Model Equations for the Eiffel Tower Profile: Historical Perspective and a New Equation

Here is the French title

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Abstract

Equations modeling the shape of the Eiffel Tower are investigated. One model, based on equilibrium of moments, gives the wrong tower curvature. A second model, based on constancy of vertical axial stress, does provide a fair approximation to the tower's skyline profile of twenty-nine contiguous panels. However, neither model can be traced back to Eiffel's writings. Reported here is a new model embodying Eiffel's concern for wind loads on the tower, as documented in his communication to the French Civil Engineering Society on March 30, 1885. The result is a nonlinear, integro-differential equation solved to yield an exponential profile. An analysis of actual panel coordinates reveals a profile closely approximated by two piecewise continuous exponentials with different growth rates. This is explained by specific safety factors for wind loading that Eiffel & Company incorporated in the design of the free-standing tower. *To cite this article: P. Weidman, I. Pinelis, C. R. Mecanique 331 (2003).*

Résumé

Equations modeling the shape of the Eiffel Tower are investigated. One model, based on equilibrium of moments, gives the wrong tower curvature. A second model, based on constancy of vertical axial stress, does provide a fair approximation to the tower's skyline profile of twenty-nine contiguous panels. However, neither model can be traced back to Eiffel's writings. Reported here is a new model embodying Eiffel's concern for wind loads on the tower, as documented in his communication to the French Civil Engineering Society on March 30, 1885. The result is a nonlinear, integro-differential equation solved to yield an exponential profile. An analysis of actual panel coordinates reveals a profile closely approximated by two piecewise continuous exponentials with different growth rates. This is explained by specific safety factors for wind loading that Eiffel & Company incorporated in the design of the free-standing tower. *Pour citer cet article : P. Weidman, I. Pinelis, C. R. Mecanique 331 (2003).*

Key words: Eiffel tower ; model equations ; wind loads ; nonlinear integral equations

Mots-clés : Mot-clé1 de la liste ; Mot-clé2 ; Mot-clé3

1. Introduction

It is well known that the Eiffel Tower was not designed according to a mathematical formula. It was designed using graphical methods to construct a tower of sufficient strength to support its immense weight and empirical results garnered from past experience to account for wind loading. This notwithstanding, the Eiffel Tower is thought by many to be of exponential form.

The present investigation into model equations for the shape of the Eiffel Tower commenced in November 2001 when the lead author received a complimentary copy of the second edition of *Advanced Engineering Mathematics* [1]. On its cover are photographs of various stages of construction of the Eiffel Tower and the fronticepiece presents a nonlinear integral equation for the tower shape, advertised as the "Eiffel Tower Equation" on a website [2] run by Christophe and Geraldine Chouard. The equation had not been solved in closed form, and the Chouards offered a challenge to find the solution "written as a combination of usual functions" and report it to them. Though we found one solution, it does not conform to the shape of the Eiffel Tower. Following our failure to find any solution of the "Eiffel Tower Equation" having proper tower curvature, we questioned whether the assumptions on which the equation was formulated could be attributed to Eiffel, as claimed. Our expanded study lead to another popular model, but neither of the two models could be traced to the writings of Eiffel. Eventually, after translating some of Gustave Eiffel's orginal documents, we learned the basis for tower construction and developed a new equation for the skyline profile, one that embraces Eiffel's deep concern for the effects of wind loading on the tower.

This article documents our discovery with an historical perspective. Circumstances leading to the proposition of erecting a 300 m tower for the 1889 Exposition in Paris is given in Section 2 along with pertinent facts about the tower. In Section 3 two existing model equations, one linear and the other nonlinear, are reviewed and analyzed. In Section 4 an integro-differential equation is derived based on a communication by Eiffel to the French Society of Civil Engineers on March 30, 1885. The only relevant solution is exponential, justifying the lore promulgated by both lay and scientific persons. The work in the Abstract and Sections 1-4 is due to P. D. Weidman while the theorem in the Appendix is due to I. Pinelis.

2. Prelude and Facts

In a notice published in the government's *Journal Officiel* of May 2, 1886, French architects and engineers were invited to bid on plans to construct semi-permanent buildings for the 1889 exposition and, in particular, to consider "the possibility of erecting on the Champ de Mars an iron tower with a base of 125 meters square and 300 meters high," this height being the nearest round metric equivalent to 1000 feet. Ultimately, Gustave Eiffel's proposal for a tower of wrought iron weighing approximately 7000 tons, costing $1.6 million, was selected and the contract was signed on January 18, 1887. Eiffel & Company had already conceived and advertised the idea of constructing a 300 meter tower beginning with an original conceptual drawing by the company's engineers Emile Nouguier and Maurice Koechlin in 1884 shown in Fig. 1a; for comparison beside the tower are sketches of the Notre Dame, the Statue of Liberty, the Arc de Triomphe, three columns the height of the column in Place Vendôme, and a six-story apartment building. The shape and structure underwent numerous modifications by Gustave Eiffel and architect Stephane Sauvestre toward the final design shown in Fig. 1b. In particular, the 40 panels exhibited in Fig. 1a were pared down to the 29 panels seen in Fig. 1b. The construction lasting two years, two months and five days was completed on March 31, 1889 — only a month before the May 5 opening of the Exposition.

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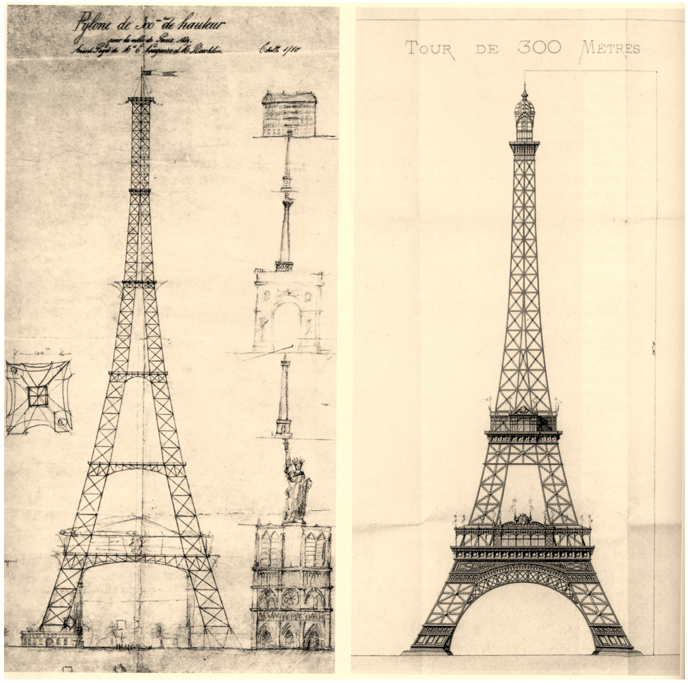
(a)



(b)

Figure 1. (a) Pylon 300 meters high for the city of Paris, 1889, a preliminary concept by Mssrs. E. Nouguier and M. Koechlin, June 6, 1884 and (b) final design of the 300 meter tower sketched by G. Eiffel given as an attachment to the contract signed on January 18, 1887. Taken from Heinle [3].

An engraving by Deroy [4] published in 1889 juxtaposes the 300 m tower with thirty-four other high towers in existence at the time. The dominance of the Eiffel Tower over all structures including its nearest rival, the all masonry 169 m Washington Monument, is remarkable. Apart from the viewing platforms, Eiffel's study at the top, the summit dome and various antennae for civil and national communication, the skyline profile is determined by the location of four sets of 29 panels symmetrically placed on each side of the tower. The panels are numbered from bottom to top, with heights, inclinations, and profile coordinates given in Table 1.

Table 1. Panel heights, inclinations, and profile coordinates derived therefrom.

Panel	Angle	h (m)	x (m)	w (m)	Panel	Angle	h (m)	x (m)	w (m)
29	$90^\circ\,00'\,00''$	11.280	0.000	5.000	14	$84^\circ\,06'\,24''$	10.600	128.076	11.786
28	$87^\circ\,12'\,31''$	5.833	11.280	5.000	13	$82^\circ\,25'\,29''$	10.700	138.676	12.880
27	$87^\circ\,12'\,31''$	6.165	17.113	5.284	12	$82^\circ\,25'\,29''$	11.300	149.376	14.303
26	$87^\circ\,12'\,31''$	6.517	23.278	5.585	11	$77^\circ\,07'\,52''$	4.900	160.676	15.806
25	$87^\circ\,12'\,31''$	6.888	29.795	5.903	10	$77^\circ\,07'\,52''$	10.000	165.576	16.925
24	$87^\circ\,12'\,31''$	7.280	36.683	6.239	9	$76^\circ\,54'\,10''$	10.200	175.576	19.210
23	$87^\circ\,12'\,31''$	7.695	43.963	6.594	8	$75^\circ\,48'\,33''$	11.000	185.776	21.583
22	$87^\circ\,12'\,31''$	8.133	51.658	6.969	7	$74^\circ\,30'\,10''$	11.000	196.776	24.365
21	$87^\circ\,12'\,31''$	8.596	59.791	7.365	6	$72^\circ\,19'\,36''$	11.000	207.776	27.415
20	$87^\circ\,12'\,31''$	9.086	68.387	7.784	5	$68^\circ\,21'\,27''$	7.000	218.776	30.919
19	$87^\circ\,12'\,31''$	9.603	77.473	8.227	4	$65^\circ\,48'\,48''$	11.000	225.776	33.697
18	$86^\circ\,51'\,24''$	10.000	87.076	8.696	3	$65^\circ\,48'\,48''$	11.000	236.776	38.637
17	$85^\circ\,41'\,18''$	10.000	97.076	9.245	2	$65^\circ\,48'\,48''$	11.000	247.776	43.578
16	$85^\circ\,41'\,18''$	10.500	107.076	9.999	1	$65^\circ\,48'\,48''$	11.000	258.776	48.519
15	$84^\circ\,35'\,02''$	10.500	117.576	10.790	0	—	—	269.776	53.459

In this study, x in Table 1 is taken as the downward coordinate from the top of the 29^{th} panel and w is the local tower half width. Note there are five sections of the tower having two or more consecutive panels of equal inclination, so that the entire polygon contains only fourteen sections differently inclined. The legs of the free-standing tower are supported from below by four huge caissons and the tower is held in place by four *ceintures*, or structural belts, at various heights. The first *ceinture* is the large restaurant and viewing platform at 91 m above ground; the second is the mid-level viewing platform at 149 m; the third is an intermediate platform at 228 m, and the fourth is the top viewing platform at 309 m.

3. Existing Mathematical Models

Clearly, any mathematical equation for the tower profile will necessarily be some approximation to its true convex polygon shape. However, that does not prevent interested persons from seeking an analytical model that might elucidate some basic physics of tower construction. Although the exterior profile is relatively smooth and elegant, the internal tower structure consists of a three-level hierarchy of iron girders, trusses and struts which many Parisians during the time of construction considered to be *très très gauche*. The first approximation for any simple model is to assume that the tower is composed of material of uniform density ρ. Lakes [6] has calculated this material would have a density $\rho = 1.2\times10^{-3}\rho_0$, where ρ_0 is the density of iron. We estimate this to be about one-tenth the density of the lightest balsa wood.

3.1. *A Website Equation*

Logging onto the Chouard's website one finds the opening sentence [2]:
"Gustave Eiffel was proud of his good-looking Tower whose shape resulted from mathematical calculation, as he said. 'At any height on the Tower, the moment of the weight of the higher part of the Tower, up to the top, is equal to the moment of the strongest wind on this same part.' Writing the differential equation of this equilibrium allows us to find the 'harmonious equation' that describes the shape of the Tower."
This is followed by a presentation of the nonlinear integral equation

$$a f(x) \int_0^x f^2(t)dt = x \int_0^x f(t)dt - \int_0^x t f(t)dt \tag{1}$$

where $f(x)$ is the tower half width, x the distance from the top, and a is a constant.
The derivation of Eq. (1) follows readily from the website sketch reproduced here in Fig. 2.

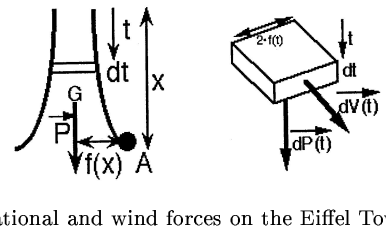
(a)  (b)

Figure 2. Schematic of gravitational and wind forces on the Eiffel Tower according to Chouard [2].

Fig. 2a shows the downward coordinate x to tower level A, the tower half width $f(x)$, the weight P of the tower above level A, and downward coordinate t to a horizontal section of thickness dt. The tower is assumed to be constructed of material of uniform specific weight k. The element of horizontal section in Fig. 2b has weight $dP(t) = 4\,k\,f^2(t)dt$ and is acted upon by a horizontal wind force $dV(t) = 2\,K\,f(t)dt$

where K is a constant. Thus element contributions to the moment about point A are $dP(t)f(x)$ counter-clockwise and $dV(t)\,(x-t)$ clockwise. Equating the resultant of these two moments for that part of the tower above point A yields

$$f(x)\int_0^x 4kf^2(t)dt = \int_0^x 2Kf(t)(x-t)dt. \tag{2}$$

Defininig $a = 2k/K$ gives Eq. (1). Note the tacit assumption that the wind results in a uniform pressure over the face of the tower, which therefore produces a force proportional to the area of that face projected on a vertical plane.

We now seek solutions that yield the tower profile. Substituting power law solution Ax^α into Eq. (1) shows that it may be satisfied for proper choice of A and α, the result being

$$f(x) = \sqrt{\frac{8}{15a}}\,x^{1/2}. \tag{3}$$

This solution cannot describe the Eiffel Tower profile for the simple fact that it gives a concave shape, while the tower is convex.

Since Eq. (1) is nonlinear, other solutions may exist. We explore this possibility by analyzing the differential analog of the integral equation. Since the constant a may be removed by an affine transformation, we set $a = 1$ without loss of generality. Next, define the volume variable

$$y(x) = \int_0^x f^2(t)dt \tag{4}$$

and insert into Eq. (1) twice differentiated to obtain

$$2yy'y''' - yy''^{\,2} + 8y'^{\,2}y'' = 4y'^{\,2}. \tag{5}$$

A series of transformations are used to try to identify a special solution. Indeed,

$$y' = y^{1/2}g^{2/3}(z), \quad z = \ln y; \qquad g' = v(g);$$

reduces (5) to the nonlinear first-order equation for $v(g)$, namely

$$vv' + \frac{9}{2}v = 3g^{-1/3} - \frac{45}{16}g. \tag{6}$$

The goal to see if a Bernoulli, Ricatti or other special nonlinear equation might appear was unsuccessful, so we suspect there is no closed form solution of (6), other than that given by Eq. (3) which has the incorrect curvature. An analysis given in the Appendix shows that no solution $y(x)$ of the differential equation (5) corresponds to a profile function $f(x)$ which possesses the monotonicity and curvature properties of the actual profile of the tower.

An extensive investigation by the lead author, many details of which will be provided in Section 4, has failed to uncover any claim that Eiffel designed his tower based on an equilibrium of moments.

3.2. *A Popular Model*

A model often cited to explain the shape of the Eiffel Tower is predicated on a uniform compressive stress at every tower elevation. The derivation below follows the notation of Banks [7] wherein y is the vertical coordinate from ground level, x the tower half width, $A(y)$ the cross-sectional area of the tower, $\sigma(y)$ the vertical compressive stress, ρ the uniform material density, and g is gravity. Wind loading is not a component of this model. A balance of vertical forces in the free-body diagram for a horizontal section dy of the tower yields

$$\rho g A = -\frac{d}{dy}(\sigma A) \tag{7}$$

as the condition for vertical equilibrium. At this juncture Banks [7] states: “For reasons of safety, it is necessary to keep the compressive stress, σ, a constant.”

Then, writing (7) as

$$2\beta A = -\frac{dA}{dy} \tag{8}$$

where $\beta = \rho g/2\sigma$, and integrating, one obtains the vertical distribution of cross-sectional area

$$A = A_0 e^{-2\beta y}. \tag{9}$$

Since $A = (2x)^2$, the tower profile is

$$x = x_0 \; e^{-\beta y} \tag{10}$$

where $x_0 = \sqrt{A_0}/2$ is the tower half-width at ground level.

Thus the tower is infinitely high and spraddles out exponentially from the top down. In contrast to the result in Section 3.1, this solution exhibits proper tower curvature. A least-squares fit of exponential form $Ae^{\gamma x}$ to the tower coordinates listed in Table 1 is provided in Fig. 3, with the values of A, γ, and the coefficient of determination R^2 given in Table 2 of §4.1. A generalization of this analysis is given by

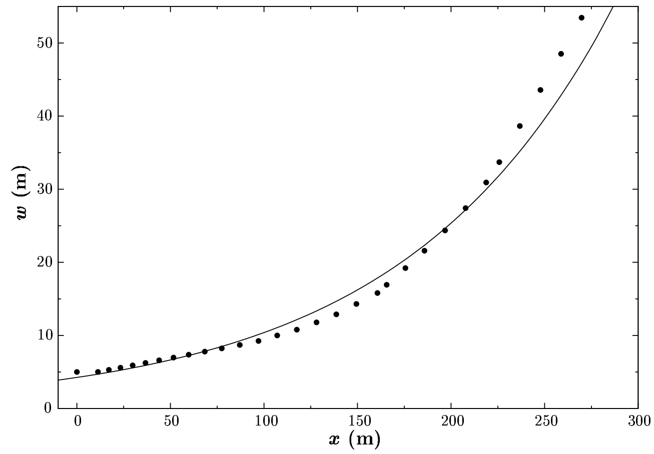


Figure 3. Least-squares exponential fit to the tower coordinates given in Table 1.

Puig-Adam [8] who considered the same problem with a weight placed on top of the tower; in this case the uniform compressive stress is maintained for an exponential profile that has finite width at the top where the weight is supported.

Although this solution gives a reasonable approximation to the tower shape, we are unable to find any documentation showing that Eiffel & Company designed the tower to have a height-independent axial compressive stress, with or without a fixed weight at the summit.

4. A New Model Equation

In his autobiography, Eiffel states the problems of wind resistance had been encountered for a long time in the construction of large-scale metal structures [9]:

“In the design of these, through lack of sufficient knowledge of the complex forces exerted by the wind, their builders were reduced to including in their calculations safety coefficients which had no scientific basis.”

At that time none of the great problems concerning wind loads on a tall structure were understood [10]:

"Does the pressure increase or decrease with surface area? What is the pressure on oblique planes? Where is the center of pressure and how is it displaced?"

Further concern for the effects of wind loading is found in an interview with the French newspaper *Le Temps* on February 14, 1887 in which Eiffel was quoted as saying [11]:

"What is the main obstacle I had to overcome in designing the tower? Its resistance to wind. And I submit that the curves of its four piers as produced by our calculations, rising from an enormous base and narrowing toward the top, will give a great impression of strength and beauty."

So what guided Eiffel & Company in the design of the free-standing tower? How much reliance was given to their past experience in the fabrication of viaduct supports for constructing a 300 meter tower? Was there some underlying physics that gave rise to the tower profile? Answers to these questions come to light in the *mêmoire* Eiffel communicated to the French Society of Civil Engineers on March 30, 1885 under the title [12]: *Projet d'une Tour en Fer de 300 Mêtres de Hauteur Destinée a L'Exposition de 1889.*

For three decades Eiffel & Company designed numerous bridges throughout greater Europe, in the French colonies, and elsewhere. In spite of their lightweight appearance, they were known to withstand large loads and experiments were performed to advertise their structural integrity. In an experiment carried out by Baron Saladin on his estate at Bossancourt, a four ton single-axle cart crossed over his bridge in the presence of the regional representative of the Highways Department when it was established that the maximum bridge deflection was 18 mm [13].

Germane to our discussion is the fact that all pier supports for viaducts and bridges constructed by Eiffel & Company had three elements in common: (i) the sides of the supports were for the most part straight from foundation to the top; (ii) each face was composed of horizontal stiffeners for rigidity and diagonal trellis bars to resist wind load forces; and (iii) the top was affixed to a horizontal bridge or viaduct. Now for the second time, the first being the complex inner structure of the Statue of Liberty, Eiffel & Company was faced with the construction of a free-standing tower which, because of its severe height, would have to withstand unknown wind forces.

After introductory remarks and an acknowledment of his collaborators Nouguier, Koechlin and Sauvestre, Eiffel states in §1 of his communication [14]:

"If, on the contrary, we are dealing with a very high pier such as our tower, where there is no longer any horizontal wind stress on the deck at the top, but only wind stress on the pier itself, things are different, and it is enough, in order to eliminate the use of the trellis members, to give the uprights a curve such that the tangents to the uprights, brought to points located at the same height, always meet at the crossing point of the resultant of the stress exerted by the wind on the section of the pier above the points being discussed."

This is sufficient, along with Eiffel's assumption that the effect of the wind may be estimated as a uniform pressure acting on the tower, to formulate a mathematical equation for the tower profile. But what was the motivation for such a statement? The answer appears in §3 where Eiffel writes [15]:

"I arrive now at the conditions of resistance: ...

Let us suppose, for a moment, that now we have laid out on the faces of a simple truss forming a resisting wall, the shearing forces of the wind, the horizontal components of which are:

$$P',\ P'',\ P''',\ P^{iv}.$$

One knows that in order to calculate the forces acting on the three pieces cut by the plane MN, we need to determine the resultant P of all exterior forces acting above the section, and to decompose this resultant into three forces passing through the cut pieces.

If the shape of the system is such that, for each horizontal cut MN, the two extended truss frames intersect on the exterior force P, the forces in the lattice bar will be zero and we will be able to exclude this member."

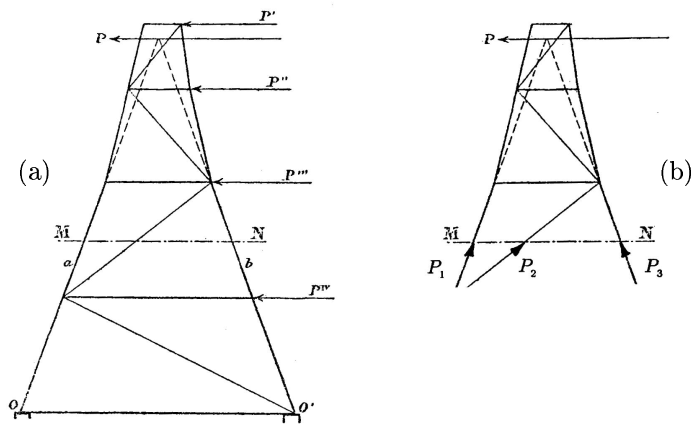


Figure 4. Original sketch (a) of a simple truss form by Eiffel [16]; our annotated sketch (b) of the free-body diagram showing the resultant forces P_1, P_2 and P_3 acting along the structural elements cut by section MN.

Eiffel offered no equations to confirm that the force in the cut trellis bar in Fig. 4a is zero, probably because it was self-evident to the civil engineers attending the presentation. Reference to the free-body diagram in Figure 4b readily confirms the accuracy of his statement. Forces on the structure above section MN are resolved into the resultant horizontal wind force P acting through the apex formed by the upward extension (dashed lines) of opposing uprights, forces P_1 and P_3 acting through those members, and the force P_2 acting along the lattice bar. The condition for rotational equilibrium is that the sum of the moments about any fixed point must vanish. Since P, P_1 and P_3 all pass through the apex, the moment about that point has a contribution only from P_2, which therefore must be zero. Thus Eiffel discovered a method of construction which could withstand wind loads without the aid of lattice bars. This form has the twofold benefit of reducing the tower weight and offering less surface area to the wind. Eiffel was very proud of this fact for in §3 he continues [16]:

"It is the application of this principle which constitutes one of the particularities of our system, and that we believe interesting to signal to the attention of the Society.

One arrives in this manner that the direction of each of the elements of the sides will result in a curve following that traced on the sketch (figure 1, plate 91), and in reality the exterior curve of the tower reproduces, at a determined scale, the same curve of the moments produced by the wind."

The statement that the tower's profile conforms to the moment distribution wrought by the wind was given without justification; we will return to this point shortly. For now, however, the mathematical model is determined with the aid of Fig. 5.

For a smooth skyline profile, the resultant wind force P in Fig. 5 would act at the centroid $\overline{x}$ of the covered surface above the tangency points C projected on the vertical plane normal to a horizontal wind. In one model (see §4.1), Eiffel assumed a uniform wind would impart a uniform stress loading on the face exposed to the wind. We retain the notation of Section 3.1 that $f(x)$ is the tower half-width and x is the downward coordinate from the uppermost panel of half width $f(x_0) = 5$ m. The centroid of the tower rising above section C-C is given by

$$\overline{x} = \frac{\int_{x_0}^{x} t f(t) dt}{\int_{x_0}^{x} f(t) dt}. \tag{11}$$

Eiffel's statement that tangents at C intersect at $\overline{x}$ yields the equation

$$f(x) = f'(x)(x - \overline{x}) \tag{12}$$

for the right tangent line in Fig. 5.

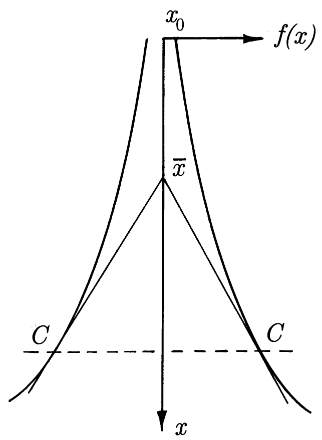


Figure 5. Schematic of the coordinates $f(x)$ used for the Eiffel Tower profile showing the initial coordinate x_0 and the centroid of projected surface area $\bar{x}$ for that part of the tower above section C-C.

Combining (11) and (12) furnishes the nonlinear integro-differential equation

$$f(x)\int_{x_0}^{x} f(t)\,dt = f'(x)\int_{x_0}^{x}(x-t)f(t)\,dt \tag{13}$$

which may be considered the continuous model for the skyline profile of the Eiffel Tower that embodies Eiffel's concern for wind resistance. To obtain the differential analog of (13) we introduce the area variable

$$y(x) = \int_{x_0}^{x} f(t)dt \tag{14}$$

and differentiate (13) to obtain

$$f(x)y'(x) = f''(x)\left[xy(x) - \int_{x_0}^{x} tf(t)dt\right]. \tag{15}$$

Elimination of the common integral appearing in (13) and (15) furnishes the nonlinear differential equation

$$yy''' = y'y''. \tag{16}$$

Dividing both sides by yy'', assuming for the moment y'' is everywhere nonzero, and integrating yields the second-order linear equation

$$y'' \pm \gamma^2 y = 0 \tag{17}$$

for positive constant γ.

For $\gamma^2 = 0$ the solution $y = Ax + B$ yields $f(x) = A$; this constant-width solution does not satisfy the original integral equation (13) and is therefore discounted. The trigonometric solution for the positive sign in (17) leads to $f(x) = A\sin\psi$ where $\psi = \alpha x + \phi$; inserting this result into (13) reveals that the only solution is the trivial solution $f(x) = 0$. Finally, for the positive sign in (17) one obtains the tower shape $f(x) = Ae^{\gamma x} + Be^{-\gamma x}$; in this case it is readily shown that the only solution of (13) is the one for which $B = 0$ and $x_0 = -\infty$. Thus the only solution satisfying the nonlinear integral differential equation, based on an analysis of its differential analogue, is

$$f(x) = Ae^{\gamma x}. \tag{18}$$

The solution is consistent with the assumption y'' is nonzero for all finite values of x. Note that solution (10) is identical to (18), but whereas the former models a tower with constant axial stress due to its weight, the latter has nothing whatsoever to do with the tower weight.

Of course the tower defined by the panels is not infinitely tall; as shown in Table 2, the panels terminate at $x = 0$ where the tower is 10 m wide. The top panel is not the true summit, however, although it does support the fourth *ceinture* that serves as the uppermost viewing platform. A major justification for building the iron structure was that its high elevation would provide an ideal location for a meteorological laboratory to record wind speed and direction, air temperature and humidity, and rainfall accumulation. In fact, the original design included provision for a comfortable room, centrally positioned on the top platform, in which Eiffel could carry out his scientific observations.

We now turn our attention to Eiffel's statement that the tower would take the same shape, within a "determined scale" as the moment distribution wrought by the wind. For Eiffel's assumed uniform wind stress denoted here as p_0, the wind moment at location x is given by

$$\begin{aligned} M(x) &= \int_{x_0}^{x} (x-t)\, p_0\, 2\, f(t)\, dt \\ &= 2p_0 \left[x \int_{x_0}^{x} f(t)dt - \left[t \int_{x_0}^{t} f(\xi)d\xi \right]_{x_0}^{x} + \int_{x_0}^{x} \left(\int_{x_0}^{t} f(\xi)d\xi \right) dt \right] \\ &= 2p_0 \int_{x_0}^{x} \int_{x_0}^{t} f(\xi)\, d\xi\, dt \end{aligned}$$

where integration by parts has been used. Thus any tower shape $f(t)$ that reproduces itself in two integrations is a candidate for Eiffel's claim. Indeed, for the exponential profile given in Eq. (18) valid for $x_0 = -\infty$, one finds

$$M(x) = \left(\frac{2p_0}{\gamma^2} \right) A e^{\gamma x} \tag{19}$$

showing that the scale relating the wind moment distribution to the tower shape is exactly $2p_0/\gamma^2$.

4.1. *Analysis of the Tower Shape*

We have seen in Fig. 3 that an exponential fit to the tower's skyline profile is not especially good. The origin of this discrepancy lies in the liberal safety factors built into the lower part of the tower. Eiffel & Company were well aware that the wind load on a viaduct proper was much larger than on its supporting piers and, by analogy, the dense metalwork of the expansive first and second level observation decks would present a large resistance to the wind. The solution to this problem is found near the end of §3 of the *mêmoire* where Eiffel writes [17]:

"As for the intensity, we have admitted two hypotheses: the first supposes that the wind over the whole height of the tower results in a constant force of 300 kilograms per meter squared; the other is that this intensity grows from the base, where it is 200 kilograms, to the summit, where it attains 400 kilograms.

As for the exposed surfaces, we have not hesitated, in spite of its apparent exaggeration, to admit the hypothesis that, on the upper half of the tower, the entire trellis structure was replaced by plain walls; that on the intermediate part, where the voids take on more importance, each original face was taken to be four times the surface of real iron; below (the first stage gallery and parts above the arcs), we have taken the exterior surface as uniform walls; finally, at the base of the tower, we have taken the uprights as uniform surfaces hit two times by the wind.

These hypotheses are more favorable compared to those that are generally adopted for viaducts."

Clearly, the lower tower section was handled with special care, since it supports the largest wind load moments. Concern for wind loads on the upper tower section was taken into account by assuming the surface to be uniformly covered, thereby taking the full force of the wind. This being the condition for our continuous model, solution (18) should provide the correct upper half tower profile for fitted values of A

and γ. The caveat, of course, is that panels 19-28 are all precisely slanted to the same $87° \, 12' \, 31''$. Eiffel & Company seems to have balanced simplicity of construction with aesthetics: there is little discernable loss of beauty, in the eyes of a beholder at ground level, in viewing a section of ten uniform, steeply-inclined panels near the top.

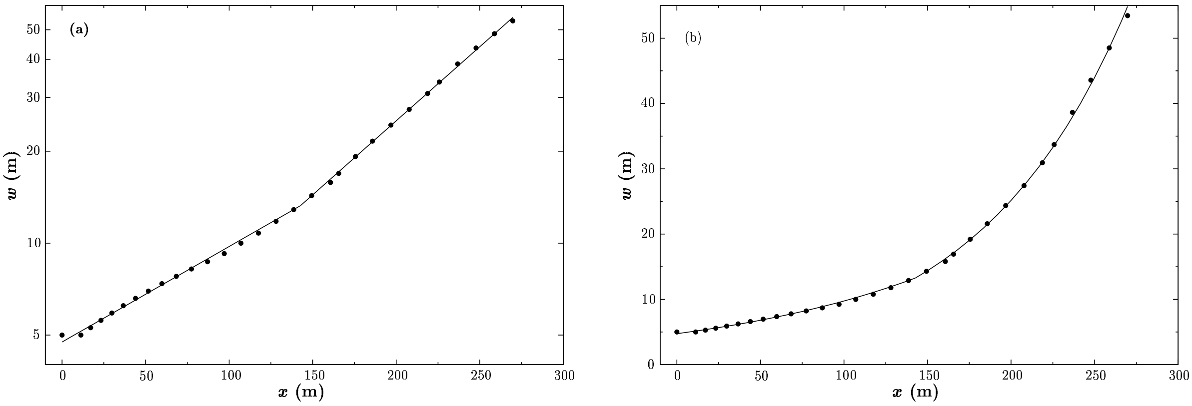


Figure 6. Composite (a) linear-log and (b) linear-linear fits to the Tower coordinates.

An analysis of the linear-log plot of panel coordinates in Fig. 6a reveals two exponentials. The solid lines are fits of the form (18) to overlapping lower (panels 1-13) and upper (panels 12-29) tower sections; the values A, γ, and coefficients of determination R^2 are given in Table 2.

Table 2. Constants A, exponents γ and coefficients of determination R^2 for least-squares exponential fits of the form (18) to the entire tower and overlapping portions of the upper and lower halves.

Figure No.	**Tower Section**	A	γ	R^2
Fig. 3	entire (panels 1-29)	4.2547	0.00892	0.9932
Fig. 6	upper (panels 1-13)	4.7439	0.00721	0.9978
Fig. 6	lower (panels 12-29)	2.6958	0.01117	0.9996

The fitted curves intersect at $x = 142.7$ m, near the mid-point of panel 13 just above the second observation deck. This is very close to the position $x = 140.4$ m shown in figure 1 on Plate 91 of Eiffel's *mêmoire* where the tower surface area first becomes exaggerated for reasons of safety.

In spite of the straight section composed of ten contiguous panels, the upper half of the tower is approximately exponential in agreement with our model. The appearance of a second exponential for the lower half of the tower, however, must be considered fortuitous. The agreement between fitted shapes and the actual tower coordinates plotted on a linear scale in Fig. 6b is remarkably good. We therefore cannot refrain from making the following observation: While events of the French Revolution are captured by Charles Dickens in his poignant novel *A Tale of Two Cities*, the centennial of the French Revolution is commemorated by Eiffel's graceful tower, the skyline profile of which is *A Tail of Two Exponentials.*

In conclusion, our study reveals that the tower design was not predicated on an equilibrium of moments nor the constancy of axial stress. It evolved out of Eiffel's respect for wind loading which could be reduced through a structural design eliminating the trellis bars previously used on straight-sided piers supporting large viaducts. Indeed, Eiffel infers that his design is a product of Nature when in §10 of his *mêmoire* he sates [18]: "Before they meet at such an impressive height, the uprights appear to spring out of the ground, molded in a way by the action of the wind itself."

Appendix: A proof concerning the curvature of $f(x)$

Assume that $x_0 \in [-\infty, \infty)$, $f(x) > 0$ and $y(x) = \int_{x_0}^{x} f(u)^2 du < \infty$ for all $x > x_0$, and y satisfies the differential equation (equivalent to (5))

$$y \cdot (2y'y''' - y''^2) = 4y'^2(1 - 2y'') \tag{A.1}$$

on the interval (x_0, ∞).

Theorem 1. *Suppose that the half-width $f(x)$ of the tower monotonically increases from the top to the bottom, that is, when x increases from x_0 to ∞. Then $f'' < 0$ on (x_0, ∞), so that the function f is everywhere concave, which is the shape opposite to the actual shape of the tower.*

Proof. Note that $y > 0$ and $y' = f^2 > 0$ on (x_0, ∞). Also, $y' = f^2$ implies $f = (y')^{1/2}$, and so,

$$4f'' = (y')^{-3/2}(2y'y''' - y''^2). \tag{A.2}$$

Comparing equations (A.1) and (A.2), we see for any $x \in (x_0, \infty)$ that $f''(x) < 0$ if and only if $y''(x) > 1/2$, and $f''(x) = 0$ if and only if $y''(x) = 1/2$.

We claim that, if $f''(c) < 0$ for some $c \in (x_0, \infty)$, then $f'' < 0$ on $[c, \infty)$. Indeed, by the just mentioned relation between the signs of f'' and $y'' - 1/2$, if $f''(c) < 0$ for some $c \in (x_0, \infty)$ then $y''(c) > 1/2$, and then it suffices to check that $y'' > 1/2$ on $[c, \infty)$. But otherwise there would exist some $x \in [c, \infty)$ such that $y''(x) \le 1/2$, whence

$$a := \inf\{x \in [c, \infty) \colon y''(x) \le 1/2\} < \infty.$$

Moreover, then $a \ge c$, $y'' > 1/2$ on $[c, a)$ and, by continuity, $y''(a) = 1/2$, so that $y''(x) - y''(a) > 0$ for all $x \in [c, a)$. Also, the equality $y''(a) = 1/2$ and the conditions $y''(c) > 1/2$ and $a \ge c$ imply $a > c$. Hence,

$$y'''(a) = \lim_{x \uparrow a} \frac{y''(x) - y''(a)}{x - a} \le 0.$$

It follows that

$$2y'(a)y'''(a) - y''(a)^2 \le -y''(a)^2 = -1/4 < 0,$$

whence, by (A.1), one has $1 - 2y''(a) < 0$, which contradicts the condition $y''(a) = 1/2$. Thus, the claim is true.

Let now

$$c_0 := \inf\{c \in (x_0, \infty) \colon f''(c) < 0\}.$$

Then, by the above claim, $f'' < 0$ on (c_0, ∞). Moreover, $f'' \ge 0$ on (x_0, c_0).

Assume that Theorem 1 does not hold. Then $c_0 > x_0$, for otherwise one would have $f'' < 0$ on (x_0, ∞). It follows by (A.1) and (A.2) that on (x_0, c_0) one has $y'' \le 1/2$ and also $2y'y''' \ge y''^2 \ge 0$, whence $y''' \ge 0$, so that y'' is nondecreasing. Note also that $y'' = 2ff' \ge 0$. Thus, there exists

$$\beta := y''(x_0+) \in [0, 1/2].$$

Moreover, it follows that $\beta = 1/2$ if and only if $y'' \equiv 1/2$ (and hence $y''' \equiv 0$) in a right neighborhood (r.n.) of x_0; but this would contradict equation (A.1). Hence, $\beta \in [0, 1/2)$.

Take now any $\beta_1 \in (\beta, 1/2)$ and let $\gamma := 1 - 2\beta_1$, so that $\gamma > 0$. Then, in a r.n. of x_0, one has $y'' \le \beta_1$, whence $1 - 2y'' \ge \gamma$ and, by (A.1),

$$2y'y''' = y''^2 + 4\frac{y'^2}{y}(1 - 2y'') \ge 4\gamma \cdot \frac{y'^2}{y},$$

$$y''' \ge 2\gamma \cdot \frac{y'}{y} = 2\gamma \cdot (\ln y)',$$

$$\infty > y'' - \beta \ge 2\gamma \cdot (\ln y - \ln(0+)) = \infty,$$

which is a contradiction (here we used the fact that $y(x_0+) = 0$).

This concludes the proof of Theorem 1.

Thus, we have proved that Eq. (1) cannot model the true shape of the Eiffel Tower.

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