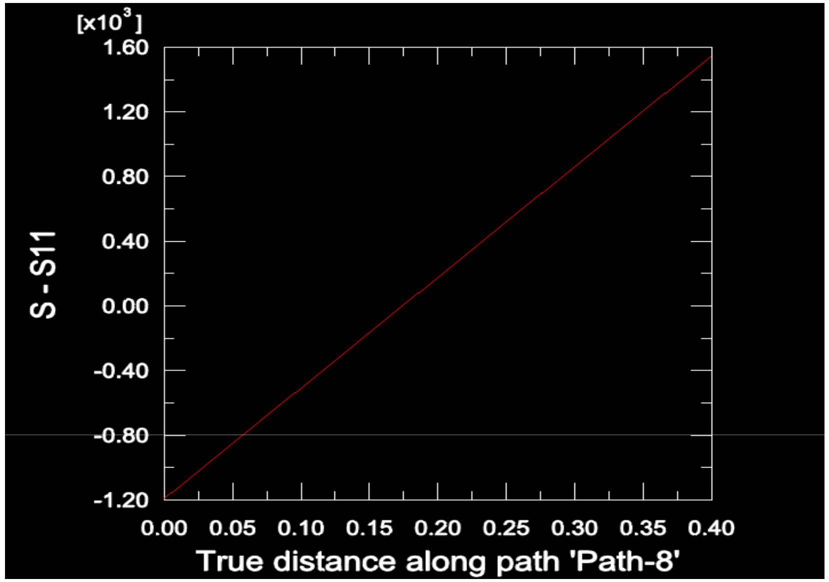


Material: Steel
Density = 7800 kg/m^3
Modulus = 2 x 10^11 Pa

- Point loads not as good as distributed loads and could is introducing some error
- Theory does not take into account bending due to the moment

Case 1

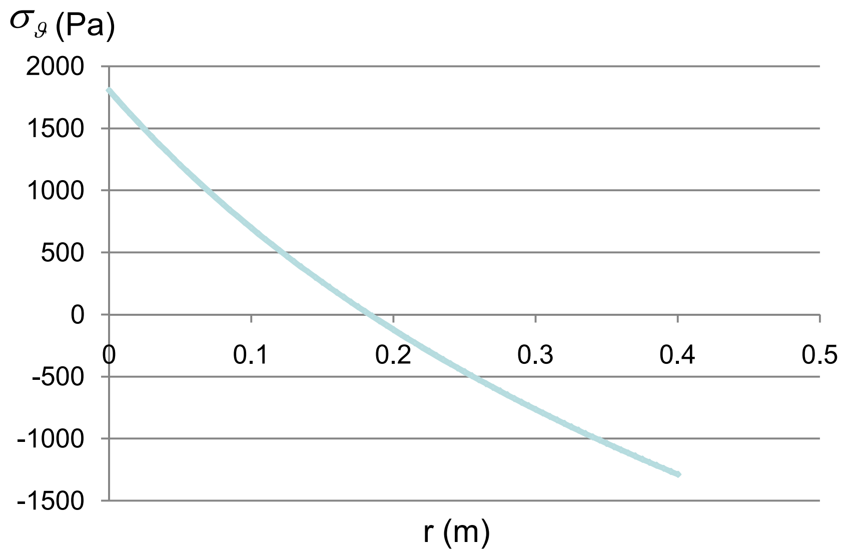




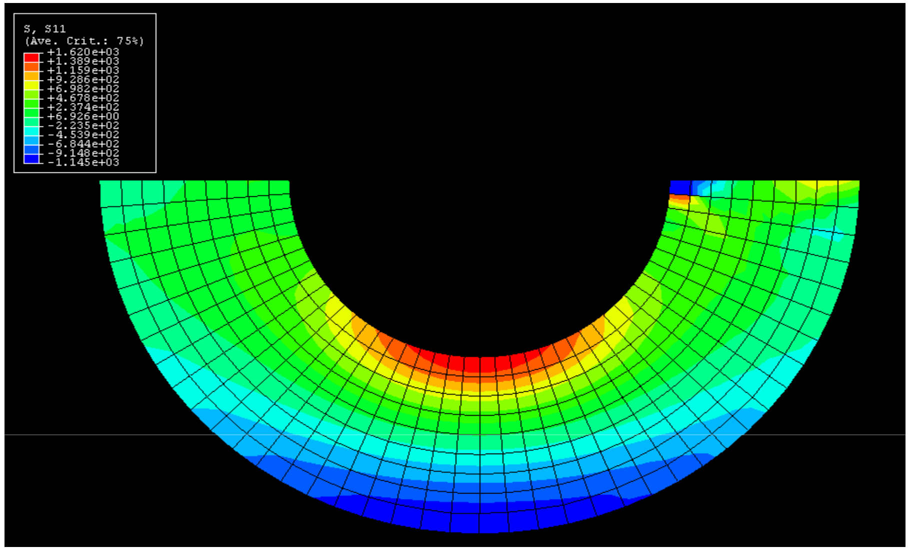
Neutral axis = 0.175 m (0.165 m)

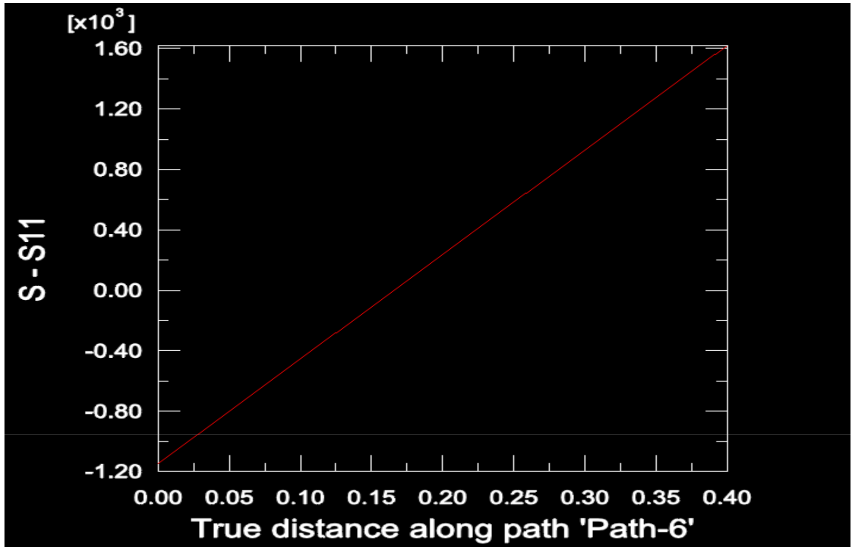
σ_ϑ along the bottom = -1200 Pa (-1287Pa)

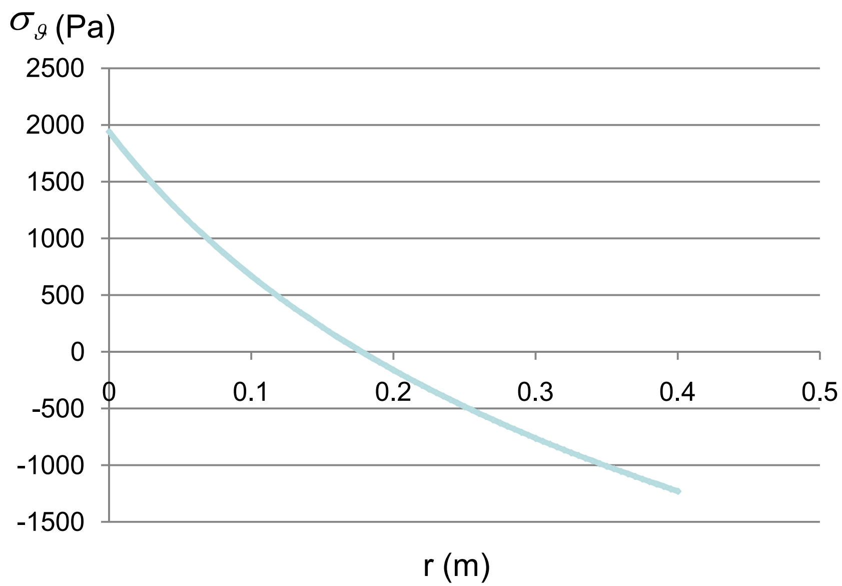
σ_ϑ along the top = 1540 Pa (1804 Pa)



Case 2





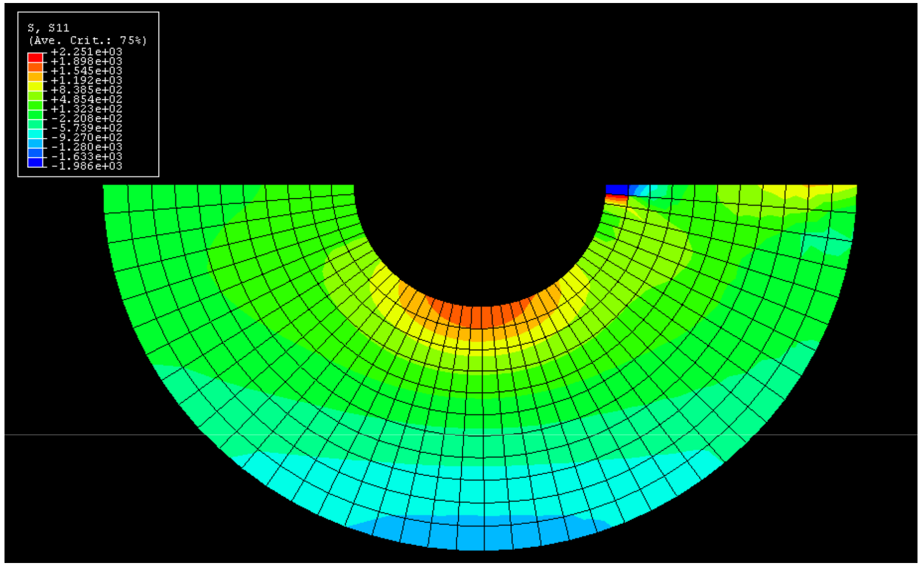


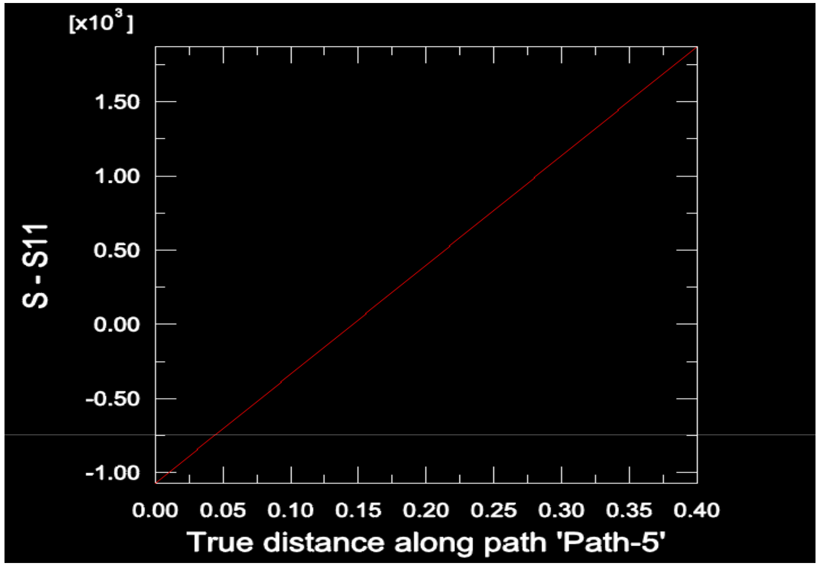
Neutral axis = 0.163 m (0.180 m)

σ_{ϑ} along the bottom = -1141 Pa (-1229Pa)

σ_{ϑ} along the top = 1600 Pa (1939 Pa)

Case 3

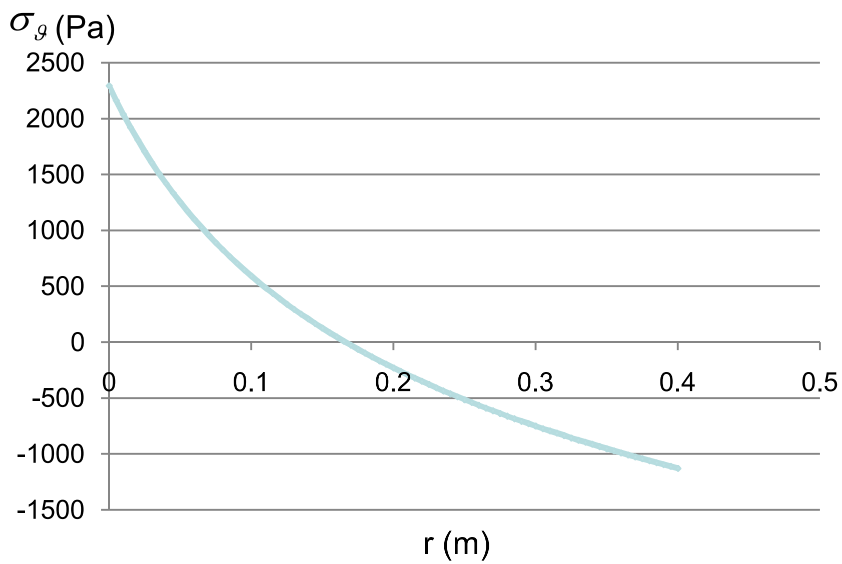




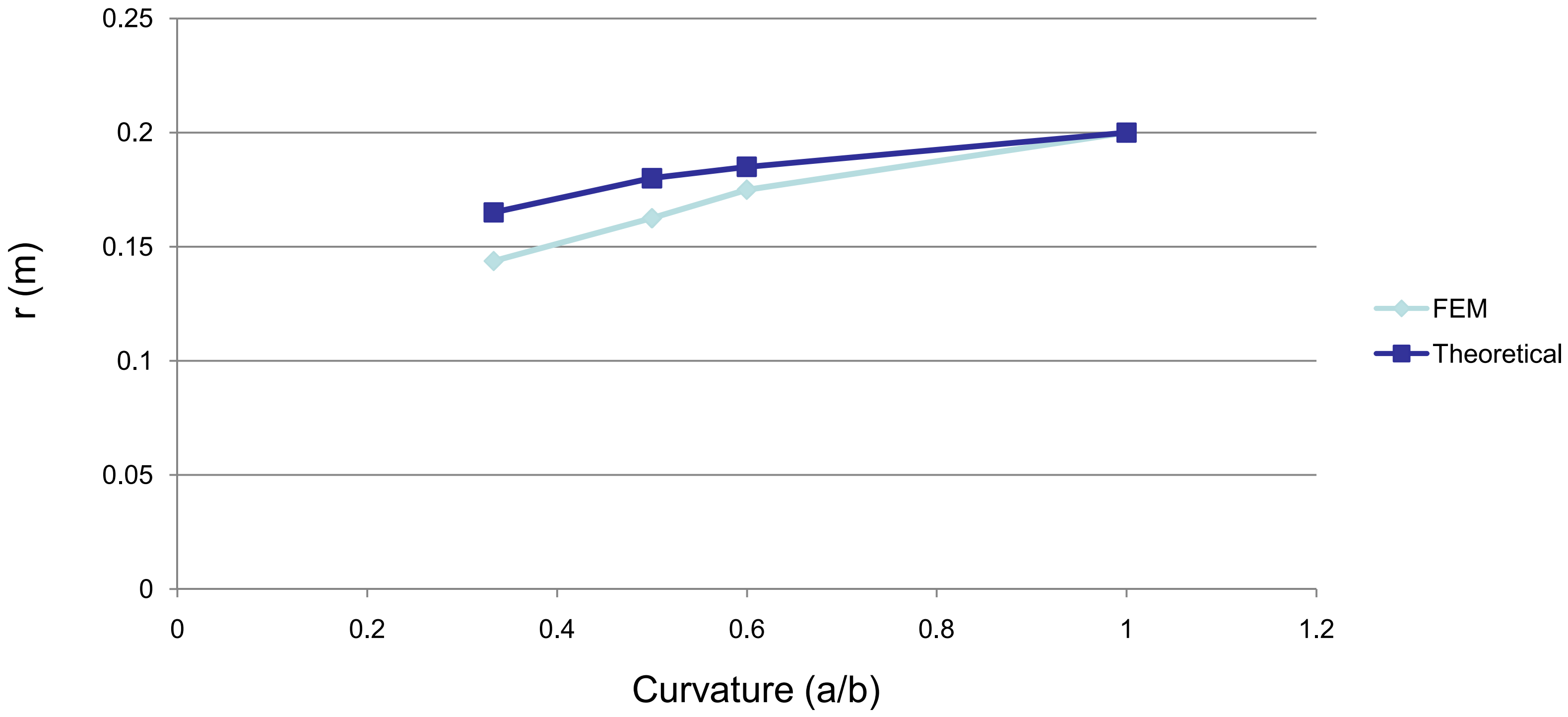
Neutral axis = 0.144 m (0.165 m)

σ_ϑ along the bottom = -1063 Pa (-1130Pa)

σ_ϑ along the top = 1875 Pa (2292 Pa)

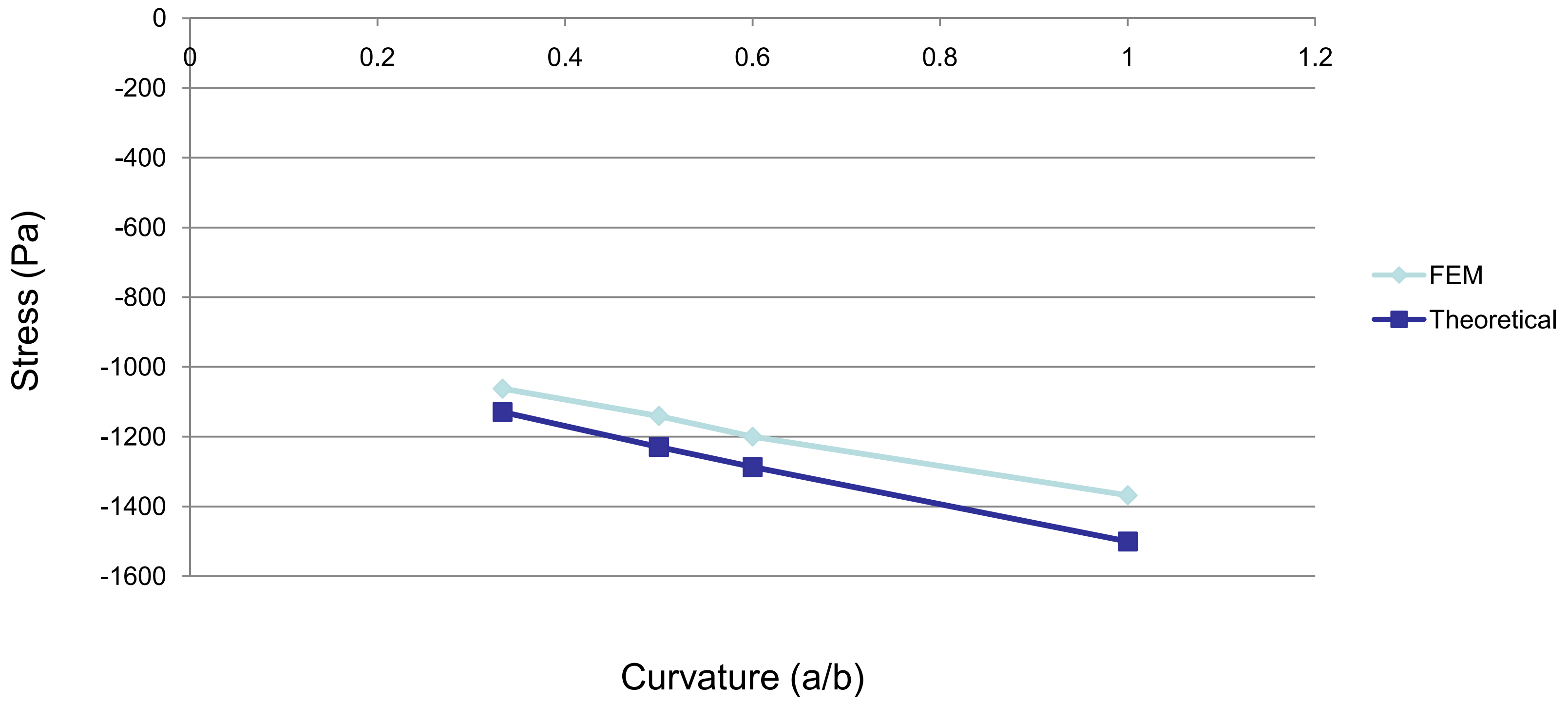


Placement of the neutral axis



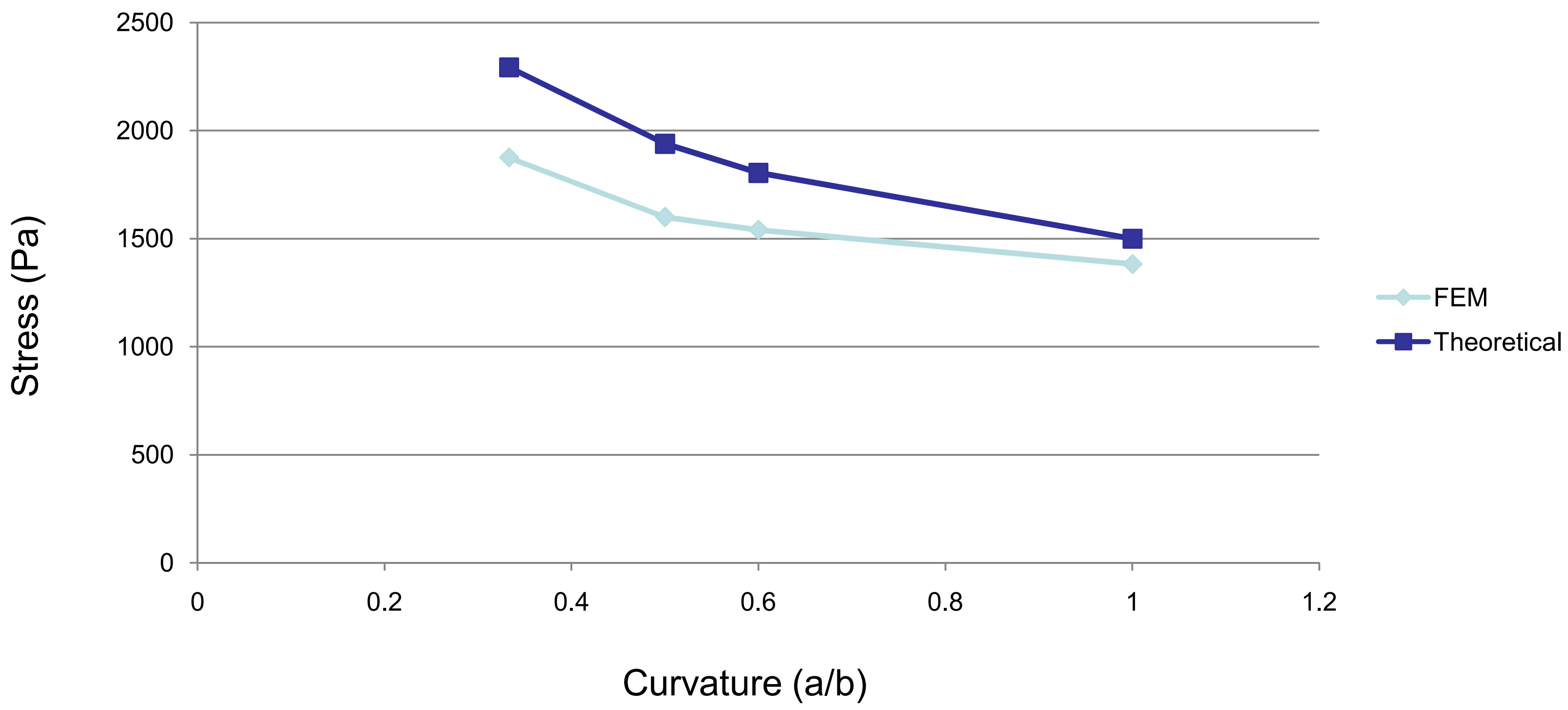
- As curvature increases the neutral axis shifts farther away from the center axis
- The results converge to the straight beam case

Magnitude of σ_ϑ along bottom



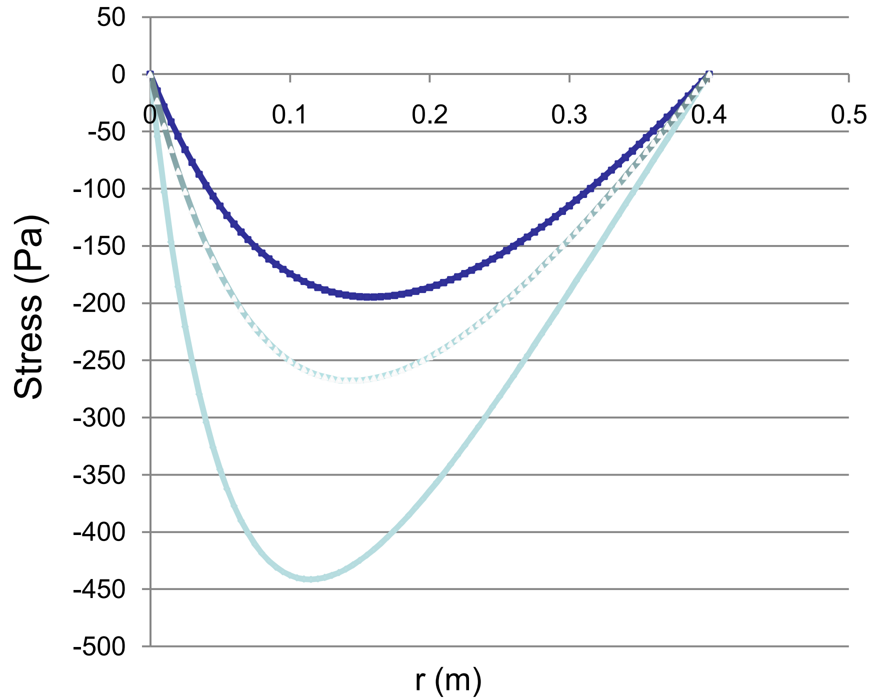
- As curvature increases σ_ϑ along the bottom decreases
- Stress highest for the straights beam

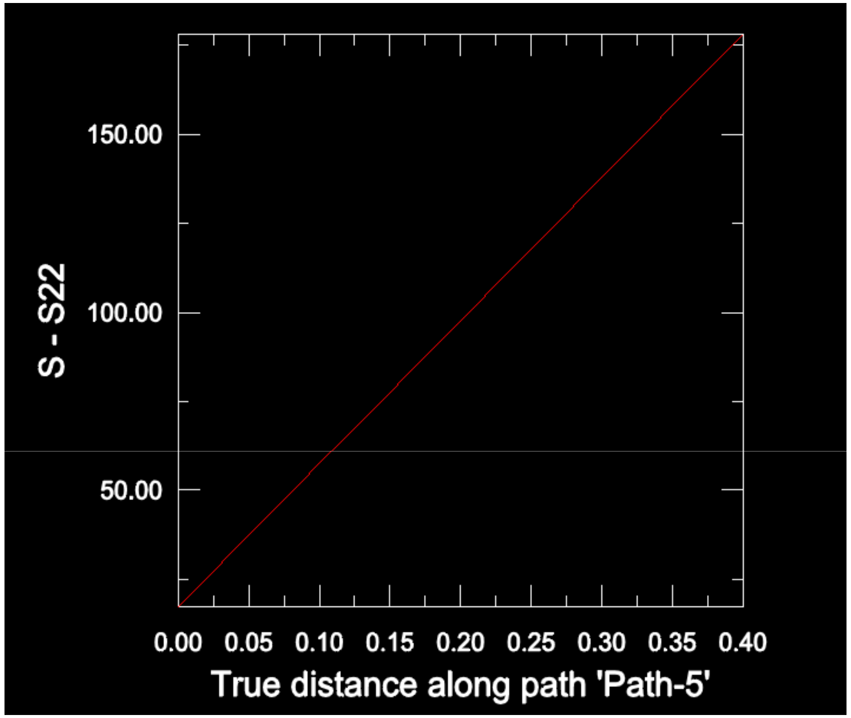
Magnitude of σ_ϑ along top



- As curvature increases σ_ϑ along the top increases and causes a stress concentration along this edge
- As the beam gets flatter it approaches the stress in a straight beam

Influence on σ_r .





- From top to bottom
  - Low curvature, medium curvature, high curvature

- σ_r is zero for a straight beam

- Order of magnitude lower than σ_ϑ

Conclusion

- Beams with low curvature are closely approximated by straight beam theory
- High curvature beams see large stresses in the theta direction on the inside edge
- The neutral axis shifts farther away from the center as curvature increases
- When designing hooks, chains, and arches this should be kept in mind

References

Haslach, H. Armstrong, R. ”Deformable Bodies and Their Material Behavior”. (2004). John Wiley & Sons: USA. pp. 125-127, 137-138, 183-187.