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磁电热弹耦合材料三维断裂力学问题超奇异积分法

Hypersingular Integral Equation Method to 3D Crack in Fully Coupled Electromagnetothermoelastic Multiphase Composites

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摘 要

近年来，磁电热弹耦合材料在航空航天、船舶、石油化工及重载发动机等领域应用日益广泛。材料通常存在裂纹、夹杂等缺陷，苛刻工作环境使材料在缺陷处产生应力集中，进而变为裂纹并不断扩展，最终导致结构失效破坏。本文从材料学、力学、计算数学、热力学等学科交叉融合角度，采用以理论分析为基础，以数值模拟为工具的思路，建立了多场耦合作用下三维断裂力学问题求解理论。应用超奇异积分法，推导出裂纹前沿广义奇性应力(增量)场解析解，编制了 FORTRAN 程序，研究了广义(应力、应变能)强度因子变化规律。研究工作包括：

1.运用张量运算，导出适用于超奇异积分法的磁电热弹耦合材料广义基本解；引入广义位移间断概念，将三维单裂纹问题转化为解超奇异积分方程问题；得到广义奇性应力场解析解；定义了广义(应力、应变能)强度因子及广义能量释放率，将奇异积分方程转化为代数方程组；通过典型算例，研究了广义强度因子变化规律。

2.在单裂纹方法基础上，导出磁电热弹耦合材料三维多裂纹问题的各裂纹面直角坐标系下广义基本解；将该问题转化为解超奇异积分方程问题，推导出广义奇性应力场解析解；定义了广义(应力、应变能)强度因子和广义能量释放率，为奇异积分方程建立了数值解法；通过典型算例，研究了平行双裂纹问题广义强度因子变化规律。

3.在线弹性裂纹研究基础上，引入时域超奇异积分、广义位移间断应变率、广义伪体积力和广义伪面力概念，将热-黏弹塑性对控制方程和边界条件的非线性影响作为伪载荷处理，使得粘弹塑项以伪体积力和伪面力形式出现；将磁电热弹耦合材料三维单裂纹扩展问题转化为解时域超奇异积分方程问题；得到广义应力增量场解析表达式，定义了广义 J 积分，为奇异积分方程建立了数值解法；进行了考核性计算，验证了本方法收敛性和准确性。

4.在以上研究基础上，导出与界面垂直裂纹面坐标系下磁压电耦合双材料广义基本解；将磁压电耦合双材料与界面垂直三维单裂纹问题、磁压电耦合材料三维单裂纹问题分别转化为解超奇异积分方程问题，分析了广义应力奇性指数，得到广义奇性应力解析解，定义了广义(应力、应变能)强度因子和广义能量释放率；为两类问题分别建立了数值方法，通过典型算例，研究了广义强度因子变化规律。

关键词： 磁电热弹耦合材料，三维裂纹，超奇异积分方程，广义(应力、应变能)强度因子。

Abstract

Nowadays, Fully coupled electromagnetothermoelastic multiphase composites (FC-EMTE-MCs) are being used more and more widely in space planes (sensors), supersonic airplanes (gas turbines), rockets, missiles, nuclear fusion reactor, power generation, petroleum and petrochemical industries as well as submarines (smart skin systems) due to their capabilities to respond in an advantageous manner to changing environment (e.g. When missile accelerated to the 1 mach, it will meet sound barrier, and when missile accelerated to the (3)5 mach, it will meet heat barrier). However, relatively little works have been made for the 3D crack (growth) problem because of the present limitations both practical (computing time) and theoretical (accurate 3D formulations of dislocation shielding and image force).These require us to provide with general precious and accurate theoretical method by use of mathematics tools, and found efficiently numerical method. This work presents hypersingular integral equation (HIE) method proposed by the author for modeling 3D crack (growth) under extended loads through the intricate theoretical analysis and numerical simulations. The main contributions of the dissertation are as follows:

1. The extended fundamental solutions of FC-EMTE-MCs in literatures are further studied by tensor transfer method and elastic theory. The compact formula of extended fundamental solutions (elastic displacement, electrical potential, magnetic potential and thermal potential) are obtained, which can be used as extended Green's functions in the crack problems analysis. Then, applying main part method and extended boundary conditions, an arbitrary shaped 3D crack in FC-EMTE-MCs subjected to the extended loads (mechanical loads, electrical loads, magnetic loads and thermal loads) are reduced to solve a set of HIEs. The unknown functions are the extended displacement discontinuities (elastic displacement discontinuities, electric potential discontinuities, magnetic potential discontinuities and thermo flow potential discontinuities) of the crack surface. Also, the singularity of the extended displacement discontinuities is analyzed. The analytical solutions of the extended singular stresses (singular mechanical stresses, electric displacements, magnetic displacements and thermal stresses), the extended SIFs (mechanical SIFs corresponding to the crack mode I, II and III, electric SIFs K_4, magnetic SIFs K_5 and thermal SIFs K_6)、the extended strain energy factors(SEDFs) and the extended energy release rate are given. In addition, the numerical method for rectangular 3D crack is proposed by the body force method and some numerical solutions are calculated. For the special case, the results are compared with those obtained in literatures. Finally, the numerical solutions of extended SIFs (SEDFs) with varying the shape of crack are discussed.

2. Based on the above method, the arbitrary angle (stochastic) 3D cracks in FC-EMTE-MCs subjected to the electro-magneto-thermo-elastic coupled loads are reduced to solve a set of HIEs coupled with extended boundary conditions. The unknown functions are the extended displacement discontinuities of the multiple cracks surface. Then, the analytical solutions of the extended singular

stresses, the extended SIFs, the extended SEDFs and the extended energy release rate are presented, respectively. Also, the numerical method for multiple 3D cracks is proposed by the extended body force method, and some numerical solutions are calculated under extended boundary condition to illustrate the accuracy and efficiency of the method. Finally, the relationship between extended SIFs (SEDFs) and the shape of flaw, the distance between two interface cracks, the properties of the materials and the electro-magneto-elastic coupling effects is discussed.

3. FC-EMTE-MCs typically exhibit pronounced nonlinear response under sufficiently higher electromagnetothermoelastic loading conditions. These extended loads are causing higher local extended stresses than under corresponding static loads and may induce crack initiation, crack growth and finally lead to fracture or failure of structures. Due to the importance of the reliability of these structures, viscoplastic fracture and failure analysis have received considerable interest in the recent decade. In this part, the general solutions of extended incremental displacement rate (creep viscoplastic incremental displacement rate, electrical incremental potential rate, magnetic incremental potential rate and thermal incremental potential rate) are obtained by time domain boundary element method. Then, applying main part method and extended boundary conditions, a 3D crack growth problem subjected to the extended incremental loads rate (incremental mechanical load rate, incremental electrical load rate, incremental magnetic load rate and incremental thermal load rate) is reduced to solve a set of HIEs. The unknown functions are the extended incremental displacement discontinuities gradient (incremental creep viscoplastic displacement discontinuities gradient, incremental electrical potential gradient, incremental magnetic potential gradient and incremental thermal potential gradient) of the crack surface. Also, the singularity of the extended incremental displacement discontinuities gradient is analyzed. The analytical solution of the extended incremental singular stresses gradient (incremental singular stress gradient, incremental singular electric displacement gradient, incremental singular magnetic displacement gradient and incremental singular thermal stress gradient) and extended incremental $\dot{J}_i^g$ integral near the crack front are given. In addition, the numerical method of the HIE for a 3D crack subjected to extended incremental loads rate is proposed and some numerical solutions are calculated. For the special case, the results are compared with those obtained in literature. Finally, the results show that the present method yields smooth variations of extended crack incremental opening displacements (CIOD) along the crack front accurately.

4. The extended fundamental solutions of fully coupled electromagnetoelastic (anisotropic) multiphase composites in literatures are further studied by tensor transfer method and elastic theory. The compact formula of extended fundamental solution (elastic displacement, electrical potential and magnetic potential) is obtained. Then, applying main part method and extended boundary conditions, a 3D crack perpendicular to the interface of anisotropic electromagnetoelastic multiphase composites subjected to the extended loads (mechanical loads, electrical loads and magnetic loads) are reduced to solve a set of HIEs. The unknown functions are the extended displacement discontinuities (elastic displacement discontinuities, electric potential discontinuities, and magnetic potential discontinuities) of

the crack surface. Also, the singularity of the extended displacement discontinuities is analyzed. The analytical solutions of the extended singular stresses (singular mechanical stresses, singular electric displacements and singular magnetic displacements), the extended SIFs (mechanical SIFs corresponding to the crack mode I, II and III, electric SIF K_4 and magnetic SIF K_5), the extended SEDFs and the extended strain release rate are carried out. In addition, a numerical method for rectangular 3D crack is proposed by the body force method and some numerical solutions are calculated. For the special case, the results are compared with those obtained in literature. Finally, the numerical solutions of extended SIFs with varying the shape of crack are discussed.

Key words: Fully coupled electromagnetothermalelastic multiphase composites, Three-dimensional crack, Hypersingular integral equation method, Stress intensity factor, Strain energy intensity factor.

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主要符号表

$\sum_{iJ}$	广义应力矩阵
σ_{ij}	二阶应力张量
D_i	电位移向量
B_i	磁感应强度向量
ϑ_i	热流强度向量
f_J	广义体力向量
f_i	体力向量
f_e	自由电荷密度向量
f_m	电流密度向量(根据电磁理论 f_m=0)
f_ϑ	热流强度向量
E_{iJKl}	磁电热弹系数矩阵
C_{iJKl}	四阶广义弹性系数张量
Π_{iJ}	二阶广义热应力系数张量
c_{ijkl}	四阶弹性系数张量
e_{lij}	三阶压电系数张量
d_{lij}	三阶压磁系数张量
$\in_{il}$	二阶介电系数张量
g_{il}	二阶电磁系数张量
μ_{il}	二阶磁性系数张量
ι_{ij}	二阶热应力系数张量
ς_{il}	二阶热电因子系数张量
η_{il}	二阶热磁因子系数张量
λ_{ij}	二阶热传导因子系数张量
Z_{Kl}	二阶广义应变场矩阵
U_K	广义弹性位移向量
u_i	弹性位移向量
ϕ	电势向量
φ	磁势向量
ϑ	热势向量
ρ	材料密度
U_{IJ}	磁电热弹耦合材料三维空间问题位移广义基本解
T_{IJ}	磁电热弹耦合材料三维空间问题面力广义基本解
Γ	广义边界
T_J	广义弹性力
t_j	弹性力
q	电荷强度
b	磁通量强度
ρ	热流强度
$\tilde{U}_J$	广义位移间断向量
$\tilde{u}_j$	位移间断向量
$\tilde{\phi}_j$	广义电势间断向量
$\tilde{\varphi}_j$	广义磁势间断向量
$\tilde{\Upsilon}_j$	广义热势间断向量
$\dot{F}_i$	伪体积力增量
$\dot{T}_i$	伪面力增量
$\dot{u}_i$	位移增量
$\dot{\varepsilon}_i$	应变增量
$\dot{\sigma}_{ij}$	应力增量
$\dot{f}_i$	弹性体积力
$\dot{\sigma}^n_{ji,j}$	弹性体积力增量
$\dot{\varepsilon}^n_{ij}$	广义应变增量
$\dot{\varepsilon}^p_{ij}$	塑性(粘塑性)应变增量

$\dot{\varepsilon}_{ij}^{c}$ 蠕变应变增量

$\dot{\varepsilon}_{ij}^{T}$ 热应变增量

α 平均热膨胀系数

$\dot{T}$ 温度增量

$\dot{\varepsilon}_{ij}^{M}$ 温度变化引起材料变化而产生的应变增量

E 杨氏模量

v 泊松比

μ 剪切模量

E' 等价杨氏模量

v' 等价泊松比

μ' 等价剪切模量.

ESI 广义奇性应力指数

ESSF 广义奇性应力场

ESIF 广义应力强度因子

ESEDF 广义应变能强度因子

ENRR 广义能量释放率

K_I 广义应力强度因子

K_1 I 型应力强度因子

K_2 II 型应力强度因子

K_3 III 型应力强度因子

K_4 广义电应力强度因子

K_5 广义磁应力强度因子

K_6 广义热应力强度因子

F_I 无量纲广义应力强度因子

F_1 无量纲 I 型应力强度因子

F_2 无量纲 II 型应力强度因子

F_3 无量纲 III 型应力强度因子

F_4 无量纲广义电应力强度因子

F_5 无量纲广义磁应力强度因子

F_6 无量纲广义热应力强度因子

G 广义能量释放率

G_1^{σ}，G_2^{σ}，G_3^{σ} 能量释放率（应力影响部分）

G_4^{σ} 广义能量释放率（电场影响部分）

G_5^{σ} 广义能量释放率（磁场影响部分）

G_6^{σ} 广义能量释放率（热应力影响部分）

$S\big|_{\theta}$ 广义应变能强度因子

$\bar{S}\big|_{\theta}$ 无量纲广义应变能强度因子

$\dot{J}_i^{g}$ 广义 J 积分

σ_{3I} 裂纹前沿广义奇异应力场

σ_{3i} 裂纹前沿奇异应力场（弹性力影响部分）

D_3 裂纹前沿广义奇异应力场（电力场影响部分）

B_3 裂纹前沿广义奇异应力场（磁力场影响部分）

ϑ_3 裂纹前沿广义奇异应力场（热力场影响部分）

$\dot{\sigma}_{3I}$ 裂纹前沿广义奇异应力增量场

$\dot{\sigma}_{3i}$ 裂纹前沿奇异应力增量场（弹性力影响）

$\dot{D}_3$ 裂纹前沿广义奇异应力增量场（电力场影响）

$\dot{B}_3$ 裂纹前沿广义奇异应力增量场（磁力场影响）

$\dot{\vartheta}_3$ 裂纹前沿广义奇异应力增量场（热力场影响）

$\dot{\tilde{U}}_J$ 广义位移间断应变率

$\dot{\tilde{u}}_j$ 位移间断应变率

$\dot{\tilde{\phi}}_j$ 广义电势间断应变率

$\dot{\tilde{\varphi}}_j$ 广义磁势间断应变率

$\dot{\tilde{\vartheta}}_j$ 广义热势间断应变率

$⨎$ 超奇异积分符号

S_{KiJ} 线弹性积分核（裂纹边界）

D_{KiJ} 线弹性积分核（空间区域）

$\bar{S}_{KiJ}$ 粘塑性积分核（裂纹边界及伪面力）

$\bar{D}_{KiJ}$ 粘塑性积分核（空间区域及伪体积力）

W_{iJl} 粘塑性积分核（空间区域及广义应变增量）

$\dot{U}_{IJ}$ 广义位移增量基本解

$\dot{T}_{IJ}$ 广义面力增量基本解

Ω 材料区域

$S\ (S^{+} \bigcup S^{-})$ 单裂纹边界

$S_{\hat{i}}(S_{\hat{i}}^{+} \bigcup S_{\hat{i}}^{-})$ 多裂纹第 $\hat{i}$ 裂纹边界

$\theta_i^{\hat{i}}(x_i\ , x_i^{\hat{i}})$ 局部系坐标 $x_i^{\hat{i}}$ 与整体系坐标 x_i 夹角

$O^{\hat{i}}$ 多裂纹第 $\hat{i}$ 裂纹中心点

θ 裂纹前沿沿半径方向角度

S_{ε} 裂纹前沿微小区域

KK，LL FORTRAN 程序中配置点

M，N FORTRAN 程序中多项式指数

b/a 矩形平片裂纹形状比

h/a 裂纹间垂直距离比

$p(\xi_1, \xi_2, \xi_3)$ 源点（参考点）

$q(x_1, x_2, x_3)$ 场点（观测点）

r 源点到裂纹前沿场点距离

第一章 绪论

1.1 研究目的及意义

断裂力学作为研究材料和结构断裂失效与裂纹扩展规律的力学学科，从达.芬奇工作算起已有几百年历史，但真正作为一门定量科学并应用于人们对客观世界破坏规律的认识，则是从二十世纪 50 年代开始的[1-6]。断裂力学由 Griffith(格里菲斯)、Inglis(殷格里斯)开创，经 Irwin(欧文)、Orowan(奥罗万)等发展而形成完整的线弹性断裂力学理论。在著名固体力学家 Rice(瑞斯)、Orowan(奥罗万)、Hutchinson(哈钦森)、Sih(薛昌明)等努力下，弹塑性断裂力学也趋于成熟。断裂力学与断裂物理相结合而形成宏微观断裂力学，已成为跨世纪发展的前沿学科之一。

破坏力学是当前国内外发展起来的一门概括性学科，它泛指对各种工程结构(机械、土木、航空航天、核电、电子元件等)和工程材料(金属、陶瓷、高分子、岩土、材料、生物材料等)破坏行为(断裂、损伤、疲劳、腐蚀、磨损等)的力学规律研究。破坏力学与材料、能源、国防、土木、机械、信息等重大工程科学领域密切相关，可对国民经济发展和劳动事故防范产生重大影响。

从二次世界大战结束至今破坏力学经历了宏观断裂力学阶段(断裂力学)和宏微观断裂力学阶段(断裂物理)。宏观断裂力学主要特征是以断裂等破坏终极现象作为防范目标，并提出了以断裂韧性理论为中心的破坏准则体系；引入宏观缺陷，但不考虑细—微观缺陷；对材料的本构行为采用较复杂的连续介质描述，但材料结构单元仍不具有细微观结构；将材料的破坏抗力唯象地反映在带裂纹标准试件的断裂指标上。宏观缺陷的引入带动了近 40 年来破坏力学发展成为一门独立学科。微观断裂力学是自上世纪自 80 年代萌生，并逐渐发展起来。其主要特征为追溯从变形、损伤至断裂的全过程；引入多层次的缺陷几何结构；对材料的本构关系行为采用宏-细-微观相结合的描述，即在材料结构单元中体现特定的微结构；材料的破坏抗力体现为可预测的力学指标[7, 8]。

在过去二十年里，随着新技术迅猛发展，材料和机械设计日趋智能化，工程材料被要求具有诸如耐磨损、耐碰撞、热防护等更多的特殊性能。这些应用中，同一种材料结构的不同部分可能承受不同的载荷条件（力、电、热、磁、光、声、化学等）。金属合金、先进晶体、多相合金/陶瓷材料已经不能很好的满足实际需要，因此需要制作一类材料参数（泊松比、拉梅常数）能连续变化的材料（如航天工业中气轮机热防护系统，要求材料具有热电特性，耐高温和腐蚀性、足够高的强度、刚度、热传导和对磨损的承受能力）。热障涂层材料（TBC s）[9-11]在一定程度上可以满足以上要求，但是 TBCs 材料的缺点是在陶瓷和金属的结合部位结合强度低而残余应力高，这就导致这类材料非常容易产生表面裂纹和分层或开焊[12-15]。

在上世纪 80 年代末，Niino 和 Maeda[16]在分析 TBCs 材料缺点的时候，引入了功能梯度材料（FGMs）概念。功能梯度材料是具有各向异性特性，陶瓷/金属复合组合，它的热-力特性是在整个材料空间变化的。这就使 FGMs 材料参数特性或者在整个内部连续变化或者在不同的区域不同。FGMs 材料具有耐高温和耐腐蚀特性的同时，还具有高强度和刚度、在涂层/界面底层

具有高的结合强度，而残余热应力却很小。实验室中有很多方法可以制作 FGMs 材料，常用的技术包括：热喷射技术[17, 18]、粉末冶金(P/M) 技术[19]、化学沉积法(CVD)[20]、自蔓延高温制备技术(SHS)[21, 22]、激光融合技术[23]。最初设计功能梯度材料的目的是为了航天应用中的热防护材料，随着科技的发展，目前功能梯度材料也应用于其它领域：换能器[24]（化学工厂构件、太阳能发生器、热交换器、高效燃烧系统及熔解反应器）；军事领域[25-33]（装甲、武器和车辆结构）；微电子领域[34]（半导体传感器）；多孔梯度材料滤波器[35]；高速切削工具[36]；生物医学材料[37-39]。尽管功能梯度材料的应用潜力是巨大的，但是在实际应用过程中，还有很多工作要做，这就要求不仅要考虑到功能梯度材料的制备，而且要弄清他们的力学和断裂行为的本质（如老龄飞机损伤和寿命预测，下几代航天器用新材料和结构的损伤容限和基于物理的、一体化的航天器材料和结构剪裁问题。复合材料结构加载后，发生微裂纹和层间开裂。失效始于底层，从材料中的孤立的空洞成核开始，经过空洞取向，形成微裂纹，发展为宏观裂纹，直至整个结构破坏）。在过去的十几年里，许多学者把精力投入到非各向同性介质的裂纹处应力和应变场的理论分析和数值模拟工作中，对于裂纹的产生和扩展原理已经有了很深入的研究[40-45]。

从 Brewster、Curie 兄弟、Vaslek，到 Davis、Vandenboomagaard、Tinkham、Aboudi[46-49]、Zheng 和 Zhou[50-57]、He[58]、Gao[59, 60]、Fang[61-79]、Wang[80-85]、Yang[7, 8]、Li[86-91]、Sih [92-111]、Jiang[33]、Wu[112, 113]；磁-电-热-弹耦合问题一直受到众多学者关注。近十年来， 随着纳米技术的发展，材料制备手段不断提高，对材料的研究也日趋深化，磁电热弹耦合材料在航天器、超音速飞机、火箭、导弹、潜艇、核反应堆、石油化工及重载发动机等方面的广泛应用。工作环境对材料磁-电-热-弹耦合性及多场(磁、电、热、力)灵敏性的要求也更为严格。苛刻的工作环境（如结构速度提高到 1 马赫时，会碰到“音障”，速度提高到 5(3)马赫时，又会碰到“热障”），将使材料在缺陷处产生应力集中，进而变为裂纹，裂纹不断扩展，最终导致结构失效及破坏。同时结构对材料稳定性要求也非常严格。因此磁电热弹耦合材料破坏及断裂力学问题的研究在理论及工程应用中都具有重要意义，也是力学学科前沿领域的重要研究方向之一[4]。磁电热弹耦合材料及其性质的研究也受到越来越多的学者的关注[46, 59, 60, 114, 115]。

但鉴于数值方法(计算量方面)、理论模型(三维裂纹<扩展>模型)、数学(公式庞杂)及力学和物理学(问题复杂)上的困难，在磁电热弹耦合材料三维裂纹方面，尤其是在粘弹塑性三维裂纹扩展方面，研究成果较少。这就客观上要求我们应用数学工具建立一种可靠(精度、准确性和可靠性方面)的理论模型，并给出相应的高效(精度、准确性及机时方面)数值方法。

本文目的就是从材料学、力学、计算数学、热学、电磁学等学科交叉融合的角度出发，采用以理论分析为基础，以数值模拟为工具的研究思路，建立磁电热弹耦合材料断裂力学行为理论体系；针对三维裂纹(扩展)问题，应用超奇异积分方法，通过对材料断裂力学行为的理论和数值方面系统和深入研究，得到多场耦合作用下裂纹前沿广义奇性应力指数、广义奇异应力（增量）场、广义应变能强度因子和广义能量释放率解析表达式，推导出广义应力强度因子和广义应变能强度因子数值计算式；应用 FORTRAN 语言，通过编制专用程序，对典型各类裂纹问题进行计算，得到各类裂纹广义应力强度因子和广义应变能强度因子数值结果，以及它们随材料

和裂纹参数变化规律；分析裂纹扩展规律，裂纹屏蔽和相互作用规律，得出各类裂纹断裂评定准则。最终得到磁电热弹耦合材料一般裂纹(扩展)问题的超奇异积分方程解法，进一步发展和丰富超奇异积分理论在复合材料断裂力学中的应用。

1.2 国内外研究进展

公元前 314 年，希腊哲学家西奥佛雷特斯在他的著作中首次提到了当加热电气石矿时会产生电效应(热电效应)；1824 年苏格兰物理学家 Brewster(布儒斯特)首次引入了热电性的提法（pyroelectricity）；19 世纪末，英国物理学家开尔文勋爵(1878)和 Voight(1897)对热电理论进行了研究。

1880 年法国物理学家皮埃尔.居里与哥哥雅克.保罗.居里在研究热电现象和晶体对称性的时候，在 α 石英晶体上最先发现了压电效应，即在某些晶体的特定方向施加压力，晶体的一些对应的表面上分别出现正负束缚电荷，其电荷密度跟作用力大小成比例，这种现象称为“直接压电效应”。1881 年，G.Lippma 依据热力学原理及其能量守恒和电荷守恒定律预言了逆压电效应即“电致伸缩”效应的存在，也就是在压电晶体上外加电场时，晶体中会产生应力而使晶体发生变形。同年，P.居里和 J.居里兄弟用试验证实了压电晶体在外加电场作用下会出现应变或应力[116, 117]。

1916 年朗之万用石英晶体做成水下发射和接收的换能器，并用回波法探测沉船和海底。1921 年研制成功石英谐振器和滤波器，开创了压电晶体在频率控制和通讯方面的应用。1935-1938 年，苏联的 G.Busch 和 P.Schemer 研制出水溶性压电晶体磷酸二氢钾(KH_2PO_4，简称 KDP)和磷酸二氢氨(简称 ADP)，在二十世纪四十年代得到了广泛应用，对压电材料的发展起了很大的推动作用。

Л.Д.朗道和 E.M.栗弗席兹根据热力学和对称性理论预言，在自旋有序的磁性物质内，可能存在磁电效应。1960 年 Д. H.阿斯特罗夫最先在实验中观察到反铁磁体 Cr_2O_3 单晶的电致磁电效应。1961 年 G.T.拉多和 V.J.福伦又观察到 Cr_2O_3 单晶的磁致磁电效应。当温度升高到磁有序温度以上时，Cr_2O_3 晶体由反铁磁性转变为顺磁性后,磁电效应随之消失。铁电材料在应用过程中，常承受较强的直流或交流电场，这使在铁电材料内电极以及制备过程产生的缺陷或裂纹附近，产生较强的电场集中与应力集中。实际使用中，常会出现由于电场载荷作用而引致的断裂行为(电致断裂)，以及交变电场作用下的疲劳裂纹扩展 (电致疲劳)。 电致断裂与电致疲劳问题的出现，限制了铁电材料的广泛应用和微电子元器件性能的提高。20 世纪 60 年代后，激光和红外技术的发展极大促进了热电效应及其应用的研究，丰富和发展了热电理论

Davis(1974)、Vandenboomagaard, et al., (1976)和 Tinkham (1974)等学者[118-123]最早系统研究压电/压磁问题；Aboudi(2001)、Gao, et al.,(2003)等学者[46, 59, 60, 114, 115]对磁电热弹耦合材料的基础性理论进行研究；作为对 Maxwell 所创始的电磁弹性力学基本理论的一个最新发展，Zhen 和 Zhou 建立了对电磁材料结构非线性力学行为研究的完整的理论框架，该理论包括能预测铁磁材料典型实验现象的磁力表达式[50, 51]， 而且还有多场耦合的非线性本构关系[52, 53]。他们还对电磁材料结构行为给出了很好的定量预测[54, 55], 在超导研究方面，Zhou 等对力-电-磁-

热耦合作用下磁通跳跃现象给出成功的预测[56, 57] ；Pan[124-139]对磁压电耦合材料各类格林函数问题进行了全面系统深入的研究；Yang[7, 8]对力电失效宏微观断裂力学进行了卓有成效的研究；Li[86-91]从晶格角度对铁电智能材料纳观缺陷进行了深入研究，从晶格层面导出的Eshelly 张量计算方法对研究智能材料断裂力学问题研究起到了重要作用。Sih [92-95]最早提出应变能强度因子理论，并这一理论应用到磁电热弹耦合材料断裂力学的研究中，做了许多具有开创性的成果[96-111]。Jiang[33]、Wu[112, 113]等学者也从不同方面对磁电热弹耦合材料研究做出了突出贡献。磁电热弹耦合材料的研究一直受到国内外众多学者的广泛关注。以上这些成果为磁电热弹耦合材料断裂问题研究奠定了良好理论基础。

1.2.1 边界元方法

边界元法最早是从研究位势问题的边界积分方程开始发展起来的，位势问题在数学分析上属于椭圆型方程问题，它代表一种定常状态，诸如稳态温度场，无旋稳定流动，静电场等。它们都满足 Laplace 方程或 Poisson 方程。Kellogg 是用积分方程方法求解 Laplace 问题的先驱者。

现代边界积分方法可以追溯到 1903 年 Fredholm 的工作，Fredholm (1903)得出势问题的积分公式；Kellogg (1929)、Muskhelishvili (1953)、Kupradze(1966) 把它应用到弹性问题中；Ritz(1909)提出有限元法；1926 年 Trefftz 提出边界元法初步概念。随着计算机技术的快速发展，积分方程方法才在一些流体力学和位势问题中广泛应用。这方面开创性的工作是在 20 世纪 60 年代由 Jaswon[140]，Symin[141]，Hess 和 Smith[142], Massonnet[143]等做出的；Jaswon(1963)、Seem(1963)应用间接边界元法计算了位势问题；以后边界元法在位势领域的应用日趋广泛和成熟，继位势问题后，1967 年 Rizzo [144]应用直接边界元法计算了二维线弹性问题；1969 年 Cruse [145]在边界元法应用到三维弹性问题中。

1971 年 Cruse、Buren[146]最早将边界积分方程法应用于断裂力学，采用分割区域法将裂纹问题化为一般的边值问题进行处理，从而避免了边界元法解决断裂问题时遇到的不确定性；Banerjee、Butterfield、Brebbia、Dominguez(1977)等提出了边界元法离散化理论；1978 年 C.A.brebbia 首次正式用边界元提法；1986 年 Ghosh 等[147]把二维弹性问题基本解超奇异积分转化为弱奇异积分；1994 年 Lei[148, 149]对二维弹性 Reissem 板建立新类型边界积分方程。

在我国，边界元法最初是独立于西方体系自主发展起来的，1965 年冯康[150]最早提出，最初称为边界积分方法，后经余德浩等不断完善成为一个完整体系。在力学应用方面，黄克服、王敏中、杜庆华、王元淳、嵇醒、匡振邦、汤任基、田宗若、柳春图、王自强、王邦银、姚振汉、张雄、袁明武、陈俊杉、陈孟成、乐金朝、秦太验等众多学者把边界元法应用到各自的研究领域，在提高奇异积分计算精度的研究中，发展出许多方法：带权 Gauss 积分公式、刚体位移间接法、Green 函数法、位错法、对偶边界积分法、位错密度法、位移二次插值法等。经过三十多年的发展，在固体力学方面，边界元法被推广应用于动态弹性问题，塑性问题，蠕变问题和断裂问题等领域。

1971 年 J.L.Swedlow 和 T.A.Cruse[151]最早研究了非线性边界元法，提出了含有塑性初应变体积分项的 Somigliana 恒等式的推广形式，但没有给出内部应力的表达式和算例；1973 年

P.C.Ricardella [152] 首次给出了二维弹塑性问题边界元法的数值结果及封闭形式的塑性应变项积分的正确表示式。同年，Mendelson[153]讨论了弹塑性问题的不同积分公式，给出了内应力表达式，但塑性应变项的处理是错误的。后来 Mukherjee 和 Kumar[154, 155]部分纠正了[153]文献中的错误。1977 年 Chaudonneret 首次把边界元的直接法应用于粘塑性分析，给出具有初应力形式的积分方程，但 Mukherjee 和 Chaudonneret 所给内点应力的积分表达式均不正确。1978 年 Bui[156]才首次提出了非线性项积分导数的正确处理方法，证明了与求导有关的附加项的存在，第一次得到了非线性应力的积分表达式。继此工作以后，Mukherjee 和 Kumar[157]完成了与时间相关的非线性幂律蠕变分析。1979 年 Telles 和 Brebbia[158]得到了三维及二维弹塑性问题的完整边界元公式。1980 年 Telles 和 Brebbia[159, 160]分别用弹塑性边界元法的初应变和初应力的形式，并使用线性插值的边界单元和内部单元，求解了具体的弹塑性问题，数值结果显示了边界元法优于有限元法的特点及其潜力。同年 Morjaria 和 Mukherjee[161]通过使用线性边界单元和 Euler 型时间积分方案改进了[157]的数值方法，并在[162]中利用平面开口问题的基本解和双调和间接法的边界元公式分析了裂纹板的时间相关变形。1982 年 Telles 和 Brebbia[163]采用 Perzyna 弹/粘本构关系模型和需限制时间步长的 Euler 时间积分方案，实现了能统一地进行塑性，蠕变和粘弹性分析的粘塑性边界元法。1979 年 Banerjee 等[164]使用常值边界单元和内部单元及初应力算法，首次完成了三维弹塑性问题的边界元数值分析，Banerjee 等[165]通过使用二次等参元和体单元，退化坐标变换，对初应力和初应变进行外插修正的迭代求解，大大提高了三维弹塑性和粘塑性边界元分析的精度和效率。早期的热弹塑性边界元的研究或仅局限于边界积分方程的理论推导，或局限于热弹塑性—蠕变问题中的一小部分[166]。

总体上看，边界元法(BEM)与有限元法(FEM)相比，具有数据准备简单、变量少、计算精度高的优点，广泛的应用到线弹性断裂(LEFM)的问题求解中。但由于要涉及到时域积分等问题，边界元法在热弹塑性—蠕变断裂方面的应用，还处于研究和发展阶段。

1.2.2 超奇异积分方法

对于规则问题，基于基本场变量的边界积分方程（通常称为常规边界积分方程）是适定的，常规的位势边界积分方程可以求得边界位势和位势法向梯度。如果寻求边界位移梯度，一个途径是沿边界对位势函数求导数，这产生了新的误差。另一种途径是试图建立导数场边界积分方程，然而导数场边界积分方程通常衍生出超奇异积分问题，引导出 Hadamard 主值型积分的概念。

1910-1923 年 Hadamard[167, 168]把比主值型奇异积分方程奇异性还要多的积分方程称为 Hadamard 奇异积分方程（即超奇异积分方程）；Fazil Erdogan[3, 169-174]一直在丰富和发展该方法在断裂力学中的应用；1982 年 Ioakimidis[175, 176]首次把 Hadamard 的超奇异积分应用到线弹性断裂力学中，推导出受法向载荷作用的任意形状平片裂纹问题的超奇异积分方程，并使用 Kutt 有限部分积分法进行数值化；1985 年 Sohn 与 Hong[177]利用位移间断分片线性表示为无限体 I 型裂纹的超奇异积分方程，并建立了相应的数值法。1990 年 Tang 与 Qin[178-180]系统的分析和研究了超奇异积分在三维均质材料断裂问题中的应用，并提出一套完整的求解理论；

Qin[181-186]把这一理论不断完善，并应用到三维复合材料断裂问题的分析中[187-190]；其它学者也不同程度的丰富和发展了该方法[148, 149, 191, 192]。

超奇异积分方程与边界元方法结合是研究断裂力学问题的一种非常有效的方法，结合因子法可以得到精度较高的数值结果。因此将裂纹问题归结为求解一超奇异积分方程组，进行奇异性处理，转化为线性方程组，利用因子法数值求解已成为解决断裂力学问题一种重要途径[179]。

1.2.3 磁电热弹耦合材料裂纹

上世纪七十年代 Davis[121, 122]、Vandenboomgaard[118, 120]、Tinkham[123]和 Vanderhoeven[119]等学者最早进行了磁压电材料性质方面的研究。因其具有许多特殊性质(高强度低密度、磁电力场耦合），所以被应用到航天器、超音速飞机、火箭、导弹、核反应堆及重载发动机等众多领域中。近二十年来，随着加工和制造技术的发展，磁电热弹耦合材料的应用也日益广泛。

在实际工作环境中，由于受到高温、高压的作用使磁电热弹耦合材料可能处于塑性或蠕变状态，而结构的制造和加工过程中又不可避免的会带有各种类型的缺陷（以裂纹为主），而这些裂纹在热弹塑性和蠕变状态下的其始扩展以及由此导致的断裂是结构和构件失效的重要原因。并且工作环境对材料磁、电、热、弹耦合性及多场灵敏性也提出了更高的要求。苛刻的工作环境，使材料在缺陷处产生应力集中，进而变为裂纹，裂纹不断扩展，最终导致结构失效及破坏。同时结构对材料稳定性和可靠性的要求也非常严格。

但是由于问题的复杂性及力学和数学上的困难，磁电热弹耦合材料破坏的研究多限于基本理论分析，断裂和破坏方面的数值结果很少，且大多数研究都是针对平面问题的；因此对磁电热弹耦合材料断裂与破坏问题的研究，尤其是三维空间问题的研究，就成为摆在广大断裂力学学者面前迫切需要解决的问题。在二维磁压电裂纹方面，Tian [193]给出了与界面平行裂纹的广义应力强度因子计算式； Sih [100, 107, 108] 研究了 $BaTiO_3$—$CoFe_2O_4$ 材料裂纹问题；Feng [194] 研究了裂纹扩展问题；Li [195] 与 Soh [196] 分析了三型界面裂纹问题；Zhou [197]分析了双界面裂纹问题；Hu [198] 研究了界面裂纹扩展问题；Wang [199] 与 Liu [114] 分析了反平面界面裂纹问题；Chue [200]分析了 $BaTiO_3$—$CoFe_2O_4$ 材料界面裂纹问题；Jiang [131] 研究了内含裂纹问题；Zhao[201]分析了椭圆裂纹问题。在功能梯度材料方面：Maheendra[202]研究了涂层和基底结合处的裂纹问题；Prabhakar[203]系统研究了界面裂纹问题；Zhao[204]应用奇异积分方法研究疲劳和断裂问题；Carl[205]对裂纹尖端应力场及其断裂参数进行了评估；Serkan[206]对断裂和接触问题进行了研究；Chan [207-209]应用超奇异偏微分方程研究了存在应变梯度效应的断裂问题；但磁电热弹耦合材料三维裂纹方面报道很少[210, 211]。

1.3 本文主要工作

根据以上综述，本文做了如下几方面研究工作：

1. 在已有文献资料基础上，应用张量变换，整理、推导、比较和验证了磁电热弹耦合材料空间广义位移基本解，推导出适合于超奇异积分方程方法中使用的磁电热弹耦合材料三维广义基本解显式表达式，为应用超奇异积分方程方法分析磁电热弹耦合材料三维断裂力学问题奠定基础；引入裂纹面广义位移间断概念，利用 Somigliana 公式、边界元法和主部分析法，通过一系列庞杂数学推导，将磁电热弹耦合材料三维单裂纹问题转化为求解一以裂纹面广义位移间断为未知函数的超奇异积分方程组问题；使用奇异积分方程的主部分析法，系统研究了超奇异积分方程未知解性态，在严格地满足裂纹表面及其前沿受力条件下，求得了未知解性态指数；在此基础上，使用渐近分析法，精确得到了平片裂纹前沿光滑点附近广义奇性应力场，并定义了广义应力强度因子、广义应变能强度因子和广义能量释放率；提出广义位移间断基本函数概念，为其建立了有限部积分边界元数值方法。在此基础上，应用因子法、有限部积分方法和体积力法，将该类裂纹问题超奇异积分方程组转化为代数方程组，并对其中出现的各类奇异积分作了专门处理，为其建立了数值方法，给出了具体计算式。同时给出了对应于广义位移间断基本函数概念的裂纹前沿广义应力强度因子和广义应变能强度因子数值计算式，从而完整的建立了超奇异积分方程数值方法；为了验证方法的可靠性和准确性，本文作了考核性计算，编写了 FORTRAN 专用程序（7000 余条），对典型问题进行了数值计算，研究了广义应力强度因子和广义应变能强度因子随材料及裂纹形状参数变化规律。这部分内容将在本文的第二章中介绍。

2. 根据前面所得到的磁电热弹耦合材料三维单裂纹分析方法，应用直角坐标系下的张量变换规则，对磁电热弹耦合材料三维空间问题广义位移基本解进行进一步整理和变换，得到以各个裂纹面上直角坐标系下的适合于超奇异积分方程方法中应用的空间问题广义基本解显示表达式。在扩展后的边界积分方程方法基础上，经过一系列庞杂的数学推导，获得了磁电热弹耦合材料平行三维多裂纹问题、三维多裂纹问题超奇异积分方程组。在此基础上，严格从三维理论出发，使用二维超奇异积分的主部分析方法，对以上两类裂纹问题裂纹前沿广义应力奇性指数、广义奇性应力场进行了分析，定义了多裂纹问题的广义应力强度因子、广义应变能强度因子和广义能量释放率。为两类问题的超奇异积分方程组建立了数值方法，编写了专用 FORTRAN 程序（7000 余条），进行了考核性计算，得到磁电热弹耦合材料线弹性一般裂纹问题的应力强度因子和广义应变能强度因子随材料参数及裂纹形状位置变化规律，分析了裂纹间的屏蔽和相互作用。这部分内容在本文第三章中给出。

3. 利用上面所导出的磁电热弹耦合材料线弹性三维裂纹问题分析方法，在引入了时域超奇异积分概念和广义位移间断应变率概念基础上，通过将广义热弹塑—蠕变对控制微分方程和边界条件的非线性影响作为伪载荷处理，使得塑性项以伪体积力和伪面力形式出现，这样就可以采用三维磁电热弹耦合材料弹性问题基本解作为弹塑性问题的基本解来求解三维弹塑性问题。应用扩展后的边界积分方程方法，将磁电热弹耦合材料三维裂纹扩展问题转化为求解一组时域超奇异积分方程问题，得到广义应力增量场的解析表达式，定义了广义 J 积分。进而通过对超奇异积分过程中遇到的各类奇异积分分别进行了数值处理，为该方程组建立数值求解方法，编写了专用的 FORTRAN 程序（9000 余条），进行了考核性计算。 通过现有数值方法计算结果和已有结果进行比较，验证了本数值方法的收敛性和精度。 这部分内容在本文第四章中给出。

4. 在以上工作基础上，通过对现有磁压电耦合双材料三维空间问题基本解进行整理、比较和验证的基础上，得到了适合于超奇异积分方程方法中使用的与界面垂直裂纹面坐标系上基本解的显示表达式；推导出了磁压电耦合双材料与界面垂直的三维单裂纹问题及磁压电耦合材料三维单裂纹问题的超奇异积分方程组；严格从三维理论出发，使用二维超奇异积分的主部分析方法，对裂纹前沿的广义应力奇性指数、广义奇性应力场进行了分析。定义了该类裂纹问题的广义应力强度因子、广义应变能强度因子和广义能量释放率。为该方程组建立数值求解方法，编写了专用的 FORTRAN 程序（7000 余条），进行了考核性计算。 通过现有数值方法计算结果和已有结果进行比较，验证了本数值方法的可靠性和准确性。得到了广义应力强度因子和广义应变能强度因子随材料参数及裂纹位置变化规律。这部分内容在本文第五章中给出。

本文以上工作，是对磁电热弹耦合材料三维断裂力学问题超奇异积分方程方法研究的一个推进和发展，首次将超奇异积分方程方法引入到磁电热弹耦合材料三维断裂力学的研究中，丰富和发展了该理论在功能梯度复合材料三维断裂力学领域中的应用，具有重要的理论价值和应用前景。这对于国内外开展这方面的研究有重要的参考价值。

第二章 磁电热弹耦合材料三维单裂纹问题

2.1 引言

从布鲁斯特(1824)引入热电性提法，Curie 兄弟(1880)发现压电效应，到 Davis、Vandenboomagaard、Tinkham[118-123]等学者对压电/压磁问题研究。到 Tiersten、Taylor、Rosen、Crawly、Yang 、Huang、Li、Pan、Qin、Fang[74, 138, 212-228]对磁压电问题的研究。进而到 Liu、Tan、Aboudi、Gao、Zheng、Zhou 、Chen、Ding[46, 50-55, 114, 115, 229, 230]对磁电热弹问题的研究。磁电热弹耦合问题一直受到众多学者的关注。

二十世纪末，磁电热弹耦合材料破坏与断裂问题吸引了国内外众多学者的关注。1990 年后，随着磁电热弹耦合材料在航空航天（发动机）、火箭导弹、船舶潜艇（智能蒙皮）、核反应堆、石油化工及车辆重载发动机等领域等的广泛应用。出于理论和工程应用上的需要（苛刻的工作环境，结构对材料的稳定性要求非常严格），更多的国内外学者加入到磁电热弹耦合材料断裂与破坏问题的研究上来，取得了一些研究成果。2000 年以后，这种趋势更加明显，每年都有大量的研究成果发表。磁电热弹耦合材料破坏与断裂力学问题的研究已成为物理学界、力学界、材料学界关注的热点之一。随着研究的深入，2004 年以来，每年的研究成果数量和质量较以前更加丰富，每年都有一些重要的理论突破。

作者就掌握的文献资料对磁电热弹断裂力学研究做如下简要分析：在磁电热弹耦合材料的性质方面[50, 51, 54-57, 214, 231-237]等做了许多研究。但是由于问题的复杂性及力学和数学上存在的困难，对于磁电热弹耦合材料的研究成果多限于基本理论分析，断裂和破坏方面的数值结果很少，且大多数成果都是针对平面问题的。因此对磁电热弹耦合材料断裂与破坏问题的研究，尤其是三维空间问题的研究，就成为摆在广大断裂力学学者面前，迫切需要解决的问题。在磁电热弹耦合材料断裂与破坏问题研究中，需要涉及到磁电、磁热、压磁、热电、压电、磁电热、磁热弹和磁电热弹耦合问题。掌握这些材料断裂与破坏问题研究现状，对磁电热弹耦合材料断裂与破坏问题的研究有着重要意义。同时对于磁电热弹问题的研究也具有非常重要的参考价值。按研究材料的耦合程度分：Podil'chuk[238-243] 对压磁耦合材料进行了研究；Sih、Tian、Feng、Hu、Li[100, 107, 108, 193, 194, 244-251]对磁压电耦合材料进行了研究；Liu、Pan、Tian、Wang、Zhou、Chou[114, 131, 193, 196, 197, 199, 200, 252-254]对磁压电耦合双材料进行了研究。Gao、Pan、Qin、Ueda、Cheng[59, 60, 138, 217, 234, 255-258]对磁电热弹耦合材料进行了研究。按平面(空间)问题分：在二维方面，一些裂纹问题得到了解决，典型的研究成果有，Huang 等[259-263]对椭圆裂纹和夹杂进行了研究。Zhou 等[197, 234, 255]对共线界面裂纹及对称平行裂纹进行了研究。Jiang[32, 33]研究了压电螺位错与含界面裂纹涂层夹杂干涉及反平面问题。在三维方面，研究成果很少。Ding、Cheng、Pan[138, 214, 215, 229, 230]给出了磁电热弹耦合材料及磁压电耦合材料的 Green 函数。

在本章中，首先利用广义位移基本解和边界元法，以广义位移间断为未知函数，将磁-电-热-弹耦合场作用下磁电热弹耦合材料平片单裂纹问题转化为求解一超奇异积分方程组问题。进而，利用主部分析法，分析了裂纹前沿广义奇性应力场及广义应力奇性指数，得到了裂纹前沿广义应力场的解析表达式，定义了广义应力强度因子、广义应变能强度因子和广义能量释放率。然后，利用有限部积分理论，对超奇异积分方程组的进行处理，得到了广义应力强度因子和广义应变能强度因子的数值公式。最后，通过编制 FORTRAN 专用程序（7000 余条），通过对矩形平片裂纹问题进行计算，得到了广义应力强度因子和广义应变能强度因子随材料参数和裂纹形状变化的变化规律，给出该类裂纹问题的断裂评定准则。

2.2 基本理论

2.2.1 基本方程

磁电热弹耦合材料平衡方程可表示为[46, 211]：

$$\sum\nolimits_{iJ,i} + f_J = 0 \tag{2-1}$$

为了计算和表示方便，对本章中应用到的符号做如下规定：如无单独声明，大写字母的取值范围为1~6，小写字母的取值范围为1~3；下标中的‘，’表示对相应的坐标求偏微分。其中，式（2-1）中符号$\sum\nolimits_{iJ}$表示广义应力位移，其具体表达式可定义为：

$$\sum\nolimits_{iJ} = \begin{cases} \sigma_{ij} & J = j = 1,2,3 \\ D_i & J = 4 \\ B_i & J = 5 \\ \vartheta_i & J = 6 \end{cases} \tag{2-2}$$

式（2-1）中符号f_J为广义体积力，其具体表达式可定义为：

$$f_J = \begin{cases} f_j & J = j = 1,2,3 \\ -f_e & J = 4 \\ -f_m & J = 5 \\ -f_\vartheta & J = 6 \end{cases} \tag{2-3}$$

磁电热弹耦合材料的物理方程可表示为：

$$\sum\nolimits_{iJ} = E_{iJKl} Z_{Kl} \tag{2-4}$$

其中式（2-4）中符号E_{iJKl}表示磁电热弹系数矩阵，其具体表达式可表示为：

$$E_{iJKl}=\begin{bmatrix} C_{iJKl} & -\Pi_{iJ} & 0 \\ 0 & 0 & -\lambda_{iJ} \end{bmatrix} \tag{2-5}$$

式（2-5）中符号 C_{iJKl}、Π_{iJ} 分别表示 广义弹性系数和广义热应力常数，其具体表达式分别为：

$$C_{iJKl}=\begin{cases} c_{ijkl} & J,K=1,2,3 \\ e_{lij} & J=1,2,3;K=4 \\ e_{ikl} & J=4;K=1,2,3 \\ d_{lij} & J=1,2,3;K=5 \\ d_{ikl} & J=5;K=1,2,3 \\ -g_{il} & J=4;K=5 or J=5;K=4 \\ -\in_{il} & J,K=4 \\ -\mu_{il} & J,K=5 \end{cases} \tag{2-6}$$

$$\Pi_{iJ}=\begin{cases} \iota_{ij} & J=1,2,3;K=6 \\ \varsigma_{il} & J=4;K=6 \\ \eta_{il} & J=5;K=6 \end{cases} \tag{2-7}$$

式（2-4）中符号 Z_{Kl} 表示广义应变场矩阵，其具体表达式为：

$$Z_{Kl}=[U_{K,l} \quad \vartheta_{,l}]^{T} \tag{2-8}$$

其中式（2-8）中符号 U_K 表示广义位移，其具体表达式为：

$$U_{K}=\begin{cases} u_i & K=k=1,2,3 \\ \phi & K=4 \\ \varphi & K=5 \\ \vartheta & K=6 \end{cases} \tag{2-9}$$

式（2-1）～（2-9）中其它符号的含义及其对表达式的更详细解释请参见本文主要符号表中说明或文献[211, 264, 265]中描述。

2.2.2 边界积分方程

假设磁电热弹耦合材料中内含一任意形状平片裂纹 S，建立一固定直角坐标系（整体坐标系）x_i $(i=1,2,3)$。裂纹位于 x_1x_2 平面内，并且裂纹面与 x_3 坐标轴垂直。Γ 为外边界，Ω 为磁电热弹耦合材料所占据的空间区域，$S^{\pm}$ 为裂纹表面边界，应用边界元法和广义形式的Somigliana恒等式，裂纹内部 p 点的广义位移可表示为：

$$U_I=-\int_{\Gamma}(T_{IJ}U_J+U_{IJ}T_J)ds+\int_{\Omega}U_{IJ}b_J dv+\int_{S^{+}+S^{-}}(U_{IJ}T_J-T_{IJ}U_J)ds \tag{2-10}$$

其中，式（2-10）中符号T_J表示广义弹性力，其具体表达式可表示为：

$$T_J=\begin{cases} t_j=\sigma_{jl}n_l \ for\ J=j=1,2,3 \\ q=D_l n_l \ \ for \quad J=4 \\ b=B_l n_l \ \ for \quad J=5 \\ \rho=\vartheta_{,J}n_i \ for \quad J=6 \end{cases} \tag{2-11}$$

式（2-10）中符号U_{IJ}、T_{IJ}分别表示广义位移基本解、广义面力基本解。两者之间的相互关系可表示为：

$$T_{IJ}(\xi-x)=E_{kJMn}U_{IM,n}(\xi-x)n_k \tag{2-12}$$

为应用超奇异积分方程方法，特引入裂纹面广义位移间断函数$\tilde{U}_J$，其具体表达式可定义为：

$$\tilde{U}_J=\begin{cases} \tilde{u}_j=u_j^+-u_j^- \quad J=j=1,2,3 \\ \tilde{\phi}_j=\phi_j^+-\phi_j^- \qquad J=4 \\ \tilde{\varphi}_j=\varphi_j^+-\varphi_j^- \qquad J=5 \\ \tilde{\Upsilon}_j=\Upsilon_j^+-\Upsilon_j^- \qquad J=6 \end{cases} \tag{2-13}$$

设p点为源点，q点为场点，利用以下关系式

$$T_{IJ}(p,q^+)=-T_{IJ}(p,q^-)=T_{IJ}^+(p,q)$$

$$U_{IJ}(p,q^+)=U_{IJ}(p,q^-)$$

式（2-10）中广义弹性位移可重新表示为：

$$U_I=\int_\Gamma (U_{IJ}T_J-T_{IJ}U_J)ds-\int_{S^+}T_{IJ}^+\tilde{U}_J ds+\int_\Omega U_{IJ}f_J dv \tag{2-14}$$

把本构关系代入上式，广义应力的具体表达式可表示为：

$$\sum\nolimits_{ij}=\int_\Gamma (D_{KiJ}T_K-S_{KiJ}U_K)ds-\int_{S^+}S_{KiJ}^+\tilde{U}_K ds+\int_\Omega D_{KiJ}f_K dv \tag{2-15}$$

式（2-15）中符号S_{KiJ}和D_{KiJ}为积分核函数，其具体表达式可表示为：

$$S_{KiJ}(p,q)=-E_{iJMn}T_{MK,n}(p,q) \tag{2-16}$$

$$D_{KiJ}(p,q)=-E_{iJMn}U_{MK,n} \tag{2-17}$$

式（2-15）为有限体裂纹问题的完整边界积分方程；$\int_\Gamma (D_{KiJ}T_K-S_{KiJ}U_K)ds$是反映空间边界影响的积分项，是通常意义下Cauchy主值积分，奇异性较小，数值求解容易；$\int_{S^+}S_{KiJ}^+\tilde{U}_K ds$是反

应裂纹边界的积分项，为超奇异积分，具有高奇异性，不能应用通常求解方法进行求解，是研究重点。实际工程应用中，只要空间无限大体裂纹问题解决后，对于有限体裂纹问题可应用叠加法求解。利用式（2-10）～（2-17）及边界条件便可得到边界积分方程。源点 $p \to P$(趋近于边界Γ)，裂纹面上的边界积分方程可以表示为：

$$C_{IJ}U_I = \int_{\Gamma}(U_{IJ}T_J - T_{IJ}U_J)ds + \int_{\Omega}U_{IJ}f_J dv + \int_{S^+}T_{IJ}^+\tilde{U}_J ds \tag{2-18}$$

其中C_{IJ}是与边界点P相关的一个常数，把裂纹表面的弹性，电，磁和热边界条件代入上式，可得到超奇异积分方程：

$$\text{⨎}_{S^+} S_{KiJ}^+\tilde{U}_K ds = \int_{\Gamma}(D_{KiJ}T_K - S_{KiJ}U_K)ds + \int_{\Omega}D_{KiJ}f_K dv \tag{2-19}$$

其中⨎ 为超奇异积分算子，需要通过有限部积分方法求解。其定义式为：

$$\int_s^r \frac{f(x)}{(x-s)^n}dx = \int_s^r (x-s)^{-n}[f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(s)}{k!}(x-s)^k]dx + \sum_{k=0}^{n-2}\frac{f^{(k)}(s)}{(-n+k+1)k!}(r-s)^{-n+k+1} + \frac{f^{(n-1)}(s)}{(n-1)!}\ln(r-s) \tag{2-20}$$

其中 $f^{(k)}(s) = \dfrac{d^k f}{dx^k}(s)$， $f(x)$ 是实函数，在[s,r]内连续，在s点的邻域内 $f(x) \in C^{n-1,\alpha}$，$(n \ge 1, 0 < \alpha \le 1)$ 即 $f(x)$ 为n-1阶可导，且 $f^{(n-1)}(x)$ 满足Holder-Lipschitz(H-L)条件。

2.2.3 主部分析法

超奇异积分方程的解析解法非常困难，通常使用数值方法求解。裂纹问题的超奇异积分方程通常包括未知函数的超奇异积分项和普通意义常规积分项，由于正常积分可采用高精度数值积分技术，因此决定求解精度的是超奇异积分。为提高数值计算精度，应正确找出积分方程未知函数在裂纹前沿的性态，然后构造高精度数值求积公式，本节简单介绍超奇异积分的主部分析方法，并对超奇异积分方程的未知函数进行性态分析。超奇异积分方程的性态由超奇异积分核决定，只需对奇异主部进行分析，超奇异积分方程的主部可写为以下形式：

$$\frac{1}{\pi}\int_a^b \frac{\tilde{U}_i(\xi)}{(x-\xi)^2}d\xi = B_i(x) \tag{2-21}$$

式（2-21）中 $x \in (\varepsilon, a+\varepsilon)$ 或 $x \in (\varepsilon, b-\varepsilon)$， ε 为正无穷小量， $B_i(x)$ 为在裂纹端点的有界函数，未知函数可表示为；

$$\tilde{U}_i(\xi) = F_i(\xi)Y_i(\xi) = F_i(\xi)(b-\xi)^{\alpha_i}(\xi-a)^{\beta_i} \tag{2-22}$$

其中式（2-22）中 $F_i\ (\xi)$ 为新的未知函数，在裂纹前沿有界，函数 $Y_i\ (\xi)=(b-\xi)^{\alpha_i}(\xi-a)^{\beta_i}$ 为裂纹的规则函数，指数 α_i， β_i 反映了未知函数在端点 a,b的渐进性质，称它们为奇异积分的性态指数，因为应力在裂纹端点可积，故指数应该满足 $0<\alpha$， $\beta<2$ 。将未知函数表达式代入主部方程，以上积分式当 x 在a端点邻域时为：

$$\frac{1}{\pi}\int_a^b \frac{\tilde{U}_i\ (\xi)}{(x-\xi)^2}d\xi \approx \lim_{x\to a,\varepsilon\to 0}\int_a^{a+\varepsilon}\frac{F_i\ (a)(b-\xi)^{\alpha_i}(\xi-a)^{\beta_i}}{(x-\xi)^2}d\xi \tag{2-23}$$

对上述表达式（2-23）依次作极限运算可以得到：

$$\frac{1}{\pi}\int_a^b \frac{\tilde{U}_i\ (\xi)}{(x-\xi)^2}d\xi \approx -(\xi-a)^{\beta_i-1}F_i\ (a)(b-a)^{\alpha_i}\beta_i ctg\beta_i\pi \qquad x\in(\varepsilon,a+\varepsilon) \tag{2-24}$$

同理，表达式（2-23）边的超奇异积分在裂纹端点b邻域中，可近似表示为

$$\frac{1}{\pi}\int_a^b \frac{\tilde{U}_i\ (\xi)}{(x-\xi)^2}d\xi \approx \lim_{x\to b,\varepsilon\to 0}\int_{b-\varepsilon}^{b}\frac{F_i\ (a)(\xi-a)^{\alpha_i}(b-a)^{\beta_i}}{(x-\xi)^2}d\xi \tag{2-25}$$

作极限运算可以得到：

$$\frac{1}{\pi}\int_a^b \frac{\tilde{U}_i\ (\xi)}{(x-\xi)^2}d\xi \approx -(b-x)^{\alpha_i-1}F_i\ (b)(b-a)^{\beta_i}\alpha_i ctg\alpha_i\pi \qquad x\in(\varepsilon,b-\varepsilon) \tag{2-26}$$

将以上结果依次代入主部方程（2-21），并做 x 趋于端点a,b的极限运算，由于 $B_i\ (x)$ 在裂纹端点有界，因而可以求得性态指数满足的特征方程，从而求出性态指数的具体值，积分方程中的未知函数可表示为具体形式。对于均匀材料 α_i， β_i 的值为1/2，但是对于非均匀材料和与界面相关的裂纹，奇异指数需根据具体的超奇异积分方程求得。本文中涉及到的各类裂纹的奇异指数可应用本方法的思路，拓展解决，详细做法见各章的相关内容。

2.2.4 有限部积分概念

1923年Hadamard在研究双曲型偏微分方程时提出了有限部积分的概念，后来有Kutt对该积分理论进行完善，具体定义如下：

$$J=\int_s^r \frac{f(\xi)}{(\xi-s)^n}d\xi \tag{2-27}$$

其中n为大于或等于1的整数。上式中函数 $f(\xi)$ 在[r,s]区间连续，且在x邻域中k阶可导。上述积分在常规意义和Cauchy意义下都是发散的。有限部积分就是将发散积分的发散部分去掉，只保留其有界部分，使积分只按有限部分计算。 按照Hadamard的定义，以下单边有限部分应按下列公式计算。

$$\int_s^r \frac{f(\xi)}{(\xi-s)^n}dx = \int_s^r (\xi-s)^{-n}[f(\xi)-\sum_{k=0}^{n-1}\frac{f^{(k)}(s)}{k!}(\xi-s)^k]d\xi + \sum_{k=0}^{n-2}\frac{f^{(k)}(s)}{(-n+k+1)k!}(r-s)^{-n+k+1}+\frac{f^{(n-1)}(s)}{(n-1)!}\ln(r-s) \tag{2-28}$$

上式也可以用极限形式表示

$$\int_s^r \frac{f(\xi)}{(\xi-s)^n}d\xi = \lim_{\varepsilon\to 0}[\int_{s+\varepsilon}^r (\xi-s)^{-n}f(\xi)d\xi + \sum_{k=0}^{n-1}\frac{f^{(k)}(\xi)}{k!(k+1-n)}+\frac{f^{(n-1)}(\xi)}{(n-1)}\ln\varepsilon] \tag{2-29}$$

当n=2，3时

$$\begin{aligned}\int_a^b \frac{f(\xi)}{(\xi-a)^2}d\xi &= \int_a^c \frac{f(\xi)}{(\xi-a)^2}d\xi + \int_c^b \frac{f(\xi)}{(\xi-a)^2}d\xi \\ &= \lim_{\varepsilon\to 0}[\int_a^{c-\varepsilon}\frac{f(\xi)}{(\xi-a)^2}d\xi + \int_{c+\varepsilon}^b \frac{f(\xi)}{(\xi-a)^2}d\xi - \frac{2f\ (\varepsilon)}{\varepsilon}]\end{aligned} \tag{2-30}$$

$$\begin{aligned}\int_a^b \frac{f(\xi)}{(\xi-a)^3}d\xi &= \int_a^c \frac{f(\xi)}{(\xi-a)^3}d\xi + \int_c^b \frac{f(\xi)}{(\xi-a)^3}d\xi \\ &= \lim_{\varepsilon\to 0}[\int_a^{c-\varepsilon}\frac{f(\xi)}{(\xi-a)^3}d\xi + \int_{c+\varepsilon}^b \frac{f(\xi)}{(\xi-a)^3}d\xi - \frac{3f\ (\varepsilon)}{\varepsilon^2}]\end{aligned} \tag{2-31}$$

这就是连续函数 $f(\xi)$ 的二、三阶双边有限部积分的定义。根据以上定义[208]，下面给出几个有用的有限部积分。设函数可分解为 $f(\xi)=F_1(\xi)F_2(\xi)$ 。其中 $F_1(\xi)$ 为有界函数，$F_2(\xi)=(1-\xi^2)^{1/2}$，关于函数 $f(\xi)$ 的二阶有限部积分，可以由以上定义按下式计算：

$$\begin{aligned}\int_{-1}^1 \frac{f(\xi)}{(\xi-a)^2}d\xi &= \int_{-1}^1 [F_1(\xi)-F_1(a)-(1-a)F_1'(\xi)]\frac{F_2(\xi)}{(\xi-a)^2}d\xi + F_1(a)\int_{-1}^1 \frac{f(\xi)}{(\xi-a)^2}d\xi \\ &\quad + F_1'(\xi)\int_{-1}^1 \frac{F_2(\xi)}{\xi-a}d\xi \qquad -1<x<1\end{aligned} \tag{2-32}$$

由此可知，上式右边第一项为正常积分，可由常规数值积分方法求出，后两个积分可分别由有限部积分和主值积分计算，使用上面的定义，它们分别可表示为：

$$\int_{-1}^1 \frac{\sqrt{1-\xi^2}}{\xi-a}d\xi = -\pi a\,,\quad \int_{-1}^1 \frac{1}{(\xi-a)^2}d\xi = -\frac{1}{1-a}-\frac{1}{1+a}\,,\quad \int_{-1}^1 \frac{\sqrt{1-\xi^2}}{(\xi-a)^2}d\xi = -\pi \tag{2-33}$$

Chebyshev多项式的有限部积分为：

$$\int_{-1}^1 \frac{F_n(\xi)\sqrt{1-\xi^2}}{(\xi-a)^2}d\xi = -\pi(n+1)F_n(a) \qquad n\ge 0 \tag{2-34}$$

这里的$F_n(x)$是第二类Chebyshev多项式。由此，Erdogan 建立了一套方便有效的Gauss型求积方法。在断裂力学混合边值问题中，其控制方程可写成以下的一般表达式

$$\int_a^b [k_s(x,\xi)+k(x,\xi)]f(\xi)d\xi = g(x) \qquad a<x<b \tag{2-35}$$

其中积分核函数$k(x,\xi)$在[a,b]平方可积，并且$g(x)$是一已知的有界函数，积分核函数$k_s(x,\xi)$具有二阶强奇异性。问题的权函数$F_2(\xi)$决定于强奇异积分核函数$k_s(x,\xi)$。根据超奇异积分方程主部分析的结果，未知函数$f(\xi)$在积分端点[a,b]有界，且可表示为以下形式：

$$f(\xi)=F_1(\xi)F_2(\xi) \tag{2-36}$$

式中的$F_1(\xi)$在[a,b]上为未知的有界函数，而$F_2(\xi)$为以下形式的未知函数

$$F_2(\xi)=(b-\xi)^{1/2}(\xi-a)^{1/2} \tag{2-37}$$

若式（2-27）中的超奇异积分在端点a,b有界或可积，则以上的超奇异积分方程便可以使用有限部积分的定义进行求解。通常可使用将函数$F\,(\xi)$进行近似展开为截断级数的方法求解：

$$F_1(\xi)\approx \sum_{n=0}^{N} a_n \Psi_n(\xi) \tag{2-38}$$

式（2-38）式中系数a_n可以由加权残数法来决定，$\Psi_n(\xi)$为任意的完备函数系。把（2-30）和（2-32）带入（2-29）中可得到以下关系式：

$$\sum_{n=0}^{N} a_n G_n(x)\approx g(x) \qquad a<x<b \tag{2-39}$$

其中

$$G_n(x)=\int_a^b k_s(x,\xi)\Psi_n(\xi)F_2(\xi)d\xi+\int_a^b k(x,\xi)\Psi_n(\xi)F_2(\xi)d\xi \tag{2-40}$$

系数a_n决定于以下的代数方程组

$$\sum_{n=0}^{N} a_n\int_a^b G_n(x)\varphi_j(x)W_j(x)dx=\int_a^b g(x)\varphi_j(x)W_j(x)dx \qquad j=0,1,2.....N \tag{2-41}$$

$\varphi_j(x)$是完备函数系（如正交多项式），而$W_j(x)$是该函数的权函数。通常将完备函数系（$\Psi_n(x),\varphi_j(x)$）选为相互正交的多项式，在做积分时可以得到简化。常用的正交多项式有三角函数系、Legendre多项式、Chebyshev多项式和$\delta(x)$函数等，也可选为比较简单的普通多项式x^j，如果将$W_j(x),\varphi_j(x)$分别选取为以下函数：

$$W_j(x)=1\text{；}\ \varphi_j(x)=\delta(x-x_j) \tag{2-42}$$

由式（2-32）便得到非常简单的代数方程组

$$\sum_{n=0}^{N} a_n G_n(x_j) = g(x_j) \tag{2-43}$$

配置点 x_j 可以任意选择，通常选择Legendre多项式、Chebyshev多项式的根。虽然对配置点的选择没有限制，但经验表明，配置点集中于端点处的对称选取数值结果较好。一般的，解方程组得到的待定系数 a_n 回代到式（2-28）后便可得到未知函数 $f(\xi)$ 的近似解。更具体的论述可参看Erdogan的相关文献和论著[3, 169-174]。

2.3 广义基本解

2.3.1 广义位移基本解

对于磁电热弹耦合材料，利用[229]中的方法，磁电热弹平衡问题控制方程可表示为：

$$\begin{cases} u_1 = (n_{41}\dfrac{\partial^6}{\partial x_3^6} + n_{51}\Delta\dfrac{\partial^4}{\partial x_3^4} + n_{61}\Delta^2\dfrac{\partial^2}{\partial x_3^2} + n_{71}\Delta^3)\dfrac{\partial \widehat{g}}{\partial x_1} - \dfrac{\partial \widehat{f}}{\partial x_2} \\ u_2 = (n_{41}\dfrac{\partial^6}{\partial x_3^6} + n_{51}\Delta\dfrac{\partial^4}{\partial x_3^4} + n_{61}\Delta^2\dfrac{\partial^2}{\partial x_3^2} + n_{71}\Delta^3)\dfrac{\partial \widehat{g}}{\partial x_2} - \dfrac{\partial \widehat{f}}{\partial x_1} \\ u_m = (n_{4m}\dfrac{\partial^6}{\partial x_3^2} + n_{5m}\Delta\dfrac{\partial^4}{\partial x_3^4} + n_{6m}\Delta^2\dfrac{\partial^2}{\partial x_3^2} + n_{7m}\Delta^3)\dfrac{\partial \widehat{g}}{\partial x_3} \\ \Upsilon = (n_1\dfrac{\partial^8}{\partial x_3^8} + n_2\Delta\dfrac{\partial^6}{\partial x_3^6} + n_3\Delta^2\dfrac{\partial^4}{\partial x_3^4} + n_4\Delta^3\dfrac{\partial^2}{\partial x_3^2} + n_5\Delta^4)\widehat{g} \end{cases} \tag{2-44}$$

式（2-44）中 $m = 3,4,5$,势函数 $\widehat{g}$ 和 $\widehat{f}$ 应满足下面方程:

$$\prod_{n=2}^{6}(\Delta + \frac{\partial^2}{\partial z_n^2})\widehat{g} = 0 \tag{2-45}$$

$$(\Delta + \frac{\partial^2}{\partial z_1^2})\widehat{f} = 0 \tag{2-46}$$

其中 $z_i = s_i z$, $s_1 = \sqrt{c_{66}/c_{44}}$, $s_6 = \sqrt{\lambda_{11}/\lambda_{33}}$ 。 s_t ($t = 2,3,4,5$)为下面方程的四个根(复数根的实部部分):

$$a_1 s^8 - a_2 s^6 + a_3 s^4 - a_4 s^2 + a_5 = 0 \tag{2-47}$$

式（2-47）中符号具体表达式见附录A。在点 $\xi(\xi_1, \xi_2, \xi_3)$ 作用广义点力，点 $x(x_1, x_2, x_3)$ 处广义位移可表示为：

$$U_{mJ}=\sum_{i=1}^{5}\frac{A_{im}}{R_i}(\frac{x_\alpha-\xi_\alpha}{\dot{R}_i}\delta_{\alpha J}+\lambda_i^{w}\delta_{3J}+\lambda_i^{\phi}\delta_{4J}+\lambda_i^{\varphi}\delta_{5J}+\frac{\xi_3-x_3}{R_i^2}\lambda_i^{\vartheta}s_i^2\delta_{6J}) \tag{2-48}$$

式（2-48）中m=3,4,5,6；$\alpha=1,2$；其它符号具体表达式见附录A。在点$\xi(\xi_1,\xi_2,\xi_3)$沿x_2方向作用单位点力，点$x(x_1,x_2,x_3)$处广义位移基本解可表示为：

$$\begin{aligned}U_{2J}=&\sum_{i=1}^{5}D_i[(x_2-\xi_2)(\frac{\delta_{\alpha J}(\xi_\alpha-x_\alpha)}{R_i\dot{R}_i^2}-\frac{\lambda_i^{w}\delta_{3J}+\lambda_i^{\phi}\delta_{4J}+\lambda_i^{\varphi}\delta_{5J}}{R_i\dot{R}_i}-\frac{\lambda_i^{\vartheta}s_i\delta_{6J}}{R_i^3})+\frac{\delta_{2J}}{\dot{R}_i}]\\&D_0[(x_1-\xi_1)\frac{(x_1-\xi_1)\delta_{2J}-(x_2-\xi_2)\delta_{1J}}{R_0\dot{R}_0^2}-\frac{D_0\delta_{2J}}{\dot{R}_0}]\end{aligned} \tag{2-49}$$

式（2-49）中符号具体表达式见附录A。当在点$\xi(\xi_1,\xi_2,\xi_3)$沿x_1方向作用单位点力，点$x(x_1,x_2,x_3)$处广义位移基本解，可通过将上式中作如下变换得到：$x_1\to x_2$，$x_2\to -x_1$。

2.3.2 广义面力基本解

对式(2-48)～（2-49）应用弹性基本理论、边界条件，经过繁杂的数学推导，可得到磁电热弹耦合材料三维裂纹面坐标系下面力基本解，其具体表达式如下：

$$\begin{aligned}T_{11}=&-c_{44}D_0s_1(R_0^{-1}\hat{R}_0^{-1}-R_0^{-3}\hat{R}_0^{-1}X_2{}^2-R_0^{-2}\hat{R}_0^{-2}X_2{}^2)+\iota_{11}\sum_{i=1}^{5}D_i\lambda_i^{v}R_i^{-3}X_1+\sum_{i=1}^{5}D_i[c_{44}s_i\\&+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}](R_i^{-1}\hat{R}_i^{-1}-R_i^{-3}\hat{R}_i^{-1}X_1{}^2-R_i^{-2}\hat{R}_i^{-2}X_1{}^2)\end{aligned} \tag{2-50}$$

$$\begin{aligned}T_{21}=&-c_{44}D_0s_1X_1X_2(R_0^{-3}\hat{R}_0^{-1}+R_0^{-2}\hat{R}_0{}^2)-\sum_{i=1}^{5}D_iX_1X_2[c_{44}s_i+c_{44}\lambda_i^{w}\\&+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}](R_i^{-3}\hat{R}_i^{-1}+R_i^{-2}\hat{R}_i^{-2})\end{aligned} \tag{2-51}$$

$$T_{31}=-\sum_{i=1}^{5}A_i^{P}[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}]R_i^{-3}X_1 \tag{2-52}$$

$$T_{41}=-\sum_{i=1}^{5}A_i^{Q}[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}]R_i^{-3}X_1 \tag{2-53}$$

$$T_{51}=-\sum_{i=1}^{5}A_i^{J}[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}]R_i^{-3}X_1 \tag{2-54}$$

$$T_{61}=-\sum_{i=1}^{5}A_i^{\Upsilon}[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}]R_i^{-3}X_1 \tag{2-55}$$

$$\begin{aligned}T_{12}=&-c_{44}D_0s_1X_1X_2(R_0^{-3}\hat{R}_0^{-1}+R_0^{-2}\hat{R}_0{}^2)-\sum_{i=1}^{5}D_iX_1X_2[c_{44}s_i+c_{44}\lambda_i^{w}\\&+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}](R_i^{-3}\hat{R}_i^{-1}+R_i^{-2}\hat{R}_i^{-2})\end{aligned} \tag{2-56}$$

$$T_{22}=-c_{44}D_0s_1(R_0^{-1}\hat{R}_0^{-1}-R_0^{-3}\hat{R}_0^{-1}X_1{}^2-R_0^{-2}\hat{R}_0^{-2}X_1{}^2)+\iota_{22}\sum_{i=1}^{5}D_i\,\lambda_i^{v}R_i^{-3}X_2+\sum_{i=1}^{5}D_i[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}](R_i^{-1}\hat{R}_i^{-1}-R_i^{-3}\hat{R}_i^{-1}X_2{}^2-R_i^{-2}\hat{R}_i^{-2}X_2{}^2) \tag{2-57}$$

$$T_{32}=-\sum_{i=1}^{5}A_i^{P}[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}]R_i^{-3}X_2 \tag{2-58}$$

$$T_{42}=-\sum_{i=1}^{5}A_i^{Q}[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}]R_i^{-3}X_2 \tag{2-59}$$

$$T_{52}=-\sum_{i=1}^{5}A_i^{J}[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}]R_i^{-3}X_2 \tag{2-60}$$

$$T_{62}=-\sum_{i=1}^{5}A_i^{\Upsilon}[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}]R_i^{-3}X_2 \tag{2-61}$$

$$T_{13}=\iota_{33}\sum_{i=1}^{5}D_i\,\lambda_i^{v}R_i^{-3}X_1+\sum_{i=1}^{5}D_i[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}]R_i^{-3}X_1 \tag{2-62}$$

$$T_{23}=\iota_{33}\sum_{i=1}^{5}D_i\,\lambda_i^{v}R_i^{-3}X_2+\sum_{i=1}^{5}D_i[c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi}]R_i^{-3}X_2 \tag{2-63}$$

$$T_{33}=-\sum_{i=1}^{5}A_i^{P}s_i\,[-c_{13}+c_{33}s_i\,\lambda_i^{w}+e_{33}s_i\,\lambda_i^{\phi}+d_{33}s_i\,\lambda_i^{\varphi}]R_i^{-3}X_3 \tag{2-64}$$

$$T_{43}=-\sum_{i=1}^{5}A_i^{Q}s_i\,[-c_{13}+c_{33}s_i\,\lambda_i^{w}+e_{33}s_i\,\lambda_i^{\phi}+d_{33}s_i\,\lambda_i^{\varphi}]R_i^{-3}X_3 \tag{2-65}$$

$$T_{53}=-\sum_{i=1}^{5}A_i^{J}s_i\,[-c_{13}+c_{33}s_i\,\lambda_i^{w}+e_{33}s_i\,\lambda_i^{\phi}+d_{33}s_i\,\lambda_i^{\varphi}]R_i^{-3}X_3 \tag{2-66}$$

$$T_{63}=-\sum_{i=1}^{5}A_i^{\Upsilon}s_i\,[-c_{13}+c_{33}s_i\,\lambda_i^{w}+e_{33}s_i\,\lambda_i^{\phi}+d_{33}s_i\,\lambda_i^{\varphi}]R_i^{-3}X_3 \tag{2-67}$$

$$T_{14}=-\sum_{i=1}^{5}D_i[-e_{31}+e_{33}s_i\,\lambda_i^{w}-\in_{33}s_i\,\lambda_i^{\phi}-g_{33}s_i\,\lambda_i^{\varphi}+\zeta_{33}]R_i^{-3}X_1 \tag{2-68}$$

$$T_{24}=-\sum_{i=1}^{5}D_i[-e_{31}+e_{33}s_i\,\lambda_i^{w}-\in_{33}s_i\,\lambda_i^{\phi}-g_{33}s_i\,\lambda_i^{\varphi}+\zeta_{33}]R_i^{-3}X_2 \tag{2-69}$$

$$T_{34}=-\sum_{i=1}^{5}D_i[-e_{31}+e_{33}s_i\,\lambda_i^{w}-\in_{33}s_i\,\lambda_i^{\phi}-g_{33}s_i\,\lambda_i^{\varphi}+\zeta_{33}]R_i^{-3}X_3 \tag{2-70}$$

$$T_{44}=-\sum_{i=1}^{5}A_i^{P}s_i[-e_{31}+e_{33}s_i\,\lambda_i^{w}-\in_{33}s_i\,\lambda_i^{\phi}-g_{33}s_i\,\lambda_i^{\varphi}+\zeta_{33}]R_i^{-3}X_3 \tag{2-71}$$

$$T_{54}=-\sum_{i=1}^{5}A_i^{Q}s_i[-e_{31}+e_{33}s_i\,\lambda_i^{w}-\in_{33}s_i\,\lambda_i^{\phi}-g_{33}s_i\,\lambda_i^{\varphi}+\zeta_{33}]R_i^{-3}X_3 \tag{2-72}$$

$$T_{64}=-\sum_{i=1}^{5}A_i^{\Upsilon}s_i[-e_{31}+e_{33}s_i\lambda_i^{w}-\in_{33}s_i\lambda_i^{\phi}-g_{33}s_i\lambda_i^{\varphi}+\zeta_{33}]R_i^{-3}X_3 \tag{2-73}$$

$$T_{15}=-\sum_{i=1}^{5}D_i s_i[-d_{31}+d_{33}s_i\lambda_i^{w}-g_{33}s_i\lambda_i^{\phi}-\mu_{33}s_i\lambda_i^{\varphi}+\eta_{33}]R_i^{-3}X_1 \tag{2-74}$$

$$T_{25}=-\sum_{i=1}^{5}D_i s_i[-d_{31}+d_{33}s_i\lambda_i^{w}-g_{33}s_i\lambda_i^{\phi}-\mu_{33}s_i\lambda_i^{\varphi}++\eta_{33}]R_i^{-3}X_2 \tag{2-75}$$

$$T_{35}=-\sum_{i=1}^{5}A_i^{P}s_i[-d_{31}+d_{33}s_i\lambda_i^{w}-g_{33}s_i\lambda_i^{\phi}-\mu_{33}s_i\lambda_i^{\varphi}+\eta_{33}]R_i^{-3}X_3 \tag{2-76}$$

$$T_{45}=-\sum_{i=1}^{5}A_i^{Q}s_i[-d_{31}+d_{33}s_i\lambda_i^{w}-g_{33}s_i\lambda_i^{\phi}-\mu_{33}s_i\lambda_i^{\varphi}+\eta_{33}]R_i^{-3}X_3 \tag{2-77}$$

$$T_{55}=-\sum_{i=1}^{5}A_i^{J}s_i[-d_{31}+d_{33}s_i\lambda_i^{w}-g_{33}s_i\lambda_i^{\phi}-\mu_{33}s_i\lambda_i^{\varphi}+\eta_{33}]R_i^{-3}X_3 \tag{2-78}$$

$$T_{65}=-\sum_{i=1}^{5}A_i^{\Upsilon}s_i[-d_{31}+d_{33}s_i\lambda_i^{w}-g_{33}s_i\lambda_i^{\phi}-\mu_{33}s_i\lambda_i^{\varphi}+\eta_{33}]R_i^{-3}X_3 \tag{2-79}$$

$$T_{16}=\sum_{i=1}^{5}D_i s_i\lambda_i^{v}(R_i^{-3}\lambda_{31}-3R_i^{-5}\lambda_{30}X_1^2-3R_i^{-5}\lambda_{32}X_1X_2+3R_i^{-5}\lambda_{33}X_3^2s_i^2) \tag{2-80}$$

$$T_{26}=\sum_{i=1}^{5}D_i\lambda_i^{v}(3R_i^{-5}\lambda_{31}X_1X_2-R_i^{-3}\lambda_{32}+3R_i^{-5}\lambda_{32}X_1^2+3R_i^{-5}\lambda_{33}X_1X_3s_i^2) \tag{2-81}$$

$$T_{46}=\sum_{i=1}^{5}A_i^{P}\lambda_i^{v}s_i^2(-3R_i^{-5}\lambda_{31}X_1X_3-3R_i^{-5}\lambda_{32}X_2X_3+R_i^{-3}\lambda_{33}-3R_i^{-5}\lambda_{33}X_3^2s_i^2) \tag{2-82}$$

$$T_{56}=\sum_{i=1}^{5}A_i^{J}\lambda_i^{v}s_i^2(-3R_i^{-5}\lambda_{31}X_1X_3-3R_i^{-5}\lambda_{32}X_2X_3+R_i^{-3}\lambda_{33}-3R_i^{-5}\lambda_{33}X_3^2s_i^2) \tag{2-83}$$

$$T_{66}=\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_i^{v}s_i^2(-3R_i^{-5}\lambda_{31}X_1X_3-3R_i^{-5}\lambda_{32}X_2X_3+R_i^{-3}\lambda_{33}-3R_i^{-5}\lambda_{33}X_3^2s_i^2) \tag{2-84}$$

T_{ij}^{I} 的第一个下标 i 表示 T_{ij}^{I} 是 x_i 坐标方向的面力分量，第二个下标 j 则表示作用于场点 q 的单位集中力指向 ξ_j 坐标方向；ξ_i 表示为源点 p 坐标，x_i 表示为场源点 q 坐标。

2.3.3 广义应力积分核函数

通过对式(2-50)～(2-84)进行求导，再应用弹性理论及边界条件，经过繁杂的数学推导，简化后可得到与界面垂直裂纹面坐标系上应力积分核 S_{KIJ} 函数，具体表达式如下：

$$\begin{aligned}S_{131}=\sum_{i=1}^{5}&(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})(c_{44}D_i s_i+c_{44}A_i^{P}\\&+e_{15}A_i^{Q}+d_{15}A_i^{J})R_i^{-3}(1-3X_1^2R_i^2)-c_{44}^2D_0s_1^2R_0^{-3}(1-3X_2^2R_0^2)\end{aligned} \tag{2-85}$$

$$S_{231}=-\sum_{i=1}^{5}3(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-5}X_1X_2-3c_{44}^2D_0s_1^2R_0^{-5}X_1X_2 \tag{2-86}$$

$$S_{331}=-\sum_{i=1}^{5}3(-c_{13}+c_{33}\lambda_i^w s_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^v s_i)(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-5}X_1X_3 \tag{2-87}$$

$$S_{431}=-\sum_{i=1}^{5}3(-e_{31}+e_{33}\lambda_i^w s_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^v s_i)(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-5}X_1X_3 \tag{2-88}$$

$$S_{531}=-\sum_{i=1}^{5}3(-d_{31}+d_{33}\lambda_i^w s_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i-\eta_{33}\lambda_i^v s_i)(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-5}X_1X_3 \tag{2-89}$$

$$S_{631}=-\sum_{i=1}^{5}3\lambda_{33}s_i^2(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-5}X_1 \tag{2-90}$$

$$S_{132}=\sum_{i=1}^{5}3(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-5}X_1X_2+c_{44}^2D_0s_1^2R_0^{-5}X_1X_2 \tag{2-91}$$

$$S_{232}=-\sum_{i=1}^{5}(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-3}(1-3X_2^2R_i^2)-c_{44}^2D_0s_1^2R_0^{-3}(1-3X_1^2R_0^2) \tag{2-92}$$

$$S_{332}=-\sum_{i=1}^{5}3(-c_{13}+c_{33}\lambda_i^w s_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^v s_i)(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-5}X_2X_3 \tag{2-93}$$

$$S_{432}=-\sum_{i=1}^{5}3(-e_{31}+e_{33}\lambda_i^w s_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^v s_i)(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-5}X_2X_3 \tag{2-94}$$

$$S_{532}=-\sum_{i=1}^{5}3(-d_{31}+d_{33}\lambda_i^w s_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i-\eta_{33}\lambda_i^v s_i)(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-5}X_2X_3 \tag{2-95}$$

$$S_{632}=-\sum_{i=1}^{5}3\lambda_{33}s_i^2(c_{44}D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)R_i^{-5}X_2+\lambda_{32}s_i^2R_i^{-5}X_3 \tag{2-96}$$

$$S_{133}=\sum_{i=1}^{5}3s_i\,(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_i\,c_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-5}X_1X_3+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)R_i^{-3}X_1 \tag{2-97}$$

$$S_{233}=\sum_{i=1}^{5}3s_i\,(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_i\,c_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-5}X_2X_3+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)R_i^{-3}X_2 \tag{2-98}$$

$$S_{333}=\sum_{i=1}^{5}s_i\,(-c_{13}+c_{33}\lambda_i^ws_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^vs_i)(-D_i\,c_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-3}+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(-c_{13}+c_{33}\lambda_i^ws_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^vs_i)R_i^{-3}X_3 \tag{2-99}$$

$$S_{433}=\sum_{i=1}^{5}s_i\,(-e_{31}+e_{33}\lambda_i^ws_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^vs_i)(-D_i\,c_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-3}+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(-e_{31}+e_{33}\lambda_i^ws_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^vs_i)R_i^{-3}X_3 \tag{2-100}$$

$$S_{533}=\sum_{i=1}^{5}s_i\,(-d_{31}+d_{33}\lambda_i^ws_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i-\eta_{33}\lambda_i^vs_i)(-D_i\,c_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-3}+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(-d_{31}+d_{33}\lambda_i^ws_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i-\eta_{33}\lambda_i^vs_i)R_i^{-3}X_3 \tag{2-101}$$

$$S_{633}=\sum_{i=1}^{5}s_i^2\lambda_i^v(-D_i\,c_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-5}(3\lambda_{31}X_1+3\lambda_{32}X_2)-\sum_{i=1}^{5}A_i^\Upsilon s_i^2\lambda_i^v\lambda_{33}\iota_{33}R_i^{-3} \tag{2-102}$$

$$S_{134}=\sum_{i=1}^{5}3s_i\,(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_i\,e_{31}+e_{33}A_i^Ps_i-\in_{33}A_i^Qs_i-g_{33}A_i^Js_i)R_i^{-5}X_1X_3+\sum_{i=1}^{5}A_i^\Upsilon\zeta_{33}(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)R_i^{-3}X_1 \tag{2-103}$$

$$S_{234}=\sum_{i=1}^{5}3s_i\,(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_i\,e_{31}+e_{33}A_i^Ps_i-\in_{33}A_i^Qs_i-g_{33}A_i^Js_i)R_i^{-5}X_2X_3+\sum_{i=1}^{5}A_i^\Upsilon\zeta_{33}(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)R_i^{-3}X_2 \tag{2-104}$$

$$S_{334}=\sum_{i=1}^{5}s_i(-c_{13}+c_{33}\lambda_i^w s_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^v s_i)(-D_i e_{31}+e_{33}A_i^P s_i-\in_{33}A_i^Q s_i-g_{33}A_i^J s_i)R_i^{-3}+\sum_{i=1}^{5}A_i^\Upsilon\zeta_{33}(-c_{13}+c_{33}\lambda_i^w s_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^v s_i)R_i^{-3}X_3 \tag{2-105}$$

$$S_{434}=\sum_{i=1}^{5}s_i(-e_{31}+e_{33}\lambda_i^w s_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^v s_i)(-D_i e_{31}+e_{33}A_i^P s_i-\in_{33}A_i^Q s_i-g_{33}A_i^J s_i)R_i^{-3}+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(-e_{31}+e_{33}\lambda_i^w s_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^v s_i)R_i^{-3}X_3 \tag{2-106}$$

$$S_{534}=\sum_{i=1}^{5}s_i(-d_{31}+d_{33}\lambda_i^w s_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i-\eta_{33}\lambda_i^v s_i)(-D_i e_{31}+e_{33}A_i^P s_i-\in_{33}A_i^Q s_i-g_{33}A_i^J s_i)R_i^{-3}+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(-d_{31}+d_{33}\lambda_i^w s_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i-\eta_{33}\lambda_i^v s_i)R_i^{-3}X_3 \tag{2-107}$$

$$S_{634}=\sum_{i=1}^{5}s_i^2\lambda_i^v(-D_i e_{31}+e_{33}A_i^P s_i-\in_{33}A_i^Q s_i-g_{33}A_i^J s_i)R_i^{-5}(3\lambda_{31}X_1+3\lambda_{32}X_2)-\sum_{i=1}^{5}A_i^\Upsilon s_i^2\lambda_i^v\lambda_{33}\zeta_{33}R_i^{-3} \tag{2-108}$$

$$S_{135}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_i d_{31}+d_{33}A_i^P s_i-g_{33}A_i^Q s_i-\mu_{33}A_i^J s_i)R_i^{-5}X_1X_3+\sum_{i=1}^{5}A_i^\Upsilon\eta_{33}(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)R_i^{-3}X_1 \tag{2-109}$$

$$S_{235}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_i d_{31}+d_{33}A_i^P s_i-g_{33}A_i^Q s_i-\mu_{33}A_i^J s_i)R_i^{-5}X_2X_3+\sum_{i=1}^{5}A_i^\Upsilon\eta_{33}(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)R_i^{-3}X_2 \tag{2-110}$$

$$S_{335}=\sum_{i=1}^{5}s_i(-c_{13}+c_{33}\lambda_i^w s_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^v s_i)(-D_i d_{31}+d_{33}A_i^P s_i-g_{33}A_i^Q s_i-\mu_{33}A_i^J s_i)R_i^{-3}+\sum_{i=1}^{5}A_i^\Upsilon\eta_{33}(-c_{13}+c_{33}\lambda_i^w s_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^v s_i)R_i^{-3}X_3 \tag{2-111}$$

$$S_{435}=\sum_{i=1}^{5}s_i(-e_{31}+e_{33}\lambda_i^w s_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^v s_i)(-D_i d_{31}+d_{33}A_i^P s_i-g_{33}A_i^Q s_i-\mu_{33}A_i^J s_i)R_i^{-3}+\sum_{i=1}^{5}A_i^\Upsilon\eta_{33}(-e_{31}+e_{33}\lambda_i^w s_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^v s_i)R_i^{-3}X_3 \tag{2-112}$$

$$\begin{aligned}S_{535}=&\sum_{i=1}^{5}s_i(-d_{31}+d_{33}\lambda_i^{w}s_i-g_{33}\lambda_i^{\phi}s_i-\mu_{33}\lambda_i^{\varphi}s_i-\eta_{33}\lambda_i^{v}s_i)(-D_i d_{31}\\&+d_{33}A_i^{P}s_i-g_{33}A_i^{Q}s_i-\mu_{33}A_i^{J}s_i)R_i^{-3}\\&+\sum_{i=1}^{5}A_i^{\Upsilon}\eta_{33}(-d_{31}+d_{33}\lambda_i^{w}s_i-g_{33}\lambda_i^{\phi}s_i-\mu_{33}\lambda_i^{\varphi}s_i-\eta_{33}\lambda_i^{v}s_i)R_i^{-3}X_3\end{aligned}\tag{2-113}$$

$$\begin{aligned}S_{635}=&\sum_{i=1}^{5}s_i^2\lambda_i^{v}(-D_i d_{31}+d_{33}A_i^{P}s_i-g_{33}A_i^{Q}s_i-\mu_{33}A_i^{J}s_i)R_i^{-5}(3\lambda_{31}X_1+3\lambda_{32}X_2)\\&-\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{v}\lambda_{33}\eta_{33}R_i^{-3}\end{aligned}\tag{2-114}$$

$$\begin{aligned}S_{136}=&-\sum_{i=1}^{5}A_i^{\Upsilon}(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})\lambda_{31}(R_i^{-3}-3R_i^{-5}X_1^2)\\&+\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{32}(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})3R_i^{-5}X_1X_2\end{aligned}\tag{2-115}$$

$$\begin{aligned}S_{236}=&-\sum_{i=1}^{5}A_i^{\Upsilon}(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})\lambda_{32}(R_i^{-3}-3R_i^{-5}X_2^2)\\&+\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{31}(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})3R_i^{-5}X_1X_2\end{aligned}\tag{2-116}$$

$$S_{336}=-\sum_{i=1}^{5}A_i^{\Upsilon}s_i(-c_{13}+c_{33}\lambda_i^{w}s_i+e_{33}\lambda_i^{\phi}s_i+d_{33}\lambda_i^{\varphi}s_i+\iota_{33}\lambda_i^{v}s_i)R_i^{-3}X_3\tag{2-117}$$

$$S_{436}=\sum_{i=1}^{5}A_i^{\Upsilon}s_i(-e_{31}+e_{33}\lambda_i^{w}s_i-\in_{33}\lambda_i^{\phi}s_i-g_{33}\lambda_i^{\varphi}s_i+\zeta_{33}\lambda_i^{v}s_i)R_i^{-3}X_3\tag{2-118}$$

$$S_{536}=\sum_{i=1}^{5}A_i^{\Upsilon}s_i(-d_{31}+d_{33}\lambda_i^{w}s_i-g_{33}\lambda_i^{\phi}s_i-\mu_{33}\lambda_i^{\varphi}s_i-\eta_{33}\lambda_i^{v}s_i)R_i^{-3}X_3\tag{2-119}$$

$$S_{636}=-\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{v}\lambda_{33}3(\lambda_{31}X_1+\lambda_{32}X_2)R_i^{-5}\tag{2-120}$$

从 S_{KIJ} 的具体表达式可见，式中出现了 $\dfrac{1}{r^n}$ (n=2,3,4,5,7,9)的因子，当内点靠近边界时，在靠近的边界上被积函数变化剧烈，数值积分时应考虑其对精度的影响，

2.4 超奇异积分方程

2.4.1 数学模型

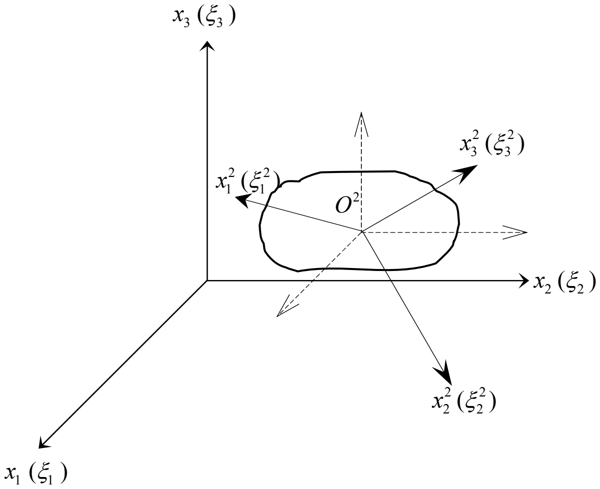


图 2-1 磁电热弹耦合材料内含任意形状平片裂纹问题示意图

如图 2-1 所示，假设磁电热弹耦合材料空间区域Ω中含有一任意形状和空间位置的平片裂纹。选择整体固定坐标系为空间直角坐标系 x_i (i=1,2,3)， 在裂纹面 S ($S^+ \bigcup S^-$) 表面分别作用大小相等方向相反的耦合载荷（力载荷 $p_j(P)$，电载荷 $q(P)$，磁载荷 $b(P)$ 和热载荷 $\rho(P)$ ）。为了研究问题的方便，我们建立局部直角坐标系 ξ_i (i=1,2,3)。

2.4.2 超奇异积分方程

应用[189]中方法，经过庞杂的数学推导，该裂纹问题可转化为求解一超奇异积分方程组问题，超奇异积分方程组的具体表达式如下：

$$\fint_{S^+}[r^{-3}(c_{44}^2 D_0 s_1^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2 t_i^2)\tilde{u}_\beta(q)+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2 t_i^1\tilde{u}_6(q)]ds(q)=-p_\alpha(p) \tag{2-121}$$

$$\fint_{S^+}[r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha(q)+r^{-3}\sum_{n=3}^{6}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n(q)+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}v_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6(q)]ds(q)=-p_m(p) \tag{2-122}$$

$$\fint_{S^+}[r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta(q)+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}v_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6(q)]ds(q)=-p_6(p) \tag{2-123}$$

其中 p_i, q_0, b_0 和 ϑ_0 分别代表裂纹表面力载荷，电载荷，磁载荷和热载荷。超奇异积分方程中的材料常数分别为：

$$t_i^1=(c_{44}^2 D_i s_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J) \tag{2-124}$$

$$t_i^2=c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi} \tag{2-125}$$

$$t_i^3 = -c_{13} + s_i(c_{33}\lambda_i^w + e_{33}\lambda_i^\phi + d_{33}\lambda_i^\varphi + \iota_{33}\lambda_i^\vartheta) \tag{2-126}$$

$$t_i^4 = s_i(e_{33}\lambda_i^w - \in_{33}\lambda_i^\phi - g_{33}\lambda_i^\varphi + \varsigma_{33}\lambda_i^\vartheta) - e_{31} \tag{2-127}$$

$$t_i^5 = s_i(d_{33}\lambda_i^w - g_{33}\lambda_i^\phi - \mu_{33}\lambda_i^\varphi + \eta_{33}\lambda_i^\vartheta) - d_{31} \tag{2-128}$$

$$\rho_i^2 = c_{44}D_i s_i + c_{44}A_i^P + e_{15}A_i^Q + d_{15}A_i^J \tag{2-129}$$

$$\rho_i^3 = s_i(c_{33}A_i^P + e_{33}A_i^Q + d_{33}A_i^J) - D_i c_{13} \tag{2-130}$$

$$\rho_i^4 = s_i(e_{33}A_i^P - \in_{33}A_i^Q - g_{33}A_i^J) - D_i e_{31} \tag{2-131}$$

$$\rho_i^5 = s_i(d_{33}A_i^P - g_{33}A_i^Q - \mu_{33}A_i^J) - D_i d_{31} \tag{2-132}$$

$$\rho_i^6 = \delta_{3m}\iota_{33} + \delta_{4m}\varsigma_{33} + \delta_{5m}\eta_{33} + \delta_{6m} \tag{2-133}$$

$$\rho_i^1 = \delta_{3m}\iota_{33} + \delta_{4m}\varsigma_{33} + \delta_{5m}\eta_{33} \tag{2-134}$$

分析以上超奇异积分方程组可以看出，该超奇异积分方程组与文献[187, 188]中裂纹问题的超奇异积分方程组具有相同的结构。

2.4.3 广义奇性指数

为研究裂纹前沿奇异性问题，假设一局部直角坐标系 $x_1\ x_2\ x_3$，坐标轴 x_1 沿裂纹前沿 q_0 点切线方向，坐标轴 x_2 沿裂纹面内法线方向，坐标轴 x_3 沿与裂纹面垂直的方向。裂纹面 q_0 点广义位移可表示为：

$$[\tilde{u}_i(q)\quad \tilde{\phi}(q)\quad \tilde{\varphi}(q)\quad \tilde{\Upsilon}(q)] = g_k(q_0)\xi_2^{\lambda_k} \qquad 0 < \mathrm{Re}(\lambda_k) < 1 \tag{2-135}$$

其中 $g_k(q_0)$ 为裂纹前沿 q_0 点相关的非零常数，λ_k 裂纹前沿奇异性指数。假设裂纹前沿有一小半圆区域 S_ε，应用文献中[189]提供的有限部分析方法，可以得到如下关系式：

$$\oint_{S_\varepsilon} \frac{\tilde{u}_1}{r^3} d\xi_1\xi_2 \cong -2\pi\lambda_1 g_1(q_0) x_2^{\lambda_1 - 1}\cot(\lambda_1\pi) \tag{2-136}$$

$$\oint_{S_\varepsilon} \frac{(x_2-\xi_2)^2\tilde{u}_1}{r^5} d\xi_1\xi_2 \cong -\frac{4}{3}\pi\lambda_1 g_1(q_0) x_2^{\lambda_1 - 1}\cot(\lambda_1\pi) \tag{2-137}$$

$$\oint_{S_\varepsilon} \frac{(x_1-\xi_1)^2 \tilde{u}_1}{r^5} d\xi_1\xi_2 \cong -\frac{2}{3}\pi\lambda_1 g_1(q_0) x_2^{\lambda_1-1}\cot(\lambda_1\pi) \tag{2-138}$$

$$\oint_{S_\varepsilon} \frac{(x_1-\xi_1)(x_2-\xi_2)\tilde{u}_1}{r^5} d\xi_1\xi_2 \cong 0 \tag{2-139}$$

$$\oint_{S_\varepsilon} \frac{(x_2-\xi_2)\tilde{u}_2}{r^3} d\xi_1\xi_2 \cong 2g_2(q_0) x_2^{\lambda_2}\cot(\lambda_2\pi) \tag{2-140}$$

把以上关系式代入超奇异积分方程组，可以得到：

$$\cot(\lambda_1\pi)=0,\cot(\lambda_2\pi)=0,\cot(\lambda_3\pi)=0,\cot(\lambda_4\pi)=0,\cot(\lambda_5\pi)=0,\cot(\lambda_6\pi)=0 \tag{2-141}$$

进一步整理可以得到裂纹奇异性指数：

$$\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=\lambda_6=\lambda=1/2 \tag{2-142}$$

2.4.4 广义应力强度因子

磁电热弹耦合材料三维裂纹的广义应力强度因子可定义为：

$$K_1=\lim_{r\to 0}\sqrt{2r}\sigma_{33}(r,\theta)\Big|_{\theta=0},K_2=\lim_{r\to 0}\sqrt{2r}\sigma_{32}(r,\theta)\Big|_{\theta=0},K_3=\lim_{r\to 0}\sqrt{2r}\sigma_{31}(r,\theta)\Big|_{\theta=0} \tag{2-143}$$

$$K_4=\lim_{r\to 0}\sqrt{2r}D_3(r,\theta)\Big|_{\theta=0},K_5=\lim_{r\to 0}\sqrt{2r}B_3(r,\theta)\Big|_{\theta=0},K_6=\lim_{r\to 0}\sqrt{2r}\vartheta_3(r,\theta)\Big|_{\theta=0} \tag{2-144}$$

K_1表示I型应力强度因子，K_2表示II型应力强度因子，K_3表示III型应力强度因子，K_4表示电应力强度因子，K_5表示磁应力强度因子，K_6热应力强度因子。其中r 表示p点到裂纹前沿 q_0点距离。

2.4.5 广义应性应力场

如图2-2所示，对于裂纹前沿点p，应用主部分析法可以得到如下关系式：

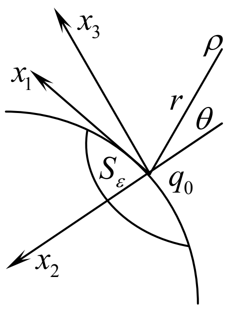


图 2-2 磁电热弹耦合材料裂纹表面局部区域示意图

$$\int_{s_\varepsilon} \frac{1}{R_0^3}(1-\frac{3(x_2-\xi_2)^2}{R_0^2})\tilde{u}_1 d\xi_1\xi_2 \cong \frac{\pi g_1(q_0)}{\sqrt{rr_0}}\cos\frac{\theta_0}{2} \tag{2-145}$$

$$\int_{s_\varepsilon} \frac{1}{R_0^3}(1-\frac{3v_0^2 x_3^2}{R_0^2})\tilde{u}_1 d\xi_1\xi_2 \cong \frac{\pi g_1(q_0)}{\sqrt{rr_0}}\cos\frac{\theta_0}{2} \tag{2-146}$$

$$\int_{s_\varepsilon} \frac{3v_j x_3(x_2-\xi_2)}{R_j^5}\tilde{u}_3 d\xi_1\xi_2 \cong \frac{\pi q_3(q_0)}{\sqrt{rr_j}}\sin\frac{\theta_j}{2} \tag{2-147}$$

$$\int_{s_\varepsilon} \frac{x_2^2 x_3}{R_j^7}\tilde{u}_6 d\xi_1\xi_2 \cong \frac{\pi g_6(q_0)}{30\sin\theta_i r r_j\sqrt{rr_j}}(2\cos^{-1}\frac{\theta_j}{2}-3\cos\frac{\theta_j}{2}-\cos\frac{5\theta_j}{2}) \tag{2-148}$$

$$\int_{s_\varepsilon} \frac{x_2 x_3}{R_j^5}\tilde{u}_6 d\xi_1\xi_2 \cong -\frac{4\pi g_6(q_0)}{3\sqrt{rr_j}}\cos\frac{\theta_j}{2} \tag{2-149}$$

$$\int_{s_\varepsilon} \frac{x_2}{R_j^5}\tilde{u}_6 d\xi_1\xi_2 \cong -\frac{2\pi g_6(q_0)}{3r^2 r_j^2\sqrt{rr_j}}\sin^{-1}\frac{\theta_j}{2} \tag{2-150}$$

其中 $r_0=\sqrt{\cos^2\theta+s_1^2\sin^2\theta}$, $\theta_0=\tan^{-1}(s_1\tan\theta)$, $r_j e^{i\theta_j}=\cos\theta+is_j\sin\theta$ 。利用以上关系式，可以得到裂纹前沿广义奇异应力场的表达式。

$$\sigma_{13}=c_{44}^2 D_0 s_1^2\frac{\pi g_1}{\sqrt{rr_0}}\cos\frac{\theta_0}{2} \tag{2-151}$$

$$\sigma_{23}=\sum_{i=1}^{5}\frac{\pi}{\sqrt{rr_i}}\{g_6\lambda_{33}[15\lambda_{33}s_i^2\cot\theta_i(3\cos\frac{\theta_i}{4}-3\sin^2\frac{\theta_i}{2}\cos\frac{\theta_i}{2}-\frac{1}{4}\sin^2\theta_i\cos\frac{3\theta_i}{2})-\frac{2\lambda_{33}}{r^2r_i^2}\sin^{-1}\frac{\theta_i}{2}$$
$$+\frac{\lambda_{32}}{2rr_i\sin\theta}(2\cos^{-1}\frac{\theta_i}{2}-3\cos\frac{\theta_i}{2}-\cos\frac{5\theta_i}{2})]+\rho_i^2\pi\cos\frac{\theta_i}{2}(t_2g_2+4\sum_{m=3}^{5}t_i^m g_m)\} \tag{2-152}$$

$$\dot{\sigma}_{3n}=\sum_{i=1}^{5}\rho_i^n s_i\pi[\frac{1}{\sqrt{rr_i}}(t_i^2g_2\cot\theta_i\cos^{-1}\frac{\theta_i}{2}+6\cot\frac{\theta_i}{2}\sum_{m=3}^{5}t_i^m g_m)+g_6s_i\lambda_i^{\vartheta}\sqrt{rr_i}\cot\theta_i\cos\frac{\theta_i}{2})]$$
$$+2\pi A_i^{\Upsilon}\iota_{33}[\sqrt{rr_i}\cos\frac{\theta_i}{2}(g_2\cot\theta_i+\sum_{m=3}^{5}t_i^m g_m)+\frac{s_i^2\lambda_{33}\lambda_i^{\vartheta}g_6}{\sqrt{rr_i}}\cos\frac{\theta_i}{2}],n=3-5 \tag{2-153}$$

$$\vartheta_3=\sum_{i=1}^{5}\frac{A_i^{\Upsilon}\pi}{\sqrt{rr_i}}[\cos\frac{\theta_i}{2}(\lambda_{32}+4\lambda_{33}s_i^2)\sum_{m=2}^{5}t_i^m g_m+\frac{4\lambda_{32}t_i^2g_2s_i^2\lambda_i^{\vartheta}}{3}(2\lambda_{32}\cos\frac{\theta_i}{2}+\frac{3\lambda_{33}}{r^2r_i^2}\sin^{-1}\frac{\theta_i}{2})$$
$$15rr_ig_6\lambda_{32}s_i^2\lambda_i^{\vartheta}\cot\theta_i(3\cos\frac{\theta_i}{4}-6\cos\frac{\theta_i}{2}\sin^2\frac{\theta_i}{2}-\frac{1}{4}\sin^2\theta_i\cos\frac{3\theta_i}{2})(2s_i^2\lambda_{33}+\frac{\lambda_{32}}{rr_i\sin\theta_i})] \tag{2-154}$$

其中 $\dot{\sigma}_{3n}=[\sigma_{33}\quad D_3\quad B_3]$,利用以上关系式子，可以得到广义应力强度因子与广义应力场之间的关系式：

$$\sigma_{13}=\frac{K_3}{\sqrt{2rr_0}}\cos\frac{\theta_0}{2} \tag{2-155}$$

$$\hat{\sigma}_{n3}=\frac{K_1f_{n1}+K_2f_{n2}+K_4f_{n4}+K_5f_{n5}+K_6f_{n6}}{\sqrt{2r}},n=2-6 \tag{2-156}$$

其中 $\hat{\sigma}_{n3}=[\sigma_{23}\quad\sigma_{33}\quad D_3\quad B_3\quad\vartheta_3\quad]$,核函数 f 的具体表达式为：

$$f_{2n}=\sum_{i=1}^{4}\frac{\rho_i^2}{\sqrt{r_i}}\cos\frac{\theta_i}{2}(t_i^2\delta_{1n}+4t_i^3\delta_{2n}+4t_i^4\delta_{4n}+4t_i^5\delta_{5n}),n=1,2,4,5$$

$$f_{n1}=\sum_{i=1}^{5}A_i^{\Upsilon}r\sqrt{r_i}\cos\frac{\theta_i}{2}\cot\theta_i(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})+\frac{\rho_i^3s_i}{\sqrt{r_i}}t_2\cot\theta_i\cos^{-1}\frac{\theta_i}{2},n=3-5$$

$$f_{mn}=\sum_{i=1}^{5}\rho_i^m s_i(A_i^{\Upsilon}r\sqrt{r_i}\cos\frac{\theta_i}{2}+6\cot\frac{\theta_i}{2})(\iota_{33}+\varsigma_{33}\delta_{4m}+\eta_{33}\delta_{5m})(t_i^3\delta_{2n}$$
$$+t_i^4\delta_{4n}+t_i^5\delta_{5n});m=3-5,n=2,4,5$$

$$f_{26}=\sum_{i=1}^{5}\frac{\lambda_{33}}{\sqrt{r_i}}[15\lambda_{33}s_i^2\cot\theta_i(3\cos\frac{\theta_i}{4}-3\sin^2\frac{\theta_i}{2}\cos\frac{\theta_i}{2}-\frac{1}{4}\sin^2\theta_i\cos\frac{3\theta_i}{2})-\frac{2\lambda_{33}}{r^2r_i^2}\sin^{-1}\frac{\theta_i}{2}$$
$$+\frac{\lambda_{32}}{2rr_i\ \sin\theta}(2\cos^{-1}\frac{\theta_i}{2}-3\cos\frac{\theta_i}{2}-\cos\frac{5\theta_i}{2})]$$

$$f_{n6}=\sum_{i=1}^{5}\rho_i^3s_i^2\lambda_i^{\vartheta}r\sqrt{r_i}\cot\theta_i\cos\frac{\theta_i}{2}+A_i^{\Upsilon}\frac{s_i^2\lambda_{33}\lambda_i^{\vartheta}}{\sqrt{r_i}}\cos\frac{\theta_i}{2}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n}),n=3-5$$

$$f_{61}=\sum_{i=1}^{5}\frac{A_i^{\Upsilon}}{\sqrt{r_i}}[\cos\frac{\theta_i}{2}(\lambda_{32}+4\lambda_{33}s_i^2)t_2+\frac{4\lambda_{32}t_2s_i^2\lambda_i^{\vartheta}}{3}(2\lambda_{32}\cos\frac{\theta_i}{2}+\frac{3\lambda_{33}}{r^2r_i^2}\sin^{-1}\frac{\theta_i}{2})]$$

$$f_{6n}=\sum_{i=1}^{5}\frac{A_i^{\Upsilon}}{\sqrt{r_i}}\cos\frac{\theta_i}{2}(\lambda_{32}+4\lambda_{33}s_i^2)(t_i^3\delta_{3n}+t_i^4\delta_{4n}+t_i^5\delta_{5n}),n=3-5$$

$$f_{66}=\sum_{i=1}^{5}45A_i^{\Upsilon}r^{\frac{3}{2}}\lambda_{32}s_i^2\lambda_i^{\vartheta}\cot\theta_i(\cos\frac{\theta_i}{4}-\cos\frac{\theta_i}{2}\sin\theta_i-\frac{\sin^2\theta_i}{12}\cos\frac{3\theta_i}{2})(2s_i^2\lambda_{33}+\frac{\lambda_{32}}{rr_i\sin\theta_i})$$

从上面关系式可以看出：应力强度因子 K_2，应力强度因子 K_3 和热应力强度因子 K_6 是相互耦合的。应力强度因子 K_1，电应力强度因子 K_4，磁应力强度因子 K_5 和热应力因子 K_6 之间是相互耦合的。

2.4.6 广义能量释放率

磁电热弹耦合材料裂纹能量释放率可以表示为：

$$G=G_1^{\sigma}+G_2^{\sigma}+G_3^{\sigma}+G_4^{D}+G_5^{B}+G_6^{\Upsilon}\tag{2-157}$$

其中：

$$G_1^{\sigma}=\frac{\pi K_1[K_4^*(K_{34}\mathrm{K}_{55}-K_{35}\mathrm{K}_{54})+K_{45}(K_{54}K_1^*-K_{34}K_5^*)+K_{44}(K_{35}K_5^*-K_{55}K_1^*)]}{2[K_{35}(K_{43}\mathrm{K}_{54}-K_{44}\mathrm{K}_{53})+K_{34}(K_{45}\mathrm{K}_{53}-K_{43}\mathrm{K}_{55})+K_{33}(K_{44}\mathrm{K}_{55}-K_{45}\mathrm{K}_{54})]}$$

$$G_2^{\sigma}=\frac{\pi K_2^2}{2(K_{11}-K_{12})}$$

$$G_3^{\sigma}=\frac{\pi K_3\{K_3(K_{11}-K_{12})K_{16}+K_{26}[(\dot{K}_{11}+2\dot{K}_{12})K_2-(K_{11}-K_{12})K_4]\}}{2(K_{11}-K_{12})[K_{26}(\dot{K}_{11}+2\dot{K}_{12})-(K_{11}-K_{12})K_{66}]}$$

$$G_4^D = \frac{\pi K_4[K_4^*(K_{35}\mathrm{K}_{53} - K_{33}\mathrm{K}_{55}) + K_{45}(K_{33}K_5^* - K_{53}K_1^*) + K_{43}(K_{55}K_1^* - K_{35}K_5^*)]}{2[K_{35}(K_{43}\mathrm{K}_{54} - K_{44}\mathrm{K}_{53}) + K_{34}(K_{45}\mathrm{K}_{53} - K_{43}\mathrm{K}_{55}) + K_{33}(K_{44}\mathrm{K}_{55} - K_{45}\mathrm{K}_{54})]}$$

$$G_5^B = \frac{\pi K_5[K_4^*(K_{33}\mathrm{K}_{54} - K_{34}\mathrm{K}_{53}) + K_{44}(K_{53}K_1^* - K_{33}K_5^*) + K_{43}(K_{34}K_5^* - K_{54}K_1^*)]}{2[K_{35}(K_{43}\mathrm{K}_{54} - K_{44}\mathrm{K}_{53}) + K_{34}(K_{45}\mathrm{K}_{53} - K_{43}\mathrm{K}_{55}) + K_{33}(K_{44}\mathrm{K}_{55} - K_{45}\mathrm{K}_{54})]}$$

$$G_6^{\Upsilon} = \frac{\pi K_6[(\dot{K}_{11} + 2\dot{K}_{12})(K_2 + K_3) - (K_{11} - K_{12})K_6]}{2[(K_{11} - K_{12})K_{66} - \aleph_{16}(\dot{K}_{11} + 2\dot{K}_{12})]}$$

符号 $\hat{\mathrm{K}}_2, \hat{\mathrm{K}}_6, \mathrm{K}_1^*, \mathrm{K}_5^*, \mathrm{K}_4^*$ 的具体含义详见附录A。

2.4.7 广义应变能强度因子

根据应变能强度理论[93, 94], 磁电热弹耦合材料内部单元体积 $dV = dx_1dx_2dx_3$ 在广义三维应力作用下的应变能为：

$$\begin{aligned} dW = (&\frac{\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2}{2E} - v\frac{\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}}{E} + \frac{\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2}{2\mu} \\ &+ \frac{D_{11}^2 + D_{22}^2 + D_{33}^2}{2E'} - v'\frac{D_{11}D_{22} + D_{22}D_{33} + D_{33}D_{11}}{E'} + \frac{D_{12}^2 + D_{13}^2 + D_{23}^2}{2\mu'} \\ &+ \frac{B_{11}^2 + B_{22}^2 + B_{33}^2}{2E'} - v'\frac{B_{11}B_{22} + B_{22}B_{33} + B_{33}B_{11}}{E'} + \frac{B_{12}^2 + B_{13}^2 + B_{23}^2}{2\mu'} \\ &+ \frac{\vartheta_{11}^2 + \vartheta_{22}^2 + \vartheta_{33}^2}{2E'} - v'\frac{\vartheta_{11}\vartheta_{22} + \vartheta_{22}\vartheta_{33} + \vartheta_{33}\vartheta_{11}}{E'} + \frac{\vartheta_{12}^2 + \vartheta_{13}^2 + \vartheta_{23}^2}{2\mu'})dV \end{aligned} \tag{2-158}$$

其中 E, v, μ, E', v', μ' 分别表示杨氏模量、泊松比、剪切模量，等价杨氏模量，等价泊松比，等价剪切模量. 把上式带入（2-151）～（2-154）以得到广义应变能量强度函数，表达式如下：

$$\frac{dW}{dV} = \frac{K_n^2}{2\sqrt{2r}}(\frac{a_{3n}^2}{E} + \frac{a_{1n}^2 + a_{2n}^2}{\mu} + \frac{a_{4n}^2 + a_{5n}^2 + a_{6n}^2}{E'}) \tag{2-159}$$

广义应变能强度函数具有 $r^{-0.5}$ 奇异性，进一步整理可得到广义应变能强度因子表达式：

$$S|_\theta = K_n^2(\frac{a_{3n}^2}{E} + \frac{a_{1n}^2 + a_{2n}^2}{\mu} + \frac{a_{4n}^2 + a_{5n}^2 + a_{6n}^2}{E'}) \tag{2-160}$$

其中 a_{mn} (m,n=1,2,3,4,5,6) 为和材料性质有关的已知参数，具体表达式见附录A。

2.5 数值方法

2.5.1 数值方法

应用文献[187, 266]中的方法，可以对超奇异积分方程组（2-121）～（2-123）进行数值化处理。利用前面对裂纹前沿奇异性分析结果，广义位移间断未知函数可以用下式表示：

$$\tilde{U}_J(\xi_1,\xi_2)=F_J(\xi_1,\xi_2)\xi_2^{0.5}W(\xi_1,\xi_2) \tag{2-161}$$

其中

$$W(\xi_1,\xi_2)=\sqrt{(a^2-\xi_1^2)(2b-\xi_2)}$$

$$F_J(\xi_1,\xi_2)=\sum_{s=0}^{S}\sum_{h=0}^{H}a_{Jsh}\xi_1^s\xi_2^h$$

a_{Jmn} 是一待求未知常量。把（2-161）代入超奇异积分方程组中，超奇异积分方程组可以转化为一组线性代数方程组，常数 a_{Jmn} 边可以通过下式求解。

$$\sum_{s=0}^{S}\sum_{h=0}^{H}a_{Ish}I_{IJsh}(x_1,x_2)=-p_J(x_1,x_2) \tag{2-162}$$

其中

$$I_{12sh}=-3\int_S r^{-3}(K_{11}+K_{12})r_{,1}r_{,2}\dot{S}_2 \tag{2-163}$$

$$I_{11sh}=\int_S r^{-3}[K_{11}(1-3r_{,2}^2)+K_{12}(1-3r_{,1}^2)]\dot{S}_1 \tag{2-164}$$

$$I_{mnsh}=\int_S r^{-3}K_{mn}\dot{S}_n \tag{2-165}$$

$$I_{21sh}=-\int_S 3r^{-3}(K_{11}+K_{12})r_{,1}r_{,2}\dot{S}_1 \tag{2-166}$$

$$I_{22sh}=\int_S r^{-3}[K_{11}(1-3r_{,1}^2)+K_{12}(1-3r_{,2}^2)]\dot{S}_2 \tag{2-167}$$

$$I_{61sh}=\int_S r^{-3}[\dot{K}_{11}(1-3r_{,1}^2)-\dot{K}_{12}r_{,1}r_{,2}]\dot{S}_1 \tag{2-168}$$

$$I_{66sh}=\int_S 3r^{-4}K_{66}\lambda_{3\alpha}r_{,\alpha}\dot{S}_6 \tag{2-169}$$

$$I_{m6sh}=\int_S 3r^{-4}\dot{K}_{m6}\lambda_{3\alpha}r_{,\alpha}\dot{S}_6 \tag{2-170}$$

$$I_{62sh}=\int_S r^{-3}[\dot{K}_{12}(1-3r_{,2}^2)-3\dot{K}_{11}r_{,1}r_{,2}]\dot{S}_2 \tag{2-171}$$

$$I_{\alpha 6sh}=\int_S 3r^{-4}K_{\alpha 6}r_{,\alpha}\dot{S}_6 \tag{2-172}$$

$$I_{m\alpha sh}=\int_S r^{-2}K_{m1}r_{,\alpha}\dot{S}_\alpha \tag{2-173}$$

其中

$$\dot{S}_n=\xi_1^s\xi_2^{\lambda_n+h}W\,d\xi_1\,d\xi_2$$

$$K_{11}=c_{44}^2D_0s_0^2$$

$$K_{12}=\sum_{i=1}^5\rho_i^2t_i^2$$

$$K_{16}=\sum_{i=1}^5\lambda_{33}s_i^2t_i^1$$

$$\dot{K}_{\alpha\beta}=\sum_{i=1}^5A_i^\Upsilon\lambda_{3\beta}t_i^2$$

$$K_{mt}=\sum_{i=1}^5\rho_i^m t_i^t$$

$$K_{m\alpha}=\sum_{i=1}^5A_i^\Upsilon t_i^2\rho_i^1$$

$$\dot{K}_{m6}=\sum_{i=1}^5s_i^2\lambda_i^\vartheta\rho_i^m$$

$$K_{m6}=\sum_{i=1}^5A_i^\Upsilon s_i^2\lambda_i^\vartheta\lambda_{33}\rho_i^6$$

方程（2-163）～（2-173）是奇异性的，在进行数值处理前必须先对奇异性进行处理，利用泰勒展开和极坐标方法，如图2-3所示，可以得到如下关系式：

$$\xi_1^s(a^2-x_1^2)^{0.5}=\begin{cases}(a^2-x_1^2)^{0.5}-x_1r_1\cos\theta_1/(a^2-x_1^2)^{0.5}-Q_1r_1^2\\ s=0\\ x_1^s(a^2-x_1^2)^{0.5}+x_1^{s-1}[sa^2-(s+1)x_1^2]r_1\cos\theta_1/(a^2-x_1^2)^{0.5}+Q_2r_1^2\\ s>0\end{cases} \tag{2-174}$$

$$\begin{aligned}\xi_2^{\lambda_J+h}(2b-\xi_2)^{0.5}=&x_2^{\lambda_J+h}(2b-x_2)^{0.5}+\{x_2^{\lambda_J+h-1}[4b(\lambda_J+h)-\\&(2\lambda_J+2h+1)x_2]r_1\sin\theta_1+Q_3^Jr_1^2\}/2(2b-x_2)^{0.5}\end{aligned} \tag{2-175}$$

其中

$$Q_1 = a^2(x_1+\xi_1)[(a^2-x_1^2)^{0.5}((a^2-x_1^2)^{0.5}+(a^2-\xi_1^2)^{0.5})(\xi_1(a^2-x_1^2)^{0.5} \\ +x_1(a^2-\xi_1^2)^{0.5})]^{-1}\cos^2\theta_1 \tag{2-176}$$

$$Q_2=\begin{cases}[s(s-1)a^4x_1^{s-2}-(s^2+1)a^2x_1^s+s(s+1)x_1^{s+2}]\cos^2\theta_1/2(a^2-x_1^2)^{3/2} \\ \xi_1=x_1 \\ r_1^{-2}\{\xi_1^s(a^2-\xi_1^2)^{0.5}-x_1^s(a^2-x_1^2)^{0.5}-x_1^{s-1}[sa^2-(s+1)x_1^2](\xi_1-x_1)/(a^2-x_1^2)^{0.5}\} \\ \xi_1\neq x_1\end{cases} \tag{2-177}$$

$$Q_3^J=\begin{cases}2(a^2-x_1^2)^{-1.5}x_2^{\lambda_J+h-2}\sin^2\theta_1\{16(\lambda_J+h)(\lambda_J+h-1)b^2-8b(\lambda+n)(2\lambda_J+2h-1)x_2 \\ +[4(\lambda_J+h)^2-1]x_2^2\}，\xi_2=x_2 \\ r_1^{-2}\{\xi_2^{\lambda_J+h}(2b-\xi_2)^{0.5}-x_2^{\lambda_J+h}(2b-x_2)^{0.5}-(2b-x_2)^{0.5}x_2^{\lambda_J+h-1}[2b(\lambda_J+h) \\ -(\lambda_J+h+0.5)x_2]\,(\xi_2-x_2)\}，\xi_2\neq x_2\end{cases} \tag{2-178}$$

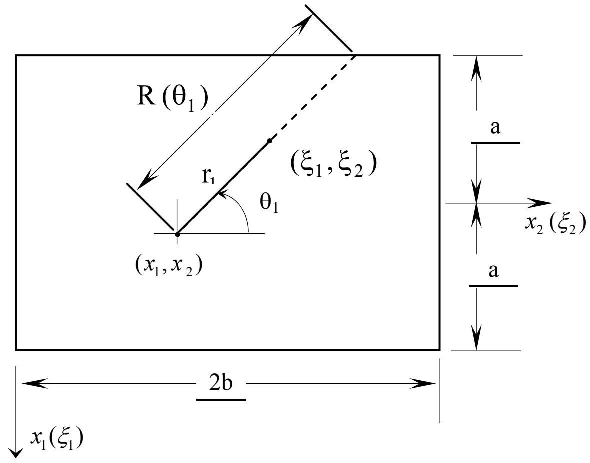


图 2-3 积分参数示意图

利用式（2-174）～（2-175），方程（2-163）～（2-173）中的奇异核函数可以表示为：

$$\xi_1^m\xi_2^{\lambda_J+h}\sqrt{(a^2-\xi_1^2)(2b-\xi_2)}=D_0^J+D_1^J\,r_1+D_2^J\,r_1^2 \tag{2-179}$$

其中 D_0^J, D_1^J 和 D_2^J 为已知函数，通过有限部积分概念和泰勒展开，式（2.163～2.731）可以简化为以下表达式：

$$I_{11sh}=\int_0^{2\pi}[K_{11}(1-3\sin^2\theta)+K_{12}(1-3\cos^2\theta)]S_1 \tag{2-180}$$

$$I_{12sh}=\int_0^{2\pi}1.5(K_{11}+K_{12})\sin 2\theta S_2 \tag{2-181}$$

$$I_{21sh}=\int_0^{2\pi}1.5(K_{11}+K_{12})\sin 2\theta S_1 \tag{2-182}$$

$$I_{22sh}=\int_0^{2\pi}[K_{11}(1-3\cos^2\theta)+K_{12}(1-3\sin^2\theta)]S_2 \tag{2-183}$$

$$I_{61sh}=\int_0^{2\pi}[\dot{K}_{11}(1-3\cos^2\theta)-0.5\dot{K}_{12}\sin 2\theta]S_1 \tag{2-184}$$

$$I_{\alpha 6sh}=\int_0^{2\pi}3(K_{16}\cos\theta\delta_{1\alpha}+K_{26}\sin\theta\delta_{2\alpha})S_6 \tag{2-185}$$

$$I_{62sh}=\int_0^{2\pi}[\dot{K}_{12}(1-3\sin^2\theta)-\dot{K}_{11}\cos\theta\sin\theta]S_2 \tag{2-186}$$

$$I_{66sh}=\int_0^{2\pi}3K_{66}(\lambda_{31}\cos\theta+\lambda_{32}\sin\theta)S_6 \tag{2-187}$$

$$I_{m6sh}=\int_0^{2\pi}3\dot{K}_{m6}(\lambda_{31}\cos\theta+\lambda_{32}\sin\theta)S_6 \tag{2-188}$$

$$I_{m\alpha sh}=\int_0^{2\pi}K_{m1}\cos\theta S_\alpha \tag{2-189}$$

$$I_{mnsh}=\int_0^{2\pi}K_{mn}S_n \tag{2-190}$$

其中核函数 $S_n=(D_1^n\ln R-R^{-1}D_0^n+\int_0^R D_2^n dr_1)d\theta$，n=1,2,3,4,5,6。

2.5.2 数值结果及讨论

为了检验上述数值方法的正确性，以磁电热弹耦合材料三维矩形裂纹为例，用FORTRAN语言编写典型算例的计算程序（7000余条）。磁电热弹耦合材料的材料参数[225]如表2-1所示。

表2-1磁电热弹耦合材料参数

弹性常数 GPa					磁电常数 Ns/VC		压电常数 Cm^{-2}		
c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	g_{11}	g_{33}	e_{15}	e_{31}	e_{33}
286	173	170	269.5	45.3	0.005E-9	0.003E-9	11.6	-4.4	18.6
压磁常数 N/Am			磁性常数 Ns^2C^{-2}		介电常数 C^2/Nm^2		热应力		
d_{15}	d_{31}	d_{33}	μ_{11}	μ_{33}	$\in_{11}$	$\in_{33}$	ι_{11}	ι_{22}	ι_{33}
550	580.3	699.7	-590E-6	157E-6	0.08E-9	0.093E-9	0	0	0
热电常数			热磁常数				热传导因子		

ς_{11}	ς_{22}	ς_{33}	η_{11}	η_{22}	η_{33}	λ_{11}	λ_{22}	λ_{33}
0	0	0	0	0	0	0	0	0

为了便于计算和比较，对表2-1中的各个材料参数进行无量纲化处理，具体公式可表示为：

$$\overline{c}_{ij}=c_{ij}/c_{11},\overline{e}_{ij}=e_{ij}/\sqrt{c_{11}e_{11}},\overline{\in}_{ij}=\in_{ij}/\in_{11},\overline{d}_{ij}=d_{ij}/\sqrt{c_{11}\mu_{11}},\overline{g}_{ij}=d_{ij}/\sqrt{\in_{11}\mu_{11}} \tag{2-191}$$

$$\overline{\mu}_{ij}=\mu_{ij}/\mu_{11},\overline{\phi}=\sqrt{\in_{11}/c_{11}}\phi,\overline{\varphi}=\sqrt{\mu_{11}/c_{11}}\varphi,\overline{D}_i=D_i/\sqrt{c_{11}\in_{11}};\overline{B}_i=B_i/\sqrt{c_{11}\mu_{11}} \tag{2-192}$$

把以上公式代入表2-1，便可以得到无量纲化的材料参数，如表2-2所表示。

表2-2 无量纲磁电热弹耦合材料参数

c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	g_{11}	g_{33}	$\in_{11}$	$\in_{33}$
1	0.6049	0.5944	0.9423	0.1584	-0.2301E-4i	-0.1381E-4i	1	1.1625
d_{15}	d_{31}	d_{33}	μ_{11}	μ_{33}	e_{15}	e_{31}	e_{33}	
-0.0423i	-0.0447i	-0.0539i	1	0.2661	2.4251	-0.9199	3.8885	

与材料相关的其他各参数的无量纲化后结果如表2-3所示，利用这些参数，我们就可以计算任意裂纹问题的广义应力强度因子了。

表2-3 与材料性质相关无量纲参数

n_1	n_2	n_3	n_4	n_{41}	n_{51}	n_{61}	n_{71}	n_{42}	n_{52}	n_{62}	n_{72}
4.39951	5.5476	-1.7959	0	1	3.3130	-0.7791	-0.0490	2.4251	2.4814	-0.2419	-0.1639
n_{43}	n_{53}	n_{63}	n_{73}	α_1	α_2	α_3	α_4	β_{11}	β_{12}	β_{13}	β_{14}
-0.0423i	0.1775i	0.3845i	-0.0099i	40.8361	1.8653	-4.3378	28.4925	19.4882	2.4812	-0.9632	14.6629
β_{21}	β_{22}	β_{23}	β_{24}	β_{31}	β_{32}	β_{33}	β_{34}	A_1^P	A_2^P	A_3^P	A_4^P
1.7496	-0.1681	-2.3976	2.7683	-5.0874i	-0.1056i	0.0442i	-3.1662i	-0.8493	0.1981	0.3747	0.2765
A_1^Q	A_2^Q	A_3^Q	A_4^Q	A_1^J	A_2^J	A_3^J	A_4^J	D_1	D_2	D_3	D_4
0.1336	0.0240	0.0591	-0.2167	-9.2924i	1.0905i	-10.616i	18.8177i	5.8703	0.1022	0.0947	-6.3481
λ_1^w	λ_2^w	λ_3^w	λ_4^w	λ_1^ϕ	λ_2^ϕ	λ_3^ϕ	λ_4^ϕ	λ_1^φ	λ_2^φ	λ_3^φ	λ_4^φ
0.2479	1.4187	2.1105	0.2969	0.0223	-0.0961	5.2533	0.0560	-0.0647	-0.0604	0.0969	0.0641

a_1	a_2	a_3	a_4	a_5	s_0	$s_{1,2}$	$s_{3,4}$	$s_{5,6}$
-0.6840	-0.7078	7.9102	13.608	6.0377	1.1168	±1.9253i	0.9376±.1052i	-0.9376±0.1052i

$s_{7,8}$	K_{11}	K_{54}	K_{12}	K_{34}	K_{33}	K_{43}
±1.7336	0.0196	0.1987×10^{-3}+19.1566i	2.8986 -0.03298i	-2.7031-0.4022$\times10^{-4}$ i	1.9849 -0.6217$\times10^{-4}$ i	-1.3770+1.5938$\times10^{-8}$i

K_{55}	K_{34}	K_{35}	K_{53}	K_{45}	K_{44}
0.07316+5.2261$\times10^{-6}$i	-2.7031-0.4022$\times10^{-4}$ i	1.0579$\times10^{-6}$+0.0274i	0.3071$\times10^{-3}$+3.7552i	-2.7120$\times10^{-8}$+0.0813i	-19.2065+1.0310$\times10^{-8}$i

同样对广义应力强度因子的计算公式也进行相应无量纲化处理，具体公式可表示为：

$$F_{1,\lambda}=K_{1,\lambda}/\sigma_{33}^{\infty}b^{1-\lambda},F_1=K_1/\sigma_{33}^{\infty}\sqrt{b} \tag{2-193}$$

$$F_{2,\lambda}=K_{2,\lambda}/\sigma_{31}^{\infty}b^{1-\lambda},F_{2}=K_{2}/\sigma_{31}^{\infty}\sqrt{b} \tag{2-194}$$

$$F_{3,\lambda}=K_{3,\lambda}/\sigma_{32}^{\infty}b^{1-\lambda},F_{3}=K_{3}/\sigma_{32}^{\infty}\sqrt{b} \tag{2-195}$$

$$F_{4,\lambda}=K_{4,\lambda}/D_{33}^{\infty}b^{1-\lambda},F_{4}=K_{4}/D_{33}^{\infty}\sqrt{b} \tag{2-196}$$

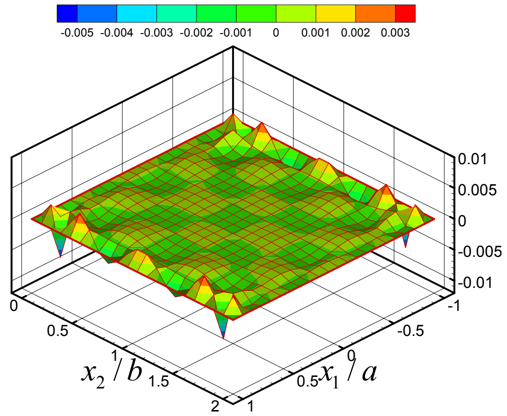
$$F_{5,\lambda}=K_{5,\lambda}/B_{33}^{\infty}b^{1-\lambda},F_{5}=K_{5}/B_{33}^{\infty}\sqrt{b} \tag{2-197}$$

对广义应变能强度因子进行无量纲化，具体公式可表示为：

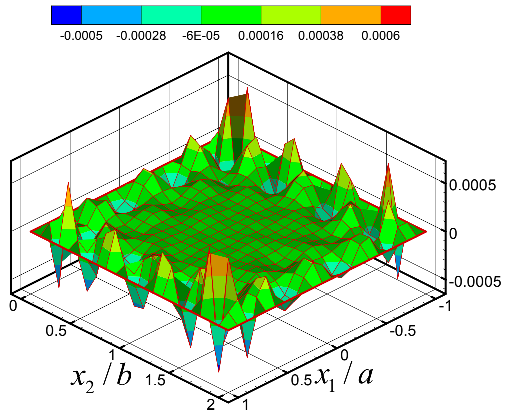
$$\bar{S}_{\lambda}\Big|_{\theta}=(F_{n,\lambda})^{2}\left(\frac{a_{3n}^{2}}{E}+\frac{a_{1n}^{2}+a_{2n}^{2}}{\mu}+\frac{a_{4n}^{2}+a_{5n}^{2}+a_{6n}^{2}}{E'}\right) \tag{2-198}$$

$$\bar{S}\Big|_{\theta}=(F_{n})^{2}\left(\frac{a_{3n}^{2}}{E}+\frac{a_{1n}^{2}+a_{2n}^{2}}{\mu}+\frac{a_{4n}^{2}+a_{5n}^{2}+a_{6n}^{2}}{E'}\right) \tag{2-199}$$

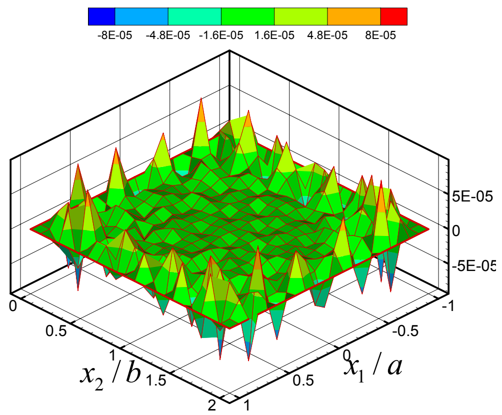
配置点为*KK=LL=20×20*，裂纹形状比*b/a=1*。裂纹表面最小剩余应力随多项式指数变化规律如图2-4和2-5所示：在图2-4中，最小剩余应力$\sigma_C=(\sigma_{33}/\sigma_{33}^{\infty})+1$随多项式指数增大而减小，数值结果精度随多项式指数增加而增大，具体数值为：*M=N=9*，$\sigma_c=5.27\times10^{-5}$；*M=N=13*，$\sigma_c=1.07\times10^{-6}$。



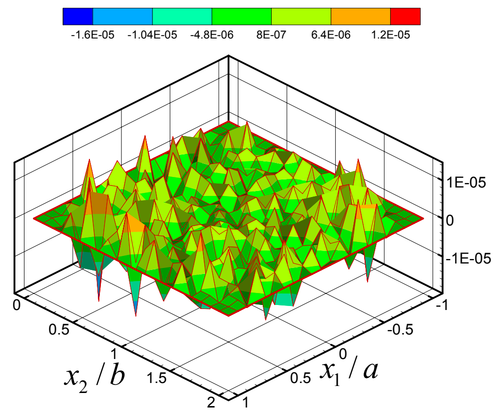
(a) M=N=7



(b) M=N=9



(c) M=N=11



(d) M=N=13

图 2-4 复合边界条件下剩余应力($\sigma_{31}/\sigma_{31}^{\infty}+1=0$，KK=LL=20×20，b/a=1)

从图2-5可以看出，在配置点为*KK=LL=20×20*，多项式指数*M=N=13*时，剩余应力在裂纹面上为一稳定值（图中蓝色区域），在裂纹边缘值有较大波动（图中红色区域）。图中变化规律显示，应用本文中数值方法计算三维裂纹问题具有良好的稳定性和可靠性。

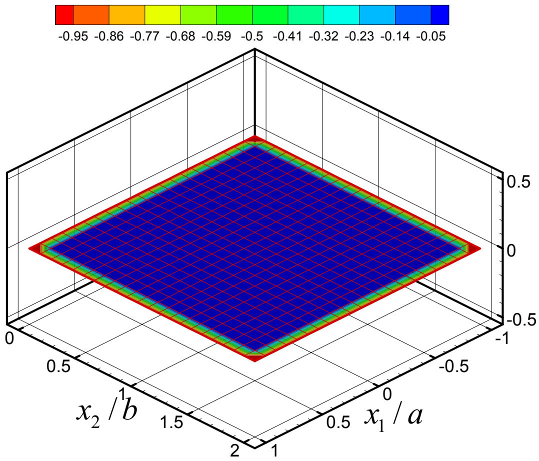


图 2-5 剩余应力变化规律($\sigma_{3l}/\sigma_{3l}^{\infty}+1=0$，KK=LL=20×20，b/a=1)

裂纹形状比 *b/a=1*，配置点 *KK=LL=20×20*，载荷为 σ_{33}^{∞} 时，广义应力强度因子随多项式指数、坐标变化规律如表 2-4 及图 2-6 所示：在 $x_2=0,2b$ 边，$F_{I,\lambda}$ 随 x_1/a 呈对称分布，$x_1/a=0$ 取最大值 $F_{1,\lambda\max}\big|_{x_1/a=0}$=0.7536。随多项式指数增加，应力强度因子是收敛的。结果与其他文献比较显示，本文数值方法所得的数值结果是收敛的，有很好的精度。

表2-4在 $x_2=0,2b$ 边 $F_{I,\lambda}$, 随多项式指数增加变化规律（KK=LL=20×20，b/a=1）

x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
M= 13	.7536	.7518	.7464	.7374	.7244	.7067	.6829	.6512	.6092	.5503	.4493
M= 11	.7536	.7518	.7464	.7374	.7245	.7067	.6829	.6512	.6089	.5501	.4520
M= 9	.7536	.7518	.7464	.7374	.7244	.7067	.6829	.6512	.6092	.5503	.4493
M= 7	.7529	.7514	.7466	.7383	.7246	.7069	.6827	.6513	.6071	.5505	.4433
[266]	.7534	.7517	.7465	.7375	.7245	.7065	.6826	.6510	.6086	.5494	.4543

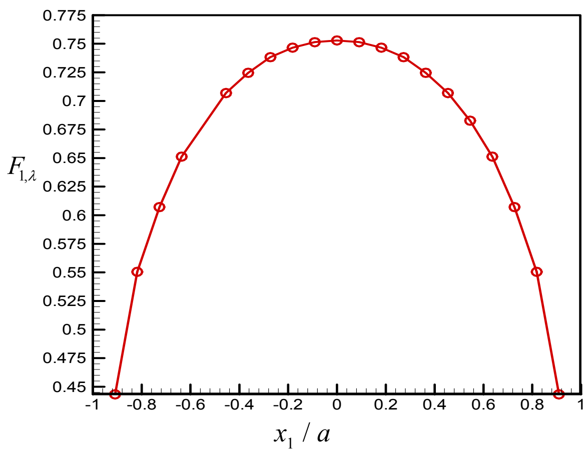


图 2-6 在 $x_2=0,2b$ 边无量纲广义应力强度因子 $F_{I,\lambda}$ 随坐标变化规律（KK=LL=20×20,M=N=13,b/a=1）

载荷为 σ_{31}^{∞} 时，裂纹形状比 *b/a=1*，配置点 *KK=LL=20×20*，广义应力强度因子随多项式指数、坐标变化规律如表 2-5、2-6 及图 2-7、2-8 所示：$F_{2,\lambda}$ (在 $x_1=\pm a$ 随 x_2/b)、$F_{3,\lambda}$ (在 $x_2=0,2b$ 随 x_1/a)与 $F_{1,\lambda}$ (在 $x_2=0,2b$ 随 x_1/a)有相似的变化规律。$F_{2,\lambda}$ 随 $x_2/b\in$（0，2b）成对称分布，在 x_2/b =*b* 取最大值 $F_{2,\lambda\max}\big|_{x_2/b=b}$ =9.8786；$F_{3,\lambda}$ 随 $x_1/a\in$（-a，a）成对称分布，在 x_1/a =0 取最大值 $F_{3,\lambda\max}\big|_{x_1/a=0}$ =0.9903。

表2-5在 $x_1=\pm a$ 边 $F_{2,\lambda}$ 随多项式指数增加变化规律（KK=LL=20×20，b/a=1）

x_2/b	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
M=13	.5541	.6576	.7232	.7700	.8039	.8290	.8479	.8617	.8710	.8763	.8781
M=11	.5565	.6588	.7229	.7697	.8039	.8290	.8479	.8616	.8709	.8763	.8780
M=9	.5564	.6613	.7230	.7691	.8039	.8294	.8478	.8611	.8707	.8765	.8786
M=7	.5514	.6655	.7252	.7678	.8020	.8292	.8493	.8629	.8711	.8752	.8765

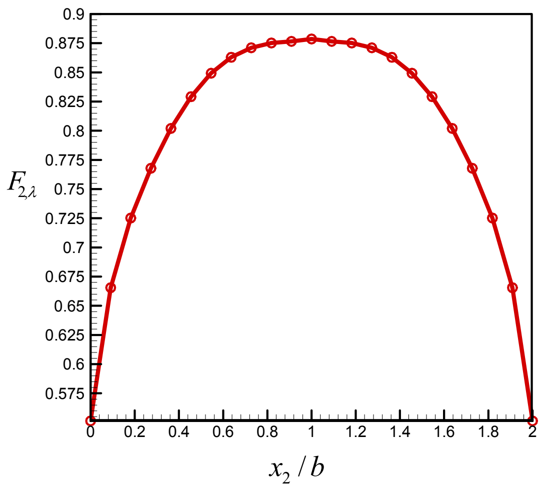


图 2-7 在 $x_1=\pm a$ 边无量纲广义应力强度因子 $F_{2,\lambda}$ 随坐标变化规律（KK=LL=20×20,M=N=13,b/a=1）

表2-6在 $x_2=0,2b$ 边 $F_{3,\lambda}$ 随多项式指数增加变化规律（KK=LL=20×20, M=N=13，b/a=1）

x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
M=13	.9903	.9880	.9809	.9686	.9506	.9263	.8944	.8526	.7978	.7240	.6020
M=11	.9905	.9882	.9809	.9686	.9508	.9266	.8946	.8529	.7991	.7258	.5998
M=9	.9905	.9881	.9808	.9686	.9507	.9265	.8948	.8540	.8009	.7262	.5936
M=7	.9900	.9877	.9808	.9689	.9512	.9271	.5958	.8555	.8021	.7233	.5799

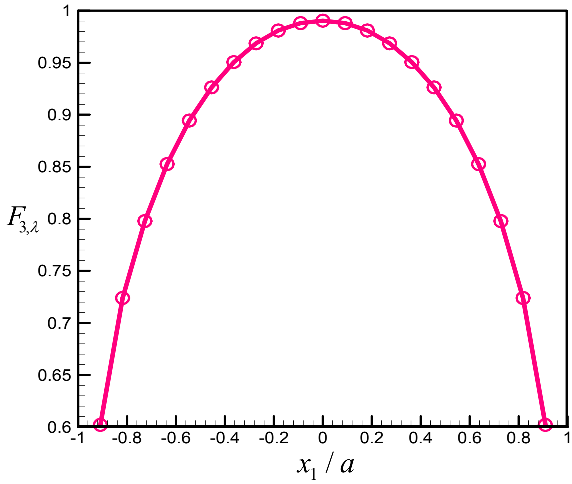
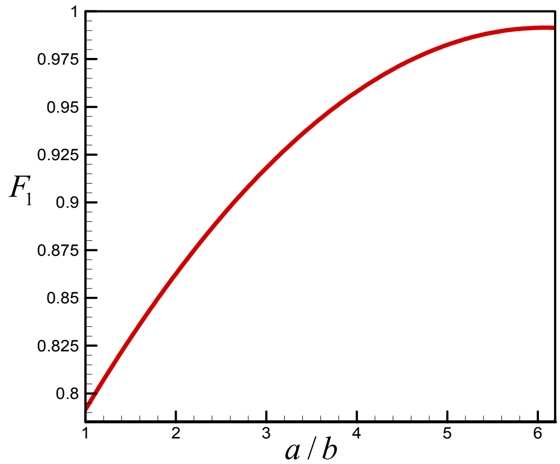


图 2-8 在 $x_2 = 0, 2b$ 边 无量纲广义应力强度因子 $F_{3,\lambda}$ 随坐标变化规律（KK=LL=20×20,M=N=13,b/a=1）

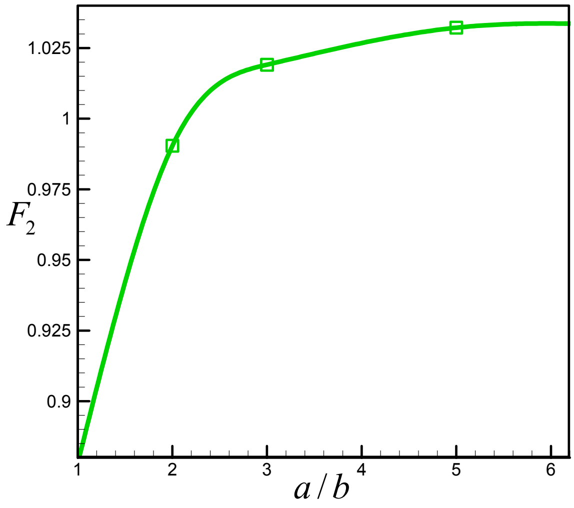
配置点 *KK=LL=20×20*，多项式指数 *M=N=13*，最大应力强度因子 F_1, F_2, F_3, F_4 , F_5 随裂纹形状比规律如表 2-7 及图 2-9 所示：应力强度因子随裂纹形状比增加而增大，增加梯度递减至零，当 $b/a \geq 8$ 时，接近二维问题裂纹数值结果，$F_{1\max}\big|_{a/b\to\infty}$ =1.0022，$F_{2\max}\big|_{a/b\to\infty}$ =1.0332，$F_{3\max}\big|_{a/b\to\infty}$ =1.5370。

表2-7 无量纲应力强度因子随形状变化比增加规律（KK=LL=20×20,M=N=13）

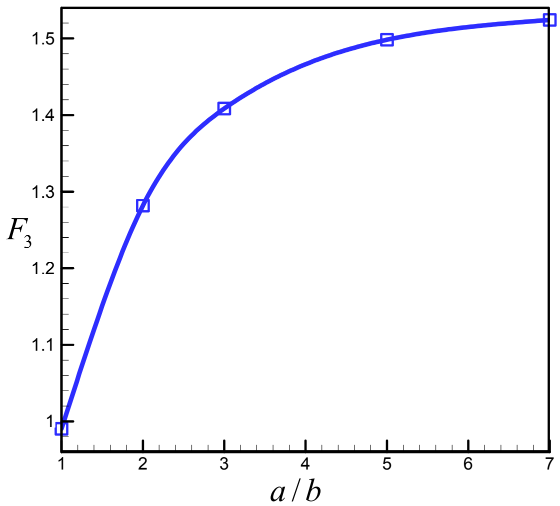
	本文结果						[248]	[244]	[243]
b/a	1	2	3	5	7	∞	∞	∞	∞
F_1 F_4 F_5	.7536	.9183	.9279	.9379	1.0020	1.0022	1.000	1.0053	1.000
F_2	.8781	.9904	1.0191	1.0322	1.0328	1.0332	——	——	——
F_3	.9903	1.2817	1.4084	1.4983	1.5242	1.5370	——	——	——



a) F_1



b) F_2



c) F_3

图 2-9 无量纲广义应力强度因子 F_I 随裂纹形状比变化规律（KK=LL=20×20,M=N=13）

载荷为 σ_{3i}^{∞}，配置点 KK=LL=20×20 时，表 2-8，2-9，2-10，2-10 分别表示多项式指数为 13，11，9，7 时应变能强度因子随 θ 和 x_1/a 变化规律：在 $x_2=0,2b$ 边，$\bar{S}_\lambda\big|_\theta$ 随 x_1/a 呈对称分布，$x_1/a=0$ 取最大值。随多项式指数增加，数值结果是收敛的。图 2-10，2-11，2-12，2-13 同样表示应变能强度因子随 θ 和 x_1/a 变化规律，$\bar{S}_\lambda\big|_\theta$ 在 180° -360° 间变化规律与 0° -180° 间变化规律成对称分布，这与实际情况相吻合。

表2-8无量纲广义应变能强度因子随 θ， x_1/a 变化规律（KK=LL=20×20, M=N=13,b/a=1）

θ \ x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
0	.5501	.5476	.5401	.5272	.5087	.4842	.4525	0.4122	0.3615	.2970	.2042
10	.5637	.5611	.5534	.5402	.5213	.4962	.4638	0.4225	0.3705	.3044	.2092
20	.5681	.5655	.5577	.5445	.5254	.5000	.4674	0.4258	0.3734	.3068	.2109
30	.5616	.5591	.5513	.5382	.5194	.4943	.4620	0.4208	0.3690	.3033	.2084
40	.5439	.5414	.5339	.5212	.5029	.4786	.4472	.4073	.3572	.2935	.2016
50	.5158	.5134	.5063	.4942	.4768	.4537	.4239	.3861	.3385	.2781	.1910
60	.4797	.4775	.4708	.4595	.4433	.4217	.3940	.3587	.3144	.2583	.1774
70	.4388	.4367	.4306	.4202	.4053	.3855	.3600	.3277	.2872	.2360	.1620
80	.3970	.3952	.3896	.3802	.3666	.3486	.3254	.2961	.2594	.2132	.1463
90	.3587	.3570	.3520	.3434	.3310	.3146	.2936	.2671	.2339	.1923	.1320
100	.3277	.3262	.3216	.3136	.3023	.2872	.2680	.2436	.2134	.1754	.1205
110	.3073	.3059	.3015	.2941	.2834	.2693	.2511	.2283	.1999	.1644	.1130
120	.2997	.2983	.2941	.2868	.2764	.2626	.2449	.2226	.1950	.1603	.1103
130	.3058	.3043	.3000	.2926	.2820	.2680	.2500	.2273	.1991	.1638	.1128
140	.3248	.3233	.3187	.3109	.2997	.2849	.2658	.2418	.2119	.1743	.1201
150	.3548	.3532	.3483	.3398	.3276	.3115	.2909	.2647	.2320	.1908	.1316
160	.3928	.3910	.3856	.3763	.3629	.3452	.3224	.2936	.2574	.2117	.1460
170	.4349	.4329	.4269	.4167	.4020	.3825	.3574	.3255	.2854	.2348	.1619
180	.4768	.4746	.4681	.4569	.4409	.4196	.3921	.3572	.3133	.2577	.1777

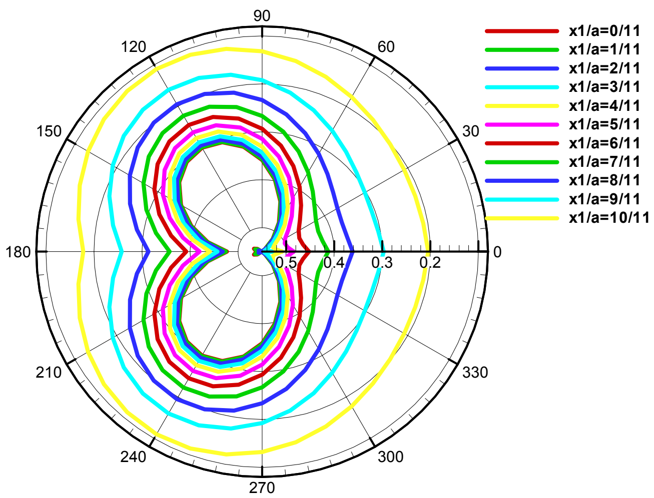


图 2-10 无量纲应变能强度因子随 θ 和 x_1/a 变化规律（KK=LL=20×20,M=N=13,b/a=1）

表2-9无量纲广义应变能强度因子随 θ， x_1/a 变化规律（KK=LL=20×20, M=N=11,b/a=1）

θ \ x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
0	.5502	.5477	.5400	.5272	.5089	.4843	.4526	.4122	.3618	.2979	.2047
10	.5638	.5612	.5534	.5402	.5215	.4963	.4638	.4225	.3708	.3053	.2098
20	.5682	.5656	.5577	.5444	.5255	.5002	.4675	.4258	.3737	.3077	.2115
30	.5617	.5591	.5513	.5382	.5195	.4944	.4621	.4209	.3694	.3041	.2090
40	.5439	.5415	.5339	.5212	.5030	.4787	.4473	.4074	.3575	.2943	.2022
50	.5159	.5135	.5063	.4942	.4770	.4538	.4240	.3861	.3388	.2789	.1914
60	.4798	.4776	.4708	.4595	.4434	.4219	.3940	.3587	.3148	.2591	.1777
70	.4388	.4368	.4306	.4202	.4054	.3856	.3601	.3277	.2876	.2367	.1621
80	.3971	.3953	.3896	.3802	.3667	.3487	.3255	.2962	.2598	.2139	.1463
90	.3588	.3571	.3520	.3434	.3311	.3148	.2937	.2672	.2344	.1929	.1318
100	.3278	.3263	.3215	.3136	.3024	.2874	.2681	.2438	.2138	.1760	.1201
110	.3074	.3060	.3015	.2941	.2835	.2694	.2512	.2284	.2004	.1650	.1126
120	.2998	.2984	.2941	.2868	.2765	.2627	.2450	.2228	.1954	.1610	.1099
130	.3058	.3044	.3000	.2926	.2821	.2681	.2501	.2274	.1996	.1644	.1123
140	.3249	.3234	.3187	.3109	.2998	.2850	.2659	.2419	.2123	.1750	.1198
150	.3549	.3533	.3482	.3398	.3278	.3117	.2909	.2648	.2324	.1916	.1314
160	.3929	.3911	.3856	.3763	.3631	.3453	.3225	.2936	.2578	.2125	.1460
170	.4350	.4330	.4269	.4167	.4021	.3826	.3575	.3255	.2858	.2356	.1620
180	.4769	.4747	.4681	.4569	.4410	.4197	.3922	.3573	.3137	.2585	.1780

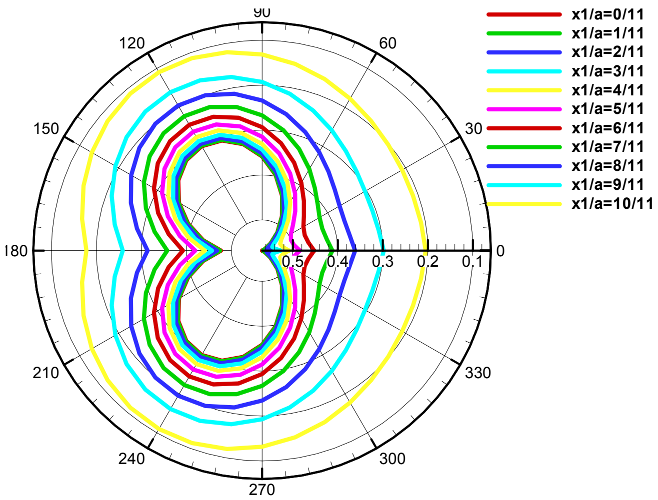


图 2-11 无量纲应变能强度因子随 θ 和 x_1/a 变化规律（KK=LL=20×20,M=N=11,b/a=1）

表2-10无量纲广义应变能强度因子随 θ，x_1/a 变化规律（KK=LL=20×20, M=N=9,b/a=1）

θ \ x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
0	.5504	.5477	.5399	.5270	.5087	.4843	.4527	.4126	.3626	.2986	.2020
10	.5639	.5612	.5532	.5401	.5213	.4964	.4639	.4228	.3717	.3061	.2071
20	.5683	.5656	.5576	.5443	.5254	.5002	.4676	.4261	.3746	.3084	.2088
30	.5619	.5592	.5512	.5381	.5194	.4945	.4622	.4212	.3702	.3048	.2063
40	.5441	.5415	.5338	.5210	.5029	.4788	.4474	.4077	.3584	.2950	.1995
50	.5160	.5135	.5062	.4941	.4769	.4539	.4241	.3865	.3397	.2795	.1888
60	.4799	.4776	.4707	.4594	.4433	.4219	.3941	.3591	.3156	.2597	.1751
70	.4389	.4368	.4305	.4202	.4053	.3856	.3602	.3281	.2884	.2371	.1597
80	.3972	.3953	.3896	.3801	.3666	.3487	.3256	.2966	.2606	.2142	.1440
90	.3588	.3571	.3519	.3433	.3311	.3148	.2938	.2676	.2352	.1932	.1296
100	.3278	.3262	.3215	.3136	.3023	.2873	.2682	.2442	.2146	.1763	.1180
110	.3074	.3060	.3015	.2941	.2835	.2693	.2513	.2289	.2011	.1652	.1105
120	.2998	.2984	.2940	.2868	.2764	.2627	.2451	.2232	.1962	.1612	.1078
130	.3059	.3044	.3000	.2926	.2821	.2681	.2502	.2279	.2003	.1646	.1103
140	.3249	.3233	.3186	.3109	.2997	.2850	.2660	.2424	.2131	.1753	.1177
150	.3550	.3533	.3482	.3397	.3277	.3117	.2910	.2652	.2332	.1919	.1292
160	.3930	.3911	.3855	.3762	.3630	.3454	.3226	.2940	.2586	.2129	.1437
170	.4351	.4330	.4268	.4166	.4020	.3826	.3575	.3259	.2866	.2362	.1597
180	.4770	.4747	.4680	.4568	.4409	.4198	.3923	.3576	.3145	.2592	.1756

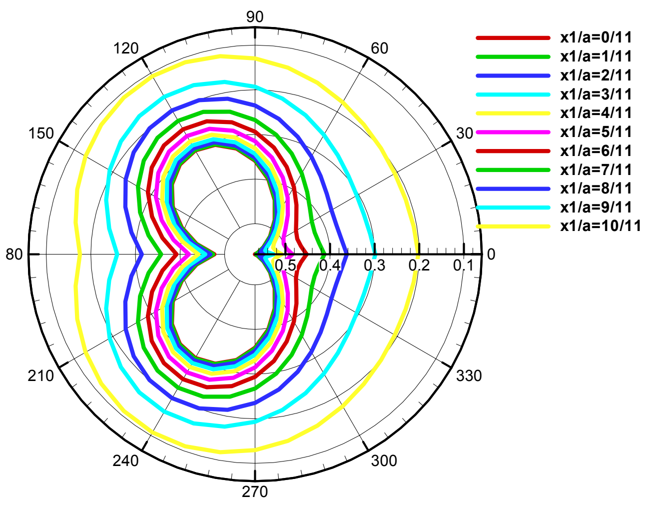


图 2-12 无量纲应变能强度因子随 θ 和 x_1/a 变化规律（KK=LL=20×20,M=N=9,b/a=1）

表2-11无量纲广义应变能强度因子随 θ， x_1/a 变化规律（KK=LL=20×20, M=N=7,b/a=1）

θ \ x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
0	.5492	.5470	.5401	.5281	.5095	.4847	.3407	.4129	.3628	.2985	.1954
10	.5627	.5605	.5535	.5412	.5221	.4967	.3519	.4232	.3718	.3060	.2004
20	.5671	.5648	.5578	.5454	.5262	.5006	.3555	.4265	.3747	.3084	.2020
30	.5606	.5584	.5514	.5392	.5201	.4948	.3502	.4216	.3704	.3047	.1995
40	.5430	.5408	.5340	.5221	.5036	.4791	.3355	.4081	.3585	.2949	.1929
50	.5150	.5129	.5064	.4950	.4775	.4542	.3122	.3869	.3398	.2793	.1825
60	.4789	.4770	.4709	.4603	.4439	.4222	.2823	.3596	.3157	.2592	.1691
70	.4381	.4363	.4307	.4209	.4059	.3859	.2485	.3287	.2885	.2366	.1540
80	.3965	.3948	.3897	.3807	.3671	.3490	.2139	.2971	.2608	.2135	.1386
90	.3583	.3567	.3520	.3438	.3314	.3151	.1822	.2682	.2353	.1924	.1245
100	.3274	.3259	.3215	.3140	.3026	.2877	.1567	.2448	.2148	.1753	.1132
110	.3071	.3057	.3015	.2944	.2837	.2697	.1398	.2295	.2014	.1642	.1059
120	.2995	.2981	.2941	.2871	.2767	.2630	.1336	.2238	.1965	.1602	.1033
130	.3055	.3041	.3000	.2929	.2824	.2684	.1387	.2285	.2006	.1637	.1057
140	.3244	.3230	.3187	.3113	.3001	.2853	.1545	.2429	.2135	.1744	.1130
150	.3544	.3529	.3482	.3402	.3281	.3119	.1794	.2657	.2336	.1912	.1243
160	.3923	.3906	.3856	.3768	.3635	.3456	.2109	.2945	.2590	.2124	.1385
170	.4342	.4324	.4269	.4173	.4026	.3829	.2457	.3263	.2870	.2358	.1541
180	.4760	.4741	.4681	.4577	.4416	.4200	.2804	.3580	.3149	.2590	.1696

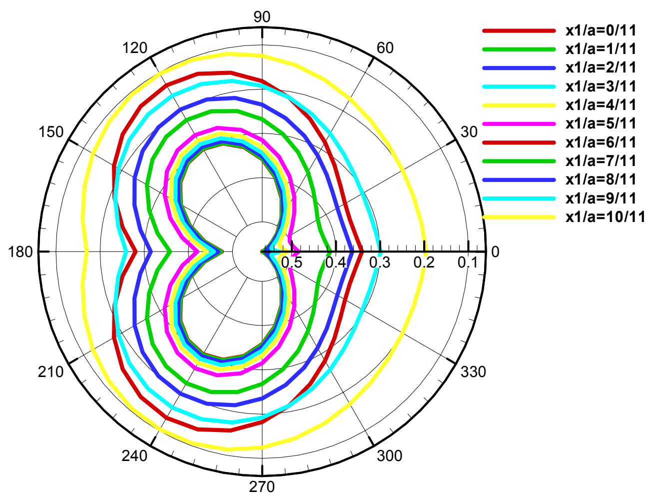


图 2-13 无量纲应变能强度因子随 θ 和 x_1/a 变化规律（KK=LL=20×20,M=N=7,b/a=1）

配置点 $KK=LL=20\times20$，多项式指数 $M=N=13$，最大应变能强度因子 $\bar{S}\big|_{\theta}$ 随裂纹形状比规律如表 2-12 及图 2-14 所示：应变能强度因子随裂纹形状比增加而增大，增加梯度递减至零，当 $b/a\geq 8$ 时，接近二维问题数值结果。

表2-12无量纲广义应变能强度因子随 θ，b/a变化规律（KK=LL=20×20, M=N=13）

θ \ a/b	1	2	3	5	7	∞
0	0.5501	0.8361	0.9366	1.0122	1.0693	1.0793
10	0.5637	0.8548	0.9561	1.0321	1.0906	1.1007
20	0.5681	0.8610	0.9625	1.0387	1.0979	1.1079
30	0.5616	0.8524	0.9535	1.0295	1.0883	1.0983
40	0.5439	0.8284	0.9284	1.0038	1.0611	1.0712
50	0.5158	0.7901	0.8885	0.9630	1.0178	1.0278
60	0.4797	0.7406	0.8370	0.9102	0.9616	0.9716
70	0.4388	0.6844	0.7786	0.8504	0.8977	0.9077
80	0.3970	0.6269	0.7188	0.7892	0.8321	0.8420
90	0.3587	0.5738	0.6637	0.7328	0.7714	0.7813
100	0.3277	0.5305	0.6189	0.6870	0.7217	0.7316
110	0.3073	0.5017	0.5891	0.6565	0.6882	0.6981
120	0.2997	0.4902	0.5774	0.6446	0.6745	0.6843
130	0.3058	0.4974	0.5852	0.6525	0.6818	0.6917
140	0.3248	0.5224	0.6115	0.6795	0.7096	0.7195
150	0.3548	0.5628	0.6537	0.7227	0.7549	0.7648
160	0.3928	0.6143	0.7074	0.7777	0.8129	0.8228
170	0.4349	0.6716	0.7671	0.8389	0.8778	0.8878
180	0.4768	0.7290	0.8269	0.9001	0.9430	0.9530

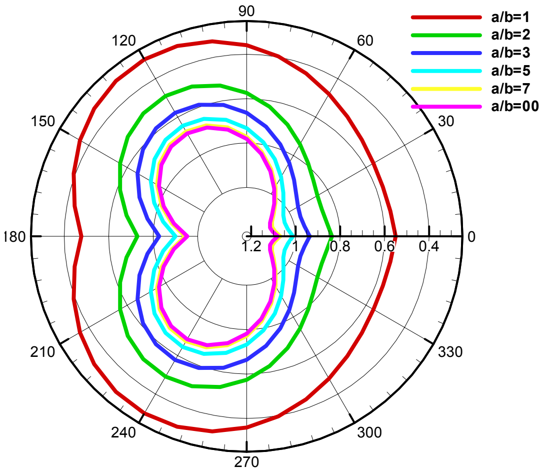


图 2-14 无量纲应变能强度因子随 θ 和 b/a 变化规律（KK=LL=20×20, M=N=13）

2.6 本章小结

1. 应用主部分析法和广义位移基本解，将磁电热弹耦合材料任意形状平片裂纹问题转化为求解一组以裂纹表面位移间断为未知函数的超奇异积分方程问题。

2. 采用二维超奇异积分的主部分析法得到了裂纹前沿的广义应力奇异性指数，进而，通过主部分析精确的求得了裂纹前沿光滑点附近的奇性应力场解析表达式。定义了广义应力强度因子、广义应变能强度因子和广义能量释放率。

3. 通过将裂纹表面位移间断未知函数表达为位移间断基本密度函数与多项式之积，使用有限部积分概念对超奇异积分方程组建立了数值方法。

4. 基于超奇异积分方程未知解的理论分析结果，得到了广义应力强度因子和广义应变能强度因子的数值计算公式，用 FORTRAN 语言编写了专用程序，对典型算例进行了计算；由数值结果可知，广义（应力、应变能）强度因子数值结果是收敛的，而且有很好的精度。最大（应力、应变能）强度因子随裂纹形状比增加而增大，当 $a/b \geq 8$ 时，接近二维裂纹问题数值结果。

5. 工程中遇到的多场耦合作用下的材料单裂纹问题，可以应用本文提供的裂纹评定准则来分析和研究结构的可靠性。在应力集中区域内，裂纹对材料结构强度和可靠性的影响十分明显，在工程应用中应予以考虑。

第三章 磁电热弹耦合材料三维多裂纹问题

3.1 引言

多裂纹问题是固体材料中普遍存在的问题，因此吸引了许多学者的关注并对此进行了大量的研究。早期多裂纹问题研究多致力于求解析解（借助于复变函数方法或通过积分变换方法），得出许多无限大弹性板中多裂纹问题解析解。如两条等长裂纹在 I，II，III 型载荷条件下的解(1962 年 Barenblatt[267]、Erdogan[169])。三条共线裂纹在 I，II，III 型载荷条件下的解（1964 年 Sih[95]）。等长共线裂纹在 I，II ，III 型情况下的解(1957 年 Irwin[268]、1959 年 Koiver、1973 年 Tada[269])。等长堆叠裂纹（即两平行而共对称轴的裂纹）在 III 型条件下的解(1973 年 Tada[269])。还有一些多裂纹问题的解是用反演积分所表示的封闭形式解(1971 年 Snedden[270])，或是无穷级数所表示的封闭形式解(1973 年 Isida[271])。然而不幸的是，这些解析解都局限于特殊分布的裂纹及其简单载荷条件，在大多数情况下，多裂纹问题，由于其固有的复杂性，都无法或难于求得解析解而只能借助于数值方法。

超奇异积分方程法是常用来求解无限大弹性体中含裂纹问题的有效方法，Kishida 和 Asano[272]应用奇异积分方程分析了三条平行裂纹的相互影响，应用该方法求解多裂纹问题还有 Narendra 和 Cleary[273]。特别值的一提的是 Narendra 和 Cleary[273]提出了一种处理复杂裂纹问题的最有效方法，这种方法考虑裂纹与一条或多条位于不同的介质区域内部或交界面上的裂纹，杂质或包含物的相互作用，这些裂纹可以是分叉的，弯曲的，也可以是平坦的，裂纹面可以承受任意载荷作用的。该方法应用高效的表面积分方法，应用边缘位错分布来表示裂纹，最后得到的奇异积分方程应用高斯——切比雪夫积分公式求解。

本章中，应用超奇异积分法对多场载荷作用下的磁电热弹耦合材料三维多裂纹问题进行了理论和数值分析。首先，将该问题转化为求解一组以广义位移间断为未知函数的超奇异积分方程问题，得到了裂纹前沿广义奇异应力场解析表达式，定义了广义应力强度因子、广义应变能强度因子和广义能量释放率；进而，应用有限部积分概念及体积力法，为超奇异积分方程组建立了数值求解方法，用 FORTRAN 语言编制了计算广义（应力、应变能）强度因子的专用程序；然后，以三维双裂纹为例，通过典型算例计算，得到了广义（应力、应变能）强度因子随裂纹位置、裂纹形状及材料参数变化的规律。最后，分析了裂纹间干扰及屏蔽作用及其在工程实际中的应用。

3.2 边界积分方程

如图3-1所示，假设磁电热弹耦合材料空间区域Ω中含有任意形状和空间位置的平片多裂纹。选择直角坐标系x_i (i=1,2,3)为整体坐标系， 在裂纹面$S_{\hat{i}}(S_{\hat{i}}^{+}\cup S_{\hat{i}}^{-})$($\hat{i}$=1,2……n)表面分别作用大小相等方向相反的载荷（力载荷$p_j^{\hat{i}}(P^{\hat{i}})$，电载荷$q^{\hat{i}}(P^{\hat{i}})$，磁载荷$b^{\hat{i}}(P^{\hat{i}})$和热载荷

$\rho^{\hat{i}}(P^{\hat{i}})$）。为了研究问题的方便，我们建立局部直角坐标系 $x_i^{\hat{i}}$，裂纹 $S_{\hat{i}}$ 在局部系坐标轴 $x_1^{\hat{i}}x_2^{\hat{i}}$ 构成的平面内，同时与局部系坐标轴 $x_3^{\hat{i}}$ 垂直。局部系坐标 $x_i^{\hat{i}}$ 与整体系坐标 x_i 的夹角为 $\theta_i^{\hat{i}}(x_i, x_i^{\hat{i}})$。裂纹中心点的坐标为 $O^{\tilde{i}}(\xi_1^{\tilde{i}}, \xi_2^{\tilde{i}}, \xi_3^{\tilde{i}})$。利用Somigliana公式，磁电热弹耦合材料中任意一点 $p(x_1, x_2, x_3)$ 处的广义位移 $U_I(p)$ 可表示为：

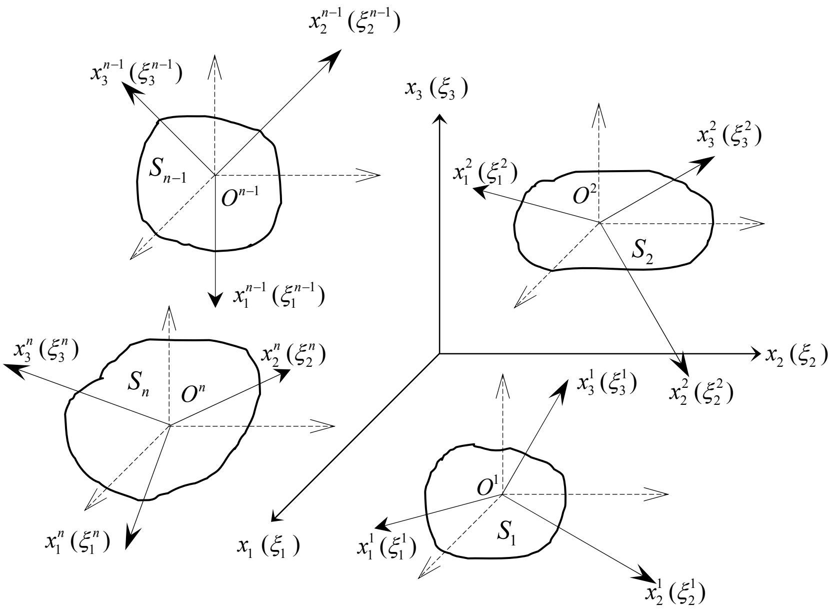


图 3-1 磁电热弹耦合材料内含任意形状和位置平片多裂纹示意图

$$
\begin{aligned}
U_I(p) = &-\int_{\Gamma}(T_{IJ}^0(p,q)U_J^0(q) + U_{IJ}^0(p,q)T_J^0(q))ds(q) + \int_{\Omega}U_{IJ}^0(p,q)f_J(q)dV(q) + \\
&\underbrace{\left(\begin{aligned}
&\int_{S_1^+ + S_1^-}(U_{IJ}^1(p,q)T_J^1(q) - T_{IJ}^1(p,q)U_J^1(q))ds(q) \\
&+\int_{S_2^+ + S_2^-}(U_{IJ}^2(p,q)T_J^2(q) - T_{IJ}^2(p,q)U_J^2(q))ds(q) + \cdots\cdots \\
&\cdots + \int_{S_{n-1}^+ + S_{n-1}^-}(U_{IJ}^{n-1}(p,q)T_J^{n-1}(q) - T_{IJ}^{n-1}(p,q)U_J^{n-1}(q))ds(q) \\
&+\int_{S_n^+ + S_n^-}(U_{IJ}^n(p,q)T_J^n(q) - T_{IJ}^n(p,q)U_J^n(q))ds(q)
\end{aligned}\right)}_{n}
\end{aligned}
\tag{3-1}
$$

式（3-1）中符号 $T_J^{\tilde{i}}(q)$ ($\tilde{i} = 0,1,2......n$)为广义面力分量，具体表达式为：

$$
T_J^{\tilde{i}} = \begin{cases}
p_j^{\tilde{i}} = \sigma_{jl}^{\tilde{i}}n_l & J = j = 1,2,3 \\
q^{\tilde{i}} = D_l^{\tilde{i}}n_l & J = 4 \\
b^{\tilde{i}} = B_l^{\tilde{i}}n_l & J = 5 \\
\rho^{\tilde{i}} = \vartheta_{,J}^{\tilde{i}}n_i & J = 6
\end{cases}
\tag{3-2}
$$

广义位移和面力基本解 $U_{IJ}^{\tilde{i}}(p,q)$ 和 $T_{IJ}^{\tilde{i}}(p,q)$ 为已知函数，他们之间的关系可由下式表示：

$$
T_{IJ}^{\tilde{i}}(\xi - x) = E_{kJMn}U_{IM,n}^{\tilde{i}}(\xi - x)n_k \tag{3-3}
$$

引入裂纹面 $S_{\hat{i}}$ 上广义位移间断 $\tilde{U}_J^{\hat{i}}(p)$，其具体定义式为：

$$\tilde{U}_J^{\hat{i}}=\begin{cases}\tilde{u}_j^{\hat{i}}=u_j^{\hat{i}+}-u_j^{\hat{i}-} & J=j=1,2,3\\ \tilde{\phi}_j^{\hat{i}}=\phi_j^{\hat{i}+}-\phi_j^{\hat{i}-} & J=4\\ \tilde{\varphi}_j^{\hat{i}}=\varphi_j^{\hat{i}+}-\varphi_j^{\hat{i}-} & J=5\\ \tilde{\vartheta}_j^{\hat{i}}=\tilde{\vartheta}_j^{\hat{i}+}-\tilde{\vartheta}_j^{\hat{i}-} & J=6\end{cases} \tag{3-4}$$

利用关系式 $T_{IJ}^{\hat{i}}(p,q^+)=-T_{IJ}^{\hat{i}}(p,q^-)=T_{IJ}^{\hat{i}+}(p,q)$ 和 $U_{IJ}^{\hat{i}}(p,q^+)=U_{IJ}^{\hat{i}}(p,q^-)$，广义位移 $U_I(p)$ 可以被重新表示为：

$$U_I(p)=\int_\Gamma (U_{IJ}^0(p,q)T_J^0(q)-T_{IJ}^0(p,q)U_J^0(q))ds(q)+\int_\Omega U_{IJ}^0(p,q)f_J(q)dV(q)$$
$$-\underbrace{\left(\begin{aligned}&\int_{S_1^+}T_{IJ}^{1+}(p,q)\tilde{U}_J^1(q)ds(q)+\int_{S_2^+}T_{IJ}^{2+}(p,q)\tilde{U}_J^2(q)ds(q)\\&+\int_{S_3^+}T_{IJ}^{3+}(p,q)\tilde{U}_J^3(q)ds(q)+\cdots\cdots\cdots\cdots\\&+\int_{S_{n-1}^+}T_{IJ}^{(n-1)+}(p,q)\tilde{U}_J^{n-1}(q)ds(q)+\int_{S_n^+}T_{IJ}^{n+}(p,q)\tilde{U}_J^n(q)ds(q)\end{aligned}\right)}_{n} \tag{3-5}$$

广义应力张量 $\sum_{ij}$ 可表示为：

$$\sum\nolimits_{ij}(p)=-\underbrace{\left(\begin{aligned}&\int_{S_1^+}S_{KiJ}^{1+}(p,q)\tilde{U}_K^1(q)ds(q)+\int_{S_2^+}S_{KiJ}^{2+}(p,q)\tilde{U}_K^2(q)ds(q)\\&+\int_{S_3^+}S_{KiJ}^{3+}(p,q)\tilde{U}_K^3(q)ds(q)+\cdots\cdots+\int_{S_{n-1}^+}S_{KiJ}^{n-1+}(p,q)\tilde{U}_K^{n-1}(q)ds(q)\\&+\int_{S_n^+}S_{KiJ}^{n+}(p,q)\tilde{U}_K^n(q)ds(q)\end{aligned}\right)}_{n}$$
$$+\int_\Gamma (D_{KiJ}^0(p,q)T_K^0(q)-S_{KiJ}^0(p,q)U_K^0(q))ds(q)+\int_\Omega D_{KiJ}^0(p,q)f_K(q)dV(q) \tag{3-6}$$

式（3-6）中积分核函数 $S_{KiJ}^{\hat{i}}$ 和 $D_{KiJ}^{\hat{i}}$ 的具体表达式为：

$$S_{KiJ}^{\hat{i}}(p,q)=-E_{iJMn}T_{MK,n}^{\hat{i}}(p,q) \tag{3-7}$$

$$D_{KiJ}^{\hat{i}}(p,q)=-E_{iJMn}U_{MK,n}^{\hat{i}}(p,q) \tag{3-8}$$

裂纹边界包括两部分：外边界 Γ 和裂纹面 $S_n(S_n^+\bigcup S_n^-)$ 处边界。应用边界条件，让源点 p 不断向边界趋近，当达到边界时，我们用P点来代替，从式(3-5)可以得到下面边界积分方程，

$$C_{IJ}U_I(P)=-\underbrace{\begin{pmatrix}\int_{S_1^+}T_{IJ}^{1+}(P,q)\tilde{U}_J^1(q)ds(q)+\int_{S_2^+}T_{IJ}^{2+}(P,q)\tilde{U}_J^2(q)ds(q)+\\ \int_{S_3^+}T_{IJ}^{3+}(P,q)\tilde{U}_J^3(q)ds(q)+\cdots\cdots\cdots\cdots+\int_{S_{n-1}^+}T_{IJ}^{(n-1)+}(P,q)\tilde{U}_J^{n-1}(q)ds(q)\\ +\int_{S_n^+}T_{IJ}^{n+}(P,q)\tilde{U}_J^n(q)ds(q)\end{pmatrix}}_{n} \tag{3-9}$$

$$+\int_\Gamma(U_{IJ}^0(P.q)T_J^0(q)-T_{IJ}^0(P,q)U_J^0(q))ds(q)+\int_\Omega U_{IJ}^0(P,q)f_J(q)dV(q)$$

其中常数C_{IJ}与边界点P相关，应用裂纹面的广义边界条件，可以得到以下超奇异积分方程组。

$$\underbrace{\begin{pmatrix}⨍_{S_1^+}S_{KiJ}^{1+}(P,q)\tilde{U}_K^1(q)ds(q)+⨍_{S_2^+}S_{KiJ}^{2+}(P,q)\tilde{U}_K^2(q)ds(q)+⨍_{S_3^+}S_{KiJ}^{3+}(P,q)\tilde{U}_K^3(q)ds(q)+\cdots\cdots+\\ +⨍_{S_{n-1}^+}S_{KiJ}^{(n-1)+}(P,q)\tilde{U}_K^{n-1}(q)ds(q)+⨍_{S_n^+}S_{KiJ}^{n+}(P,q)\tilde{U}_K^n(q)ds(q)\end{pmatrix}}_{n} \tag{3-10}$$

$$=\int_\Gamma(D_{KiJ}^0(P,q)T_K^0(q)-S_{KiJ}^0(P,q)U_K^0(q))ds(q)+\int_\Omega D_{KiJ}^0(P,q)f_K(q)dV(q)$$

其中 $⨍$ 必须通过有限部积分才可以求解，上式中的第一个积分的奇异指数为r^{-3}，是一个超奇异积分。通过求解(3-9)～（3-10)式，以上各个未知函数就可以求解了。Q点为空间 Ω 任意一点，则该点在整体坐标系下沿坐标轴方向的位移可表示为：

$$U_K(Q)=-\int_{S_{\hat{i}}^+}T_{KI}(P_{\hat{i}},Q)\tilde{U}_I(P_{\hat{i}})dS_{\hat{i}}=-\int_{S_{\hat{i}}^+}T_{KI}^{\hat{i}}(P_{\hat{i}},Q)\tilde{U}_I^{\hat{i}}(P_{\hat{i}})dS_{\hat{i}} \tag{3-11}$$

其中$\tilde{U}_I(P_{\hat{i}})=U_I(P_{\hat{i}}^+)-U_I(P_{\hat{i}}^-)$和$\tilde{U}_I^{\hat{i}}(P_{\hat{i}})=U_I^{\hat{i}}(P_{\hat{i}}^+)-U_I^{\hat{i}}(P_{\hat{i}}^-)$分别表示在第 I 裂纹处整体系和局部系下的广义位移间断。详细的定义可表示为如下关系式

$$\tilde{U}_I(P_{\hat{i}})=\begin{cases}\tilde{u}_i(P_{\hat{i}})=u_i^+(P_{\hat{i}})-u_i^-(P_{\hat{i}}) & I=i=1,2,3\\ \tilde{\phi}_I(P_{\hat{i}})=\phi_I^+(P_{\hat{i}})-\phi_I^-(P_{\hat{i}}) & I=4\\ \tilde{\varphi}_I(P_{\hat{i}})=\varphi_I^+(P_{\hat{i}})-\varphi_I^-(P_{\hat{i}}) & I=5\\ \tilde{\vartheta}_I(P_{\hat{i}})=\vartheta_I^+(P_{\hat{i}})-\vartheta_I^-(P_{\hat{i}}) & I=6\end{cases} \tag{3-12}$$

$$\tilde{U}_I^{\hat{i}}(P_{\hat{i}})=\begin{cases}\tilde{u}_i^{\hat{i}}(P_{\hat{i}})=u_i^{\hat{i}+}(P_{\hat{i}})-u_i^{\hat{i}-}(P_{\hat{i}}) & I=i=1,2,3\\ \tilde{\phi}_I^{\hat{i}}(P_{\hat{i}})=\phi_I^{\hat{i}+}(P_{\hat{i}})-\phi_I^{\hat{i}-}(P_{\hat{i}}) & I=4\\ \tilde{\varphi}_I^{\hat{i}}(P_{\hat{i}})=\varphi_I^{\hat{i}+}(P_{\hat{i}})-\varphi_I^{\hat{i}-}(P_{\hat{i}}) & I=5\\ \tilde{\vartheta}_I^{\hat{i}}(P_{\hat{i}})=\vartheta_I^{\hat{i}+}(P_{\hat{i}})-\vartheta_I^{\hat{i}-}(P_{\hat{i}}) & I=6\end{cases} \tag{3-13}$$

$T_{KI}(P_{\hat{i}},Q)$和$T_{KI}^{\hat{i}}(P_{\hat{i}},Q)$分别代表整体系$x_i$和局部系$x_i^{\hat{i}}$下广义面力基本解。它们与广义位移基本解的关系式如下：

$$T_{KI}(P_{\hat{i}},Q)=E_{IIMn}U_{KM,n}(P_{\hat{i}},Q)n_I \tag{3-14}$$

$$T_{KI}^{\hat{i}}(P_{\hat{i}},Q)=E_{IIMn}U_{KM,n}^{\hat{i}}(P_{\hat{i}},Q)n_I=\mathrm{H}_{KT}^{\hat{i}}\mathrm{H}_{ILIL}^{\hat{i}}T_{TL}(P_{\hat{i}},Q)=\mathrm{H}_{KT}^{\hat{i}}\mathrm{H}_{IL}^{\hat{i}}E_{IILn}U_{KL,n}(P_{\hat{i}},Q)n_I \tag{3-15}$$

其中$U_{\hat{K}M}$为整体系下广义位移基本解，具体表达式子可参考文献[274]，坐标旋转变换张量$\mathrm{H}_{KT}^{\hat{i}\hat{j}}$，$\mathrm{H}_{KT}^{\hat{i}}$及坐标平动转换张量$x_i^{\hat{i}\hat{j}}$，$x_i^{\hat{i}}$的具体定义式如下：

$$\mathrm{H}_{KT}^{\hat{i}\hat{j}}=\begin{cases}\iota_{KT}^{\hat{i}\hat{j}} & K,T=1,2,3\\ 1 & K,T=4,5,6\end{cases} \tag{3-16}$$

$$x_i^{\hat{i}\hat{j}}=\mathrm{H}_{KT}^{\hat{i}\hat{j}}(x_i^{\hat{i}}-O_i^{\hat{i}\hat{j}}) \tag{3-17}$$

$$\mathrm{H}_{KT}^{\hat{i}}=\begin{cases}\iota_{KT}^{\hat{i}} & K,T=1,2,3\\ 1 & K,T=4,5,6\end{cases} \tag{3-18}$$

$$x_i^{\hat{i}}=\mathrm{H}_{KT}^{\hat{i}\hat{j}}(x_i^{\hat{i}}-O_i) \tag{3-19}$$

其中 $\iota_{KT}^{\hat{i}\hat{j}}=\cos(x_K^{\hat{i}},x_T^{\hat{j}})$为局部系下坐标轴$x_i^{\hat{i}}$与局部系下坐标轴$x_i^{\hat{j}}$夹角余弦。$\iota_{KT}^{\hat{i}}=\cos(x_K^{\hat{i}},x_T)$为局部系下坐标轴$x_i^{\hat{i}}$与整体系下坐标轴 x_i 夹角余弦。利用位移-应变关系和Hooke定律，可以得到局部系$x_i^{\hat{i}}$中$Q(\xi_1,\xi_2,\xi_3)$点用广义基本解表示的广义应力分量

$$\sum\nolimits_{IJ}(Q)=-\int_{S_{\hat{i}}^+}S_{KIJ}^{\hat{i}}(P_{\hat{i}},Q)\tilde{U}_K^{\hat{i}}(Q)dS_{\hat{i}}^+ \tag{3-20}$$

式（3-20）中核函数$S_{KiJ}^{\hat{i}}$具体表达式如下：

$$S_{KIJ}^{\hat{i}}(P_{\hat{i}},Q)=-E_{IJMn}T_{MK,n}^{\hat{i}}(P_{\hat{i}},Q) \tag{3-21}$$

3.3 广义基本解

3.3.1 广义位移基本解

在已有资料[138, 214, 215, 229, 230, 275, 276]基础上，应用张量变换，整理、推导、比较和验证了磁电热弹耦合材料空间位移基本解，推导出适合于超奇异积分方程方法中使用的磁电热弹耦合材料三维广义位移基本解显式表达式，具体表达式如下：

1．点$\xi_j^{\bar{i}}$处共同作用$P_z^{\bar{i}}$、$Q^{\bar{i}}$、$J^{\bar{i}}$、$\vartheta^{\bar{i}}$，点$x_j^{\bar{i}}$广义位移基本解为：

$$U_{mJ}^{\hat{i}}=\sum\nolimits_{i=1}^{5}R_{i\hat{i}}^{-1}A_{im}(\dot{R}_{i\hat{i}}^{-1}(x_\alpha-\xi_\alpha)\delta_{\alpha J}+\lambda_i^w\delta_{3J}+\lambda_i^\phi\delta_{4J}+\lambda_i^\varphi\delta_{5J}+R_{i\hat{i}}^{-2}(\xi_3-x_3)\lambda_i^\vartheta s_i^2\delta_{6J}) \tag{3-22}$$

式（3-22）中其它参数具体表达式见附录B。

2．点 $\xi_j^{\check{i}}$ 在 $x_{j2}^{\check{i}}$ 方向作用单位点力，点 $x_j^{\check{i}}$ 广义位移基本解可表示为：

$$\begin{aligned}U_{2J}^{\hat{i}} =& \sum_{i=1}^{5} D_i((x_2-\xi_2)(\delta_{\alpha J} R_{i\hat{i}}^{-1} \dot{R}_{i\hat{i}}^{-2}(\xi_\alpha - x_\alpha) - R_{i\hat{i}}^{-1}\dot{R}_{i\hat{i}}^{-1}(\lambda_i^w \delta_{3J} + \lambda_i^\phi \delta_{4J} + \lambda_i^\varphi \delta_{5J}) - \lambda_i^\vartheta R_i^{\hat{i}-3} s_i \delta_{6J} \\ &+ \dot{R}_{i\hat{i}}^{-1}\delta_{2J}) + D_0((x_1-\xi_1) R_{0\hat{i}}^{-1}\dot{R}_{0\hat{i}}^{-2}((x_1-\xi_1)\delta_{2J} - (x_2-\xi_2)\delta_{1J}) - \dot{R}_{0\hat{i}}^{-1}\delta_{2J})\end{aligned} \tag{3-23}$$

式（3-23）中其它参数具体表达式见附录B。

3.3.2 广义面力基本解

对位移场基本解公式（3-22）～（3-23)应用弹性基本理论、边界条件，经过推导简化，可得到磁电热弹耦合材料三维裂纹面 $\hat{i}$ 坐标系下面力基本解，其具体表达式如下：

$$\begin{aligned}T_{11}^{\hat{i}} =& -c_{44} D_0 s_0 (R_{0\hat{i}}^{-1}\hat{R}_{0\hat{i}}^{-1} - R_{0\hat{i}}^{-3}\hat{R}_{0\hat{i}}^{-1} X_{2\hat{i}}{}^2 - R_{0\hat{i}}^{-2}\hat{R}_{0\hat{i}}^{-2} X_{2\hat{i}}{}^2) + \iota_{11}\sum_{i=1}^{5} D_i\, \lambda_i^v R_{i\hat{i}}^{-3} X_{1\hat{i}} + \sum_{i=1}^{5} D_i [c_{44} s_i \\ &+ c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi](R_{i\hat{i}}^{-1}\hat{R}_{i\hat{i}}^{-1} - R_{i\hat{i}}^{-3}\hat{R}_{i\hat{i}}^{-1} X_{1\hat{i}}{}^2 - R_{i\hat{i}}^{-2}\hat{R}_{i\hat{i}}^{-2} X_{1\hat{i}}{}^2)\end{aligned} \tag{3-24}$$

$$\begin{aligned}T_{21}^{\hat{i}} =& -c_{44} D_0 s_0 X_{1\hat{i}} X_{2\hat{i}} (R_{0\hat{i}}^{-3}\hat{R}_{0\hat{i}}^{-1} + R_{0\hat{i}}^{-2}\hat{R}_{0\hat{i}}{}^2) - \sum_{i=1}^{5} D_i X_{1\hat{i}} X_{2\hat{i}} [c_{44} s_i + c_{44}\lambda_i^w \\ &+ e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi](R_{i\hat{i}}^{-3}\hat{R}_{i\hat{i}}^{-1} + R_{i\hat{i}}^{-2}\hat{R}_{i\hat{i}}^{-2})\end{aligned} \tag{3-25}$$

$$T_{31}^{\hat{i}} = -\sum_{i=1}^{5} A_i^P [c_{44} s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{1\hat{i}} \tag{3-26}$$

$$T_{41}^{\hat{i}} = -\sum_{i=1}^{5} A_i^Q [c_{44} s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{1\hat{i}} \tag{3-27}$$

$$T_{51}^{\hat{i}} = -\sum_{i=1}^{5} A_i^J [c_{44} s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{1\hat{i}} \tag{3-28}$$

$$T_{61}^{\hat{i}} = -\sum_{i=1}^{5} A_i^\Upsilon [c_{44} s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{1\hat{i}} \tag{3-29}$$

$$\begin{aligned}T_{12}^{\hat{i}} =& -c_{44} D_0 s_0 X_{1\hat{i}} X_{2\hat{i}} (R_{0\hat{i}}^{-3}\hat{R}_{0\hat{i}}^{-1} + R_{0\hat{i}}^{-2}\hat{R}_{0\hat{i}}{}^2) - \sum_{i=1}^{5} D_i X_{1\hat{i}} X_{2\hat{i}} [c_{44} s_i + c_{44}\lambda_i^w \\ &+ e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi](R_{i\hat{i}}^{-3}\hat{R}_{i\hat{i}}^{-1} + R_{i\hat{i}}^{-2}\hat{R}_{i\hat{i}}^{-2})\end{aligned} \tag{3-30}$$

$$\begin{aligned}T_{22}^{\hat{i}} =& -c_{44} D_0 s_0 (R_{0\hat{i}}^{-1}\hat{R}_{0\hat{i}}^{-1} - R_{0\hat{i}}^{-3}\hat{R}_{0\hat{i}}^{-1} X_{1\hat{i}}{}^2 - R_{0\hat{i}}^{-2}\hat{R}_{0\hat{i}}^{-2} X_{1\hat{i}}{}^2) + \iota_{22}\sum_{i=1}^{5} D_i\, \lambda_i^v R_{i\hat{i}}^{-3} X_{2\hat{i}} + \sum_{i=1}^{5} D_i [c_{44} s_i \\ &+ c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi](R_{i\hat{i}}^{-1}\hat{R}_{i\hat{i}}^{-1} - R_{i\hat{i}}^{-3}\hat{R}_{i\hat{i}}^{-1} X_{2\hat{i}}{}^2 - R_{i\hat{i}}^{-2}\hat{R}_{i\hat{i}}^{-2} X_{2\hat{i}}{}^2)\end{aligned} \tag{3-31}$$

$$T_{32}^{\hat{i}} = -\sum_{i=1}^{5} A_i^P [c_{44} s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{2\hat{i}} \tag{3-32}$$

$$T_{42}^{\hat{i}} = -\sum_{i=1}^{5} A_i^Q [c_{44} s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{2\hat{i}} \tag{3-33}$$

$$T_{52}^{\hat{i}} = -\sum_{i=1}^{5} A_i^J [c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{2\hat{i}} \tag{3-34}$$

$$T_{62}^{\hat{i}} = -\sum_{i=1}^{5} A_i^\Upsilon [c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{2\hat{i}} \tag{3-35}$$

$$T_{13}^{\hat{i}} = \iota_{33}\sum_{i=1}^{5} D_i \lambda_i^v R_{i\hat{i}}^{-3} X_{1\hat{i}} + \sum_{i=1}^{5} D_i [c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{1\hat{i}} \tag{3-36}$$

$$T_{23}^{\hat{i}} = \iota_{33}\sum_{i=1}^{5} D_i \lambda_i^v R_{i\hat{i}}^{-3} X_{2\hat{i}} + \sum_{i=1}^{5} D_i [c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{2\hat{i}} \tag{3-37}$$

$$T_{33}^{\hat{i}} = -\sum_{i=1}^{5} A_i^P s_i [-c_{13} + c_{33}s_i \lambda_i^w + e_{33}s_i \lambda_i^\phi + d_{33}s_i \lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{3\hat{i}} \tag{3-38}$$

$$T_{43}^{\hat{i}} = -\sum_{i=1}^{5} A_i^Q s_i [-c_{13} + c_{33}s_i \lambda_i^w + e_{33}s_i \lambda_i^\phi + d_{33}s_i \lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{3\hat{i}} \tag{3-39}$$

$$T_{53}^{\hat{i}} = -\sum_{i=1}^{5} A_i^J s_i [-c_{13} + c_{33}s_i \lambda_i^w + e_{33}s_i \lambda_i^\phi + d_{33}s_i \lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{3\hat{i}} \tag{3-40}$$

$$T_{63}^{\hat{i}} = -\sum_{i=1}^{5} A_i^\Upsilon s_i [-c_{13} + c_{33}s_i \lambda_i^w + e_{33}s_i \lambda_i^\phi + d_{33}s_i \lambda_i^\varphi] R_{i\hat{i}}^{-3} X_{3\hat{i}} \tag{3-41}$$

$$T_{14}^{\hat{i}} = -\sum_{i=1}^{5} D_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi + \zeta_{33}] R_{i\hat{i}}^{-3} X_{1\hat{i}} \tag{3-42}$$

$$T_{24}^{\hat{i}} = -\sum_{i=1}^{5} D_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi + \zeta_{33}] R_{i\hat{i}}^{-3} X_{2\hat{i}} \tag{3-43}$$

$$T_{34}^{\hat{i}} = -\sum_{i=1}^{5} D_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi + \zeta_{33}] R_{i\hat{i}}^{-3} X_{3\hat{i}} \tag{3-44}$$

$$T_{44}^{\hat{i}} = -\sum_{i=1}^{5} A_i^P s_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi + \zeta_{33}] R_{i\hat{i}}^{-3} X_{3\hat{i}} \tag{3-45}$$

$$T_{54}^{\hat{i}} = -\sum_{i=1}^{5} A_i^Q s_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi + \zeta_{33}] R_{i\hat{i}}^{-3} X_{3\hat{i}} \tag{3-46}$$

$$T_{64}^{\hat{i}} = -\sum_{i=1}^{5} A_i^\Upsilon s_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi + \zeta_{33}] R_{i\hat{i}}^{-3} X_{3\hat{i}} \tag{3-47}$$

$$T_{15}^{\hat{i}} = -\sum_{i=1}^{5} D_i s_i [-d_{31} + d_{33}s_i \lambda_i^w - g_{33}s_i \lambda_i^\phi - \mu_{33}s_i \lambda_i^\varphi + \eta_{33}] R_{i\hat{i}}^{-3} X_{1\hat{i}} \tag{3-48}$$

$$T_{25}^{\hat{i}} = -\sum_{i=1}^{5} D_i s_i [-d_{31} + d_{33}s_i \lambda_i^w - g_{33}s_i \lambda_i^\phi - \mu_{33}s_i \lambda_i^\varphi + +\eta_{33}] R_{i\hat{i}}^{-3} X_{2\hat{i}} \tag{3-49}$$

$$T_{35}^{\hat{i}} = -\sum_{i=1}^{5} A_i^P s_i [-d_{31} + d_{33}s_i \lambda_i^w - g_{33}s_i \lambda_i^\phi - \mu_{33}s_i \lambda_i^\varphi + \eta_{33}] R_{i\hat{i}}^{-3} X_{3\hat{i}} \tag{3-50}$$

$$T_{45}^{\hat{i}}=-\sum_{i=1}^{5}A_i^{Q}s_i[-d_{31}+d_{33}s_i\lambda_i^{w}-g_{33}s_i\lambda_i^{\phi}-\mu_{33}s_i\lambda_i^{\varphi}+\eta_{33}]R_{i\hat{i}}^{-3}X_{3\hat{i}} \tag{3-51}$$

$$T_{55}^{\hat{i}}=-\sum_{i=1}^{5}A_i^{J}s_i[-d_{31}+d_{33}s_i\lambda_i^{w}-g_{33}s_i\lambda_i^{\phi}-\mu_{33}s_i\lambda_i^{\varphi}+\eta_{33}]R_{i\hat{i}}^{-3}X_{3\hat{i}} \tag{3-52}$$

$$T_{65}^{\hat{i}}=-\sum_{i=1}^{5}A_i^{\Upsilon}s_i[-d_{31}+d_{33}s_i\lambda_i^{w}-g_{33}s_i\lambda_i^{\phi}-\mu_{33}s_i\lambda_i^{\varphi}+\eta_{33}]R_{i\hat{i}}^{-3}X_{3\hat{i}} \tag{3-53}$$

$$T_{16}^{\hat{i}}=\sum_{i=1}^{5}D_i\,s_i\lambda_i^{v}(R_{i\hat{i}}^{-3}\lambda_{31}-3R_{i\hat{i}}^{-5}\lambda_{30}X_{1\hat{i}}^{2}-3R_{i\hat{i}}^{-5}\lambda_{32}X_{1\hat{i}}X_{2\hat{i}}+3R_{i\hat{i}}^{-5}\lambda_{33}X_{3\hat{i}}^{2}s_i^{2}) \tag{3-54}$$

$$T_{26}^{\hat{i}}=\sum_{i=1}^{5}D_i\,\lambda_i^{v}(3R_{i\hat{i}}^{-5}\lambda_{31}X_{1\hat{i}}X_{2\hat{i}}-R_{i\hat{i}}^{-3}\lambda_{32}+3R_{i\hat{i}}^{-5}\lambda_{32}X_{1\hat{i}}^{2}+3R_{i\hat{i}}^{-5}\lambda_{33}X_{1\hat{i}}X_{3\hat{i}}s_i^{2}) \tag{3-55}$$

$$T_{46}^{\hat{i}}=\sum_{i=1}^{5}A_i^{P}\lambda_i^{v}s_i^{2}(-3R_{i\hat{i}}^{-5}\lambda_{31}X_{1\hat{i}}X_{3\hat{i}}-3R_{i\hat{i}}^{-5}\lambda_{32}X_{2\hat{i}}X_{3\hat{i}}+R_{i\hat{i}}^{-3}\lambda_{33}-3R_{i\hat{i}}^{-5}\lambda_{33}X_{3\hat{i}}^{2}s_i^{2}) \tag{3-56}$$

$$T_{56}^{\hat{i}}=\sum_{i=1}^{5}A_i^{J}\lambda_i^{v}s_i^{2}(-3R_{i\hat{i}}^{-5}\lambda_{31}X_{1\hat{i}}X_{3\hat{i}}-3R_{i\hat{i}}^{-5}\lambda_{32}X_{2\hat{i}}X_{3\hat{i}}+R_{i\hat{i}}^{-3}\lambda_{33}-3R_{i\hat{i}}^{-5}\lambda_{33}X_{3\hat{i}}^{2}s_i^{2}) \tag{3-57}$$

$$T_{66}^{\hat{i}}=\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_i^{v}s_i^{2}(-3R_{i}^{-5}\lambda_{31}X_{1\hat{i}}X_{3\hat{i}}-3R_{i\hat{i}}^{-5}\lambda_{32}X_{2\hat{i}}X_{3\hat{i}}+R_{i\hat{i}}^{-3}\lambda_{33}-3R_{i\hat{i}}^{-5}\lambda_{33}X_{3\hat{i}}^{2}s_i^{2}) \tag{3-58}$$

3.3.3 广义应力积分核函数

通过对式（3-24）～（3-58)的广义面力场基本解进行求导，经过繁杂的推导，简化后可得到相应的应力积分核 S_{KIJ}，其具体表达式如下：

$$\begin{aligned}S_{131}^{\hat{i}}=\sum_{i=1}^{5}\ &(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})(c_{44}D_i\,s_i+c_{44}A_i^{P}\\&+e_{15}A_i^{Q}+d_{15}A_i^{J})R_{i\hat{i}}^{-3}(1-3X_{1\hat{i}}^{2}R_{i\hat{i}}^{2})-c_{44}^{2}D_0s_0^{2}R_{0\hat{i}}^{-3}(1-3X_{2\hat{i}}^{2}R_{0\hat{i}}^{2})\end{aligned} \tag{3-59}$$

$$\begin{aligned}S_{231}^{\hat{i}}=-\sum_{i=1}^{5}\ &3(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})(c_{44}D_i\,s_i+c_{44}A_i^{P}\\&+e_{15}A_i^{Q}+d_{15}A_i^{J})R_i^{-5}X_1X_2-3c_{44}^{2}D_0s_0^{2}R_{0\hat{i}}^{-5}X_{1\hat{i}}X_{2\hat{i}}\end{aligned} \tag{3-60}$$

$$\begin{aligned}S_{331}^{\hat{i}}=-\sum_{i=1}^{5}\ &3(-c_{13}+c_{33}\lambda_i^{w}s_i+e_{33}\lambda_i^{\phi}s_i+d_{33}\lambda_i^{\varphi}s_i+\iota_{33}\lambda_i^{v}s_i)(c_{44}D_i\,s_i\\&+c_{44}A_i^{P}+e_{15}A_i^{Q}+d_{15}A_i^{J})R_{i\hat{i}}^{-5}X_{1\hat{i}}X_{3\hat{i}}\end{aligned} \tag{3-61}$$

$$\begin{aligned}S_{431}^{\hat{i}}=-\sum_{i=1}^{5}\ &3(-e_{31}+e_{33}\lambda_i^{w}s_i-\in_{33}\lambda_i^{\phi}s_i-g_{33}\lambda_i^{\varphi}s_i+\zeta_{33}\lambda_i^{v}s_i)(c_{44}D_i\,s_i\\&+c_{44}A_i^{P}+e_{15}A_i^{Q}+d_{15}A_i^{J})R_{i\hat{i}}^{-5}X_{1\hat{i}}X_{3\hat{i}}\end{aligned} \tag{3-62}$$

$$S_{531}^{\hat{i}} = -\sum_{i=1}^{5} 3(-d_{31} + d_{33}\lambda_i^w s_i - g_{33}\lambda_i^\phi s_i - \mu_{33}\lambda_i^\varphi s_i - \eta_{33}\lambda_i^v s_i)(c_{44}D_i s_i + c_{44}A_i^P + e_{15}A_i^Q + d_{15}A_i^J)R_{i\hat{i}}^{-5}X_{1\hat{i}}X_{3\hat{i}} \tag{3-63}$$

$$S_{631}^{\hat{i}} = -\sum_{i=1}^{5} 3\lambda_{33}s_i^2(c_{44}D_i s_i + c_{44}A_i^P + e_{15}A_i^Q + d_{15}A_i^J)R_{i\hat{i}}^{-5}X_{1\hat{i}} \tag{3-64}$$

$$S_{132}^{\hat{i}} = \sum_{i=1}^{5} 3(c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi)(c_{44}D_i s_i + c_{44}A_i^P + e_{15}A_i^Q + d_{15}A_i^J)R_i^{-5}X_1 X_2 + c_{44}^2 D_0 s_0^2 R_{0\hat{i}}^{-5}X_{1\hat{i}}X_{2\hat{i}} \tag{3-65}$$

$$S_{232}^{\hat{i}} = -\sum_{i=1}^{5} (c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi)(c_{44}D_i s_i + c_{44}A_i^P + e_{15}A_i^Q + d_{15}A_i^J)R_{i\hat{i}}^{-3}(1-3X_{2\hat{i}}^2R_{i\hat{i}}^2) - c_{44}^2 D_0 s_0^2 R_{0\hat{i}}^{-3}(1-3X_{1\hat{i}}^2R_{0\hat{i}}^2) \tag{3-66}$$

$$S_{332}^{\hat{i}} = -\sum_{i=1}^{5} 3(-c_{13} + c_{33}\lambda_i^w s_i + e_{33}\lambda_i^\phi s_i + d_{33}\lambda_i^\varphi s_i + \iota_{33}\lambda_i^v s_i)(c_{44}D_i s_i + c_{44}A_i^P + e_{15}A_i^Q + d_{15}A_i^J)R_{i\hat{i}}^{-5}X_{2\hat{i}}X_{3\hat{i}} \tag{3-67}$$

$$S_{432}^{\hat{i}} = -\sum_{i=1}^{5} 3(-e_{31} + e_{33}\lambda_i^w s_i - \in_{33}\lambda_i^\phi s_i - g_{33}\lambda_i^\varphi s_i + \zeta_{33}\lambda_i^v s_i)(c_{44}D_i s_i + c_{44}A_i^P + e_{15}A_i^Q + d_{15}A_i^J)R_{i\hat{i}}^{-5}X_{2\hat{i}}X_{3\hat{i}} \tag{3-68}$$

$$S_{532}^{\hat{i}} = -\sum_{i=1}^{5} 3(-d_{31} + d_{33}\lambda_i^w s_i - g_{33}\lambda_i^\phi s_i - \mu_{33}\lambda_i^\varphi s_i - \eta_{33}\lambda_i^v s_i)(c_{44}D_i s_i + c_{44}A_i^P + e_{15}A_i^Q + d_{15}A_i^J)R_{i\hat{i}}^{-5}X_{2\hat{i}}X_{3\hat{i}} \tag{3-69}$$

$$S_{632}^{\hat{i}} = -\sum_{i=1}^{5} 3\lambda_{33}s_i^2(c_{44}D_i s_i + c_{44}A_i^P + e_{15}A_i^Q + d_{15}A_i^J)R_{i\hat{i}}^{-5}X_{2\hat{i}} + \lambda_{32}s_i^2R_{i\hat{i}}^{-5}X_{3\hat{i}} \tag{3-70}$$

$$S_{133}^{\hat{i}} = \sum_{i=1}^{5} 3s_i(c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi)(-D_i c_{13} + c_{33}A_i^P s_i + e_{33}A_i^Q s_i + d_{33}A_i^J s_i)R_{i\hat{i}}^{-5}X_{1\hat{i}}X_{3\hat{i}} + \sum_{i=1}^{5} A_i^\Upsilon \iota_{33}(c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi)R_{i\hat{i}}^{-3}X_{1\hat{i}} \tag{3-71}$$

$$S_{233}^{\hat{i}} = \sum_{i=1}^{5} 3s_i(c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi)(-D_i c_{13} + c_{33}A_i^P s_i + e_{33}A_i^Q s_i + d_{33}A_i^J s_i)R_{i\hat{i}}^{-5}X_{2\hat{i}}X_{3\hat{i}} + \sum_{i=1}^{5} A_i^\Upsilon \iota_{33}(c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi)R_{i\hat{i}}^{-3}X_{2\hat{i}} \tag{3-72}$$

$$S_{333}^{\hat{i}}=\sum_{i=1}^{5}s_i(-c_{13}+c_{33}\lambda_i^w s_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^v s_i)(-D_i c_{13}$$
$$+c_{33}A_i^P s_i+e_{33}A_i^Q s_i+d_{33}A_i^J s_i)R_{i\hat{i}}^{-3} \tag{3-73}$$
$$+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(-c_{13}+c_{33}\lambda_i^w s_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^v s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}}$$

$$S_{433}^{\hat{i}}=\sum_{i=1}^{5}s_i(-e_{31}+e_{33}\lambda_i^w s_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^v s_i)(-D_i c_{13}$$
$$+c_{33}A_i^P s_i+e_{33}A_i^Q s_i+d_{33}A_i^J s_i)R_{i\hat{i}}^{-3}+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(-e_{31} \tag{3-74}$$
$$+e_{33}\lambda_i^w s_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^v s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}}$$

$$S_{533}^{\hat{i}}=\sum_{i=1}^{5}s_i(-d_{31}+d_{33}\lambda_i^w s_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i-\eta_{33}\lambda_i^v s_i)(-D_i c_{13}$$
$$+c_{33}A_i^P s_i+e_{33}A_i^Q s_i+d_{33}A_i^J s_i)R_{i\hat{i}}^{-3} \tag{3-75}$$
$$+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(-d_{31}+d_{33}\lambda_i^w s_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i-\eta_{33}\lambda_i^v s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}}$$

$$S_{633}^{\hat{i}}=\sum_{i=1}^{5}s_i^2\lambda_i^v(-D_i c_{13}+c_{33}A_i^P s_i+e_{33}A_i^Q s_i+d_{33}A_i^J s_i)R_{i\hat{i}}^{-5}(3\lambda_{31}X_1$$
$$+3\lambda_{32}X_2)-\sum_{i=1}^{5}A_i^\Upsilon s_i^2\lambda_i^v\lambda_{33}\iota_{33}R_{i\hat{i}}^{-3} \tag{3-76}$$

$$S_{134}^{\hat{i}}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_i e_{31}$$
$$+e_{33}A_i^P s_i-\in_{33}A_i^Q s_i-g_{33}A_i^J s_i)R_{i\hat{i}}^{-5}X_{1\hat{i}}X_{3\hat{i}} \tag{3-77}$$
$$+\sum_{i=1}^{5}A_i^\Upsilon\zeta_{33}(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)R_{i\hat{i}}^{-3}X_{1\hat{i}}$$

$$S_{234}^{\hat{i}}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_i e_{31}$$
$$+e_{33}A_i^P s_i-\in_{33}A_i^Q s_i-g_{33}A_i^J s_i)R_{i\hat{i}}^{-5}X_{2\hat{i}}X_{3\hat{i}} \tag{3-78}$$
$$+\sum_{i=1}^{5}A_i^\Upsilon\zeta_{33}(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)R_{i\hat{i}}^{-3}X_{2\hat{i}}$$

$$S_{334}^{\hat{i}}=\sum_{i=1}^{5}s_i(-c_{13}+c_{33}\lambda_i^w s_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^v s_i)(-D_i e_{31}$$
$$+e_{33}A_i^P s_i-\in_{33}A_i^Q s_i-g_{33}A_i^J s_i)R_{i\hat{i}}^{-3} \tag{3-79}$$
$$+\sum_{i=1}^{5}A_i^\Upsilon\zeta_{33}(-c_{13}+c_{33}\lambda_i^w s_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i+\iota_{33}\lambda_i^v s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}}$$

$$S_{434}^{\hat{i}}=\sum_{i=1}^{5}s_i(-e_{31}+e_{33}\lambda_i^w s_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^v s_i)(-D_i e_{31}$$
$$+e_{33}A_i^P s_i-\in_{33}A_i^Q s_i-g_{33}A_i^J s_i)R_{i\hat{i}}^{-3} \tag{3-80}$$
$$+\sum_{i=1}^{5}A_i^\Upsilon\iota_{33}(-e_{31}+e_{33}\lambda_i^w s_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i+\zeta_{33}\lambda_i^v s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}}$$

$$S_{534}^{\hat{i}}=\sum_{i=1}^{5}s_i(-d_{31}+d_{33}\lambda_i^{w}s_i-g_{33}\lambda_i^{\phi}s_i-\mu_{33}\lambda_i^{\varphi}s_i-\eta_{33}\lambda_i^{v}s_i)(-D_i e_{31}+e_{33}A_i^{P}s_i-\in_{33}A_i^{Q}s_i-g_{33}A_i^{J}s_i)R_{i\hat{i}}^{-3}+\sum_{i=1}^{5}A_i^{\Upsilon}\iota_{33}(-d_{31}+d_{33}\lambda_i^{w}s_i-g_{33}\lambda_i^{\phi}s_i-\mu_{33}\lambda_i^{\varphi}s_i-\eta_{33}\lambda_i^{v}s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}} \tag{3-81}$$

$$S_{634}^{\hat{i}}=\sum_{i=1}^{5}s_i^{2}\lambda_i^{v}(-D_i e_{31}+e_{33}A_i^{P}s_i-\in_{33}A_i^{Q}s_i-g_{33}A_i^{J}s_i)R_{i\hat{i}}^{-5}(3\lambda_{31}X_1+3\lambda_{32}X_2)-\sum_{i=1}^{5}A_i^{\Upsilon}s_i^{2}\lambda_i^{v}\lambda_{33}\zeta_{33}R_{i\hat{i}}^{-3} \tag{3-82}$$

$$S_{135}^{\hat{i}}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})(-D_i d_{31}+d_{33}A_i^{P}s_i-g_{33}A_i^{Q}s_i-\mu_{33}A_i^{J}s_i)R_{i\hat{i}}^{-5}X_{1\hat{i}}X_{3\hat{i}}+\sum_{i=1}^{5}A_i^{\Upsilon}\eta_{33}(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})R_{i\hat{i}}^{-3}X_{1\hat{i}} \tag{3-83}$$

$$S_{235}^{\hat{i}}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})(-D_i d_{31}+d_{33}A_i^{P}s_i-g_{33}A_i^{Q}s_i-\mu_{33}A_i^{J}s_i)R_{i\hat{i}}^{-5}X_{2\hat{i}}X_{3\hat{i}}+\sum_{i=1}^{5}A_i^{\Upsilon}\eta_{33}(c_{44}s_i+c_{44}\lambda_i^{w}+e_{15}\lambda_i^{\phi}+d_{15}\lambda_i^{\varphi})R_{i\hat{i}}^{-3}X_{2\hat{i}} \tag{3-84}$$

$$S_{335}^{\hat{i}}=\sum_{i=1}^{5}s_i(-c_{13}+c_{33}\lambda_i^{w}s_i+e_{33}\lambda_i^{\phi}s_i+d_{33}\lambda_i^{\varphi}s_i+\iota_{33}\lambda_i^{v}s_i)(-D_i d_{31}+d_{33}A_i^{P}s_i-g_{33}A_i^{Q}s_i-\mu_{33}A_i^{J}s_i)R_{i\hat{i}}^{-3}+\sum_{i=1}^{5}A_i^{\Upsilon}\eta_{33}(-c_{13}+c_{33}\lambda\triangleright_i^{w}s_i+e_{33}\lambda_i^{\phi}s_i+d_{33}\lambda_i^{\varphi}s_i+\iota_{33}\lambda_i^{v}s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}} \tag{3-85}$$

$$S_{435}^{\hat{i}}=\sum_{i=1}^{5}s_i(-e_{31}+e_{33}\lambda_i^{w}s_i-\in_{33}\lambda_i^{\phi}s_i-g_{33}\lambda_i^{\varphi}s_i+\zeta_{33}\lambda_i^{v}s_i)(-D_i d_{31}+d_{33}A_i^{P}s_i-g_{33}A_i^{Q}s_i-\mu_{33}A_i^{J}s_i)R_{i\hat{i}}^{-3}+\sum_{i=1}^{5}A_i^{\Upsilon}\eta_{33}(-e_{31}+e_{33}\lambda_i^{w}s_i-\in_{33}\lambda_i^{\phi}s_i-g_{33}\lambda_i^{\varphi}s_i+\zeta_{33}\lambda_i^{v}s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}} \tag{3-86}$$

$$S_{535}^{\hat{i}}=\sum_{i=1}^{5}s_i(-d_{31}+d_{33}\lambda_i^{w}s_i-g_{33}\lambda_i^{\phi}s_i-\mu_{33}\lambda_i^{\varphi}s_i-\eta_{33}\lambda_i^{v}s_i)(-D_i d_{31}+d_{33}A_i^{P}s_i-g_{33}A_i^{Q}s_i-\mu_{33}A_i^{J}s_i)R_{i\hat{i}}^{-3}+\sum_{i=1}^{5}A_i^{\Upsilon}\eta_{33}(-d_{31}+d_{33}\lambda_i^{w}s_i-g_{33}\lambda_i^{\phi}s_i-\mu_{33}\lambda_i^{\varphi}s_i-\eta_{33}\lambda_i^{v}s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}} \tag{3-87}$$

$$S_{635}^{\hat{i}}=\sum_{i=1}^{5}s_i^{2}\lambda_i^{v}(-D_i d_{31}+d_{33}A_i^{P}s_i-g_{33}A_i^{Q}s_i-\mu_{33}A_i^{J}s_i)R_{i\hat{i}}^{-5}(3\lambda_{31}X_1+3\lambda_{32}X_2)-\sum_{i=1}^{5}A_i^{\Upsilon}s_i^{2}\lambda_i^{v}\lambda_{33}\eta_{33}R_{i\hat{i}}^{-3} \tag{3-88}$$

$$\begin{aligned} S_{136}^{\hat{i}} = & -\sum_{i=1}^{5} A_i^{\Upsilon}(c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^{\phi} + d_{15}\lambda_i^{\varphi})\lambda_{31}(R_{i\hat{i}}^{-3} - 3R_{i\hat{i}}^{-5}X_{1\hat{i}}^2) \\ & + \sum_{i=1}^{5} A_i^{\Upsilon}\lambda_{32}(c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^{\phi} + d_{15}\lambda_i^{\varphi})3R_{i\hat{i}}^{-5}X_{1\hat{i}}X_{2\hat{i}} \end{aligned} \tag{3-89}$$

$$\begin{aligned} S_{236}^{\hat{i}} = & -\sum_{i=1}^{5} A_i^{\Upsilon}(c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^{\phi} + d_{15}\lambda_i^{\varphi})\lambda_{32}(R_{i\hat{i}}^{-3} - 3R_{i\hat{i}}^{-5}X_{2\hat{i}}^2) \\ & + \sum_{i=1}^{5} A_i^{\Upsilon}\lambda_{31}(c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^{\phi} + d_{15}\lambda_i^{\varphi})3R_{i\hat{i}}^{-5}X_{1\hat{i}}X_{2\hat{i}} \end{aligned} \tag{3-90}$$

$$S_{336}^{\hat{i}} = -\sum_{i=1}^{5} A_i^{\Upsilon}s_i(-c_{13} + c_{33}\lambda_i^w s_i + e_{33}\lambda_i^{\phi}s_i + d_{33}\lambda_i^{\varphi}s_i + \iota_{33}\lambda_i^{v}s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}} \tag{3-91}$$

$$S_{436}^{\hat{i}} = \sum_{i=1}^{5} A_i^{\Upsilon}s_i(-e_{31} + e_{33}\lambda_i^w s_i - \in_{33}\lambda_i^{\phi}s_i - g_{33}\lambda_i^{\varphi}s_i + \zeta_{33}\lambda_i^{v}s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}} \tag{3-92}$$

$$S_{536}^{\hat{i}} = \sum_{i=1}^{5} A_i^{\Upsilon}s_i(-d_{31} + d_{33}\lambda_i^w s_i - g_{33}\lambda_i^{\phi}s_i - \mu_{33}\lambda_i^{\varphi}s_i - \eta_{33}\lambda_i^{v}s_i)R_{i\hat{i}}^{-3}X_{3\hat{i}} \tag{3-93}$$

$$S_{636}^{\hat{i}} = -\sum_{i=1}^{5} A_i^{\Upsilon}s_i^2\lambda_i^{v}\lambda_{33}3(\lambda_{31}X_1 + \lambda_{32}X_2)R_{i\hat{i}}^{-5} \tag{3-94}$$

3.4 超奇异积分方程

3.4.1 超奇异积分方程

1 任意形状与位置平片多裂纹问题

应用边界法和主部分析法，经过复杂数学推导可得到磁电热弹耦合材料任意角度多裂纹问题的超奇异积分方程组

$$
\hat{n}\left\{\begin{array}{l}
S_1\left[\begin{array}{l}
\fint_{S_1^+}(r^{-3}(c_{44}^2D_0s_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2t_i^2)\tilde{u}_\beta^1+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2t_i^1\tilde{u}_6^1)ds+\sum_{\hat{i}=2}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{\alpha i}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_\alpha^1\\
\fint_{S_1^+}(r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha^1+r^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n^1+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}s_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6^1)ds+\sum_{\hat{i}=2}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{mi}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_m^1\\
\fint_{S_1^+}(r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta^1+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6^1)ds+\sum_{\hat{i}=2}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{6i}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_6^1
\end{array}\right.\\
S_2\left[\begin{array}{l}
\fint_{S_2^+}(r^{-3}(c_{44}^2D_0s_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2t_i^2)\tilde{u}_\beta^2+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2t_i^1\tilde{u}_6^2)ds+\int_{S_1^+}K_{\alpha i}^{1*}\tilde{u}_i^1ds+\sum_{\hat{i}=3}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{\alpha i}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_\alpha^2\\
\fint_{S_2^+}(r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha^2+r^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n^2+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}s_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6^2)ds+\int_{S_1^+}K_{mi}^{1*}\tilde{u}_i^1ds+\sum_{\hat{i}=3}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{mi}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_m^2\\
\fint_{S_2^+}(r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta^2+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6^2)ds+\int_{S_1^+}K_{6i}^{1*}\tilde{u}_i^1ds+\sum_{\hat{i}=3}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{6i}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_6^2
\end{array}\right.\\
\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
S_{\hat{n}-1}\left[\begin{array}{l}
\int_{S_{\hat{n}-1}^+}(r^{-3}(c_{44}^2D_0s_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2t_i^2)\tilde{u}_\beta^{\hat{n}-1}+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2t_i^1\tilde{u}_6^{\hat{n}-1})ds+\int_{S_{\hat{n}}^+}K_{\alpha i}^{\hat{n}*}\tilde{u}_i^{\hat{n}}ds+\sum_{\hat{i}=1}^{\hat{n}-2}\int_{S_{\hat{i}}^+}K_{\alpha i}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_\alpha^{\hat{n}-1}\\
\fint_{S_{\hat{n}-1}^+}(r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha^{\hat{n}-1}+r^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n^{\hat{n}-1}+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}s_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6^{\hat{n}-1})ds+\int_{S_{\hat{n}}^+}K_{mi}^{\hat{n}*}\tilde{u}_i^{\hat{n}}ds+\sum_{\hat{i}=1}^{\hat{n}-2}\int_{S_{\hat{i}}^+}K_{mi}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_m^{\hat{n}-1}\\
\fint_{S_{\hat{n}-1}^+}(r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta^{\hat{n}-1}+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6^{\hat{n}-1})ds+\int_{S_{\hat{n}}^+}K_{6i}^{\hat{n}*}\tilde{u}_i^{\hat{n}}ds+\sum_{\hat{i}=1}^{\hat{n}-2}\int_{S_{\hat{i}}^+}K_{6i}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_6^{\hat{n}-1}
\end{array}\right.\\
S_{\hat{n}}\left[\begin{array}{l}
\fint_{S_{\hat{n}}^+}(r^{-3}(c_{44}^2D_0s_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2t_i^2)\tilde{u}_\beta^{\hat{n}}+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2t_i^1\tilde{u}_6^{\hat{n}})ds+\sum_{\hat{i}=1}^{\hat{n}-1}\int_{S_{\hat{i}}^+}K_{\alpha i}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_\alpha^{\hat{n}}\\
\fint_{S_{\hat{n}}^+}(r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha^{\hat{n}}+r^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n^{\hat{n}}+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}s_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6^{\hat{n}})ds+\sum_{\hat{i}=1}^{\hat{n}-1}\int_{S_{\hat{i}}^+}K_{mi}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_m^{\hat{n}}\\
\fint_{S_{\hat{n}}^+}(r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta^{\hat{n}}+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6^{\hat{n}})ds+\sum_{\hat{i}=1}^{\hat{n}-1}\int_{S_{\hat{i}}^+}K_{6i}^{\hat{i}*}\tilde{u}_i^{\hat{i}}ds=-p_6^{\hat{n}}
\end{array}\right.
\end{array}\right.
\qquad (3\text{-}95)
$$

其中 $p_i^{\hat{i}}$, $p_4^{\hat{i}}(q^{\hat{i}})$, $p_5^{\hat{i}}(b^{\hat{i}})$ 和 $p_5^{\hat{i}}(\rho^{\hat{i}})$ 可以通过材料非裂纹处解来表示。材料常数 t_i^m , ρ_i^m 及核函数 $K_{ij}^{\hat{i}*}$ 的具体表达式见附录B.

2 任意形状平行平片多裂纹问题

如图3-2所示，磁电热弹耦合材料中三维平行多裂纹，裂纹面垂直与 x_3 轴，受多场载荷作用，广义载荷在裂纹面处连续。超奇异积分方程组可表示为：

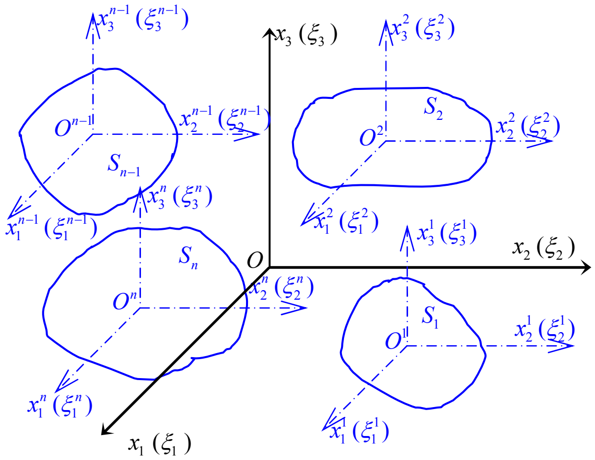


图 3-2 磁电热弹耦合材料内含任意形状平行平片多裂纹示意图

$$
\hat{n}\begin{cases}
S_1\left[\begin{array}{l}
\int\!\!\!\!\!\!=_{S_1^+}(r^{-3}(c_{44}^2D_0s_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2t_i^2)\tilde{u}_\beta^1+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2t_i^1\tilde{u}_6^1)ds+\sum_{\hat{i}=2}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{\alpha i}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_\alpha^1\\
\int\!\!\!\!\!\!=_{S_1^+}(r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha^1+r^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n^1+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}s_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6^1)ds+\sum_{\hat{i}=2}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{mi}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_m^1\\
\int\!\!\!\!\!\!=_{S_1^+}(r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta^1+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6^1)ds+\sum_{\hat{i}=2}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{6i}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_6^1
\end{array}\right.\\
S_2\left[\begin{array}{l}
\int\!\!\!\!\!\!=_{S_2^+}(r^{-3}(c_{44}^2D_0s_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2t_i^2)\tilde{u}_\beta^2+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2t_i^1\tilde{u}_6^2)ds+\int_{S_1^+}K_{\alpha i}^1\tilde{u}_i^1ds+\sum_{\hat{i}=3}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{\alpha i}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_\alpha^2\\
\int\!\!\!\!\!\!=_{S_2^+}(r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha^2+r^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n^2+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}s_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6^2)ds+\int_{S_1^+}K_{mi}^1\tilde{u}_i^1ds+\sum_{\hat{i}=3}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{mi}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_m^2\\
\int\!\!\!\!\!\!=_{S_2^+}(r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta^2+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6^2)ds+\int_{S_1^+}K_{6i}^1\tilde{u}_i^1ds+\sum_{\hat{i}=3}^{\hat{n}}\int_{S_{\hat{i}}^+}K_{6i}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_6^2
\end{array}\right.\\
\qquad\qquad\vdots\qquad\qquad\vdots\qquad\qquad\vdots\\
S_{\hat{n}-1}\left[\begin{array}{l}
\int\!\!\!\!\!\!=_{S_{\hat{n}-1}^+}(r^{-3}(c_{44}^2D_0s_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2t_i^2)\tilde{u}_\beta^{\hat{n}-1}+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2t_i^1\tilde{u}_6^{\hat{n}-1})ds+\int_{S_{\hat{n}}^+}K_{\alpha i}^{\hat{n}}\tilde{u}_i^{\hat{n}}ds+\sum_{\hat{i}=1}^{\hat{n}-2}\int_{S_{\hat{i}}^+}K_{\alpha i}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_\alpha^{\hat{n}-1}\\
\int\!\!\!\!\!\!=_{S_{\hat{n}-1}^+}(r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha^{\hat{n}-1}+r^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n^{\hat{n}-1}+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}s_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6^{\hat{n}-1})ds+\int_{S_{\hat{n}}^+}K_{mi}^{\hat{n}}\tilde{u}_i^{\hat{n}}ds+\sum_{\hat{i}=1}^{\hat{n}-2}\int_{S_{\hat{i}}^+}K_{mi}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_m^{\hat{n}-1}\\
\int\!\!\!\!\!\!=_{S_{\hat{n}-1}^+}(r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta^{\hat{n}-1}+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6^{\hat{n}-1})ds+\int_{S_{\hat{n}}^+}K_{6i}^{\hat{n}}\tilde{u}_i^{\hat{n}}ds+\sum_{\hat{i}=1}^{\hat{n}-2}\int_{S_{\hat{i}}^+}K_{6i}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_6^{\hat{n}-1}
\end{array}\right.\\
S_{\hat{n}}\left[\begin{array}{l}
\int\!\!\!\!\!\!=_{S_{\hat{n}}^+}(r^{-3}(c_{44}^2D_0s_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2t_i^2)\tilde{u}_\beta^{\hat{n}}+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2t_i^1\tilde{u}_6^{\hat{n}})ds+\sum_{\hat{i}=1}^{\hat{n}-1}\int_{S_{\hat{i}}^+}K_{\alpha i}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_\alpha^{\hat{n}}\\
\int\!\!\!\!\!\!=_{S_{\hat{n}}^+}(r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha^{\hat{n}}+r^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n^{\hat{n}}+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}s_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6^{\hat{n}})ds+\sum_{\hat{i}=1}^{\hat{n}-1}\int_{S_{\hat{i}}^+}K_{mi}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_m^{\hat{n}}\\
\int\!\!\!\!\!\!=_{S_{\hat{n}}^+}(r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta^{\hat{n}}+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6^{\hat{n}})ds+\sum_{\hat{i}=1}^{\hat{n}-1}\int_{S_{\hat{i}}^+}K_{6i}^{\hat{i}}\tilde{u}_i^{\hat{i}}ds=-p_6^{\hat{n}}
\end{array}\right.
\end{cases}
\tag{3-96}
$$

材料常数 t_i^m , ρ_i^m 及核函数 $K_{ij}^{\hat{i}}$ 的具体表达式见附录B。对于特殊情况，磁电热弹耦合材料两平行裂纹，上面的超奇异积分方程组可简化为：

$$\left\{\begin{array}{l}
\displaystyle\oint_{S_1^+}(r^{-3}(c_{44}^2 D_0 s_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2 t_i^2)\tilde{u}_\beta^1+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2 t_i^1\tilde{u}_6^1)ds+\int_{S_2^+}K_{\alpha i}^1\tilde{u}_i^2 ds=-p_\alpha^1 \\
\qquad\displaystyle\oint_{S_1^+}(r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha^1+r^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n^1+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}s_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6^1)ds+\int_{S_2^+}K_{mi}^1\tilde{u}_i^2 ds=-p_m^1 \\
\qquad\displaystyle\oint_{S_1^+}(r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta^1+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6^1)ds+\int_{S_2^+}K_{6i}^1\tilde{u}_i^2 ds=-p_6^1 \\
\displaystyle\oint_{S_2^+}(r^{-3}(c_{44}^2 D_0 s_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}\rho_i^2 t_i^2)\tilde{u}_\beta^2+3r^{-4}r_{,\alpha}\sum_{i=1}^{5}\lambda_{33}s_i^2 t_i^1\tilde{u}_6^2)ds+\int_{S_1^+}K_{\alpha i}^2\tilde{u}_i^1 ds=-p_\alpha^2 \\
\qquad\displaystyle\oint_{S_2^+}(r^{-2}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1\tilde{u}_\alpha^1+r^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^m t_i^t\tilde{u}_n^2+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}s_i^2\lambda_i^{\vartheta}\rho_i^m\tilde{u}_6^2)ds+\int_{S_1^+}K_{mi}^2\tilde{u}_i^1 ds=-p_m^2 \\
\qquad\displaystyle\oint_{S_2^+}(r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2\tilde{u}_\beta^2+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^{5}A_i^{\Upsilon}s_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6\tilde{u}_6^2)ds+\int_{S_1^+}K_{6i}^2\tilde{u}_i^1 ds=-p_6^2
\end{array}\right. \tag{3-97}$$

其中 $R_{,\alpha}=(\xi_\alpha-x_\alpha)/R$, $R=((\xi_1-x_1)^2+(\xi_2-x_2)^2+h^2)^{1/2}$ ，积分核函数 K_{mn}^{α} 的具体表达式见附录。可以看出上述超奇异积分方程与文献[187, 188]中所研究问题的超奇异积分方程组有相类似的结构。**为了使问题简化，在以下求解过程中均假设 $\theta_i^{\hat{i}}(x_i, x_i^{\hat{i}})=0$（即平行多裂纹问题）。**

3.4.2 广义奇性指数

为了分析裂纹前沿的奇异性问题，我们在裂纹间断建立一局部坐标系 $x_1\ x_2\ x_3$ 。 x_1 轴为裂纹前沿 q_0 点切线方向， x_2 轴为 q_0 点裂纹面内内法线方向， x_3 轴为与裂纹面垂直方向。这样裂纹前沿 q_0 点附近广义位移间断可表示为：

$$[\tilde{u}_i^{\hat{i}}(q)\quad \tilde{\phi}^{\hat{i}}(q)\quad \tilde{\varphi}^{\hat{i}}(q)\quad \tilde{\vartheta}^{\hat{i}}(q)]=g_k^{\hat{i}}(q_0)\xi_2^{\lambda_k}\qquad 0<R\mathrm{e}(\lambda_k)<1 \tag{3-98}$$

其中 $g_k(q_0)$ 为一与 q_0 点相关非零常量， λ_k 为裂纹前沿应力奇性指数。在裂纹面区域内，取一包含 q_0 点的半圆区域 S_ε ，应用主部分析法[189, 274], 裂纹前沿奇性指数可表示为：

$$\cot(\lambda_1^{\hat{i}}\pi)=0\,,\cot(\lambda_2^{\hat{i}}\pi)=0\,,\cot(\lambda_3^{\hat{i}}\pi)=0\,,\cot(\lambda_4^{\hat{i}}\pi)=0\,,\cot(\lambda_5^{\hat{i}}\pi)=0\,,\cot(\lambda_6^{\hat{i}}\pi)=0 \tag{3-99}$$

对上式求解，可得到奇性指数的具体结果

$$\lambda_1^{\hat{i}}=\lambda_2^{\hat{i}}=\lambda_3^{\hat{i}}=\lambda_4^{\hat{i}}=\lambda_5^{\hat{i}}=\lambda_6^{\hat{i}}=\lambda=1/2 \tag{3-100}$$

3.4.3 广义应力强度因子

广义应力强度因子可以定义为：

$$K_1^{\hat{i}}=\lim_{r\to 0}\sqrt{2r}\sigma_{33}^{\hat{i}}(r,\theta)\Big|_{\theta=0},K_2^{\hat{i}}=\lim_{r\to 0}\sqrt{2r}\sigma_{32}^{\hat{i}}(r,\theta)\Big|_{\theta=0},K_3^{\hat{i}}=\lim_{r\to 0}\sqrt{2r}\sigma_{31}^{\hat{i}}(r,\theta)\Big|_{\theta=0} \tag{3-101}$$

$$K_4^{\hat{i}}=\lim_{r\to 0}\sqrt{2r}D_3^{\hat{i}}(r,\theta)\Big|_{\theta=0},K_5^{\hat{i}}=\lim_{r\to 0}\sqrt{2r}B_3^{\hat{i}}(r,\theta)\Big|_{\theta=0},K_6^{\hat{i}}=\lim_{r\to 0}\sqrt{2r}\vartheta_3^{\hat{i}}(r,\theta)\Big|_{\theta=0} \tag{3-102}$$

3.4.4 广义奇异应力场

利用以上关系式，可得到裂纹前沿奇性应力场，其具体表达式如下：

$$\sigma_{13}^{\hat{i}}=c_{44}^2 D_0 s_0^2 \frac{\pi g_1^{\hat{i}}}{\sqrt{rr_0}}\cos\frac{\theta_0}{2} \tag{3-103}$$

$$\begin{aligned}\sigma_{23}^{\hat{i}}=&\sum_{i=1}^{5}\frac{\pi}{\sqrt{rr_i}}\{g_6^{\hat{i}}\lambda_{33}[15\lambda_{33}s_i^2\cot\theta_i(3\cos\frac{\theta_i}{4}-3\sin^2\frac{\theta_i}{2}\cos\frac{\theta_i}{2}-\frac{1}{4}\sin^2\theta_i\cos\frac{3\theta_i}{2})-\frac{2\lambda_{33}}{r^2r_i^2}\sin^{-1}\frac{\theta_i}{2}\\&+\frac{\lambda_{32}}{2rr_i\sin\theta}(2\cos^{-1}\frac{\theta_i}{2}-3\cos\frac{\theta_i}{2}-\cos\frac{5\theta_i}{2})]+\rho_i^2\pi\cos\frac{\theta_i}{2}(t_2g_2+4\sum_{m=3}^{5}t_i^m g_m^{\hat{i}})\}\end{aligned} \tag{3-104}$$

$$\begin{aligned}\dot{\sigma}_{3n}^{\hat{i}}=&\sum_{i=1}^{5}\rho_i^n s_i\pi[\frac{1}{\sqrt{rr_i}}(t_i^2g_2^{\hat{i}}\cot\theta_i\cos^{-1}\frac{\theta_i}{2}+6\cot\frac{\theta_i}{2}\sum_{m=3}^{5}t_i^m g_m^{\hat{i}})+g_6^{\hat{i}}s_i\lambda_i^{\vartheta}\sqrt{rr_i}\cot\theta_i\cos\frac{\theta_i}{2})]\\&+2\pi A_i^{\Upsilon}\iota_{33}[\sqrt{rr_i}\cos\frac{\theta_i}{2}(g_2^{\hat{i}}\cot\theta_i+\sum_{m=3}^{5}t_i^m g_m^{\hat{i}})+\frac{s_i^2\lambda_{33}\lambda_i^{\vartheta}g_6^{\hat{i}}}{\sqrt{rr_i}}\cos\frac{\theta_i}{2}],n=3-5\end{aligned} \tag{3-105}$$

$$\begin{aligned}\vartheta_3^{\hat{i}}=&\sum_{i=1}^{5}\frac{A_i^{\Upsilon}\pi}{\sqrt{rr_i}}[\cos\frac{\theta_i}{2}(\lambda_{32}+4\lambda_{33}s_i^2)\sum_{m=2}^{5}t_i^m g_m^{\hat{i}}+\frac{4\lambda_{32}t_i^2g_2^{\hat{i}}s_i^2\lambda_i^{\vartheta}}{3}(2\lambda_{32}\cos\frac{\theta_i}{2}+\frac{3\lambda_{33}}{r^2r_i^2}\sin^{-1}\frac{\theta_i}{2})\\&15rr_ig_6^{\hat{i}}\lambda_{32}s_i^2\lambda_i^{\vartheta}\cot\theta_i(3\cos\frac{\theta_i}{4}-6\cos\frac{\theta_i}{2}\sin^2\frac{\theta_i}{2}-\frac{1}{4}\sin^2\theta_i\cos\frac{3\theta_i}{2})(2s_i^2\lambda_{33}+\frac{\lambda_{32}}{rr_i\sin\theta_i})]\end{aligned} \tag{3-106}$$

定义 $\dot{\sigma}_{3n}^{\hat{i}}=[\sigma_{33}^{\hat{i}}\quad D_3^{\hat{i}}\quad B_3^{\hat{i}}]$，以上奇性应力场表达式可进一步简化为：

$$\sigma_{13}^{\hat{i}}=K_3^{\hat{i}}\cos\frac{\theta_0}{2}/\sqrt{2rr_0} \tag{3-107}$$

$$\hat{\sigma}_{n3}^{\hat{i}}=[\sigma_{23}^{\hat{i}}\quad \sigma_{33}^{\hat{i}}\quad D_3^{\hat{i}}\quad B_3^{\hat{i}}\quad \vartheta_3^{\hat{i}}\quad]=(K_1^{\hat{i}}f_{n1}+K_2^{\hat{i}}f_{n2}+K_4^{\hat{i}}f_{n4}+K_5^{\hat{i}}f_{n5}+K_6^{\hat{i}}f_{n6})/\sqrt{2r} \tag{3-108}$$

式（3-108）中其它参数具体表达式见附录B。

3.4.5 广义能量释放率

磁电热弹耦合材料三维多裂纹问题的广义能量释放率可以表示为：

$$G^{\hat{i}}=\pi(G_1^{\sigma\hat{i}}+G_2^{\sigma\hat{i}}+G_3^{\sigma\hat{i}}+G_4^{D\hat{i}}+G_5^{B\hat{i}}+G_6^{\Upsilon\hat{i}})/2 \tag{3-109}$$

其中

$$G_1^{\sigma\hat{i}}=\frac{K_1^{\hat{i}}[K_4^{*\hat{i}}(K_{34}K_{55}-K_{35}K_{54})+K_{45}(K_{54}K_1^{*\hat{i}}-K_{34}K_5^{*\hat{i}})+K_{44}(K_{35}K_5^{*\hat{i}}-K_{55}K_1^{*\hat{i}})]}{K_{35}(K_{43}K_{54}-K_{44}K_{53})+K_{34}(K_{45}K_{53}-K_{43}K_{55})+K_{33}(K_{44}K_{55}-K_{45}K_{54})},$$

$$G_2^{\sigma\hat{i}}=\frac{K_2^{2\hat{i}}}{K_{11}-K_{12}}$$

$$G_3^{\sigma\hat{i}}=\frac{K_3{}^{\hat{i}}(K_3{}^{\hat{i}}(K_{11}-K_{12})K_{16}+K_{26}((\dot{K}_{11}+2\dot{K}_{12})K_2{}^{\hat{i}}-(K_{11}-K_{12})K_4{}^{\hat{i}}))}{(K_{11}-K_{12})(K_{26}(\dot{K}_{11}+2\dot{K}_{12})-(K_{11}-K_{12})K_{66})}$$

$$G_4^{D\hat{i}}=\frac{K_4{}^{\hat{i}}(K_4^{*\hat{i}}(K_{35}K_{53}-K_{33}K_{55})+K_{45}(K_{33}K_5^{*\hat{i}}-K_{53}K_1^{*\hat{i}})+K_{43}(K_{55}K_1^{*\hat{i}}-K_{35}K_5^{*\hat{i}}))}{K_{35}(K_{43}K_{54}-K_{44}K_{53})+K_{34}(K_{45}K_{53}-K_{43}K_{55})+K_{33}(K_{44}K_{55}-K_{45}K_{54})}$$

$$G_5^{B\hat{i}}=\frac{K_5{}^{\hat{i}}[K_4^{*\hat{i}}(K_{33}K_{54}-K_{34}K_{53})+K_{44}(K_{53}K_1^{*\hat{i}}-K_{33}K_5^{*\hat{i}})+K_{43}(K_{34}K_5^{*\hat{i}}-K_{54}K_1^{*\hat{i}})]}{K_{35}(K_{43}K_{54}-K_{44}K_{53})+K_{34}(K_{45}K_{53}-K_{43}K_{55})+K_{33}(K_{44}K_{55}-K_{45}K_{54})}$$

$$G_6^{\Upsilon\hat{i}}=\frac{K_6{}^{\hat{i}}[(\dot{K}_{11}+2\dot{K}_{12})(K_2{}^{\hat{i}}+K_3{}^{\hat{i}})-(K_{11}-K_{12})K_6{}^{\hat{i}}]}{(K_{11}-K_{12})K_{66}-K_{16}(\dot{K}_{11}+2\dot{K}_{12})}$$

上式中其它参数$\mathrm{K}_1^{\hat{i}*}$, $\mathrm{K}_5^{\hat{i}*}$和$\mathrm{K}_4^{\hat{i}*}$的表达式见附录 B。

3.4.6 广义应变能强度因子

根据应变能强度理论[93, 94], 磁电热弹耦合材料内部单元体积$dV_{\hat{i}}=dx_{1\hat{i}}dx_{2\hat{i}}dx_{3\hat{i}}$在广义三维应力作用下的应变能为：

$$\begin{aligned}dW_{\hat{i}}=(&\frac{\sigma_{\hat{i}11}^2+\sigma_{\hat{i}22}^2+\sigma_{\hat{i}33}^2}{2E}-v\frac{\sigma_{\hat{i}11}\sigma_{\hat{i}22}+\sigma_{\hat{i}22}\sigma_{\hat{i}33}+\sigma_{\hat{i}33}\sigma_{\hat{i}11}}{E}+\frac{\sigma_{\hat{i}12}^2+\sigma_{\hat{i}13}^2+\sigma_{\hat{i}23}^2}{2\mu}\\&+\frac{D_{\hat{i}11}^2+D_{\hat{i}22}^2+D_{\hat{i}33}^2}{2E'}-v'\frac{D_{\hat{i}11}D_{\hat{i}22}+D_{\hat{i}22}D_{\hat{i}33}+D_{\hat{i}33}D_{\hat{i}11}}{E'}+\frac{D_{\hat{i}12}^2+D_{\hat{i}13}^2+D_{\hat{i}23}^2}{2\mu'}\\&+\frac{B_{\hat{i}11}^2+B_{\hat{i}22}^2+B_{\hat{i}33}^2}{2E'}-v'\frac{B_{\hat{i}11}B_{\hat{i}22}+B_{\hat{i}22}B_{\hat{i}33}+B_{\hat{i}33}B_{\hat{i}11}}{E'}+\frac{B_{\hat{i}12}^2+B_{\hat{i}13}^2+B_{\hat{i}23}^2}{2\mu'}\\&+\frac{\vartheta_{\hat{i}11}^2+\vartheta_{\hat{i}22}^2+\vartheta_{\hat{i}33}^2}{2E'}-v'\frac{\vartheta_{\hat{i}11}\vartheta_{\hat{i}22}+\vartheta_{\hat{i}22}\vartheta_{\hat{i}33}+\vartheta_{\hat{i}33}\vartheta_{\hat{i}11}}{E'}+\frac{\vartheta_{\hat{i}12}^2+\vartheta_{\hat{i}13}^2+\vartheta_{\hat{i}23}^2}{2\mu'})dV_{\hat{i}}\end{aligned}\tag{3-110}$$

E, v, μ, E', v', μ'分别表示杨氏模量、泊松比、剪切模量，等价杨氏模量，等价泊松比，等价剪切模量. 把上式带入（3-103）～（3-106），可得到广义应变能量强度函数，表达式如下：

$$\frac{dW_{\hat{i}}}{dV_{\hat{i}}}=\frac{K_{\hat{i}n}^2}{2\sqrt{2r}}\left(\frac{a_{\hat{i}3n}^2}{E}+\frac{a_{\hat{i}1n}^2+a_{\hat{i}2n}^2}{\mu}+\frac{a_{\hat{i}4n}^2+a_{\hat{i}5n}^2+a_{\hat{i}6n}^2}{E'}\right) \tag{3-111}$$

进一步整理可得到广义应变能强度因子表达式：

$$S_{\hat{i}}\Big|_{\theta}=K_{\hat{i}n}^2\left(\frac{a_{\hat{i}3n}^2}{E}+\frac{a_{\hat{i}1n}^2+a_{\hat{i}2n}^2}{\mu}+\frac{a_{\hat{i}4n}^2+a_{\hat{i}5n}^2+a_{\hat{i}6n}^2}{E'}\right) \tag{3-112}$$

其中 $a_{\hat{i}mn}$ (m,n=1,2,3,4,5,6) 为和材料性质有关的已知参数，具体表达式见附录B。对于两平行裂纹的特殊情况，广义应变能量强度因子可表示为：

$$S_1\Big|_{\theta}=K_{1n}^2\left(\frac{a_{13n}^2}{E}+\frac{a_{11n}^2+a_{12n}^2}{\mu}+\frac{a_{14n}^2+a_{15n}^2+a_{16n}^2}{E'}\right) \tag{3-113}$$

$$S_2\Big|_{\theta}=K_{2n}^2\left(\frac{a_{23n}^2}{E}+\frac{a_{21n}^2+a_{22n}^2}{\mu}+\frac{a_{24n}^2+a_{25n}^2+a_{26n}^2}{E'}\right) \tag{3-114}$$

其中 $a_{\alpha mn}$ (m,n=1,2,3,4,5,6)的具体表达式见附录 B。

3.5 数值方法

3.5.1 数值方法

从以上分析中可以看出，超奇异积分方程的数值求解是最大困难，而最难求解的超奇异积分部分决定了数值求解结果的精度和收敛性。同时我们注意到核函数 $K_{ij}^{\hat{i}}$ 为Gauss-Chebyshev 型积分，应用适当的方法，这部分积分尽管数学表达式复杂，但较超奇异部分容易求解。裂纹面上的广义位移间断可以表示为：

$$\tilde{U}_1^{\hat{i}}(\xi_1^{\hat{i}},\xi_2^{\hat{i}})=F_J^{\hat{i}}(\xi_1^{\hat{i}},\xi_2^{\hat{i}})\xi_2^{\hat{i}\lambda_J}W^{\hat{i}}(\xi_1^{\hat{i}},\xi_2^{\hat{i}}) \tag{3-115}$$

其中 $W^{\hat{i}}(\xi_1^{\hat{i}},\xi_2^{\hat{i}})=\sqrt{(a^2-\xi_1^{\hat{i}2})(2b-\xi_2^{\hat{i}})}$，$F_J^{\hat{i}}(\xi_1^{\hat{i}},\xi_2^{\hat{i}})=\sum_{s=0}^{S}\sum_{h=0}^{H}a^{\hat{i}}_{Jsh}\xi_1^{s\hat{i}}\xi_2^{h\hat{i}}$； $a^{\hat{i}}_{Jmn}$ 为一未知常数。把上式带入超奇异积分方程组，可得到如下代数方程组，其具体表达式如下：

$$\sum_{\hat{i}}^{n}\sum_{s=0}^{S}\sum_{h=0}^{H}a^{\hat{i}}_{Ish}(I^{\hat{i}1}_{IJsh}+I^{\hat{i}2}_{IJsh}+I^{\hat{i}3}_{IJsh})=-p^{\hat{i}}_{J}(x^{\hat{i}}_1,x^{\hat{i}}_2) \tag{3-116}$$

其中

$$I^{\hat{i}1}_{11sh}=\int_{S_{\hat{i}}} r^{-3}[K_{11}(1-3r_{,2}^{2})+K_{12}(1-3r_{,1}^{2})]\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_1+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-117}$$

$$I^{\hat{i}1}_{12sh}=-3\int_{S_{\hat{i}}} r^{-3}(K_{11}+K_{12})r_{,1}r_{,2}\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_2+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-118}$$

$$I^{\hat{i}2}_{\alpha Jsh}=\sum_{\hat{i}=2}^{\hat{n}}\int_{S_{\hat{i}}} K^{\hat{i}}_{\alpha i}\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_n+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-119}$$

$$I^{\hat{i}1}_{21sh}=-\int_{S_{\hat{i}}} 3r^{-3}(K_{11}+K_{12})r_{,1}r_{,2}\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_1+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-120}$$

$$I^{\hat{i}1}_{22sh}=\int_{S_{\hat{i}}} r^{-3}[K_{11}(1-3r_{,1}^{2})+K_{12}(1-3r_{,2}^{2})]\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_2+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-121}$$

$$I^{\hat{i}3}_{\alpha 6sh}=\int_{S_{\hat{i}}} 3r^{-4}K_{\alpha 6}r_{,\alpha}\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_6+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-122}$$

$$I^{\hat{i}1}_{m\alpha sh}=\int_{S_{\hat{i}}} r^{-2}K_{m1}r_{,\alpha}\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_\alpha+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-123}$$

$$I^{\hat{i}1}_{mnsh}=\int_{S_{\hat{i}}} r^{-3}K_{mn}\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_n+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-124}$$

$$I^{\hat{i}3}_{m6sh}=\int_{S_{\hat{i}}} 3r^{-4}\dot{K}_{m6}\lambda_{3\alpha}r_{,\alpha}\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_6+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-125}$$

$$I^{\hat{i}2}_{mnsh}=\sum_{\hat{i}=2}^{\hat{n}}\int_{S_{\hat{i}}} K^{\hat{i}}_{mi}\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_n+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-126}$$

$$I^{\hat{i}1}_{61sh}=\int_{S_{\hat{i}}} r^{-3}[\dot{K}_{11}(1-3r_{,1}^{2})-\dot{K}_{12}r_{,1}r_{,2}]\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_1+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-127}$$

$$I^{\hat{i}1}_{62sh}=\int_{S_{\hat{i}}} r^{-3}[\dot{K}_{12}(1-3r_{,2}^{2})-3\dot{K}_{11}r_{,1}r_{,2}]\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_2+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-128}$$

$$I^{\hat{i}3}_{66sh}=\int_{S_{\hat{i}}} 3r^{-4}K_{66}\lambda_{3\alpha}r_{,\alpha}\xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_6+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-129}$$

$$I^{\hat{i}2}_{66sh}=\sum_{\hat{i}=2}^{\hat{n}}\int_{S_{\hat{i}}} \xi^{\hat{i}\,s}_{1}\xi^{\hat{i}\,\lambda_n+h}_{2}W^{\hat{i}}d\xi_1^{\hat{i}}d\xi_2^{\hat{i}} \tag{3-130}$$

K_{ij} 具体表达式见附录 B。以上各式中的超奇异部分必须先进行数值处理，才可以进行积分。应用如下关系式

$$\xi^{\hat{i}\,m}_{1}\xi^{\hat{i}\,\lambda_J+h}_{2}\sqrt{(a^2-\xi^{\hat{i}\,2}_{1})(2b-\xi^{\hat{i}}_{2})}=D^{\hat{i}J}_{0}+D^{\hat{i}J}_{1}r_1+D^{\hat{i}J}_{2}r_1^2 \tag{3-131}$$

上面各式可简化为：

$$I^{\hat{i}1}_{11sh}=\int_0^{2\pi}[K_{11}(1-3\sin^2\theta^{\hat{i}})+K_{12}(1-3\cos^2\theta^{\hat{i}})](D^{\hat{i}1}_{1}\ln R-R^{-1}D^{\hat{i}1}_{0}+\int_0^R D^{\hat{i}1}_{2}dr_1)d\theta^{\hat{i}} \tag{3-132}$$

$$I^{\hat{i}1}_{12sh}=\int_0^{2\pi}1.5(K_{11}+K_{12})\sin 2\theta^{\hat{i}}(D^{\hat{i}2}_{1}\ln R-R^{-1}D^{\hat{i}2}_{0}+\int_0^R D^{\hat{i}2}_{2}dr_1)d\theta^{\hat{i}} \tag{3-133}$$

$$I^{\hat{i}1}_{21sh}=\int_0^{2\pi}1.5(K_{11}+K_{12})\sin 2\theta^{\hat{i}}(D^{\hat{i}1}_{1}\ln R-R^{-1}D^{\hat{i}1}_{0}+\int_0^R D^{\hat{i}1}_{2}dr_1)d\theta^{\hat{i}} \tag{3-134}$$

$$I^{\hat{i}1}_{22sh}=\int_0^{2\pi}[K_{11}(1-3\cos^2\theta^{\hat{i}})+K_{12}(1-3\sin^2\theta^{\hat{i}})](D^{\hat{i}2}_{1}\ln R-R^{-1}D^{\hat{i}2}_{0}+\int_0^R D^{\hat{i}2}_{2}dr_1)d\theta^{\hat{i}} \tag{3-135}$$

$$I^{\hat{i}1}_{61sh}=\int_0^{2\pi}[\dot{K}_{11}(1-3\cos^2\theta^{\hat{i}})-0.5\dot{K}_{12}\sin 2\theta^{\hat{i}}](D^{\hat{i}1}_{1}\ln R-R^{-1}D^{\hat{i}1}_{0}+\int_0^R D^{\hat{i}1}_{2}dr_1)d\theta^{\hat{i}} \tag{3-136}$$

$$I^{\hat{i}1}_{\alpha 6sh}=\int_0^{2\pi}3(K_{16}\cos\theta^{\hat{i}}\delta_{1\alpha}+K_{26}\sin\theta^{\hat{i}}\delta_{2\alpha})(D^{\hat{i}6}_{1}\ln R-R^{-1}D^{\hat{i}6}_{0}+\int_0^R D^{\hat{i}6}_{2}dr_1)d\theta^{\hat{i}} \tag{3-137}$$

$$I^{\hat{i}1}_{62sh}=\int_0^{2\pi}[\dot{K}_{12}(1-3\sin^2\theta^{\hat{i}})-\dot{K}_{11}\cos\theta^{\hat{i}}\sin\theta^{\hat{i}}](D^{\hat{i}2}_{1}\ln R-R^{-1}D^{\hat{i}2}_{0}+\int_0^R D^{\hat{i}2}_{2}dr_1)d\theta^{\hat{i}} \tag{3-138}$$

$$I^{\hat{i}1}_{66sh}=\int_0^{2\pi}3K_{66}(\lambda_{31}\cos\theta^{\hat{i}}+\lambda_{32}\sin\theta^{\hat{i}})(D^{\hat{i}6}_{1}\ln R-R^{-1}D^{\hat{i}6}_{0}+\int_0^R D^{\hat{i}6}_{2}dr_1)d\theta^{\hat{i}} \tag{3-139}$$

$$I^{\hat{i}1}_{m6sh}=\int_0^{2\pi}3\dot{K}_{m6}(\lambda_{31}\cos\theta^{\hat{i}}+\lambda_{32}\sin\theta^{\hat{i}})(D^{\hat{i}6}_{1}\ln R-R^{-1}D^{\hat{i}6}_{0}+\int_0^R D^{\hat{i}6}_{2}dr_1)d\theta^{\hat{i}} \tag{3-140}$$

$$I^{\hat{i}1}_{m\alpha sh}=\int_0^{2\pi}K_{m1}\cos\theta^{\hat{i}}(D^{\hat{i}\alpha}_{1}\ln R-R^{-1}D^{\hat{i}\alpha}_{0}+\int_0^R D^{\hat{i}\alpha}_{2}dr_1)d\theta^{\hat{i}} \tag{3-141}$$

$$I^{\hat{i}1}_{mnsh}=\int_0^{2\pi}K_{mn}(D^{\hat{i}n}_{1}\ln R-R^{-1}D^{\hat{i}n}_{0}+\int_0^R D^{\hat{i}n}_{2}dr_1)d\theta^{\hat{i}} \tag{3-142}$$

3.5.2 数值结果及讨论

在这部分，我们将应用上面所建立的数值方法来计算一些典型的三维多裂纹问题。为了计算和比较的方便，我们先对裂纹前沿广义应力强度因子 $F^{\hat{i}}_{I,\lambda}$ 及裂纹内部广义应力强度因子 $F^{\hat{i}}_{I}$ 进行无量纲处理，无量纲化公式为：

$$F^{\hat{i}}_{1,\lambda}=K^{\hat{i}}_{1,\lambda}/\sigma^{\hat{i}\infty}_{33}b^{1-\lambda} \qquad F^{\hat{i}}_{1}=K^{\hat{i}}_{1}/\sigma^{\hat{i}\infty}_{33}\sqrt{b} \tag{3-143}$$

$$F^{\hat{i}}_{2,\lambda}=K^{\hat{i}}_{2,\lambda}/\sigma^{\hat{i}\infty}_{31}b^{1-\lambda} \qquad F^{\hat{i}}_{2}=K^{\hat{i}}_{2}/\sigma^{\hat{i}\infty}_{31}\sqrt{b} \tag{3-144}$$

$$F_{3,\lambda}^{\hat{i}}=K_{3,\lambda}^{\hat{i}}/\sigma_{32}^{\hat{i}\infty}b^{1-\lambda}\qquad F_{3}^{\hat{i}}=K_{3}^{\hat{i}}/\sigma_{32}^{\hat{i}\infty}\sqrt{b}\tag{3-145}$$

$$F_{4,\lambda}^{\hat{i}}=K_{4,\lambda}^{\hat{i}}/D_{33}^{\hat{i}\infty}b^{1-\lambda}\qquad F_{4}^{\hat{i}}=K_{4}^{\hat{i}}/D_{33}^{\hat{i}\infty}\sqrt{b}\tag{3-146}$$

$$F_{5,\lambda}^{\hat{i}}=K_{5,\lambda}^{\hat{i}}/B_{33}^{\hat{i}\infty}b^{1-\lambda}\qquad F_{5}^{\hat{i}}=K_{5}^{\hat{i}}/B_{33}^{\hat{i}\infty}\sqrt{b}\tag{3-147}$$

$$F_{6,\lambda}^{\hat{i}}=K_{6,\lambda}^{\hat{i}}/\vartheta_{3}^{\hat{i}\infty}b^{1-\lambda}\qquad F_{6}^{\hat{i}}=K_{6}^{\hat{i}}/\vartheta_{3}^{\hat{i}\infty}\sqrt{b}\tag{3-148}$$

对于简单情况，三维双裂纹问题的相应的广义应力强度因子可表示为：

$$F_{1,\lambda}^{1}=K_{1,\lambda}^{1}/\sigma_{33}^{1\infty}b^{1-\lambda},F_{2,\lambda}^{1}=K_{2,\lambda}^{1}/\sigma_{31}^{1\infty}b^{1-\lambda},F_{3,\lambda}^{1}=K_{3,\lambda}^{1}/\sigma_{32}^{1\infty}b^{1-\lambda}\tag{3-149}$$

$$F_{1}^{1}=K_{1}^{1}/\sigma_{33}^{1\infty}\sqrt{b},F_{2}^{1}=K_{2}^{1}/\sigma_{31}^{1\infty}\sqrt{b},F_{3}^{1}=K_{3}^{1}/\sigma_{32}^{1\infty}\sqrt{b}\tag{3-150}$$

$$F_{4,\lambda}^{1}=K_{4,\lambda}^{1}/D_{33}^{1\infty}b^{1-\lambda},F_{5,\lambda}^{1}=K_{5,\lambda}^{1}/B_{33}^{1\infty}b^{1-\lambda},F_{6,\lambda}^{1}=K_{6,\lambda}^{1}/\vartheta_{3}^{1\infty}b^{1-\lambda}\tag{3-151}$$

$$F_{4}^{1}=K_{4}^{1}/D_{33}^{1\infty}\sqrt{b},F_{5}^{1}=K_{5}^{1}/B_{33}^{1\infty}\sqrt{b},F_{6}^{1}=K_{6}^{1}/\vartheta_{3}^{1\infty}\sqrt{b}\tag{3-152}$$

$$F_{1,\lambda}^{2}=K_{1,\lambda}^{2}/\sigma_{33}^{2\infty}b^{1-\lambda},F_{2,\lambda}^{2}=K_{2,\lambda}^{2}/\sigma_{31}^{2\infty}b^{1-\lambda},F_{3,\lambda}^{2}=K_{3,\lambda}^{2}/\sigma_{32}^{2\infty}b^{1-\lambda}\tag{3-153}$$

$$F_{1}^{2}=K_{1}^{2}/\sigma_{33}^{2\infty}\sqrt{b},F_{2}^{2}=K_{2}^{2}/\sigma_{31}^{2\infty}\sqrt{b},F_{3}^{2}=K_{3}^{2}/\sigma_{32}^{2\infty}\sqrt{b}\tag{3-154}$$

$$F_{4,\lambda}^{2}=K_{4,\lambda}^{2}/D_{33}^{2\infty}b^{1-\lambda},F_{5,\lambda}^{2}=K_{5,\lambda}^{2}/B_{33}^{2\infty}b^{1-\lambda},F_{6,\lambda}^{2}=K_{6,\lambda}^{2}/\vartheta_{3}^{2\infty}b^{1-\lambda}\tag{3-155}$$

$$F_{4}^{2}=K_{4}^{2}/D_{33}^{2\infty}\sqrt{b},F_{5}^{2}=K_{5}^{2}/B_{33}^{2\infty}\sqrt{b},F_{6}^{2}=K_{6}^{2}/\vartheta_{3}^{2\infty}\sqrt{b}\tag{3-156}$$

无量纲化广义应变能强度因子可表示为：

$$\bar{S}_{\hat{i},\lambda}\Big|_{\theta}=(F_{n,\lambda}^{\hat{i}})^{2}(\frac{a_{\hat{i}3n}^{2}}{E}+\frac{a_{\hat{i}1n}^{2}+a_{\hat{i}2n}^{2}}{\mu}+\frac{a_{\hat{i}4n}^{2}+a_{\hat{i}5n}^{2}+a_{\hat{i}6n}^{2}}{E'})\tag{3-157}$$

$$\bar{S}_{\hat{i}}\Big|_{\theta}=(F_{n}^{\hat{i}})^{2}(\frac{a_{\hat{i}3n}^{2}}{E}+\frac{a_{\hat{i}1n}^{2}+a_{\hat{i}2n}^{2}}{\mu}+\frac{a_{\hat{i}4n}^{2}+a_{\hat{i}5n}^{2}+a_{\hat{i}6n}^{2}}{E'})\tag{3-158}$$

对于简单情况，三维双裂纹问题的相应的广义无量纲应变能强度因子可表示为：

$$\bar{S}_{1,\lambda}\Big|_{\theta}=(F_{n,\lambda}^{1})^{2}(\frac{a_{13n}^{2}}{E}+\frac{a_{11n}^{2}+a_{12n}^{2}}{\mu}+\frac{a_{14n}^{2}+a_{15n}^{2}+a_{16n}^{2}}{E'})\tag{3-159}$$

$$\bar{S}_{2,\lambda}\Big|_{\theta}=(F_{n,\lambda}^{2})^{2}(\frac{a_{23n}^{2}}{E}+\frac{a_{21n}^{2}+a_{22n}^{2}}{\mu}+\frac{a_{24n}^{2}+a_{25n}^{2}+a_{26n}^{2}}{E'}) \tag{3-160}$$

$$\bar{S}_{1}\Big|_{\theta}=(F_{n}^{1})^{2}(\frac{a_{13n}^{2}}{E}+\frac{a_{11n}^{2}+a_{12n}^{2}}{\mu}+\frac{a_{14n}^{2}+a_{15n}^{2}+a_{16n}^{2}}{E'}) \tag{3-161}$$

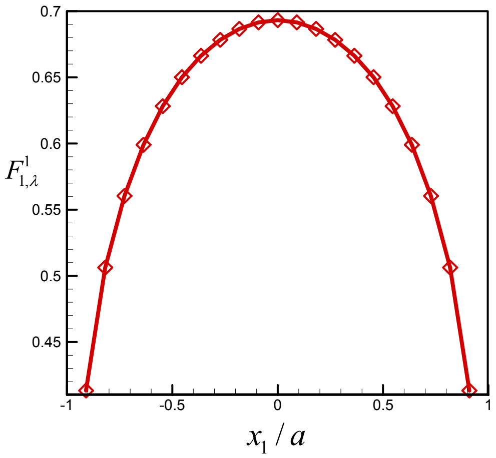
$$\bar{S}_{2}\Big|_{\theta}=(F_{n}^{2})^{2}(\frac{a_{23n}^{2}}{E}+\frac{a_{21n}^{2}+a_{22n}^{2}}{\mu}+\frac{a_{24n}^{2}+a_{25n}^{2}+a_{26n}^{2}}{E'}) \tag{3-162}$$

1. 收敛性

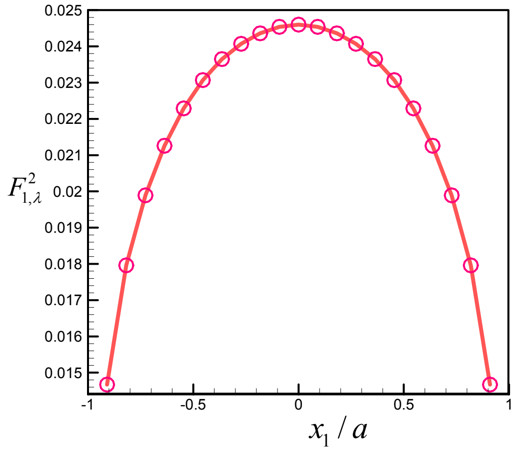
在裂纹形状比 b/a=1，两平行裂纹间垂直距离比 h/b=1，配置点 KK=LL=20×20，无限远处广义载荷为 σ_{3i}^{1} 时，广义应力强度因子随多项式指数增加时收敛性及随裂纹位置变化规律如表 3-1 和图 3-3 所示。在 $x_2=0,2b$ 边，$F_{1,\lambda}^{1}$，$F_{1,\lambda}^{2}$ 随 x_1/a 呈对称分布，$x_1/a=0$ 取最大值 $F_{1\max}^{1}\Big|_{x_1/a=0}$=0.6932，$F_{1\max}^{2}\Big|_{x_1/a=0}$=0.02460。随多项式指数增加，数值结果是收敛的。本文数值方法所得的数值结果是收敛的，有很好的精度。

表3-1 在 $x_2=0,2b$ 边无量纲广义应力强度因子 $F_{1,\lambda}^{1},F_{1,\lambda}^{2}$ 随多项式指数变化规律(KK=LL=20×20,b/a=1,h/a=1)

M	x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
13	$F_{1,\lambda}^{1}$	.6932	.6915	.6866	.6783	.6663	.6501	.6282	.5990	.5604	.5062	.4133
	$F_{1,\lambda}^{2}$	.02460	.02454	.02436	.02407	.02365	.02307	.02229	.02126	.01989	.01796	.01467
11	$F_{1,\lambda}^{1}$	.6932	.6915	.6866	.6783	.6664	.6501	.6282	.5990	.5601	.5060	.4158
	$F_{1,\lambda}^{2}$	.02460	.02454	.02436	.02407	.02365	.02307	.02229	.02126	.01988	.01796	.01475
9	$F_{1,\lambda}^{1}$	.6932	.6915	.6866	.6783	.6663	.6501	.6282	.5990	.5604	.5062	.4133
	$F_{1,\lambda}^{2}$	.02640	.02634	.02615	.02583	.02538	.02476	.02392	.02281	.02134	.01928	.01574
7	$F_{1,\lambda}^{1}$	.6926	.6912	.6868	.6792	.6666	.6503	.6280	.5991	.5585	.5064	.4078
	$F_{1,\lambda}^{2}$	.02456	.02451	.02435	.02408	.02364	.02306	.02227	.02125	.01980	.01796	.01446



1) $F_{1,\lambda}^{1}$



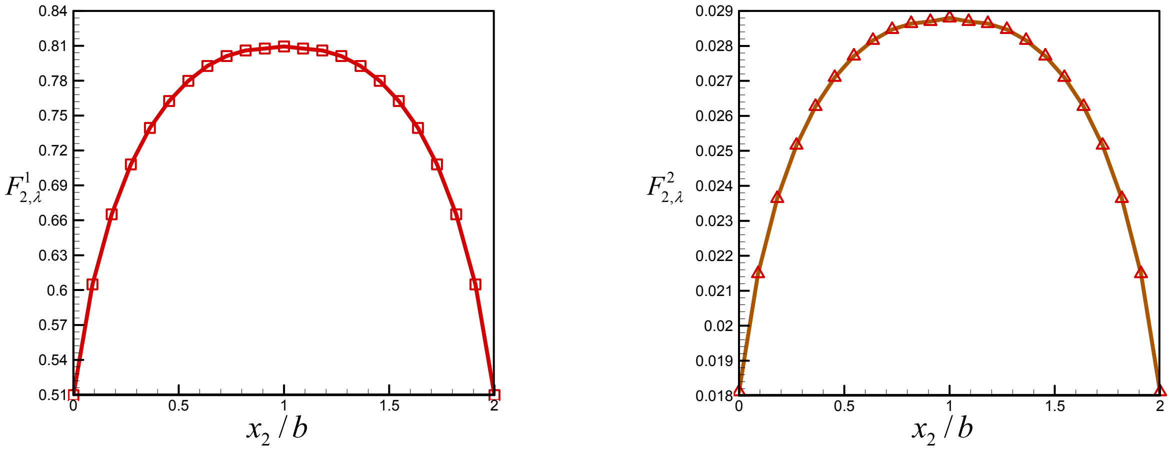
2) $F_{1,\lambda}^{2}$

图 3-3 在 $x_2=0,2b$ 边无量纲广义应力强度因子 $F_{1,\lambda}^{1},F_{1,\lambda}^{2}$ 随坐标变化规律(KK=LL=20×20,M=N=13,b/a=1,h/a=1)

载荷为 σ_{31}^{∞} 时，裂纹形状比 b/a=1，配置点 KK=LL=20×20，广义应力强度因子随多项式指数、坐标变化规律如表 3-2、3-3 及图 3-4、3-5 所示：F_2(在 $x_1=\pm a$ 随 x_2/b)、F_3(在 $x_2=0,2b$ 随 x_1/a)与 F_1 (在 $x_2=0,2b$ 随 x_1/a)有相似的变化规律。F_2 随 $x_2/b\in$（0，2b）成对称分布，在 $x_2/b=b$ 取最大值 $F_{2\max}^{1}\big|_{x_2/b=b}$=0.8077，$F_{2\max}^{2}\big|_{x_2/b=b}$=0.0287；$F_3$ 随 $x_1/a\in$（-a，a）成对称分布，在 x_1/a=0 取最大值 $F_{3\max}^{1}\big|_{x_1/a=0}$=0.9109，$F_{3\max}^{2}\big|_{x_1/a=0}$=0.0324。

表3-2 在 $x_1=\pm a$ 无量纲广义应力强度因子 $F_{2,\lambda}^{1}$, $F_{2,\lambda}^{2}$ 随多项式指数变化规律(KK=LL=20×20,b/a=1,h/a=1)

M	x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
13	$F_{2,\lambda}^{1}$	.5097	.6049	.6652	.7080	.7394	.7625	.7799	.7926	.8012	.8060	.8077
	$F_{2,\lambda}^{2}$	.01811	.02149	.02364	.02516	.02627	.02710	.02771	.02816	.02847	.02864	.02870
11	$F_{2,\lambda}^{1}$	.5119	.6060	.6649	.7080	.7395	.7625	.7799	.7925	.8011	.8060	.8076
	$F_{2,\lambda}^{2}$	.01819	.02153	.02363	.02516	.02628	.02710	.02771	.02816	.02846	.02864	.02870
9	$F_{2,\lambda}^{1}$	.5107	.6070	.6637	.7060	.7379	.7613	.7782	.7904	.7992	.8046	.8065
	$F_{2,\lambda}^{2}$	.01815	.02157	.02358	.02509	.02622	.02705	.02765	.02809	.02840	.02859	.02866
7	$F_{2,\lambda}^{1}$	.5072	.6121	.6671	.7062	.7377	.7627	.7812	.7937	.8013	.8050	.8062
	$F_{2,\lambda}^{2}$	.01802	.02175	.02370	.02510	.02621	.02710	.02776	.02820	.02847	.02861	.02865

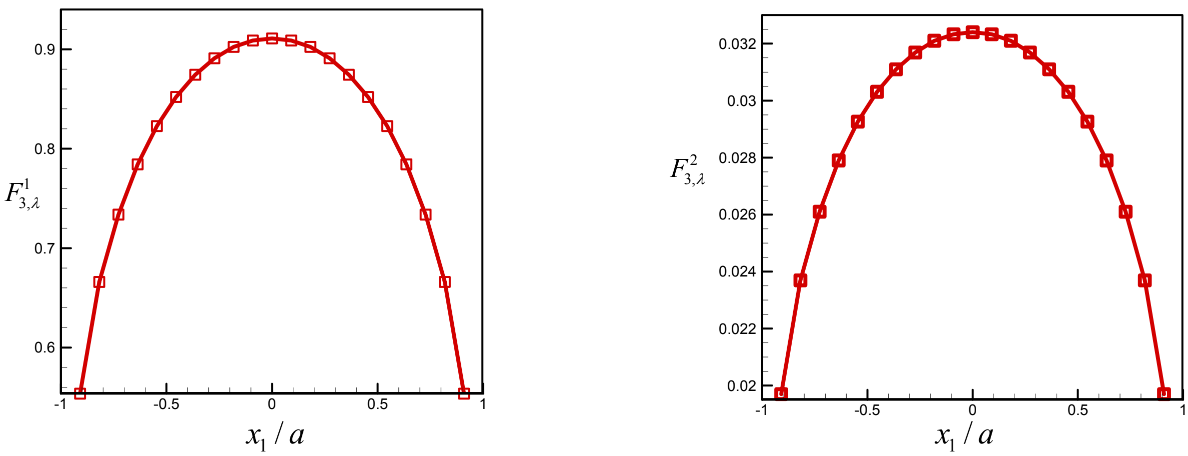


1) $F^1_{2,\lambda}$ 2) $F^2_{2,\lambda}$

图 3-4 在 $x_2=0,2b$ 边无量纲广义应力强度因子 $F^1_{2,\lambda}, F^2_{2,\lambda}$ 随坐标变化规律(KK=LL=20×20,M=N=13,b/a=1,h/a=1)

表3-3 在 $x_2=0,2b$ 边无量纲广义应力强度因子 $F^1_{3,\lambda}$，$F^2_{3,\lambda}$ 随多项式指数变化规律(KK=LL=20×20,b/a=1,h/a=1)

M	x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
13	$F^1_{3,\lambda}$	.9109	.9088	.9023	.8909	.8744	.8520	.8227	.7842	.7338	.6660	.5537
	$F^2_{3,\lambda}$	.0324	.03233	.03210	.03169	.03110	.03031	.02926	.02790	.02610	.02369	.0197
11	$F^1_{3,\lambda}$	.9111	.9090	.9023	.8909	.8746	.8523	.8229	.7845	.7350	.6676	.5517
	$F^2_{3,\lambda}$	.03241	.03233	.03209	.03169	.03111	.03032	.02927	.02791	.02614	.02375	.01962
9	$F^1_{3,\lambda}$	.9111	.9089	.9022	.8909	.8745	.8522	.8231	.7855	.7367	.6680	.5460
	$F^2_{3,\lambda}$	.03241	.03233	.03209	.03169	.03110	.03031	.02928	.02794	.02620	.02376	.01942
7	$F^1_{3,\lambda}$	.9106	.9085	.9022	.8912	.8749	.8528	.5480	.7869	.7378	.6653	.5334
	$F^2_{3,\lambda}$	.03239	.03231	.03209	.03170	.03112	.03033	.01949	.02799	.02624	.02366	.01897



1) $F^1_{3,\lambda}$ 2) $F^2_{3,\lambda}$

图 3-5 在 $x_2=0,2b$ 边无量纲广义应力强度因子 $F^1_{3,\lambda}, F^2_{3,\lambda}$ 随坐标变化规律(KK=LL=20×20,M=N=13,b/a=1,h/a=1)

表3-4 裂纹面I上无量纲广义应变能强度因子随 θ， x_1/a 变化规律(KK=LL=20×20,M=N=13,b/a=1,h/a=1)

x_1/a / θ	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
0	.3880	.4110	.4220	.4250	.4200	.4100	.3920	.3680	.3340	.2900	.2190
10	.3950	.4200	.4310	.4340	.4300	.4200	.4020	.3770	.3440	.2980	.2260
20	.3980	.4230	.4350	.4380	.4340	.4230	.4060	.3800	.3460	.3000	.2270
30	.3960	.4190	.4310	.4330	.4290	.4180	.4010	.3750	.3420	.2960	.2240
40	.3870	.4090	.4190	.4210	.4160	.4050	.3870	.3620	.3290	.2840	.2140
50	.3730	.3920	.4000	.4010	.3950	.3840	.3670	.3420	.3100	.2670	.1990
60	.3540	.3690	.3750	.3740	.3680	.3570	.3400	.3160	.2850	.2440	.1810
70	.3320	.3430	.3470	.3450	.3380	.3260	.3090	.2870	.2570	.2190	.1600
80	.3100	.3170	.3180	.3140	.3070	.2950	.2790	.2570	.2290	.1930	.1390
90	.2880	.2920	.2910	.2860	.2780	.2660	.2500	.2300	.2040	.1700	.1210
100	.2700	.2710	.2690	.2630	.2550	.2430	.2280	.2080	.1830	.1520	.1060
110	.2560	.2560	.2530	.2480	.2390	.2280	.2130	.1940	.1710	.1410	.0978
120	.2490	.2500	.2470	.2410	.2330	.2220	.2080	.1890	.1670	.1380	.0959
130	.2490	.2510	.2500	.2450	.2370	.2270	.2130	.1950	.1720	.1430	.1010
140	.2570	.2610	.2610	.2580	.2510	.2410	.2270	.2090	.1860	.1560	.1120
150	.2700	.2780	.2810	.2790	.2730	.2640	.2500	.2310	.2070	.1760	.1290
160	.2880	.3010	.3060	.3060	.3010	.2920	.2780	.2590	.2340	.2010	.1500
170	.3090	.3270	.3350	.3360	.3320	.3240	.3100	.2900	.2630	.2280	.1720
180	.3310	.3530	.3630	.3670	.3640	.3550	.3410	.3200	.2920	.2540	.1930

在裂纹形状比 b/a=1，两平行裂纹之间垂直距离比 h/a=1，配置点 KK=LL=20×20，无限远处广义载荷为 σ_{3i}^{1} 时，无量纲广义应变能强度因子 $\left.\overline{S}_{1,\lambda}\right|_{\theta}$ 、 $\left.\overline{S}_{2,\lambda}\right|_{\theta}$ 随 θ， x_1/a 变化规律如表 3-4，3-5 及图 3-6，3-7 所示。$\overline{S}_{\alpha,\lambda}$ 在 180°-360° 间变化规律与 0°-180° 间变化规律成对称分布，这与实际情况相吻合。

表3-5 裂纹面II上无量纲广义应变能强度因子随 θ， x_1/a 变化规律(KK=LL=20×20,M=N=13,b/a=1,h/a=1)

θ \ x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
0	.00059	.00059	.00058	.00056	.00054	.00052	.00048	.00044	.00039	.00032	.00022
10	.00060	.00060	.00059	.00058	.00056	.00053	.00050	.00045	.00040	.00033	.00022
20	.00061	.00060	.00060	.00058	.00056	.00053	.00050	.00045	.00040	.00033	.00023
30	.00060	.00060	.00059	.00057	.00055	.00053	.00049	.00045	.00039	.00032	.00022
40	.00058	.00058	.00057	.00056	.00054	.00051	.00048	.00044	.00038	.00031	.00022
50	.00055	.00055	.00054	.00053	.00051	.00048	.00045	.00041	.00036	.00030	.00020
60	.00051	.00051	.00050	.00049	.00047	.00045	.00042	.00038	.00034	.00028	.00019
70	.00047	.00047	.00046	.00045	.00043	.00041	.00038	.00035	.00031	.00025	.00017
80	.00042	.00042	.00042	.00041	.00039	.00037	.00035	.00032	.00028	.00023	.00016
90	.00038	.00038	.00038	.00037	.00035	.00034	.00031	.00029	.00025	.00021	.00014
100	.00035	.00035	.00034	.00034	.00032	.00031	.00029	.00026	.00023	.00019	.00013
110	.00033	.00033	.00032	.00031	.00030	.00029	.00027	.00024	.00021	.00018	.00012
120	.00032	.00032	.00031	.00031	.00030	.00028	.00026	.00024	.00021	.00017	.00012
130	.00033	.00033	.00032	.00031	.00030	.00029	.00027	.00024	.00021	.00018	.00012
140	.00035	.00035	.00034	.00033	.00032	.00030	.00028	.00026	.00023	.00019	.00013
150	.00038	.00038	.00037	.00036	.00035	.00033	.00031	.00028	.00025	.00020	.00014
160	.00042	.00042	.00041	.00040	.00039	.00037	.00034	.00031	.00028	.00023	.00016
170	.00046	.00046	.00046	.00045	.00043	.00041	.00038	.00035	.00031	.00025	.00017
180	.00051	.00051	.00050	.00049	.00047	.00045	.00042	.00038	.00033	.00028	.00019

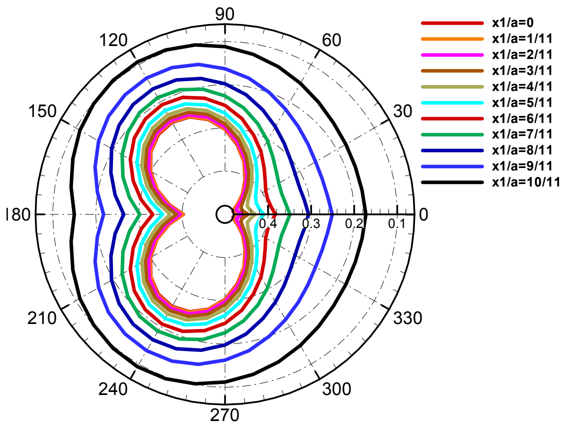


图 3-6 裂纹 I 上广义无量纲应变能强度因子变化规律(KK=LL=20×20, M=N=13，b/a=1,h/a=1)

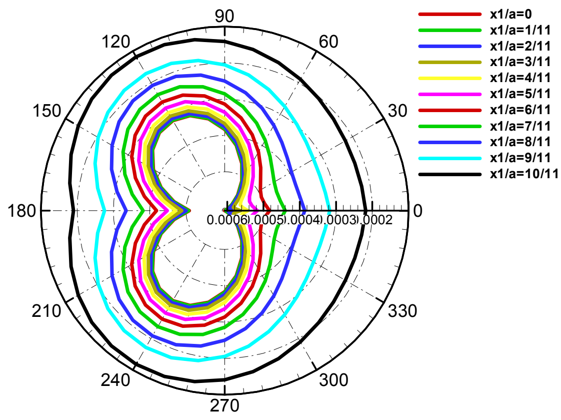


图 3-7 裂纹 II 上广义无量纲应变能强度因子变化规律(KK=LL=20×20, M=N=13，b/a=1,h/a=1)

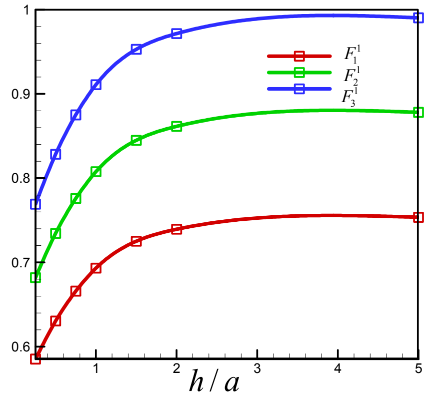
2 一般情况

配置点KK=LL=20×20，多项式指数M=N=13，裂纹形状比 (b/a) 和裂纹面间距离比 (h/a) 对广义应力强度因子影响如表3-6及图3-8、3-9所示。裂纹1上应力强度因子 F_I^1 随裂纹面间距离比增加而增大，裂纹2上应力强度因子 F_I^2 随裂纹面间距离比增加而减小；当h/a=5 时， F_I^1 收敛到一固定值(单裂纹情况)， F_I^2 则收敛到0。结果显示裂纹间的干扰和屏蔽作用随照裂纹面间距离比增加而不断减弱，当h/a≥5时，它们之间相互作用可以忽略不计。数值结果还显示，裂纹面上的广义应力强度因子 F_I^1， F_I^2 均随裂纹形状比增加而增加，且增加的梯度不断减少；当b/a=7时，F_I^1， F_I^2 收敛到一固定值(面裂纹情况)。这表示应力强度因子在b/a≥7时，增加的梯度趋于零，已经可以看作为平面问题了。

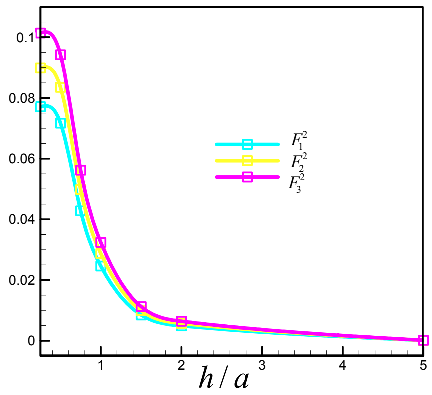
表3-6 无量纲最大广义应力强度因子 F_I^α 随裂纹形状比、裂纹面间距离比变化规律(KK=LL=20×20, M=N=13)

h/a	*h / a*	F_1^1	F_2^1	F_3^1	F_1^2	F_2^2	F_3^2
	0.25	.5853	.6820	.7692	.0771	.0899	.1014
	0.5	.6304	.7345	.8284	.0717	.0835	.0942
	0.75	.6659	.7759	.8751	.0428	.0498	.0562
1	1.0	.6932	.8077	.9109	.0246	.0287	.0324
	1.5	.7252	.8450	.9529	.0085	.0099	.0112
	2.0	.7394	.8615	.9716	.0049	.0057	.0064
	5.0	.7536	.8781	.9903	.00010	.00012	.00014
	∞	.7536	.8781	.9903	.0000	.0000	.0000
2	0.25	.7133	.7693	.9955	.0940	.1014	.1312
	0.5	.7681	.8284	1.0721	.0874	.0942	.1219
	0.75	.8115	.8752	1.1326	.0521	.0562	.0728
	1.0	.8447	.9110	1.1789	.0300	.0324	.0419
	1.5	.8836	.9530	1.2333	.0104	.0112	.0145
	2.0	.9010	.9717	1.2575	.0059	.0064	.0083

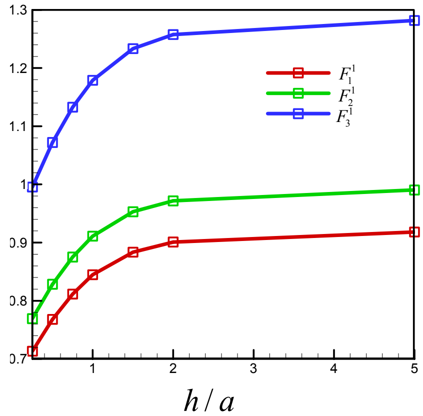
	5.0	.9183	.9904	1.2817	.00014	.00016	.00020
	∞	.9183	.9904	1.2817	.0000	.0000	.0000
3	0.25	.7207	.7916	1.0939	.0950	.1043	.1442
	0.5	.7762	.8525	1.1781	.0883	.0969	.1340
	0.75	.8199	.9005	1.2445	.0527	.0578	.0799
	1.0	.8535	.9374	1.2955	.0303	.0333	.0461
	1.5	.8929	.9806	1.3553	.0105	.0115	.0159
	2.0	.9104	.9999	1.3818	.0060	.0066	.0091
	5.0	.9279	1.0191	1.4084	.00014	.0016	.00022
	∞	.9279	1.0191	1.4084	.0000	.0000	.0000
5	0.25	.7193	.8017	1.1638	.0948	.1057	.1534
	0.5	.7747	.8634	1.2533	.0881	.0982	.1425
	0.75	.8184	.9121	1.3240	.0526	.0586	.0850
	1.0	.8519	.9494	1.3782	.0303	.0338	.0490
	1.5	.8912	.9932	1.4418	.0105	.0117	.0170
	2.0	.9086	1.0127	1.4700	.0060	.0067	.0097
	5.0	.9379	1.0322	1.4983	.00015	.00016	.00024
	∞	.9379	1.0322	1.4983	.0000	.0000	.0000
7	0.25	.7685	.8022	1.1839	.1013	.1057	.1560
	0.5	.8276	.8639	1.2750	.0941	.0982	.1450
	0.75	.8743	.9126	1.3469	.0562	.0586	.0865
	1.0	.9101	.9500	1.4020	.0324	.0338	.0498
	1.5	.9521	.9938	1.4667	.0112	.0117	.0173
	2.0	.9707	1.0133	1.4954	.0064	.0067	.0098
	5.0	1.002	1.0328	1.5242	.00016	.00016	.00024
	∞	1.002	1.0328	1.5242	.0000	.0000	.0000
∞	0.25	.7687	.8025	1.1938	.1013	.1058	.1573
	0.5	.8278	.8642	1.2857	.0941	.0983	.1462
	0.75	.8745	.9130	1.3582	.0562	.0586	.0872
	1.0	.9103	.9503	1.4138	.0324	.0338	.0503
	1.5	.9523	.9942	1.4790	.0112	.0117	.0174
	2.0	.9709	1.0137	1.5080	.0064	.0067	.0099
	5.0	1.0022	1.0332	1.5370	.00016	.00016	.00024
	∞	1.0022	1.0332	1.5370	.0000	.0000	.0000



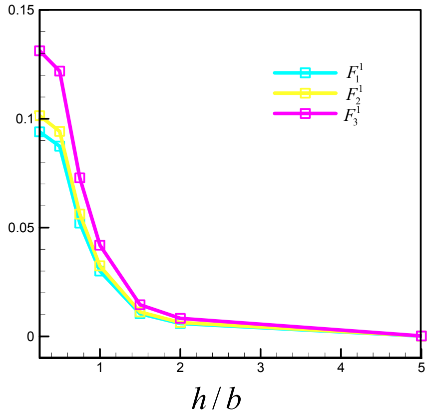
1) F_i^1,b/a=1



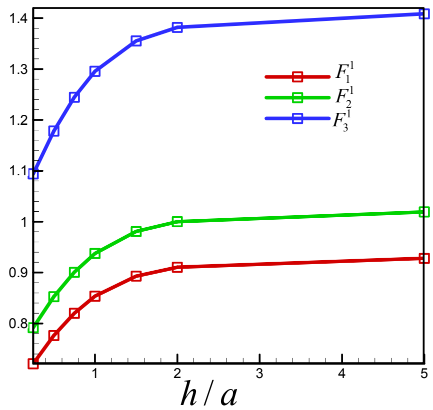
2) F_i^2,b/a=1



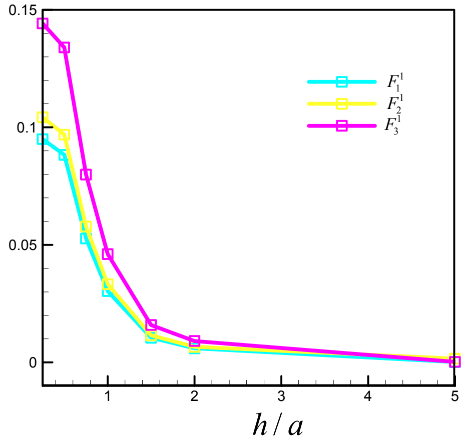
3) F_i^1,b/a=2



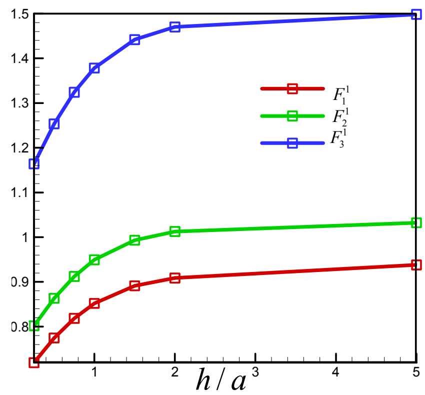
4) F_i^2,b/a=2



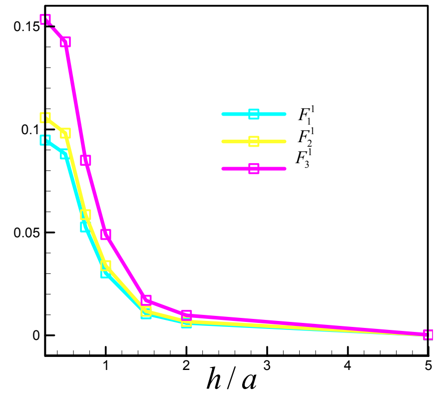
5) F_i^1,b/a=3



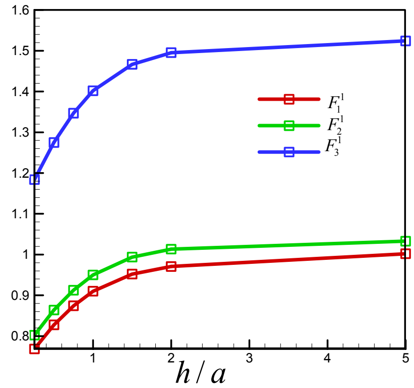
6) F_i^2,b/a=3



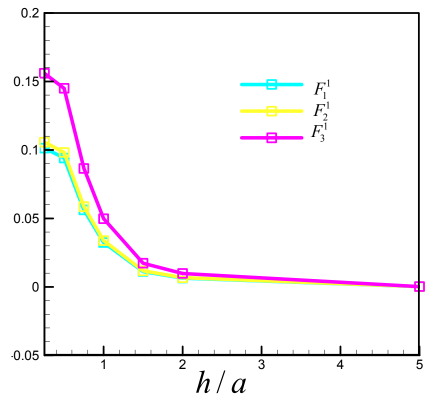
7) F_i^1,b/a=5



8) F_i^2,b/a=5

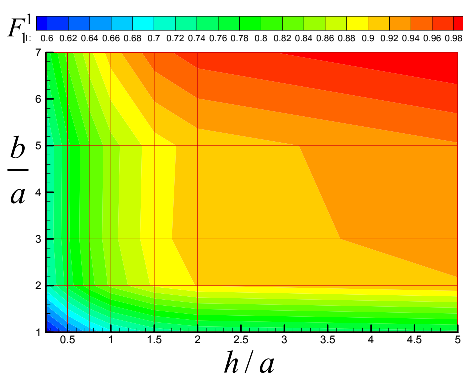


9) F_i^1,b/a=7

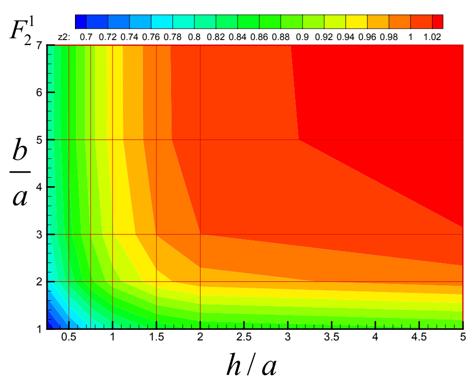


10) F_i^2,b/a=7

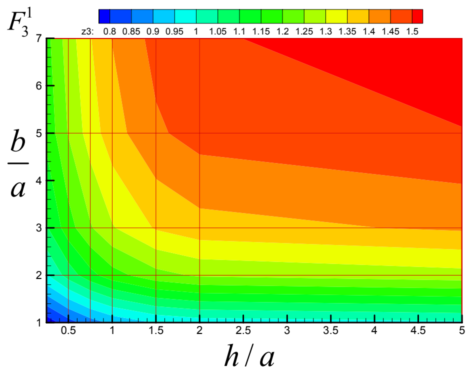
图 3-8 最大广义应力强度因子 F_I^1, F_I^2 随裂纹形状比、裂纹面间距离比变化规律(KK=LL=20×20, M=N=13)



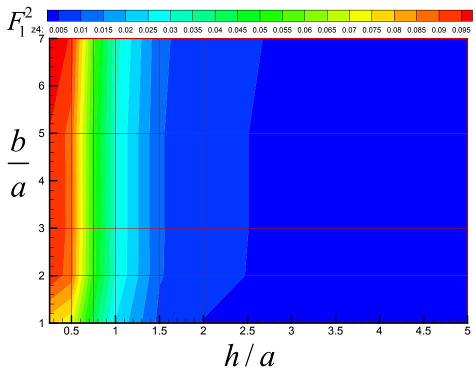
1) F_1^1



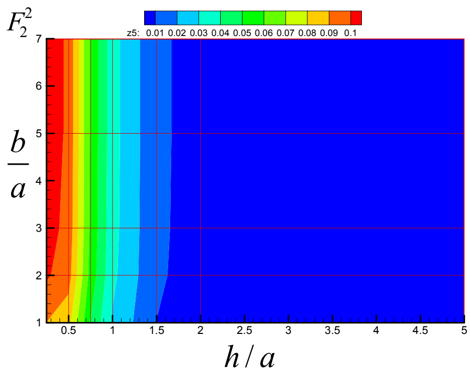
2) F_2^1



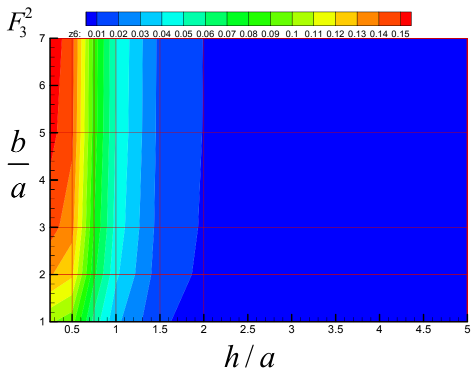
3) F_3^1



4) F_1^2



5) F_2^2



6) F_3^2

图 3-9 F_1^1 F_2^1 F_3^1 F_1^2 F_2^2 F_3^2 随裂纹形状比、裂纹面间距离比变化规律(KK=LL=20×20, M=N=13)

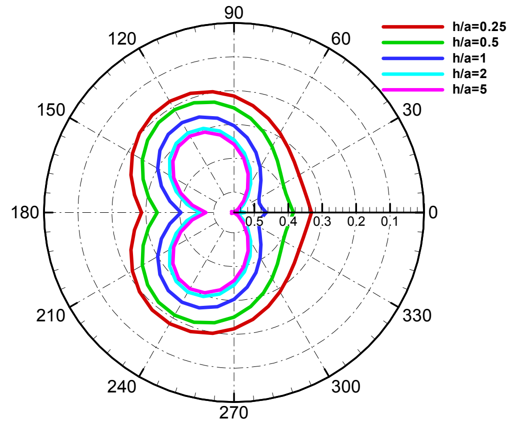
配置点 KK=LL=20×20，多项式指数 M=N=13，裂纹形状比和裂纹间距离比对广义应变能强度因子的影响如表 3-7、3-8 所示及图 3-10、3-11 所示。裂纹 1 上应变能强度因子 $\bar{S}_1\big|_\theta$ 随裂纹面间距离比增加而增大，裂纹 2 上应变能强度因子 $\bar{S}_2\big|_\theta$ 随裂纹面间距离比增加而减小；当 h/a=5 时，$\bar{S}_1\big|_\theta$ 收敛到一固定值(单裂纹情况)，$\bar{S}_2\big|_\theta$ 则收敛到 0。结果显示裂纹间的干扰和屏蔽作用随照裂纹面间距离比增加而不断减弱，当 h/a≥5 时，它们之间相互作用可以忽略不计。数值结果还显示，裂纹面上的广义应变能强度因子 $\bar{S}_1\big|_\theta$，$\bar{S}_2\big|_\theta$ 均随裂纹形状比增加而增加，且增加的梯度不断减少；当 b/a=7 时，$\bar{S}_1\big|_\theta$，$\bar{S}_2\big|_\theta$ 收敛到一固定值(面裂纹情况)。这表示应变能强度因子在 b/a≥7 时，增加的梯度趋于零，已经可以看作为平面问题了。(图 3-9 彩色版请参见 www.imechanica.org/blog/2237)。

表3-7 裂纹I上无量纲广义应变能强度因子随b/a,h/a和θ变化规律(KK=LL=20×20, M=N=13)

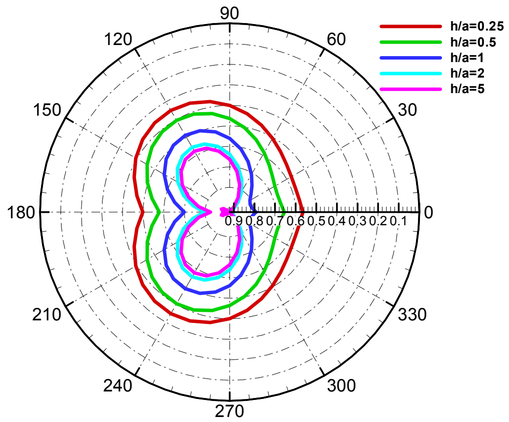
b/a	h/a	$\theta=0$	$\theta=\pi/6$	$\theta=\pi/3$	$\theta=\pi/2$	$\theta=2\pi/3$	$\theta=5\pi/6$	$\theta=\pi$
	0.25	.3401	.3388	.2894	.2164	.1808	.2141	.2876
	0.5	.3849	.3930	.3357	.2510	.2097	.2483	.3336
1	1.0	.4655	.4752	.4059	.3035	.2536	.3002	.4034
	2.0	.5296	.5406	.4617	.3453	.2885	.3416	.4590
	5.0	.5501	.5616	.4797	.3587	.2997	.3548	.4767
	0.25	.5650	.5752	.5050	.4004	.3483	.3944	.4988
	0.5	.6554	.6672	.5857	.4644	.4040	.4574	.5786
3	1.0	.7925	.8067	.7082	.5615	.4886	.5531	.6996
	2.0	.9016	.9178	.8057	.6389	.5558	.6293	.7960
	5.0	.9366	.9535	.8370	.6637	.5774	.6537	.8269
	0.25	.6066	.6168	.5457	.4401	.3878	.4347	.5404
	0.5	.7035	.7154	.6329	.5104	.4497	.5041	.6268
5	1.0	.8507	.8651	.7653	.6172	.5438	.6096	.7579
	2.0	.9678	.9841	.8707	.7021	.6187	.6935	.8623
	5.0	1.0122	1.0295	.9102	.7328	.6446	.7227	.9001
	0.25	.6407	.6519	.5764	.4631	.4057	.4539	.5662
	0.5	.7430	.7561	.6685	.5371	.4705	.5265	.6566
7	1.0	.8985	.9142	.8083	.6494	.5689	.6366	.7940
	2.0	1.0221	1.0401	.9196	.7388	.6472	.7242	.9033
	5.0	1.0693	1.0883	.9616	.7714	.6745	.7549	.9430

表3-8 裂纹II上无量纲广义应变能强度因子随b/a,h/a和θ变化规律(KK=LL=20×20, M=N=13)

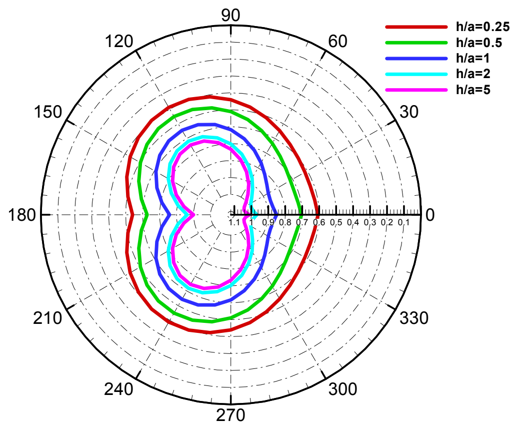
b/a	h/a	$\theta=0$	$\theta=\pi/6$	$\theta=\pi/3$	$\theta=\pi/2$	$\theta=2\pi/3$	$\theta=5\pi/6$	$\theta=\pi$
	0.25	.00576	.00588	.00503	.00376	.00314	.00372	.0050
1	0.5	.00498	.00508	.00434	.00325	.00271	.00321	.00431
	1.0	.00059	.00060	.00051	.00038	.00032	.00038	.00051
	0.25	.00982	.01000.	.00877	.00696	.00605	.00685	.00867
3	0.5	.00878	.00863	.00758	.00601	.00523	.00592	.00748
	1.0	.00100	.00102	.00090	.00071	.0062	.00070	.00088
	0.25	.01054	.01072	.00948	.00765	.00674	.00755	.00939
5	0.5	.00910	.00925	.00848	.00660	.00584	.00652	.00815
	1.0	.00108	.00109	.00097	.00078	.00069	.00077	.00096
	0.25	.01112	.01132	.01000	.00804	.00704	.00788	.00983
7	0.5	.00961	.00978	.00864	.00695	.00608	.00681	.00849
	1.0	.00113	.00116	.00102	.00082	.00072	.00070	.00100



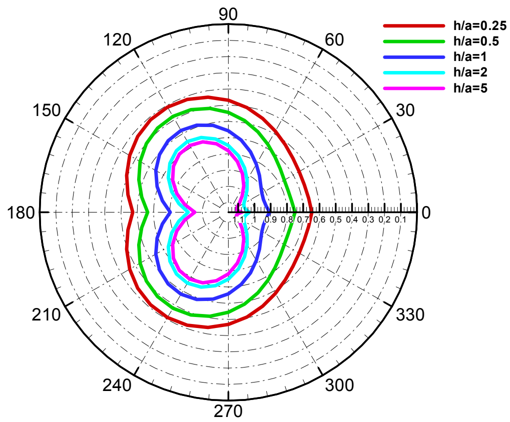
(a) b/a=1



(b) b/a=3

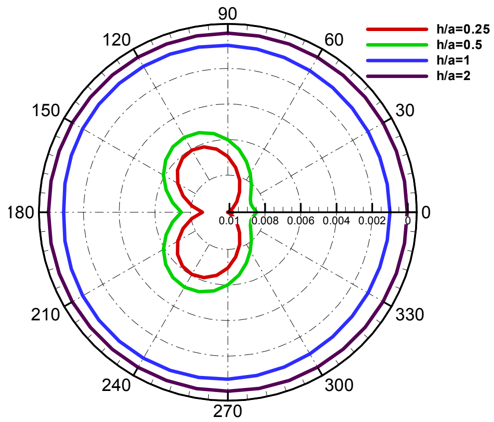


(c) b/a=5

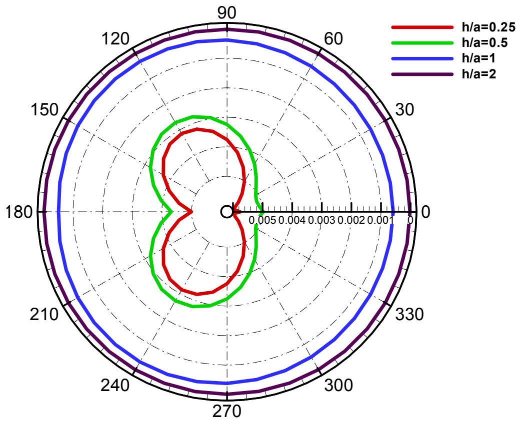


(d) b/a=7

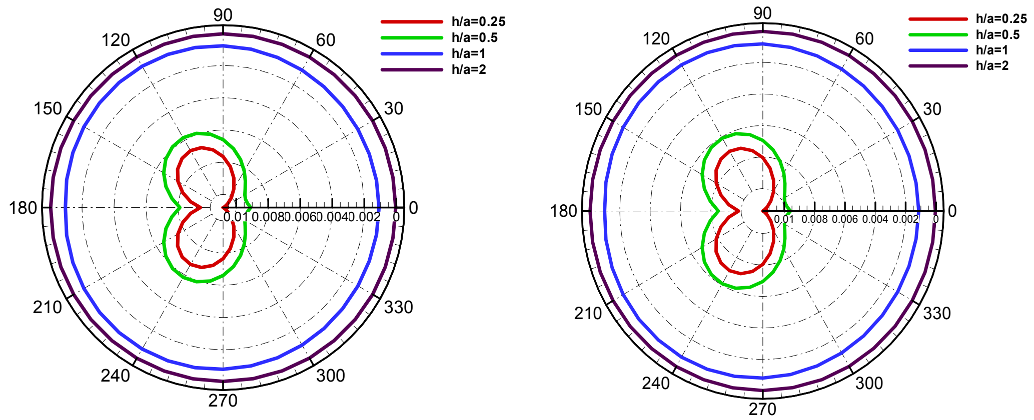
图 3-10 裂纹 I 上广义无量纲应变能强度因子随 θ 和 b/a 变化规律(KK=LL=20×20, M=N=13)



(a) b/a=1



(b) b/a=3



(c) b/a=5　　　　(d) b/a=7

图 3-11 裂纹 II 上广义无量纲应变能强度因子随 θ 和 b/a 变化规律(KK=LL=20×20, M=N=13)

3.6.本章小节

本章应用超奇异积分方程方法，分析了多场(磁、电、热、弹)载荷共同作用下的磁电热弹耦合材料三维多裂纹问题，主要结论如下：

1. 多场(磁、电、热、弹)载荷共同作用下磁电热弹耦合材料三维多裂纹问题被转化为求解一以广义位移间断为未知函数的超奇异积分方程组问题。在此基础上，通过庞杂的数学推导，得到了广义奇异应力场的解析表达式，定义了广义(应力、应变能)强度因子和广义能量释放率。

2. 通过对超奇异积分方程中的各类奇异积分的专门数值处理，将该方程转化成为一代数方程组，为超奇异积分方程建立了数值解法，推导出广义（应力、应变能）强度因子的数值计算公式。以三维双裂纹为例，用 FORTRAN 语言编写了专门程序，计算了典型算例。数值结果显示：本方法具有良好的收敛性和精度。通过分析广义（应力、应变能）强度因子与裂纹形状比、裂纹面间距离比、广义载荷及材料参数之间的变化规律，裂纹间干扰和屏蔽作用不但与裂纹形状、位置及裂纹面距离有关，而且与载荷类型及材料参数有关。

3. 工程中遇到的多场耦合作用下的材料多裂纹问题，可以应用本文提供的裂纹评定准则来分析和研究结构的可靠性。在应力集中区域内，多裂纹间干扰和屏蔽作用对材料结构强度和可靠性的影响十分明显，在工程应用中应予以考虑。

第四章 磁电热弹耦合材料粘弹塑性三维裂纹扩展问题

4.1 引言

上世纪六十年代后，许多著名科学家将研究重点转到非弹性问题的弹塑性断裂力学上来，1959 年 Barenblatt[277, 278]引入裂纹边界附近渐近场概念，使得利用弹性力学方法分析非线性弹性断裂力学成为可能。随着线弹性断裂力学研究的日趋成熟，非线性断裂力学越来越受到人们的重视，对于弹塑性静态断裂力学问题，自 1968 年 Hutchison[279], Rice 和 Rosengren(荣盛仁)[280]发表了著名的 HRR 奇异解后，Rice[281]又引入了一种与路径无关的积分——J 积分(更早提出类似概念的还有 Eshelby(伊塞尔贝)[331]和 Cherepanov(蔡来巴诺夫)[332])，使得弹塑性断裂静力学得到很大发展。1991 年柳春图和蒋持平[282]系统的阐述了板壳弯曲断裂理论及其应用。黄克智等[283-286]对弹塑性裂纹问题也进行了深人而系统研究。近二十年，新材料（如压电材料[287-289]， 功能梯度材料[227, 290-293]）的研究，使众多学者把研究方向转到功能材料的断裂破坏问题上来[47-49]。蠕变有两个相对的方面，裂尖应力松弛和裂纹扩展。在高温下，应力松弛发生在裂尖钝化和塑性区处，应力松弛阻碍裂纹扩展。蠕变裂纹扩展是由于材料存在各向异性，空穴形成而后扩展为裂纹。在高温和循环加载的恶劣工况下，结构的变形形式、受力状态都很复杂，材料常发生显著的非弹性响应，蠕变和低周疲劳及其相互作用往往是造成结构破坏的主要原因[294]。

与弹性问题不同，弹塑—蠕变为非线性问题，需要在时间域上进行积分计算。其主要困难在于非线性基本解的寻找非线性量在区域内的剖分。尽管很难找到非线性问题的基本解，但可以用线性问题的基本解代替，将非线性场变量用小增量代替，则线性问题的基本解同样适用于非线性问题[295-299]。为了将以上分析弹性线性问题方法扩展应用到非线性问题，首先需要将广义热弹塑—蠕变对基本控制微分方程和边界条件的非线性影响作为伪载荷处理。同前面章节中导出三维弹性断裂问题的超奇异积分方程方法类似，本章也将源点位于边界上的情况作为由内点趋于边界的极限过程处理，通过对源点邻域内物体边界、区域和函数积分的极限分析，将内点应力的积分方程推广至边界点，最终推导出磁电热弹耦合材料三维弹塑性—蠕变断裂问题的超奇异积分方程。

本章中，首先，引入时域超奇异积分概念和广义位移间断应变率概念，通过将热弹塑—蠕变对控制微分方程和边界条件的非线性影响作为伪载荷处理，使得塑性蠕变项以伪体积力和伪面力形式出现，这样就可以采用三维磁电热弹耦合材料弹性问题的基本解作为弹塑性蠕变问题的基本解来求解三维弹塑性蠕变问题。然后，应用扩展后的边界积分方程方法，将磁电热弹粘塑性三维裂纹扩展问题转化为求解一组时域超奇异积分方程问题，得到广义应力增量场解析表达式，定义了广义 J 积分。 进而，对超奇异积分过程中遇到的各类奇异积分分别进行了数值处理，为该方程组建立数值求解方法，编写了专用的 FORTRAN 程序（7000 余条），进行了考核性计算。 最后，通过现有数值方法计算结果和已有结果进行比较，验证了本方法的可靠性和准确性。

4.2 基本理论

解决粘弹塑性问题可以应用全量直接法和增量法，在全量直接法中除了本构关系以外，其余方程均与线弹性理论一样，只是所有变量都与时间相关。全量直接法限于线性弹塑性问题；增量法以增量形式出现，它也适用于非线性粘弹性问题。本章以增量法来进行理论推导和数值计算。

4.2.1流动理论

连续、均匀、各向同性弹塑性—蠕变体在伪体积力增量 $\dot{F}_I$，伪面力增量 $\dot{T}_I$ 和边界约束作用下保持平衡。广义位移增量 $\dot{u}_I$、广义应变增量 $\dot{\varepsilon}_I$ 和应力增量 $\dot{\sigma}_{iJ}$ 在小变形条件下满足如下的广义热弹塑性—蠕变基本方程[166, 300]；

平衡方程：

$$\dot{\sigma}_{jI,j} + \dot{F}_I = 0 \qquad x \in V \tag{4-1}$$

其中：

$$\dot{F}_I = \dot{f}_I - \dot{\sigma}^n_{jI,j}$$

$$\dot{\sigma}^n_{iJ} = E^e_{iJKl}\dot{\varepsilon}^n_{Kl}$$

$$\dot{\varepsilon}^n_{iJ} = \dot{\varepsilon}^p_{iJ} + \dot{\varepsilon}^c_{iJ} + \dot{\varepsilon}^T_{iJ} + \dot{\varepsilon}^M_{iJ}$$

$$\dot{\varepsilon}^T_{iJ} = \alpha\dot{T}\delta_{iJ}$$

$$\dot{\varepsilon}^M_{iJ} = (E^e_{iJKl})_{,T}\,\sigma_{Kl}\dot{T}$$

几何方程：

$$\dot{\varepsilon}_{iJ} = \frac{1}{2}(\dot{u}_{i,J} + \dot{u}_{J,i}) \qquad x \in V \tag{4-2}$$

物理方程：

$$\dot{\sigma}_{iJ} = E_{iJKl}\dot{\varepsilon}_{Kl} \qquad x \in V \tag{4-3}$$

边界上面力和应力关系(力边界条件)：

$$\dot{t}_I = \dot{\sigma}_{Ij}n_j = \dot{\bar{t}}_I \qquad x \in S \tag{4-4}$$

在以上各式中，x 为坐标向量，δ_{ij} 是 $Kronecker$ δ 函数，V 表示弹塑性—蠕变体，S 为弹塑性—蠕变体边界，n_j 为边界 S 的外法线方向余弦。在边界 S 上，通常有如下的边界条件：

$$\dot{u}_I = \dot{\bar{u}}_I \qquad x \in S^u \tag{4-5}$$

$$\dot{T}_I = \dot{\bar{t}}_I + \dot{\sigma}^n_{jI} n_j \qquad x \in S^t \tag{4-6}$$

其中 S^u 和 S^t 分别表示边界 S 上给定广义位移增量 $\dot{\bar{u}}_I$ 和伪面力增量 $\dot{T}_I$ 的部分。在实际问题中若在边界的同一部分上既有位移条件，又有面力条件，则此时位移条件与面力条件应该互补，即位移分量若为未知，则对应面力分量为已知，反之亦然。弹塑性—蠕变体力学问题就是要求在给定体力和边界条件下，求解以下的偏微分方程组：

$$\begin{cases} \dot{\sigma}_{jI,j} + \dot{F}_I = 0 & x \in V \\ \dot{\sigma}_{iJ} = E_{iJKl}\dot{\varepsilon}_{Kl} & x \in V \\ \dot{T}_I = \dot{\bar{t}}_I + \dot{\sigma}^n_{jI} n_j & x \in S^t \\ \dot{u}_I = \dot{\bar{u}}_I & x \in S^u \end{cases} \tag{4-7}$$

4.2.2 边界积分方程

设在三维磁电热弹耦合材料 有限体中有一任意形状平片裂纹 S $(S^{\pm})$，在裂纹上表面 S^+ 放置一个直角坐标系 $x_i (i=1,2,3)$，使 x_3 轴与 S^+ 垂直。对于弹塑-蠕变问题，在考虑伪体力增量 $\dot{F}_i$ 和伪面力增量 $\dot{T}_i$ 时，我们可以采用磁电热弹耦合材料三维弹性问题的基本解作为粘弹塑性问题的基本解来求解粘弹塑性问题。使用 Somigiliana 公式，域内某点 p 处的广义位移的边界积分方程增量形式为：

$$c_{iJ}\dot{U}_J(p) = -\int_S \dot{T}_{iJ}(p,Q)\dot{U}_J(Q)ds(Q) + \int_S \dot{U}_{iJ}(p,Q)\dot{T}_J(Q)ds(Q) + \int_\Omega \dot{U}_{iJ}(p,Q)\dot{F}_J(Q)dV(Q) \tag{4-8}$$

式中 Ω 为磁电热弹体所占据的域，$\dot{U}_{IJ}$ 和 $\dot{T}_{IJ}$ 分别为磁电热弹耦合材料广义位移增量和广义面力增量基本解。进一步整理有：

$$\begin{aligned} c_{iJ}\dot{U}_J(p) &= -\int_S \dot{T}_{iJ}(p,Q)\dot{U}_J(Q)ds(Q) + \int_S \dot{U}_{iJ}(p,Q)(\dot{\bar{t}}_i + \dot{\sigma}^n_{ji}n_j)ds(Q) + \int_\Omega \dot{U}_{iJ}(p,Q)(\dot{f}_i - \dot{\sigma}^n_{ji,j})dV(Q) \\ &= -\int_S \dot{T}_{iJ}(p,Q)\dot{U}_J(Q)ds(Q) + \int_S \dot{U}_{iJ}(p,Q)\dot{\bar{t}}_J(Q)ds(Q) + \int_S \dot{U}_{iJ}(p,Q)\dot{\sigma}^n_{KJ}(Q)n_J)ds(Q) \\ &\quad + \int_\Omega \dot{U}_{iJ}(p,Q)\dot{f}_i ds(Q) - \int_\Omega \dot{U}_{iJ}(p,Q)\dot{\sigma}^n_{ji,j}dV(Q) \end{aligned} \tag{4-9}$$

利用高斯定理并考虑到 $\dot{\sigma}_{KJ}$ 对指标 K，J 具有对称性，磁电热弹耦合材料三维弹塑-蠕变问题的位移边界积分方程可表示为：

$$c_{iJ}\dot{U}_J(p)=-\int_S \dot{T}_{iJ}(p,Q)\dot{U}_J(Q)ds(Q)+\int_S \dot{U}_{iJ}(p,Q)\dot{\bar{t}}_J(Q)ds(Q)+\int_\Omega \dot{U}_{iJ}(p,Q)\dot{f}_i dV(Q) \\ -\frac{1}{2}\int_\Omega [\dot{U}_{iJ,\xi_K}(p,Q)+\dot{U}_{iK,\xi_J}(p,Q)]\dot{\sigma}^n_{JK}(Q)dV(Q) \tag{4-10}$$

上式就是将非弹性应力增量作为初应力增量的边界积分方程。引入裂纹面上的广义位移间断应变率：

$$\dot{\tilde{U}}_J=\begin{cases}\dot{\tilde{u}}_j=\dot{u}_j^+-\dot{u}_j^- & J=j=1,2,3\\ \dot{\tilde{\phi}}_j=\dot{\phi}_j^+-\dot{\phi}_j^- & J=4\\ \dot{\tilde{\varphi}}_j=\dot{\varphi}_j^+-\dot{\varphi}_j^- & J=5\\ \dot{\tilde{\Upsilon}}_j=\dot{\Upsilon}_j^+-\dot{\Upsilon}_j^- & J=6\end{cases} \tag{4-11}$$

在裂纹面上有关系：$\dot{T}_{IJ}(p,q^+)=-\dot{T}_{IJ}(p,q^-)=\dot{T}^+_{IJ}(p,q)$ 和 $\dot{U}_{IJ}(p,q^+)=\dot{U}_{IJ}(p,q^-)$，于是广义位移增量的表达式（4-9）可改写为：

$$c_{iJ}\dot{U}_J(p)=-\int_{S^+} \dot{T}^+_{iJ}(p,Q)\dot{\tilde{U}}_J(Q)ds(Q)+\int_\Omega \dot{U}_{iJ}(p,Q)\dot{f}_i ds(Q) \\ -\frac{1}{2}\int_\Omega [\dot{U}_{iJ,\xi_K}(p,Q)+\dot{U}_{iK,\xi_J}(p,Q)]\dot{\sigma}^n_{JK}(Q)dV(Q) \tag{4-12}$$

将式（4-12）代入到本构方程（4-3）中，于是可以得到广义应力增量表达式，如下所示：

$$\Sigma_{iJ}(p)=-\int_{S^+}\bar{S}^+_{KiJ}(p,Q)\dot{\tilde{U}}_K(Q)ds(Q)+\int_\Omega \bar{D}_{KiJ}(p,Q)\dot{f}_K ds(Q)-\int_\Omega W_{iJl}(p,Q)\dot{\varepsilon}^n_{JK}(Q)ds(Q) \tag{4-13}$$

式中，积分核函数 $\bar{S}_{KiJ}(p,Q)$、$\bar{D}_{KiJ}(p,Q)$ 和 $W_{iJl}(p,Q)$ 分别表示如下：

$$\bar{S}_{KiJ}(p,Q)=E_{iJMn}\frac{\partial \dot{T}_{MK}(p,Q)}{\partial x_n}=-E_{iJMn}\frac{\partial \dot{T}_{MK}(p,Q)}{\partial \xi_n} \tag{4-14}$$

$$\bar{D}_{KiJ}(p,Q)=E_{iJMn}\frac{\partial \dot{U}_{MK}(p,Q)}{\partial x_n}=-E_{iJMn}\frac{\partial \dot{U}_{MK}(p,Q)}{\partial \xi_n} \tag{4-15}$$

$$W_{iJn}(p,Q)=E_{iJMn}\left(\frac{\partial \dot{U}_{iJ}(p,Q)}{\partial \xi_M \partial x_n}+\frac{\partial \dot{U}_{iM}(p,Q)}{\partial \xi_J \partial x_n}\right) \tag{4-16}$$

如果能够求出或得到三维磁电热弹耦合材料问题的基本解，利用上述公式即可得到含有裂纹的三维磁电热弹耦合材料体中任意一点的广义位移增量以及广义应力增量场具体表达式。$\bar{S}_{KiJ}(p,Q)$ 与 $\bar{D}_{KiJ}(p,Q)$ 的展开形式读者可参见本文前面章节内容或相关文献[301, 302]自行得出，由于篇幅限制这里就不再重复。$W_{iJn}(p,Q)$ 具体展开式为：

$$W_{32l}=c_{44}\dot{u}_{31,13}+\frac{1}{2}[c_{44}(\dot{u}_{33,11}+\dot{u}_{31,31})+e_{15}(\dot{u}_{34,11}+\dot{u}_{31,31})+d_{15}(\dot{u}_{35,11}+\dot{u}_{31,31})-\iota_{31}(\dot{u}_{36,11}+\dot{u}_{31,31})]$$

$$W_{32l} = c_{44}\dot{u}_{32,23} + \frac{1}{2}[c_{44}(\dot{u}_{32,32} + \dot{u}_{33,22}) + e_{15}(\dot{u}_{32,32} + \dot{u}_{34,22}) + d_{15}(\dot{u}_{32,32} + \dot{u}_{35,22}) - \iota_{31}(\dot{u}_{36,22} + \dot{u}_{32,32})]$$

$$W_{33l} = \frac{1}{2}[c_{13}(\dot{u}_{31,31} + \dot{u}_{33,11}) + c_{13}(\dot{u}_{32,32} + \dot{u}_{33,22}) + 2c_{33}\dot{u}_{33,33} + e_{33}(\dot{u}_{33,33} + \dot{u}_{34,33}) + d_{33}(\dot{u}_{33,33} + \dot{u}_{35,33})] - \iota_{33}\dot{u}_{36,33}$$

$$W_{34l} = \frac{1}{2}[e_{31}(\dot{u}_{34,11} + \dot{u}_{33,31}) + e_{31}(\dot{u}_{34,22} + \dot{u}_{32,32}) + e_{33}(\dot{u}_{34,33} + \dot{u}_{33,33}) - g_{33}(\dot{u}_{34,33} + \dot{u}_{35,33}) - \zeta_{33}(\dot{u}_{34,33} + \dot{u}_{36,33})] - \in_{33} u_{34,33}$$

$$W_{35l} = \frac{1}{2}[d_{31}(\dot{u}_{34,11} + \dot{u}_{33,31}) + d_{31}(\dot{u}_{34,22} + \dot{u}_{32,32}) + d_{33}(\dot{u}_{34,33} + \dot{u}_{33,33}) - \mu_{33}(\dot{u}_{34,33} + \dot{u}_{35,33}) - \eta_{33}(\dot{u}_{34,33} + \dot{u}_{36,33})] - g_{33}u_{34,33}$$

$$W_{36l} = -\lambda_{33}\dot{u}_{36,33}$$

4.3 广义基本解

应用文献 [214, 229, 230, 303-305]中提供的方法， 经过复杂的推导，可以得到磁电热弹耦合材料广义增量位移基本解，具体表达式为：

1点力增量、电荷增量、磁增量和热应力增量

在点 $\hat{p}(\xi_1,\xi_2,\xi_3,t)$ 联合作用 $\dot{P}_z,\dot{Q},\dot{J}$ 和 $\dot{\Upsilon}$ 下，点 $\hat{q}(x_1,x_2,x_3,t)$ 广义位移增量可表示为：

$$\dot{U}_{mJ} = \sum_{i=1}^{5} R_i^{-1} A_{im} (\hat{R}_i^{-1}(x_\alpha - \xi_\alpha)\delta_{\alpha J} + \kappa_{i1}^{-1} v_i \sum_{n=2}^{5} \kappa_{in}\delta_{(n+1)J} - R_i^{-2}(x_3 - \xi_3)\kappa_{i5} v_i^2 \delta_{6J})) / 4\pi \quad (4\text{-}17)$$

其中

$$\hat{R}_i = R_i + v_i z$$

$$R_i = ((x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (v_i z)^2)^{0.5}$$

$$A_i = [A_{i1}\ A_{i2}\ A_{i3}\ A_{i4}\ 0]^T = W^{-1}$$

$$v_1 = c_{66}^{0.5} c_{44}^{-0.5}$$

$$v_6 = \lambda_{11}^{0.5} \lambda_{33}^{-0.5}$$

其它参数见本章附录C

2 在 x_2 方向作用点力增量

在点 $\hat{p}(\xi_1,\xi_2,\xi_3,t)$ 作用 x_2 方向点力增量，点 $\hat{q}(x_1,x_2,x_3,t)$ 广义位移增量可表示为：

$$\dot{U}_{2J}=\sum_{i=1}^{5}(D_i((x_2-\xi_2)(\delta_{\alpha J}R_i^{-1}\hat{R}_i^{-2}(\xi_\alpha-x_\alpha)-\kappa_{i1}^{-1}v_i(R_i^{-1}\hat{R}_i^{-1}\sum_{n=2}^{5}\kappa_{in}\delta_{(n+1)J}-\kappa_{i5}v_iR_i^{-3}\delta_{6J})+\hat{R}_i^{-1}\delta_{2J})-((x_1-\xi_1)R_0^{-1}\hat{R}_0^{-2}((x_1-\xi_1)\delta_{2J}-(x_2-\xi_2)\delta_{1J})-\hat{R}_0^{-1}\delta_{2J}))/4\pi c_{44} \tag{4-18}$$

其中

$$D_i=\hat{V}^{-1}[0\,0\,0\,0\,1]^T$$

$$\hat{v}_{il}=(\kappa_{il}v_i\,\delta_{kl}+\kappa_{55}\delta_{i5}\delta_{4l})\kappa_{i1}^{-1}+v_i\,\delta_{5l}$$

把上式（4-17）~（4-18）代入W_{3Jl}，整理得到可得到W_{3Jl}的具体表达式：

$$W_{31l}=\sum_{i=1}^{5}A_i^P\{-c_{44}v_i+\frac{1}{2}[c_{44}(v_i+\lambda_i^w)+e_{15}(v_i+\lambda_i^\varphi)+d_{15}(v_i+\lambda_i^v)]\}(\frac{1}{R_i^3}+\frac{3x_1^2}{\dot{R}_i^5})$$

$$W_{32l}=\sum_{i=1}^{5}A_i^P\{-c_{44}v_i+\frac{1}{2}[c_{44}(v_i+\lambda_i^w)+e_{15}(v_i+\lambda_i^\varphi)+d_{15}(v_i+\lambda_i^v)]\}(\frac{1}{R_i^3}+\frac{3x_2^2}{\dot{R}_i^5})$$

$$W_{33l}=\sum_{i=1}^{5}A_i^P\{\frac{c_{13}(v_i+\lambda_i^w)}{2}(\frac{2}{R_i^3}+\frac{3x_1^2}{\dot{R}_i^5}+\frac{3x_2^2}{\dot{R}_i^5})+3i_{33}\lambda_i^v v_i^4x_3(\frac{1}{R_i^5}+\frac{5v_i^2x_3^2}{\dot{R}_i^7})+[\frac{3}{2}e_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^\varphi)+\frac{3}{2}d_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^v)+3c_{33}v_i^2\lambda_i^w](\frac{1}{R_i^3}+\frac{3x_3^2}{\dot{R}_i^5})\}$$

$$W_{34l}=\sum_{i=1}^{5}A_i^P\{\frac{e_{31}(v_i+\lambda_i^w)}{2}(\frac{2}{R_i^3}+\frac{3x_1^2}{\dot{R}_i^5}+\frac{3x_2^2}{\dot{R}_i^5})-\frac{3}{2}\zeta_{33}\lambda_i^v v_i^4x_3(\frac{1}{R_i^5}+\frac{5v_i^2x_3^2}{\dot{R}_i^7})+[\frac{3}{2}e_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^\varphi)-\frac{3}{2}g_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^v)-3\in_{33}\lambda_i^v v_i^2\lambda_i^w-\frac{3}{2}\zeta_{33}v_i^2\lambda_i^v](\frac{1}{R_i^3}+\frac{3v_i^2x_3^2}{\dot{R}_i^5})\}$$

$$W_{35l}=\sum_{i=1}^{5}A_i^P\{\frac{d_{31}(v_i+\lambda_i^w)}{2}(\frac{2}{R_i^3}+\frac{3x_1^2}{\dot{R}_i^5}+\frac{3x_2^2}{\dot{R}_i^5})-\frac{3}{2}\eta_{33}\lambda_i^v v_i^4x_3(\frac{1}{R_i^5}+\frac{5v_i^2x_3^2}{\dot{R}_i^7})+[\frac{3}{2}d_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^\varphi)-\frac{3}{2}\mu_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^v)-3g_{33}\lambda_i^v v_i^2\lambda_i^w-\frac{3}{2}\eta_{33}v_i^2\lambda_i^v](\frac{1}{R_i^3}+\frac{3v_i^2x_3^2}{\dot{R}_i^5})\}$$

进一步整理有：

$$W_{31l}=\sum_{i=1}^{5}\dot{w}_1(\frac{1}{R_i^3}+\frac{3x_1^2}{\dot{R}_i^5})$$

$$W_{32l}=\sum_{i=1}^{5}\dot{w}_1(\frac{1}{R_i^3}+\frac{3x_2^2}{\dot{R}_i^5})$$

$$W_{33l}=\sum_{i=1}^{5}\dot{w}_3(\frac{1}{R_i^3}+\frac{3x_1^2}{\dot{R}_i^5})+\dot{w}_3(\frac{1}{R_i^3}+\frac{3x_2^2}{\dot{R}_i^5})+\hat{w}_3\frac{1}{R_i^3}$$

$$W_{34l}=\sum_{i=1}^{5}\dot{w}_4(\frac{1}{R_i^3}+\frac{3x_1^2}{\dot{R}_i^5})+\dot{w}_4(\frac{1}{R_i^3}+\frac{3x_2^2}{\dot{R}_i^5})+\hat{w}_4\frac{1}{R_i^3}$$

$$W_{35l}=\sum_{i=1}^{5}\dot{w}_5(\frac{1}{R_i^3}+\frac{3x_1^2}{\dot{R}_i^5})+\dot{w}_5(\frac{1}{R_i^3}+\frac{3x_2^2}{\dot{R}_i^5})+\hat{w}_5\frac{1}{R_i^3}$$

$$W_{36l}=0$$

其中：

$$\dot{w}_1=\dot{w}_2=A_i^P\{-c_{44}v_i+[c_{44}(v_i+\lambda_i^w)+e_{15}(v_i+\lambda_i^\varphi)+d_{15}(v_i+\lambda_i^v)]/2\}$$

$$\dot{w}_3=A_i^P c_{13}(v_i+\lambda_i^w)/2$$

$$\hat{w}_3=3A_i^P[e_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^\varphi)/2+d_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^v)/2+c_{33}v_i^2\lambda_i^w]$$

$$\dot{w}_4=A_i^P e_{31}(v_i+\lambda_i^w)/2$$

$$\hat{w}_4=3A_i^P[e_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^\varphi)/2-g_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^v)/2-\in_{33}\lambda_i^v v_i^2\lambda_i^w-\zeta_{33}v_i^2\lambda_i^v/2]$$

$$\dot{w}_5=A_i^P d_{31}(v_i+\lambda_i^w)/2$$

$$\hat{w}_5=3A_i^P[d_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^\varphi)/2-\mu_{33}v_i^2(\lambda_i^w v_i^2+\lambda_i^v)/2-g_{33}\lambda_i^v v_i^2\lambda_i^w-\eta_{33}v_i^2\lambda_i^v/2]$$

4.4 超奇异积分方程

4.4.1 数学模型

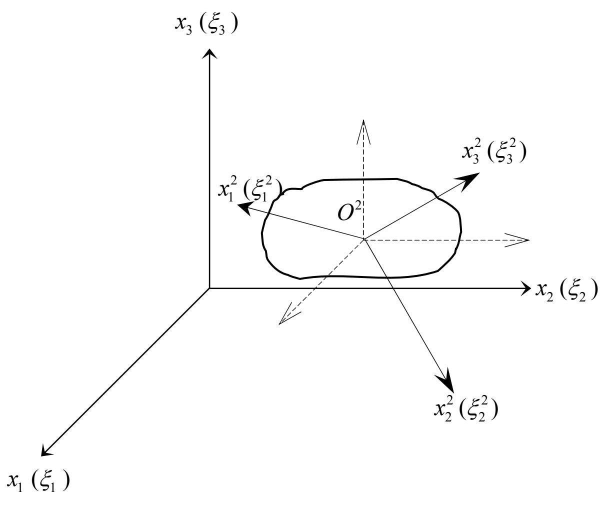


图 4-1 磁电热弹耦合材料内含任意形状和位置平片单裂纹示意图

如图 4-1 所示，假设磁电热弹耦合材料空间区域Ω中含有任意形状平片裂纹。选择整体固定坐标系为空间直角坐标系x_i (i=1,2,3)， 在裂纹面$S(S^+\bigcup S^-)$表面分别作用大小相等方向相反的载荷。

4.4.2 时域超奇异积分方程

应用粘弹塑性问题的边界法和时域主部分析法，经过复杂数学推导可得到磁电热弹耦合材料任意形状平片裂纹问题的时域超奇异积分方程组

$$\begin{aligned}&\oint_{S^+}[r^{-3}(c_{44}^2D_0v_0^2(\delta_{\bar{\alpha}\bar{\beta}}-3r_{,\bar{\alpha}}r_{,\bar{\beta}})+(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^5\rho_i^2t_i^2)\dot{\tilde{u}}_\beta(q)+3r^{-4}r_{,\alpha}\sum_{i=1}^5\lambda_{33}v_i^2t_i^1\dot{\tilde{u}}_6(q)]ds(q)\\&+\int_\Omega\sum_{i=1}^5\dot{w}_\alpha r^{-3}(1+3r_{,\alpha}^2)(\dot{\varepsilon}_{jK}^p+\dot{\varepsilon}_{jK}^c+\dot{\varepsilon}_{jK}^T+\dot{\varepsilon}_{jK}^M)dV(q)=-\dot{p}_\alpha(p)\end{aligned}\tag{4-19}$$

$$\begin{aligned}&\oint_{S^+}[r^{-2}r_{,\alpha}\sum_{i=1}^5A_i^\Upsilon t_i^2\rho_i^1\dot{\tilde{u}}_\alpha(q)+r^{-3}\sum_{n=3}^5\sum_{i=1}^5\rho_i^mt_i^t\dot{\tilde{u}}_n(q)+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^5v_i^2\lambda_i^\vartheta\rho_i^m\dot{\tilde{u}}_6(q)]ds(q)\\&+\int_\Omega\sum_{i=1}^5[r^{-3}[\dot{w}_m(\delta_{\alpha\alpha}+r_{,\alpha}^2)+\hat{w}_m](\dot{\varepsilon}_{jK}^p+\dot{\varepsilon}_{jK}^c+\dot{\varepsilon}_{jK}^T+\dot{\varepsilon}_{jK}^M)dV(q)=-\dot{p}_m(p)\end{aligned}\tag{4-20}$$

$$\oint_{S^+}[r^{-2}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})\sum_{i=1}^5A_i^\Upsilon\lambda_{3\beta}t_i^2\tilde{u}_\beta(q)+3r^{-4}\lambda_{3\alpha}r_{,\alpha}\sum_{i=1}^5A_i^\Upsilon v_i^2\lambda_i^\vartheta\lambda_{33}\rho_i^6\tilde{u}_6(q)]ds(q)=-p_6(p)\tag{4-21}$$

其中：

$$t_i^1=(c_{44}^2D_iv_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J)$$

$$t_i^2=c_{44}v_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi$$

$$t_i^3=-c_{13}+v_i(c_{33}\lambda_i^w+e_{33}\lambda_i^\phi+d_{33}\lambda_i^\varphi+\iota_{33}\lambda_i^\vartheta)$$

$$t_i^4=v_i(e_{33}\lambda_i^w-\in_{33}\lambda_i^\phi-g_{33}\lambda_i^\varphi+\varsigma_{33}\lambda_i^\vartheta)-e_{31}$$

$$t_i^5=v_i(d_{33}\lambda_i^w-g_{33}\lambda_i^\phi-\mu_{33}\lambda_i^\varphi+\eta_{33}\lambda_i^\vartheta)-d_{31}$$

$$\rho_i^2=c_{44}D_is_i+c_{44}A_i^P+e_{15}A_i^Q+d_{15}A_i^J$$

$$\rho_i^3=v_i(c_{33}A_i^P+e_{33}A_i^Q+d_{33}A_i^J)-D_ic_{13}$$

$$\rho_i^4=v_i(e_{33}A_i^P-\in_{33}A_i^Q-g_{33}A_i^J)-D_ie_{31}$$

$$\rho_i^5=v_i(d_{33}A_i^P-g_{33}A_i^Q-\mu_{33}A_i^J)-D_id_{31}$$

$$\rho_i^6 = \delta_{3m}\iota_{33} + \delta_{4m}\varsigma_{33} + \delta_{5m}\eta_{33} + \delta_{6m}$$

$$\rho_i^1 = \delta_{3m}\iota_{33} + \delta_{4m}\varsigma_{33} + \delta_{5m}\eta_{33}$$

4.4.3 广义奇性应力增量

裂纹前沿 $\hat{q}_0$ 点的广义位移间断应变率可表示为：

$$\dot{\tilde{U}}_{i,J} = g_k(q_0)\xi_2^{\lambda_k} \qquad 0 < \mathrm{Re}(\lambda_k) < 1 \tag{4-22}$$

应用广义主部分析法[189],可以得到如下关系式

$$\oint_{S_\varepsilon} r^{-3}\dot{\tilde{u}}_{i,1} d\xi_1\xi_2 \cong -2\pi\lambda_1 g_1 x_2^{\lambda_1-1}\cot(\lambda_1\pi) \tag{4-23}$$

$$\oint_{S_\varepsilon} 3r^{-5}(x_2-\xi_2)^2\dot{\tilde{u}}_{i,1} d\xi_1\xi_2 \cong -4\pi\lambda_1 g_1 x_2^{\lambda_1-1}\cot(\lambda_1\pi) \tag{4-24}$$

$$\oint_{S_\varepsilon} 3r^{-5}(x_1-\xi_1)^2\dot{\tilde{u}}_{i,1} d\xi_1\xi_2 \cong -2\pi\lambda_1 g_1 x_2^{\lambda_1-1}\cot(\lambda_1\pi) \tag{4-25}$$

$$\oint_{S_\varepsilon} r^{-3}(x_2-\xi_2)\dot{\tilde{u}}_{i,2} d\xi_1\xi_2 \cong 2 g_2 x_2^{\lambda_2}\cot(\lambda_2\pi) \tag{4-26}$$

裂纹前沿广义奇性应力增量可表示为：

$$\dot{\sigma}_{13} = -0.25 c_{44} v_0 g_1 (rr_0)^{-0.5}\cos\dot{\theta}_0 \tag{4-27}$$

$$\begin{aligned}\dot{\sigma}_{23} = {} & \sum_{i=1}^{5}\pi(rr_i)^{-0.5}(g_6\lambda_{33}(15\lambda_{33}v_i^2\cot\theta_i(3\cos 0.25\theta_i - 3\sin^2 0.5\theta_i\cos 0.5\theta_i - 0.25\sin^2\theta_i\cos 1.5\theta_i) - \\ & 2\lambda_{33}(rr_i)^{-2}\sin^{-1}0.5\theta_i + 0.5\lambda_{32}r^{-1}r_i^{-1}\sin^{-1}\theta_i(2\cos^{-1}0.5\theta_i - 3\cos 0.5\theta_i - \cos 2.5\theta_i)) + \rho_{i2}\pi\cos 0.5\theta_i \\ & (t_2 g_2 + 4\sum_{m=3}^{5}\hat{\rho}_{im}g_m) + \hat{w}_2\dot{\varepsilon}_{jK}^n\cos 0.5\theta_0)\end{aligned} \tag{4-28}$$

$$\begin{aligned}\bar{\dot{\sigma}}_{3n} = {} & \sum_{i=1}^{5}\rho_{in}v_i\pi\cot\theta_i((rr_i)^{-0.5}(\hat{\rho}_{i2}g_2\cos^{-1}\dot{\theta}_i + 6\cot\dot{\theta}_i\sum_{m=3}^{5}\hat{\rho}_{im}g_m) + g_6 v_i^2\kappa_{i5}\kappa_{i1}^{-1}(rr_i)^{0.5}\cos\dot{\theta}_i)) \\ & + 2\pi A_{i4}\iota_{33}((rr_i)^{0.5}\cos\dot{\theta}_i(g_2\cot\theta_i + \sum_{m=3}^{5}\hat{\rho}_{im}g_m) + (rr_i)^{-0.5}v_i^3\lambda_{33}\kappa_{i5}\kappa_{i1}^{-1}g_6\cos\dot{\theta}_i) + \\ & \sum_{i=1}^{5}\pi(rr_i)^{-0.5}\hat{w}_n\dot{\varepsilon}_{jK}^n\cos\dot{\theta}_0, n = 3-5\end{aligned} \tag{4-29}$$

$$\begin{aligned}\dot{\vartheta}_3 = {} & \sum_{i=1}^{5}A_{i4}\pi(rr_0)^{-0.5}(\cos\dot{\theta}_i(\lambda_{32} + 4\lambda_{33}v_i^2)\sum_{m=3}^{5}\hat{\rho}_{im}g_m + 4\lambda_{32}\hat{\rho}_{i2}g_2 v_i^3\kappa_{i5}\kappa_{i1}^{-1}(2/3\lambda_{32}\cos\dot{\theta}_i + \lambda_{33}(rr_0)^{-2} \\ & \sin^{-1}\dot{\theta}_i) + 15 rr_i g_6\lambda_{32}v_i^3\kappa_{i5}\kappa_{i1}^{-1}\cot\theta_i(3\cos 0.5\dot{\theta}_i - 6\cos\dot{\theta}_i\sin^2\dot{\theta}_i - 0.25\sin^2\theta_i\cos 3\dot{\theta}_i)(2v_i^2\lambda_{33} \\ & + \lambda_{32}(rr_0)^{-1}\sin^{-1}\theta_i))\end{aligned} \tag{4-30}$$

其中 $\bar{\dot{\sigma}}_{3n} = [\dot{\sigma}_{33} \quad \dot{D}_3 \quad \dot{B}_3]$.

4.4.4 粘弹塑性断裂参数

利用平衡方程、几何条件和边界条件，并考虑到体力、裂纹面上分布力的作用、温度场及温度对材料特性的影响，给出一种以增量形式的矢量路径无关积分 $\dot{J}_i^g$ 。由于是基于增量的本构关系，因而物体内各点的应力功密度增量不是应变增量的单值函数，而是显式地与物质点的坐标有关。$\dot{J}_i^g$ 对应于所作用外载和边界条件的增量，而非当前状态量，是一种回路-区域积分[306-311]，而非纯粹的回路积分，推导过程确保 $\dot{J}_i^g$ 是裂纹前沿附近应力，应变增量场的控制参量；并且，对于任意热弹塑性加载/卸载历程，皆能保持路径无关性。当对加载路径进行积分后，即对从初始至当前状态的所有增量步求和，$\sum \dot{J}_i^g$ 可以认为是材料特性常数。

本章给出的 $\dot{J}_i^g$ 适用于任意热弹塑性加载/卸载历程和非稳态蠕变的情况，当然也适用于扩展裂纹，并考虑到了热效应所导致的材料非均匀性，因而是一种合适的弹塑性-蠕变断裂参数。

$$\begin{aligned}\dot{J}_i^g &= \int_{S+S_{cs}}[\dot{W}n_i-(\dot{p}_ju_{j,i}+p_j\dot{u}_{j,i}+\dot{p}_j\dot{u}_{j,i})]dS-\int_{A_s}[\dot{W}_{,i}-(\sigma_{jk}+\dot{\sigma}_{jk})\dot{\varepsilon}_{jk,i}-\dot{\sigma}_{jk}\varepsilon_{jk,i}]dA\\&\quad-\int_{A_s}[(\dot{\sigma}_{j3}u_{j,i}+\sigma_{j3}\dot{u}_{j,i}+\dot{\sigma}_{j3}\dot{u}_{j,i})_{,3}+(f_j+\dot{f}_j)\dot{u}_{j,i}+\dot{f}_ju_{j,i}]dA\end{aligned} \tag{4-31}$$

其中 $\dot{W}\doteq\sigma_{ij}\dot{\varepsilon}_{ij}+0.5\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}$

4.5 数值方法

4.5.1 数值方法

广义增量位移间断应变率函数可表示为：

$$\dot{\hat{U}}_{i,J}(\xi_1,\xi_2,t)=F_{iJ}(\xi_1,\xi_2,t)\xi_2^{\lambda_J}W(\xi_1,\xi_2) \tag{4-32}$$

其中

$$W(\xi_1,\xi_2)=((a^2-\xi_1^2)(2b-\xi_2))^{0.5}$$

$$F_{iJ}(\xi_1,\xi_2,t)=\sum_{s=0}^{S}\sum_{h=0}^{H}a_{iJsh}(t)\xi_1^s\xi_2^h$$

$a_{iJmn}(t)$ 为未知常数。把上式代入超奇异积分方程，可得到以下代数方程组：

$$\sum_{s=0}^{S}\sum_{h=0}^{H}a_{lJsh}(t)I_{lJsh}^i(x_1,x_2)=-\dot{p}_J(x_1,x_2,t) \tag{4-33}$$

其中 i 为迭代次数，取值为 0～n。每次迭代时的系数 $I_{lJsh}^i(x_1,x_2)$ 具体表达式为：

$$I_{12sh}^i=-3\int_S r^{-3}(K_{11}+K_{12})r_{,1}r_{,2}\xi_1^s\xi_2^{\lambda_2+h}W\,d\xi_1\,d\xi_2 \tag{4-34}$$

$$I_{11sh}^i=\int_S r^{-3}(K_{11}(1-3r_{,2}^2)+K_{12}(1-3r_{,1}^2))\xi_1^s\xi_2^{\lambda_1+h}W\,d\xi_1\,d\xi_2 \tag{4-35}$$

$$I_{mnsh}^{i} = \int_S r^{-3} K_{mn} \xi_1^s \xi_2^{\lambda_n + h} W \, d\xi_1 \, d\xi_2 \tag{4-36}$$

$$I_{21sh}^{i} = -\int_S 3r^{-3} (K_{11} + K_{12}) r_{,1} r_{,2} \xi_1^s \xi_2^{\lambda_1 + h} W \, d\xi_1 \, d\xi_2 \tag{4-37}$$

$$I_{22sh}^{i} = \int_S r^{-3} (K_{11}(1 - 3r_{,1}^2) + K_{12}(1 - 3r_{,2}^2)) \xi_1^s \xi_2^{\lambda_2 + h} W \, d\xi_1 \, d\xi_2 \tag{4-38}$$

$$I_{61sh}^{i} = \int_S r^{-3} (\hat{K}_{11}(1 - 3r_{,1}^2) - \hat{K}_{12} r_{,1} r_{,2}) \xi_1^s \xi_2^{\lambda_1 + h} W \, d\xi_1 \, d\xi_2 \tag{4-39}$$

$$I_{66sh}^{i} = \int_S 3r^{-4} K_{66} \lambda_{3\alpha} r_{,\alpha} \xi_1^s \xi_2^{\lambda_6 + h} W \, d\xi_1 \, d\xi_2 \tag{4-40}$$

$$I_{m6sh}^{i} = \int_S 3r^{-4} \hat{K}_{m6} \lambda_{3\alpha} r_{,\alpha} \xi_1^s \xi_2^{\lambda_6 + h} W \, d\xi_1 \, d\xi_2 \tag{4-41}$$

$$I_{62sh}^{i} = \int_S r^{-3} (\hat{K}_{12}(1 - 3r_{,2}^2) - 3\hat{K}_{11} r_{,1} r_{,2}) \xi_1^s \xi_2^{\lambda_2 + h} W \, d\xi_1 \, d\xi_2 \tag{4-42}$$

$$I_{\alpha 6sh}^{i} = \int_S 3r^{-4} K_{\alpha 6} r_{,\alpha} \xi_1^s \xi_2^{\lambda_6 + h} W \, d\xi_1 \, d\xi_2 \tag{4-43}$$

$$I_{m\alpha sh}^{i} = \int_S r^{-2} K_{m1} r_{,\alpha} \xi_1^s \xi_2^{\lambda_\alpha + h} W \, d\xi_1 \, d\xi_2 \tag{4-44}$$

其它参数见本文附录 C。上面各超奇异积分必须在数值计算前进行处理。应用 Taylor’s 展开和极坐标 $\xi_1 - x_1 = r_1 \cos\theta_1$, $\xi_2 - x_2 = r_1 \sin\theta_1$ 可得到如下关系式：

$$\xi_1^s (a^2 - x_1^2)^{0.5} = \begin{cases} (a^2 - x_1^2)^{0.5} - x_1 r_1 \cos\theta_1 / (a^2 - x_1^2)^{0.5} - Q_1 r_1^2, & s = 0 \\ x_1^s (a^2 - x_1^2)^{0.5} + x_1^{s-1} (s a^2 - (s+1) x_1^2) r_1 \cos\theta_1 / (a^2 - x_1^2)^{0.5} + Q_2 r_1^2, & s > 0 \end{cases} \tag{4-45}$$

$$\xi_2^{\lambda_J + h} (2b - \xi_2)^{0.5} = x_2^{\lambda_J + h} (2b - x_2)^{0.5} + (x_2^{\lambda_J + h - 1} (4b(\lambda_J + h) - (2\lambda_J + 2h + 1) x_2) r_1 \sin\theta_1 + Q_3^J r_1^2) / 2(2b - x_2)^{0.5} \tag{4-46}$$

其中

$$Q_1 = a^2 (x_1 + \xi_1)((a^2 - x_1^2)^{0.5} ((a^2 - x_1^2)^{0.5} + (a^2 - \xi_1^2)^{0.5})(\xi_1 (a^2 - x_1^2)^{0.5} + x_1 (a^2 - \xi_1^2)^{0.5}))^{-1} \cos^2\theta_1 \tag{4-47}$$

$$Q_2 = \begin{cases} (s(s-1) a^4 x_1^{s-2} - (s^2 + 1) a^2 x_1^s + s(s+1) x_1^{s+2}) \cos^2\theta_1 / 2(a^2 - x_1^2)^{3/2}, & \xi_1 = x_1 \\ r_1^{-2} (\xi_1^s (a^2 - \xi_1^2)^{0.5} - x_1^s (a^2 - x_1^2)^{0.5} - x_1^{s-1} (s a^2 - (s+1) x_1^2)(\xi_1 - x_1) / (a^2 - x_1^2)^{0.5}), & \xi_1 \neq x_1 \end{cases} \tag{4-48}$$

$$Q_3^J = \begin{cases} 2(a^2 - x_1^2)^{-1.5} x_2^{\lambda_J + h - 2} \sin^2\theta_1 (16(\lambda_J + h)(\lambda_J + h - 1) b^2 - 8b(\lambda + n)(2\lambda_J + 2h - 1) x_2 + (4(\lambda_J + h)^2 - 1) x_2^2), & \xi_2 = x_2 \\ r_1^{-2} (\xi_2^{\lambda_J + h} (2b - \xi_2)^{0.5} - x_2^{\lambda_J + h} (2b - x_2)^{0.5} - (2b - x_2)^{0.5} x_2^{\lambda_J + h - 1} (2b(\lambda_J + h) - (\lambda_J + h + 0.5) x_2)(\xi_2 - x_2)), & \xi_2 \neq x_2 \end{cases} \tag{4-49}$$

利用关系式 (4-45)～（4−46),方程式(4-34)～（4−44）积分核函数可以表示为：

$$\xi_1^m \xi_2^{\lambda_J+h}((a^2-\xi_1^2)(2b-\xi_2))^{0.5} = D_0^J + D_1^J r_1 + D_2^J r_1^2 \tag{4-50}$$

其中 D_0^J, D_1^J 和 D_2^J 为已知函数，可以通过泰勒展开得到。应用上式，方程式(4-34)～（4-44）可以简化为以下各式

$$I_{11sh}^i = \int_0^{2\pi} (K_{11}(1-3\sin^2\theta) + K_{12}(1-3\cos^2\theta))(D_1^1 \ln R - R^{-1}D_0^1 + \int_0^R D_2^1 dr_1)d\theta \tag{4-51}$$

$$I_{12sh}^i = \int_0^{2\pi} 1.5(K_{11}+K_{12})\sin 2\theta (D_1^2 \ln R - R^{-1}D_0^2 + \int_0^R D_2^2 dr_1)d\theta \tag{4-52}$$

$$I_{21sh}^i = \int_0^{2\pi} 1.5(K_{11}+K_{12})\sin 2\theta (D_1^1 \ln R - R^{-1}D_0^1 + \int_0^R D_2^1 dr_1)d\theta \tag{4-53}$$

$$I_{22sh}^i = \int_0^{2\pi} (K_{11}(1-3\cos^2\theta) + K_{12}(1-3\sin^2\theta))(D_1^2 \ln R - R^{-1}D_0^2 + \int_0^R D_2^2 dr_1)d\theta \tag{4-54}$$

$$I_{61sh}^i = \int_0^{2\pi} (\hat{K}_{11}(1-3\cos^2\theta) - 0.5\hat{K}_{12}\sin 2\theta)(D_1^1 \ln R - R^{-1}D_0^1 + \int_0^R D_2^1 dr_1)d\theta \tag{4-55}$$

$$I_{\alpha 6sh}^i = \int_0^{2\pi} 3(K_{16}\cos\theta\delta_{1\alpha} + K_{26}\sin\theta\delta_{2\alpha})(D_1^6 \ln R - R^{-1}D_0^6 + \int_0^R D_2^6 dr_1)d\theta \tag{4-56}$$

$$I_{62sh}^i = \int_0^{2\pi} (\hat{K}_{12}(1-3\sin^2\theta) - 0.5\hat{K}_{11}\sin 2\theta)(D_1^2 \ln R - R^{-1}D_0^2 + \int_0^R D_2^2 dr_1)d\theta \tag{4-57}$$

$$I_{66sh}^i = \int_0^{2\pi} 3K_{66}(\lambda_{31}\cos\theta + \lambda_{32}\sin\theta)(D_1^6 \ln R - R^{-1}D_0^6 + \int_0^R D_2^6 dr_1)d\theta \tag{4-58}$$

$$I_{m6sh}^i = \int_0^{2\pi} 3\hat{K}_{m6}(\lambda_{31}\cos\theta + \lambda_{32}\sin\theta)(D_1^6 \ln R - R^{-1}D_0^6 + \int_0^R D_2^6 dr_1)d\theta \tag{4-59}$$

$$I_{m\alpha sh}^i = \int_0^{2\pi} K_{m1}\cos\theta(D_1^\alpha \ln R - R^{-1}D_0^\alpha + \int_0^R D_2^\alpha dr_1)d\theta \tag{4-60}$$

$$I_{mnsh}^i = \int_0^{2\pi} K_{mn}(D_1^n \ln R - R^{-1}D_0^n + \int_0^R D_2^n dr_1)d\theta \tag{4-61}$$

4.5.2 数值结果和讨论

为了检验上述数值方法的正确性，以磁电热弹耦合材料粘弹塑性三维矩形裂纹为例，用FORTRAN 语言编写典型算例的计算程序（7000 余条）[312]。如取 b/a=1,配置数为 20×20 ，多项式指数为 13×13 。裂纹前沿粘塑性应力余量函数随时间变化规律如图 4-2 所示. 从图中可以看出，粘塑性应力余量随蠕变时间增加而减小，这与文献[162, 313]中规律相吻合。

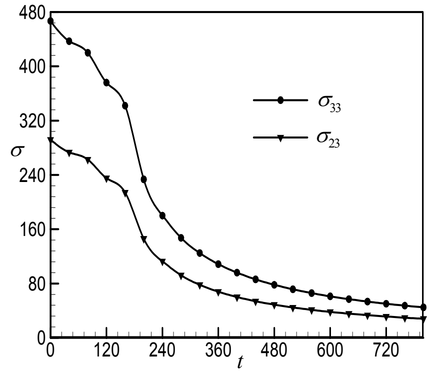


图 4-2 裂纹前沿粘塑性应力余量随蠕变时间变化规律

微小粘塑性区域内广义应力集中随r变化规律如图4-3所示，广义应力集中在一个很小距离内随r增加而增加，一旦超过该距离广义应力集中随r增加而迅速减小，减小的梯度不断降低，直至趋近于零。

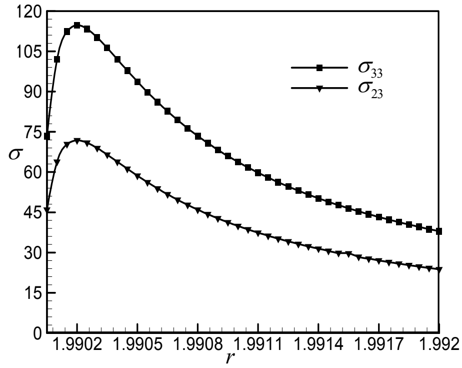


图 4-3 裂纹前沿塑性广义应力集中随 r 变化规律

图4-4显示广义位移增量函数随广义应变变化规律，变化规律与文献 [313, 314]中规律吻合良好。

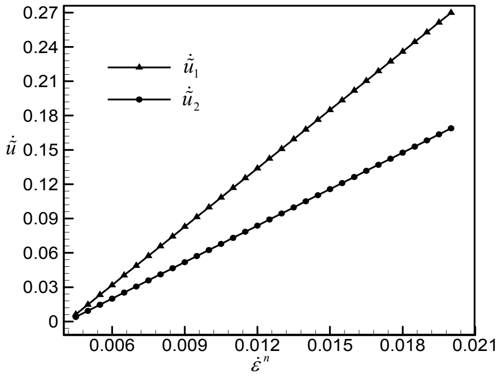


图 4-4 广义位移增量函数随广义应变变化规律

4.6 本章小节

1.通过将广义热弹塑—蠕变对控制微分方程和边界条件的非线性影响作为伪载荷处理，使得塑性项以伪体积力和伪面力形式出现，这样就可以采用三维磁电热弹耦合材料弹性问题的基本解作为弹塑性问题的基本解来求解三维弹塑性问题。应用主部分析法和广义位移增量基本解、广义时域超奇异积分方程方法，将磁电热弹粘塑性三维裂纹扩展问题转化为求解一组时域超奇异积分方程问题，得到广义应力增量场的解析表达式。

2.应用有限部积分方法，定义了三维裂纹前沿广义J积分；进而通过对时域超奇异积分过程中遇到的各类奇异积分分别进行数值处理，为该方程组建立数值求解方法，得到裂纹前沿广义增量位移的数值计算式。编写了专用FORTRAN语言程序，对典型裂纹问题进行了考核性计算；通过数值结果与文献[162, 313, 314]已有结果比较，验证了本数值方法的收敛性和精度。

3.工程中遇到的多场耦合作用下的材料裂纹扩展问题，可以应用本文提供的裂纹评定准则来分析和研究结构的可靠性。在应力集中粘塑性区域内，裂纹扩展对材料结构强度和可靠性的影响十分明显，在工程应用中应予以考虑

第五章 磁压电耦合双材料三维裂纹问题

5.1 引言

磁压电耦合双材料研究多限于基本理论分析，断裂和破坏方面数值结果相对较少，且已有研究成果多为平面问题。在二维磁压电裂纹方面，Tian [193] 给出了与界面平行裂纹的广义应力强度因子计算式； Sih [100, 107, 108]研究了 $BaTiO_3$—$CoFe_2O_4$ 材料裂纹问题；Feng [194]研究了裂纹扩展问题；Li [195] 与 Soh [196] 分析了 III 型界面裂纹问题；Zhou [197]分析了双界面裂纹问题；Hu [198]研究了界面裂纹扩展问题；Wang [199] 与 Liu [114] 分析了反平面界面裂纹问题；Chue [200]分析了 $BaTiO_3$—$CoFe_2O_4$ 材料界面裂纹问题；Jiang [131] 研究了内含裂纹问题；Zhao[201]分析了椭圆裂纹问题；但磁压电耦合双材料三维裂纹方面报道很少。

在本章中，首先根据文献Pan [138, 214, 215]和Ding [230]方法，应用张量变换，整理、推导、比较和验证了磁压电耦合双材料空间广义位移基本解，推导出适合于超奇异积分方程方法中使用的磁压电耦合材料三维广义位移基本解显式表达式，为应用超奇异积分方程方法分析磁压电耦合材料三维断裂力学问题奠定基础；以广义位移间断为未知函数，利用超奇异积分方程方法及主部分析法，将垂直于磁压电耦合双材料界面三维复合型裂纹问题转化为求解一组以裂纹表面位移间断为未知函数的超奇异积分方程组问题。通过退化得到磁压电耦合材料三维单裂纹问题的超奇异积分方程组；进而，采用二维超奇异积分的主部分析法得到了裂纹前沿的广义应力奇异指数。在此基础上，通过主部分析精确的求得了裂纹前沿光滑点附近的奇性应力场的解析表达式。并定义了广义应力强度因子、广义应变能强度因子和广义能量释放率；然后，通过将裂纹表面位移间断未知函数表达为广义位移间断基本密度函数与多项式之积，对超奇异积分方程组进行处理，使用有限部积分概念对超奇异积分方程组建立了数值方法，导出广义应力强度因子和广义应变能强度因子的数值计算公式；编制了专用的FORTRAN程序（7000条）。最后通过对典型裂纹问题的计算，考核了数值方法的正确性，得出广义应力强度因子和广义应变能强度因子随材料参数及裂纹几何形状的变化规律。

5.2 基本理论

5.2.1基本方程

为了表示简洁明确，并与前面几章所用符号脚标含义相区分，特对本章中符号和脚标做如下重新规定：张量符号脚标(除特别声明外)，小写英文字母表示变化范围为1～3，大写英文字母表示变化范围为1～5，下标逗号表示在直角坐标系下求偏导数。

磁压电耦合材料平衡方程具体表达式可表示为：

$$\sum_{iJ,i} + f_J = 0 \tag{5-1}$$

其中式（5-1）中符号 $\sum_{iJ}$ 和 f_J 分别表示广义应力位移和广义体力，其具体表达式可定义为：

$$\sum_{iJ} = \begin{cases} \sigma_{ij} & J = j = 1,2,3 \\ D_i & J = 4 \\ B_i & J = 5 \end{cases} \tag{5-2}$$

$$f_J = \begin{cases} f_j & J = j = 1,2,3 \\ -f_e & J = 4 \\ -f_m & J = 5 \end{cases} \tag{5-3}$$

磁压电耦合材料本构关系可表示为：

$$\sum_{iJ} = C_{iJKl} U_{K,l} \tag{5-4}$$

式（5-4）中符号 C_{iJKl} 和 U_K 分别表示广义磁压电系数和广义位移，其具体表达式可定义为：

$$C_{iJKl} = \begin{cases} c_{ijkl} & J,K = 1,2,3 \\ e_{lij} & J = 1,2,3; K = 4 \\ e_{ikl} & J = 4; K = 1,2,3 \\ d_{lij} & J = 1,2,3; K = 5 \\ d_{ikl} & J = 5; K = 1,2,3 \\ -g_{il} & J = 4; K = 5 or J = 5; K = 4 \\ -\in_{il} & J,K = 4 \\ -\mu_{il} & J,K = 5 \end{cases} \tag{5-5}$$

$$U_K = \begin{cases} u_k & K = 1,2,3 \\ \phi & K = 4 \\ \varphi & K = 5 \end{cases} \tag{5-6}$$

式（5-1）～（5-6）中其它符号的含义及其对表达式的更详细解释请参见本文主要符号表中说明，更详细的说明可考文献[210]。

5.2.2 边界积分方程

应用广义形式的Somigliana恒等式，磁压电耦合材料中 $\hat{p}$ 点的广义位移可表示为：

$$U_I = -\int_\Gamma (T_{IJ}U_J + U_{IJ}T_J)ds + \int_\Omega U_{IJ} f_J dv + \int_{S^+ + S^-} (U_{IJ}T_J - T_{IJ}U_J)ds \tag{5-7}$$

式（5-7）中符号Ω表示磁压弹耦合双材料空间，符号Γ表示广义边界，符号T_J表示边界上的广义力（弹性力，电荷强度和磁通量强度），符号U_{IJ}和T_{IJ}分别表示磁压电耦合双材料广义位移和面力基本解。

广义面力基本解与广义位移基本解之间关系可表示为：

$$T_{IJ}(\xi, x) = C_{kJMn}U_{IM,n}(\xi, x)n_k \tag{5-8}$$

式（5-8）中符号$\xi(\xi_1, \xi_2, \xi_2)$表示源点（参考点），符号$x(x_1, x_2, x_3)$表示场点（观测点），符号n_k表示边界处外法线向量的分量。边界上广义力T_J与广义应力间关系可表示为：

$$T_J = \begin{cases} p_j = \sigma_{jl} n_l \ J = j = 1,2,3 \\ q = D_l n_l \qquad J = 4 \\ b = B_l n_l \qquad J = 5 \end{cases} \tag{5-9}$$

式（5-9）中符号p_i, q和b分别表示由内或外载荷引起的裂纹面上的力载荷，电载荷和磁载荷。符号n_l表示边界处外法线向量的分量。

磁压电耦合双材料裂纹问题的广义位移间断$\tilde{U}_J$可定义为：

$$\tilde{U}_J = \begin{cases} \tilde{u}_j = u_j^+ - u_j^- \ J = j = 1,2,3 \\ \tilde{\phi}_j = \phi_j^+ - \phi_j^- \qquad J = 4 \\ \tilde{\varphi}_j = \varphi_j^+ - \varphi_j^- \qquad J = 5 \end{cases} \tag{5-10}$$

式（5-10）中符号$\tilde{u}_j$，$\tilde{\phi}_j$，$\tilde{\varphi}_j$分别表示为弹性位移间断、广义电势间断和广义磁势间断。为了后续计算和推导方便，设$\hat{p}(\xi_1, \xi_2, \xi_2)$，$\hat{q}(x_1, x_2, x_3)$分别代表源点和场点，广义位移基本解和广义面力基本解在场点和源点有以下关系。

$$T_{IJ}(\hat{p}, \hat{q}^+) = -T_{IJ}(\hat{p}, \hat{q}^-) = T_{IJ}^+(\hat{p}, \hat{q})$$

$$U_{IJ}(\hat{p}, \hat{q}^+) = U_{IJ}(\hat{p}, \hat{q}^-)$$

利用以上关系式，式（5-7）中广义弹性位移U_I可进一步简化为：

$$U_I = \int_\Gamma (U_{IJ}T_J - T_{IJ}U_J)ds - \int_{S^+} T_{IJ}^+ \tilde{U}_J ds + \int_\Omega U_{IJ} f_J dv \tag{5-11}$$

应用本构关系（5-4）和基本解，广义应力$\sum_{ij}$可表示为：

$$\sum_{ij} = \int_\Gamma (D_{KiJ}T_K - S_{KiJ}U_K)ds - \int_{S^+} S_{KiJ}^+ \tilde{U}_K ds + \int_\Omega D_{KiJ} f_K dv \tag{5-12}$$

其中积分核函数S_{KiJ}和D_{KiJ}可代表裂纹边界和空间区域的影响，其具体表达式为：

$$S_{KiJ}(\hat{p},\hat{q})=-C_{iJMn}T_{MK,n}(\hat{p},\hat{q})$$

$$D_{KiJ}(\hat{p},\hat{q})=-C_{iJMn}U_{MK,n}(\hat{p},\hat{q})$$

对于磁压电耦合双材料中垂直于界面的三维混合型裂纹问题，有两种边界条件，包括外部边界Γ和裂纹面边界$S^{\pm}$。应用边界条件，可以得到边界积分方程和超奇异积分方程。让源点$\hat{p}$靠近边界Γ，当到达边界时用点$\hat{P}$表示，此时边界积分方程可以表示为：

$$C_{IJ}U_I=\int_{\Gamma}(U_{IJ}T_J-T_{IJ}U_J)ds+\int_{\Omega}U_{IJ}f_Jdv+\int_{S^+}T_{IJ}^{+}\tilde{U}_Jds \tag{5-13}$$

C_{IJ}为与边界点$\hat{P}$相关常数，应用裂纹面上力、电和磁边界条件，超奇异积分方程可表示为：

$$\oint_{S^+}S_{KiJ}^{+}\tilde{U}_Kds=\int_{\Gamma}(D_{KiJ}T_K-S_{KiJ}U_K)ds+\int_{\Omega}D_{KiJ}f_Kdv \tag{5-14}$$

其中$\oint$必须通过有限部积分方法才能求解，上式中第一项具有r^{-3}奇异性。式（5-14）为有限体裂纹问题的完整边界积分方程；$\int_{\Gamma}(D_{KiJ}T_K-S_{KiJ}U_K)ds$是反映空间边界影响的积分项，是通常意义下Cauchy主值积分，奇异性较小，数值求解容易；$\oint_{S^+}S_{KiJ}^{+}\tilde{U}_Kds$是反应裂纹边界的积分项，为超奇异积分，具有高奇异性，不能应用通常求解方法进行求解，是研究重点。实际工程应用中，只要空间无限大体裂纹问题解决后，对于有限体裂纹问题可应用叠加法求解。

5.3 广义基本解

5.3.1广义位移基本解

在本章中, C_{iJKl}和C'_{iJKl}分别代表磁压电耦合材料I和II的广义磁压电常数。 对于磁压电材料I, 弹性系数c_{ijkl}、压电系数e_{lij}和介电系数$\in_{il}$具体表达式分别定义如下：

$$\begin{aligned}c_{ijkl}=&(c_{13}-c_{12})(\delta_{ij}\delta_{3k}\delta_{3l}+\delta_{3i}\delta_{3j}\delta_{kl})+(c_{44}-c_{66})(\delta_{jk}\delta_{3i}\delta_{3l}+\delta_{ik}\delta_{3j}\delta_{3l}+\delta_{il}\delta_{3j}\delta_{3k}+\delta_{jl}\delta_{3i}\delta_{3k})\\&+(c_{11}+c_{33}-2c_{13}-4c_{44})\delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l}+c_{12}\delta_{ij}\delta_{kl}+c_{66}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\end{aligned} \tag{5-15}$$

$$e_{lij}=e_{31}\delta_{ij}\delta_{3l}+e_{15}(\delta_{il}\delta_{3j}+\delta_{jl}\delta_{3i})+(e_{33}-e_{31}-2e_{15})\delta_{3i}\delta_{3j}\delta_{3l} \tag{5-16}$$

$$\in_{il}=\in_{11}\delta_{il}+(\in_{33}-\in_{11})\delta_{3i}\delta_{3l} \tag{5-17}$$

磁电系数d_{lij}和压电系数e_{lij}具有相同的结构,磁性系数μ_{il}、电磁系数g_{il}和介电系数$\in_{il}$具有相同的结构，读者可自行推导得出，$c_{66}=(c_{11}-c_{12})/2$。应用调和函数镜像法，把磁压电耦合双材料各自满足$lame'-navier$平衡方程的$Papkovich-Neuber$通解及$Kelvin$特解，代入Hooke定律及界面处广义位移边界条件中，得出在空间内部作用有集中力的空间弹性力学位移基本解。应用文献[214, 230, 315]中所提供的方法，即可得到磁压电耦合双材料的Green函数具体表达式。控制方程可表示为：

$$
\begin{aligned}
U_J=&[(n_{41}\frac{\partial^6}{\partial x_3^6}+n_{51}\Delta\frac{\partial^4}{\partial x_3^4}+n_{61}\Delta^2\frac{\partial^2}{\partial x_3^2}+n_{71}\Delta^3)\frac{\partial\widehat{g}}{\partial x_\alpha}-\frac{\partial\widehat{f}}{\partial x_{\dot\alpha}}]\delta_{\alpha J}+\delta_{mJ}(n_{4m}\frac{\partial^6}{\partial x_3^2}+n_{5m}\Delta\frac{\partial^4}{\partial x_3^4}+n_{6m}\Delta^2\frac{\partial^2}{\partial x_3^2}\\
&+n_{7m}\Delta^3)\frac{\partial\widehat{g}}{\partial x_3}+\delta_{6J}(n_1\frac{\partial^8}{\partial x_3^8}+n_2\Delta\frac{\partial^6}{\partial x_3^6}+n_3\Delta^2\frac{\partial^4}{\partial x_3^4}+n_4\Delta^3\frac{\partial^2}{\partial x_3^2}+n_5\Delta^4)\widehat{g}\ m=3,4,5
\end{aligned}
\quad (5\text{-}18)
$$

势函数$\widehat{g}$和$\widehat{f}$必须满足如下方程：

$$
\prod_{n=2}^{5}(\Delta+\frac{\partial^2}{\partial z_n^2})\widehat{g}=0 \quad (5\text{-}19)
$$

$$
(\Delta+\frac{\partial^2}{\partial z_1^2})\widehat{f}=0 \quad (5\text{-}20)
$$

其中$s_0=(c_{66}/c_{44})^{0.5}$, s_i (i=1，2，3，4) 为方程$a_1s^8-a_2s^6+a_3s^4-a_4s^2+a_5=0$的四个特征根。如果能得到在点力、点电荷和磁流共同作用下上述方程的解，则可获得广义位移u_i,ϕ和φ的基本解。根据 Pan [138, 214, 215]和 Ding [230]方法，应用张量变换规则，垂直于磁压电耦合双材料界面三维裂纹面坐标系下的广义位移基本解的具体表达式可表示为：

1.点力P_z，点电荷Q和磁流J共同作用

在点$\hat{p}(\xi_1,\xi_2,\xi_3)$作用$P_z,Q$和$J$,点$\hat{q}(x_1,x_2,x_3)$的广义位移可表示为：

$$
U_{mj}=A_j\sum_{i=1}^{5}(R_{1i}^{-1}\dot{R}_i^{-1}(x_\alpha-\xi_\alpha)\delta_{\alpha j}+\dot{\kappa}_i^{-1}R_{2i}^{-1}s_i^{-1}\sum_{k=1}^{3}\kappa_{ki}\delta_{kj})/4\pi \quad (5\text{-}21)
$$

其中

$$
R_{1i}=((x_1-\xi_1)^2+(x_2-\xi_2)^2+z_i{}^2)^{0.5}
$$

$$
R_{2i}=((x_1-\xi_1)^2+(x_2+\xi_2)^2+z_i{}^2)^{0.5}
$$

$$
A_i=W^{-1}[1\quad 1\quad -1\quad -1]^T+\sum_{k=1}^{5}(V^{-1}BW^{-1}[1\quad 1\quad -1\quad -1]^T)
$$

$$
\dot{R}_i=R_{2i}+z_i
$$

其它参数见本章附录 D。

2 在 x_2 方向作用点力

在点 $\hat{p}(\xi_1,\xi_2,\xi_3)$ 作用 x_2 方向单位点力,点 $\hat{q}(x_1,x_2,x_3)$ 的广义位移可表示为：

$$\begin{aligned}U_{2j}&=(\delta_{1j}(R_{1j}^{-1}\dot{R}_j^{-2}D_0+\sum_{i=1}^{5}R_{2i}^{-1}\dot{R}_i^{-2}D_i)\prod_{\alpha=1}^{2}(x_\alpha-\xi_\alpha)+\dot{\kappa}_j^{-1}R_{2j}^{-1}\dot{R}_j^{-2}D_j(\xi_2-x_2)\sum_{i=1}^{5}\sum_{k=1}^{3}\kappa_{ki}\delta_{kj}\\&-\delta_{2j}(\dot{R}_j^{-1}D_0+R_{1j}^{-1}\dot{R}_j^{-2}D_0(x_1-\xi_1)^2-\sum_{i=1}^{5}(\dot{R}_i^{-1}D_i(x_2-\xi_2)^2D_i-R_{2i}^{-1}\dot{R}_i^{-2}D_i(x_2-\xi_2)^2))/4\pi c_{44}\end{aligned} \tag{5-22}$$

其中

$$D_0=1-(v_1c_{44}-v_1'c_{44}')/(v_1c_{44}+v_1'c_{44}')$$

$$D_i=\dot{V}^{-1}[0\quad 0\quad 0\quad 1]^T+\sum_{j=1}^{5}(V^{-1}B(\dot{V}^{-1}[0\quad 0\quad 0\quad 1]^T)$$

其它参数见本文附录 D

5.3.2广义面力基本解

对位移场基本解公式(5-21) ～（5-22）应用弹性基本理论、边界条件，经过推导简化，可得到磁压电耦合材料三维裂纹面坐标系下面力基本解，其具体表达式如下：

$$\begin{aligned}T_{11}&=-c_{44}D_0s_0(R_0^{-1}\hat{R}_0^{-1}-R_0^{-3}\hat{R}_0^{-1}X_2{}^2-R_0^{-2}\hat{R}_0^{-2}X_2{}^2)+\sum_{i=1}^{5}D_i[c_{44}s_i\\&+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi](R_i^{-1}\hat{R}_i^{-1}-R_i^{-3}\hat{R}_i^{-1}X_1{}^2-R_i^{-2}\hat{R}_i^{-2}X_1{}^2)\end{aligned} \tag{5-23}$$

$$T_{21}=-c_{44}D_0s_0X_1X_2(R_0^{-3}\hat{R}_0^{-1}+R_0^{-2}\hat{R}_0^{2})-\sum_{i=1}^{5}D_iX_1X_2[c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi](R_i^{-3}\hat{R}_i^{-1}+R_i^{-2}\hat{R}_i^{-2}) \tag{5-24}$$

$$T_{31}=-\sum_{i=1}^{5}A_i^P[c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi]R_i^{-3}X_1 \tag{5-25}$$

$$T_{41}=-\sum_{i=1}^{5}A_i^Q[c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi]R_i^{-3}X_1 \tag{5-26}$$

$$T_{51}=-\sum_{i=1}^{5}A_i^J[c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi]R_i^{-3}X_1 \tag{5-27}$$

$$T_{12}=-c_{44}D_0s_0X_1X_2(R_0^{-3}\hat{R}_0^{-1}+R_0^{-2}\hat{R}_0^{2})-\sum_{i=1}^{5}D_iX_1X_2[c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi](R_i^{-3}\hat{R}_i^{-1}+R_i^{-2}\hat{R}_i^{-2}) \tag{5-28}$$

$$\begin{aligned}T_{22}&=-c_{44}D_0s_0(R_0^{-1}\hat{R}_0^{-1}-R_0^{-3}\hat{R}_0^{-1}X_1{}^2-R_0^{-2}\hat{R}_0^{-2}X_1{}^2)+\sum_{i=1}^{5}D_i[c_{44}s_i\\&+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi](R_i^{-1}\hat{R}_i^{-1}-R_i^{-3}\hat{R}_i^{-1}X_2{}^2-R_i^{-2}\hat{R}_i^{-2}X_2{}^2)\end{aligned} \tag{5-29}$$

$$T_{32} = -\sum_{i=1}^{5} A_i^P [c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_i^{-3} X_2 \tag{5-30}$$

$$T_{42} = -\sum_{i=1}^{5} A_i^Q [c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_i^{-3} X_2 \tag{5-31}$$

$$T_{52} = -\sum_{i=1}^{5} A_i^J [c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_i^{-3} X_2 \tag{5-32}$$

$$T_{13} = \sum_{i=1}^{5} D_i [c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_i^{-3} X_1 \tag{5-33}$$

$$T_{23} = \sum_{i=1}^{5} D_i [c_{44}s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi] R_i^{-3} X_2 \tag{5-34}$$

$$T_{33} = -\sum_{i=1}^{5} A_i^P s_i [-c_{13} + c_{33}s_i \lambda_i^w + e_{33}s_i \lambda_i^\phi + d_{33}s_i \lambda_i^\varphi] R_i^{-3} X_3 \tag{5-35}$$

$$T_{43} = -\sum_{i=1}^{5} A_i^Q s_i [-c_{13} + c_{33}s_i \lambda_i^w + e_{33}s_i \lambda_i^\phi + d_{33}s_i \lambda_i^\varphi] R_i^{-3} X_3 \tag{5-36}$$

$$T_{53} = -\sum_{i=1}^{5} A_i^J s_i [-c_{13} + c_{33}s_i \lambda_i^w + e_{33}s_i \lambda_i^\phi + d_{33}s_i \lambda_i^\varphi] R_i^{-3} X_3 \tag{5-37}$$

$$T_{14} = -\sum_{i=1}^{5} D_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi] R_i^{-3} X_1 \tag{5-38}$$

$$T_{24} = -\sum_{i=1}^{5} D_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi] R_i^{-3} X_2 \tag{5-39}$$

$$T_{34} = -\sum_{i=1}^{5} D_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi] R_i^{-3} X_3 \tag{5-40}$$

$$T_{44} = -\sum_{i=1}^{5} A_i^P s_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi] R_i^{-3} X_3 \tag{5-41}$$

$$T_{54} = -\sum_{i=1}^{5} A_i^Q s_i [-e_{31} + e_{33}s_i \lambda_i^w - \in_{33} s_i \lambda_i^\phi - g_{33}s_i \lambda_i^\varphi] R_i^{-3} X_3 \tag{5-42}$$

$$T_{15} = -\sum_{i=1}^{5} D_i s_i [-d_{31} + d_{33}s_i \lambda_i^w - g_{33}s_i \lambda_i^\phi - \mu_{33}s_i \lambda_i^\varphi] R_i^{-3} X_1 \tag{5-43}$$

$$T_{25} = -\sum_{i=1}^{5} D_i s_i [-d_{31} + d_{33}s_i \lambda_i^w - g_{33}s_i \lambda_i^\phi - \mu_{33}s_i \lambda_i^\varphi] R_i^{-3} X_2 \tag{5-44}$$

$$T_{35} = -\sum_{i=1}^{5} A_i^P s_i [-d_{31} + d_{33}s_i \lambda_i^w - g_{33}s_i \lambda_i^\phi - \mu_{33}s_i \lambda_i^\varphi] R_i^{-3} X_3 \tag{5-45}$$

$$T_{45} = -\sum_{i=1}^{5} A_i^Q s_i [-d_{31} + d_{33}s_i \lambda_i^w - g_{33}s_i \lambda_i^\phi - \mu_{33}s_i \lambda_i^\varphi] R_i^{-3} X_3 \tag{5-46}$$

$$T_{55} = -\sum_{i=1}^{5} A_i^J s_i [-d_{31} + d_{33} s_i \lambda_i^w - g_{33} s_i \lambda_i^\phi - \mu_{33} s_i \lambda_i^\varphi] R_i^{-3} X_3 \tag{5-47}$$

T_{ij}^I 的第一个下标 i 表示 T_{ij}^I 是 x_i 坐标方向的面力分量，第二个下标 j 则表示作用于场点 $\hat{q}$ 的单位集中力指向 ξ_j 坐标方向；ξ_i 表示为源点 $\hat{p}$ 坐标，x_i 表示为场点 $\hat{q}$ 坐标。

5.3.3广义应力积分核函数

通过对公式(5-23) ～（5-47）的广义面力场基本解进行求导，再应用弹性理论及边界条件，经过繁杂的推导，简化后可得到与界面垂直裂纹面坐标系上的应力积分核 S_{KIJ} 公式，其具体表达式如下：

$$\begin{aligned} S_{131} = &\sum_{i=1}^{5} (c_{44} s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi)(c_{44} D_i s_i + c_{44} A_i^P + e_{15} A_i^Q + d_{15} A_i^J) R_i^{-3} (1 - 3X_1^2 R_i^2) \\ &- c_{44}^2 D_0 s_0^2 R_0^{-3} (1 - 3X_2^2 R_0^2) \end{aligned} \tag{5-48}$$

$$\begin{aligned} S_{231} = &-\sum_{i=1}^{5} 3(c_{44} s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi)(c_{44} D_i s_i + c_{44} A_i^P + e_{15} A_i^Q + d_{15} A_i^J) R_i^{-5} X_1 X_2 \\ &- 3c_{44}^2 D_0 s_0^2 R_0^{-5} X_1 X_2 \end{aligned} \tag{5-49}$$

$$S_{331} = -\sum_{i=1}^{5} 3(-c_{13} + c_{33}\lambda_i^w s_i + e_{33}\lambda_i^\phi s_i + d_{33}\lambda_i^\varphi s_i)(c_{44} D_i s_i + c_{44} A_i^P + e_{15} A_i^Q + d_{15} A_i^J) R_i^{-5} X_1 X_3 \tag{5-50}$$

$$S_{431} = -\sum_{i=1}^{5} 3(-e_{31} + e_{33}\lambda_i^w s_i - \in_{33} \lambda_i^\phi s_i - g_{33}\lambda_i^\varphi s_i)(c_{44} D_i s_i + c_{44} A_i^P + e_{15} A_i^Q + d_{15} A_i^J) R_i^{-5} X_1 X_3 \tag{5-51}$$

$$S_{531} = -\sum_{i=1}^{5} 3(-d_{31} + d_{33}\lambda_i^w s_i - g_{33}\lambda_i^\phi s_i - \mu_{33}\lambda_i^\varphi s_i)(c_{44} D_i s_i + c_{44} A_i^P + e_{15} A_i^Q + d_{15} A_i^J) R_i^{-5} X_1 X_3 \tag{5-52}$$

$$\begin{aligned} S_{132} = &\sum_{i=1}^{5} 3(c_{44} s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi)(c_{44} D_i s_i + c_{44} A_i^P + e_{15} A_i^Q + d_{15} A_i^J) R_i^{-5} X_1 X_2 \\ &+ c_{44}^2 D_0 s_0^2 R_0^{-5} X_1 X_2 \end{aligned} \tag{5-53}$$

$$\begin{aligned} S_{232} = &-\sum_{i=1}^{5} (c_{44} s_i + c_{44}\lambda_i^w + e_{15}\lambda_i^\phi + d_{15}\lambda_i^\varphi)(c_{44} D_i s_i + c_{44} A_i^P + e_{15} A_i^Q + d_{15} A_i^J) R_i^{-3} (1 - 3X_2^2 R_i^2) \\ &- c_{44}^2 D_0 s_0^2 R_0^{-3} (1 - 3X_1^2 R_0^2) \end{aligned} \tag{5-54}$$

$$S_{332} = -\sum_{i=1}^{5} 3(-c_{13} + c_{33}\lambda_i^w s_i + e_{33}\lambda_i^\phi s_i + d_{33}\lambda_i^\varphi s_i)(c_{44} D_i s_i + c_{44} A_i^P + e_{15} A_i^Q + d_{15} A_i^J) R_i^{-5} X_2 X_3 \tag{5-55}$$

$$S_{432} = -\sum_{i=1}^{5} 3(-e_{31} + e_{33}\lambda_i^w s_i - \in_{33} \lambda_i^\phi s_i - g_{33}\lambda_i^\varphi s_i)(c_{44} D_i s_i + c_{44} A_i^P + e_{15} A_i^Q + d_{15} A_i^J) R_i^{-5} X_2 X_3 \tag{5-56}$$

$$S_{532} = -\sum_{i=1}^{5} 3(-d_{31} + d_{33}\lambda_i^w s_i - g_{33}\lambda_i^\phi s_i - \mu_{33}\lambda_i^\varphi s_i)(c_{44} D_i s_i + c_{44} A_i^P + e_{15} A_i^Q + d_{15} A_i^J) R_i^{-5} X_2 X_3 \tag{5-57}$$

$$S_{133}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_ic_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-5}X_1X_3 \tag{5-58}$$

$$S_{233}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_ic_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-5}X_2X_3 \tag{5-59}$$

$$S_{333}=\sum_{i=1}^{5}s_i(-c_{13}+c_{33}\lambda_i^ws_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i)(-D_ic_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-3} \tag{5-60}$$

$$S_{433}=\sum_{i=1}^{5}s_i(-e_{31}+e_{33}\lambda_i^ws_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i)(-D_ic_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-3} \tag{5-61}$$

$$S_{533}=\sum_{i=1}^{5}s_i(-d_{31}+d_{33}\lambda_i^ws_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i)(-D_ic_{13}+c_{33}A_i^Ps_i+e_{33}A_i^Qs_i+d_{33}A_i^Js_i)R_i^{-3} \tag{5-62}$$

$$S_{134}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_ie_{31}+e_{33}A_i^Ps_i-\in_{33}A_i^Qs_i-g_{33}A_i^Js_i)R_i^{-5}X_1X_3 \tag{5-63}$$

$$S_{234}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_ie_{31}+e_{33}A_i^Ps_i-\in_{33}A_i^Qs_i-g_{33}A_i^Js_i)R_i^{-5}X_2X_3 \tag{5-64}$$

$$S_{334}=\sum_{i=1}^{5}s_i(-c_{13}+c_{33}\lambda_i^ws_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i)(-D_ie_{31}+e_{33}A_i^Ps_i-\in_{33}A_i^Qs_i-g_{33}A_i^Js_i)R_i^{-3} \tag{5-65}$$

$$S_{434}=\sum_{i=1}^{5}s_i(-e_{31}+e_{33}\lambda_i^ws_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i)(-D_ie_{31}+e_{33}A_i^Ps_i-\in_{33}A_i^Qs_i-g_{33}A_i^Js_i)R_i^{-3} \tag{5-66}$$

$$S_{534}=\sum_{i=1}^{5}s_i(-d_{31}+d_{33}\lambda_i^ws_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i)(-D_ie_{31}+e_{33}A_i^Ps_i-\in_{33}A_i^Qs_i-g_{33}A_i^Js_i)R_i^{-3} \tag{5-67}$$

$$S_{135}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_id_{31}+d_{33}A_i^Ps_i-g_{33}A_i^Qs_i-\mu_{33}A_i^Js_i)R_i^{-5}X_1X_3 \tag{5-68}$$

$$S_{235}=\sum_{i=1}^{5}3s_i(c_{44}s_i+c_{44}\lambda_i^w+e_{15}\lambda_i^\phi+d_{15}\lambda_i^\varphi)(-D_id_{31}+d_{33}A_i^Ps_i-g_{33}A_i^Qs_i-\mu_{33}A_i^Js_i)R_i^{-5}X_2X_3 \tag{5-69}$$

$$S_{335}=\sum_{i=1}^{5}s_i(-c_{13}+c_{33}\lambda_i^ws_i+e_{33}\lambda_i^\phi s_i+d_{33}\lambda_i^\varphi s_i)(-D_id_{31}+d_{33}A_i^Ps_i-g_{33}A_i^Qs_i-\mu_{33}A_i^Js_i)R_i^{-3} \tag{5-70}$$

$$S_{435}=\sum_{i=1}^{5}s_i(-e_{31}+e_{33}\lambda_i^ws_i-\in_{33}\lambda_i^\phi s_i-g_{33}\lambda_i^\varphi s_i)(-D_id_{31}+d_{33}A_i^Ps_i-g_{33}A_i^Qs_i-\mu_{33}A_i^Js_i)R_i^{-3} \tag{5-71}$$

$$S_{535}=\sum_{i=1}^{5}s_i(-d_{31}+d_{33}\lambda_i^ws_i-g_{33}\lambda_i^\phi s_i-\mu_{33}\lambda_i^\varphi s_i)(-D_id_{31}+d_{33}A_i^Ps_i-g_{33}A_i^Qs_i-\mu_{33}A_i^Js_i)R_i^{-3} \tag{5-72}$$

从S_{KIJ}的具体表达公式可见，公式中出现了$\dfrac{1}{r^n}$(n=2,3,4,5,7,9)的因子，当内点靠近边界时，在靠近的边界上被积函数变化剧烈，数值积分时应考虑其对精度的影响，

5.4 超奇异积分方程

5.4.1 数学模型

如图 5-1 所示，考虑一由两半无限磁压电体结合构成的双材料弹性体，界面位于直角坐标系 x_i (i=1,2,3)的 ox_1x_3 平面内；左半空间区域用 II 表示，右半空间区域用 I 表示，空间区域 I 和 II 的材料常数分别用广义弹性系数 C_{iJKl} 和广义弹性系数 C'_{iJKl}。S 表示裂纹表面， 根据[138, 214, 215, 230]中法和结果进行整理和简化，可得到无限空间磁压电耦合双材料广义位移基本解的具体表达式：

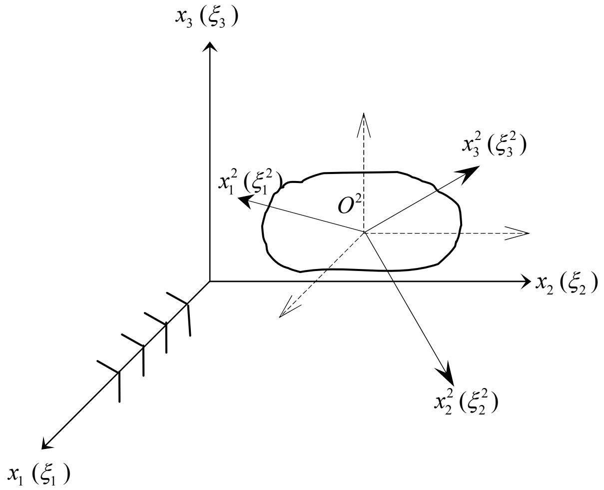


图 5-1 磁压电耦合双材料与界面垂直任意形状平片裂纹问题示意图

5.4.2 超奇异积分方程

设在无限远处受广义载荷 $\tau_{zx}^{\infty}, \tau_{zy}^{\infty}, \sigma_{zz}^{\infty}, Q_z^{\infty}, J_z^{\infty}$ 作用。由式(5-14)及本构关系，并利用裂纹面上边界条件及超奇异积分方程方法，通过繁杂的数学推导，即可得到垂直于磁压电耦合双材料界面的三维混合型裂纹问题的超奇异积分方程组，其具体表达式为：

$$\oint_{S^+}(r_{1\alpha}^{-3}((c_{44}s_1(3r_{1\alpha,\alpha}r_{1\alpha,\beta}-2\delta_{\alpha\beta})/4\pi+\sum_{i=1}^{5}D_i\rho_{i2}\hat{\rho}_{i2}(\delta_{\alpha\beta}-3r_{1\alpha,\alpha}r_{1\alpha,\beta}))+K_{\alpha\beta}(p,q))\tilde{u}_\beta(q)ds(q)=-p_\alpha(p) \quad (5\text{-}73)$$

$$\oint_{S^+}(r_{1m}^{-3}\sum_{i=3}^{5}\sum_{n=3}^{5}D_i\rho_{im}\hat{\rho}_{in}+K_{mn}(p,q))\tilde{u}_n(q)ds(q)=-p_m(p) \quad (5\text{-}74)$$

其中

$$r_{1i}=((x_1-\xi_1)^2+(x_2-\xi_2)^2+z_i^{\,2})^{0.5}$$

$r_{2i}=((x_1-\xi_1)^2+(x_2+\xi_2)^2+z_i{}^2)^{0.5}$

$\dot{r}_i=r_{2i}+z_i$

与材料性质有关的系数 $\hat{\rho}_{iJ}$, ρ_{iJ} 可表示为：

$$\hat{\rho}_{iJ}=\delta_{J2}(c_{44}s_i^{-1}+\dot{\kappa}_i^{-1}(c_{44}\kappa_{1i}+e_{15}\kappa_{2i}+d_{15}\kappa_{3i}))+\delta_{J3}(\dot{\kappa}_i^{-1}s_i(c_{33}\kappa_{1i}+e_{33}\kappa_{2i}+d_{33}\kappa_{3i})-c_{13})+\delta_{J4}(\dot{\kappa}_i^{-1}s_i(e_{33}\kappa_{1i}-\in_{33}\kappa_{2i}-g_{33}\kappa_{3i})-e_{31})+\delta_{J5}(\dot{\kappa}_i^{-1}s_i(d_{33}\kappa_{1i}-g_{33}\kappa_{2i}-\mu_{33}\kappa_{3i})-d_{31})$$

$$\rho_{iJ}=\delta_{2J}(c_{44}s_i^{-1}+\dot{\kappa}_i^{-1}(c_{44}D_is_i+c_{44}A_{i1}+e_{15}A_{i2}+d_{15}A_{i3})+\delta_{3J}(s_i(c_{33}A_{i1}+e_{33}A_{i2}+d_{33}A_{i2})-D_ic_{13})+\delta_{4J}(v_i(e_{33}A_{i1}-\in_{33}A_{i2}-g_{33}A_{i3})-D_ie_{31})-e_{31})+\delta_{5J}(s_i(d_{33}A_{i1}-g_{33}A_{i2}-\mu_{33}A_{i3})-D_id_{31})$$

从方程组(5-73) ～（5-74）可以看出，磁电热弹耦合双材料三维裂纹问题和文Qin[187, 188]中问题的超奇异积分方程组具有相似的结构。将上式退化：在不考虑磁、电中一种或两种因素影响(去除核函数 $\hat{\rho}_{iJ}$， ρ_{iJ} 中相应的项)，便可得到压磁、压电等各类材料三维裂纹问题的超奇异积分方程组。以磁压电耦合材料内含任意形状单裂纹问题为例，将式(5-73) ～（5-74）退化后可得到该问题的超奇异积分方程，具体表达式如下：

$$\oint_{S^+} r_{1\alpha}^{-3}((c_{44}s_1(3r_{1\alpha,\alpha}r_{1\alpha,\beta}-2\delta_{\alpha\beta})/4\pi+\sum_{i=1}^{5}D_i\rho_{i2}\hat{\rho}_{i2}(\delta_{\alpha\beta}-3r_{1\alpha,\alpha}r_{1\alpha,\beta})ds(p)=-p_\alpha(p)$$

$$\oint_{S^+} r_{1m}^{-3}\sum_{i=3}^{5}\sum_{n=3}^{5}D_i\rho_{im}\hat{\rho}_{in}ds(q)=-p_m(p)$$

在本章后续分析中，为了使分析结果更具有一般性，以磁压电耦合双材料与界面垂直三维裂纹问题这种更一般情况为例进行分析，其它退化情形读者可参考本分析过程自行得出，限于论文篇幅限制，就不在一一退化分析。为了使读者能更好的了解退化后的裂纹评定准则，特在数值结果与讨论部分，对应于磁压电耦合双材料问题，同时给出磁压电耦合材料三维裂纹问题的各类典型算例下的强度因子数值结果，具体内容请参见本章后面相关章节。

5.4.3 广义应力奇性指数

沿平面裂纹S前沿光滑周边 ∂S 上 $\hat{q}$ 点导入一局部三维正交自然坐标系和密切面内以 $\hat{q}$ 点为中心的微小扇形区域 S_ε，则裂纹前沿 $\hat{q}$ 点广义位移间断可定义为：

$$\tilde{U}_J(\hat{q})=g_k(\hat{q}_0)\xi_2^{\lambda_k} \qquad 0<\mathrm{Re}(\lambda_k)<1 \tag{5-75}$$

利用Qin [189]中二维超奇异积分主部分析法，可得到裂纹前沿应力奇异性指数特征方程：

$$A_i\dot{\kappa}_i^{-1}s_i^{-1}\sum_{k=1}^{3}\kappa_{ki}\lambda^2+(\cos\lambda\pi)/2\pi c_{44}+\sum_{i=1}^{5}(D_i+\sum_{n=3}^{5}\sum_{m}^{5}\rho_{im}\hat{\rho}_{in})=0 \tag{5-76}$$

$$\cos(\lambda_i \pi) = D_0 + \sum_{i=1}^{5} D_i \rho_{i2} \hat{\rho}_{i2} \tag{5-77}$$

裂纹前沿广义应力奇性指数随材料参数变化规律如图5-2所示。

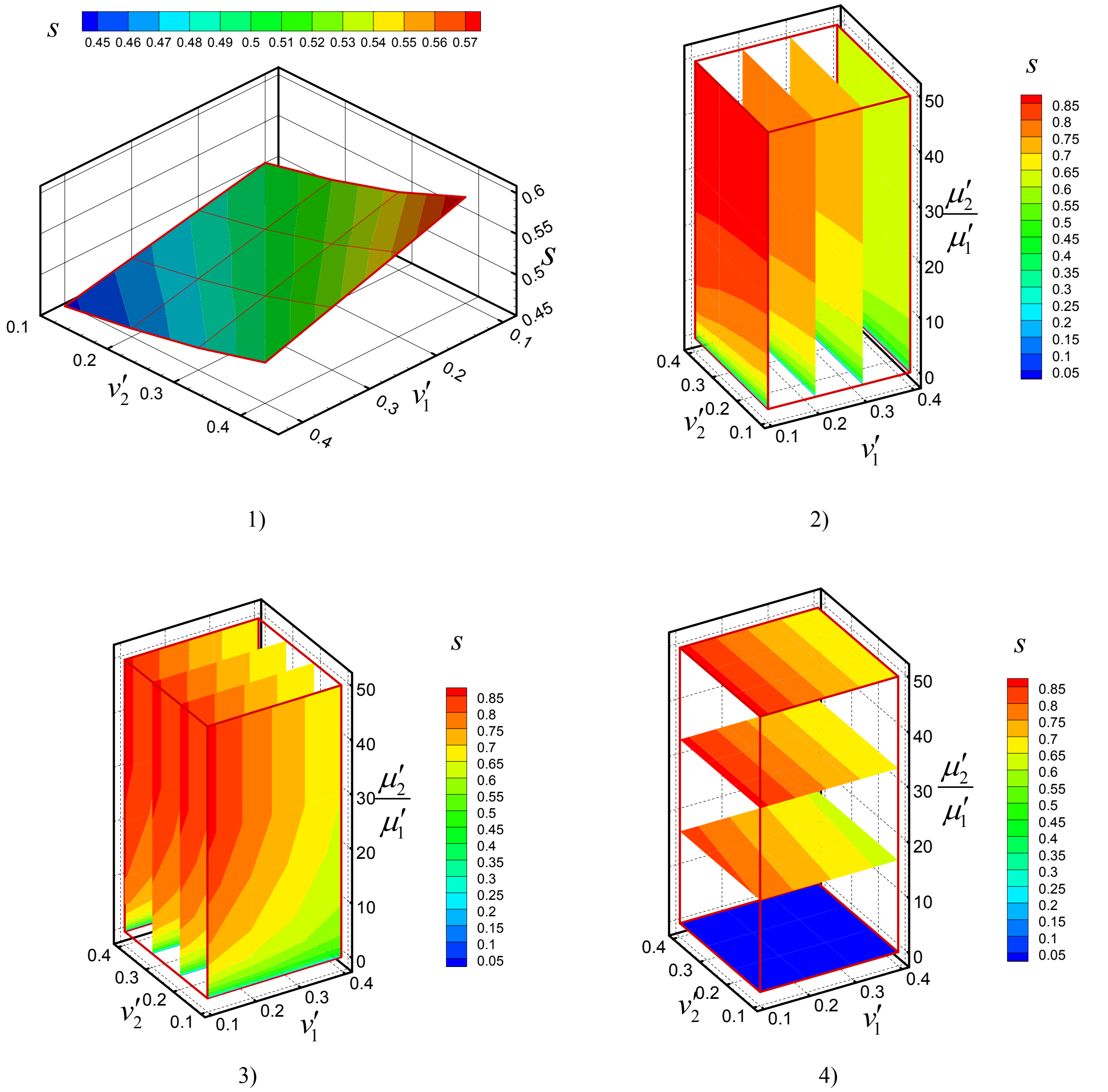


图 5-2 裂纹前沿广义应力奇性指数随材料参数变化规律

等效泊松比(ν_1', ν_2')和等效剪切模量比(μ_2'/μ_1')对广义应力奇性指数s变化规律如图 2 所示。 广义应力奇性指数随ν_1'增加而增加，随ν_2'增加而减小；广义应力奇性指数随μ_2'/μ_1'增加而增加，增加梯度不断减小，当$\mu_2'/\mu_1' = 50$已趋近于一固定值(图 5-2 彩色版请参见 www.imechanica.org/blog/2237)。

5.4.4 广义奇异应力场

综合(5-76) ～（5-77）及(5-73) ～（5-74）就可以得到裂纹前沿奇异广义应力场具体表达式：

$$\sigma_{13}=0.25c_{44}s_1 g_1(rr_0)^{-0.5}\cos\dot{\theta}_1((v_1c_{44}-v_1'c_{44}')/(v_1c_{44}+v_1'c_{44}')-1) \tag{5-78}$$

$$\sigma_{23}=\sum_{i=1}^{5}\pi^2\rho_{i2}(rr_i)^{-0.5}\cos\dot{\theta}_i\sum_{m=3}^{5}(\hat{\rho}_{i2}g_m+4\hat{\rho}_{im}g_m) \tag{5-79}$$

$$\dot{\sigma}_{3n}=\sum_{i=1}^{5}\sum_{m=3}^{5}\rho_{im}\delta_{mn}(rr_i)^{-0.5}v_i\pi(\hat{\rho}_{i2}g_2\cot\theta_i\cos^{-1}\dot{\theta}_i+6\cot\dot{\theta}_i\sum_{m=3}^{5}\hat{\rho}_{im}g_m),\quad n=3,4,5 \tag{5-80}$$

其中

$$\dot{\sigma}_{3n}=[\sigma_{33}\quad D_3\quad B_3]$$

5.4.5 广义应力强度因子

磁电热弹耦合双材料三维裂纹的广义应力强度因子可定义为：

$$K_1=\lim_{r\to0}\sqrt{2r}\sigma_{33}(r,\theta)\Big|_{\theta=0},K_2=\lim_{r\to0}\sqrt{2r}\sigma_{32}(r,\theta)\Big|_{\theta=0},K_3=\lim_{r\to0}\sqrt{2r}\sigma_{31}(r,\theta)\Big|_{\theta=0} \tag{5-81}$$

$$K_4=\lim_{r\to0}\sqrt{2r}D_3(r,\theta)\Big|_{\theta=0},K_5=\lim_{r\to0}\sqrt{2r}B_3(r,\theta)\Big|_{\theta=0} \tag{5-82}$$

K_1表示I型应力强度因子，K_2表示II型应力强度因子，K_3表示III型应力强度因子，K_4表示电应力强度因子，K_5表示磁应力强度因子。其中r表示$\hat{p}$点到裂纹前沿$\hat{q}_0$点距离。应用关系式(5-81)～（5-82），广义奇异应力(5-78)～（5-80）可以简化为：

$$\sigma_{13}=K_3\cos\dot{\theta}_1/\sqrt{2rr_0}，\quad \hat{\sigma}_{n3}=\sum_i^5 K_i\breve{f}_{ni}/\sqrt{2r},n=2-5 \tag{5-83}$$

其中$\hat{\sigma}_{n3}$及核函数$\breve{f}_{mn}$具体表达式为：

$$\hat{\sigma}_{n3}=[\sigma_{23}\quad\sigma_{33}\quad D_3\quad B_3]$$

$$\breve{f}_{mn}=\delta_{m2}\sum_{i=1}^{5}(\rho_{i2}r_i^{-0.5}\cos\dot{\theta}_i(\hat{\rho}_{i2}\delta_{1n}+4\sum_{k=3}^{5}\hat{\rho}_{i3}\delta_{kn}))+\delta_{mk}\sum_{i=1}^{5}\rho_{ik}s_i\cot\dot{\theta}_i(r_i^{-0.5}\hat{\rho}_{i2}\cos^{-1}\dot{\theta}_i\delta_{1n}$$
$$+6\hat{\rho}_{in}(\delta_{2n}+\delta_{4n}+\delta_{5n}))$$

5.4.6 广义能量释放率

磁电热弹耦合双材料三维多裂纹问题的广义能量释放率可以表示为：

$$G=\pi(G_1^{\sigma}+G_2^{\sigma}+G_3^{\sigma}+G_4^{D}+G_5^{B})/2 \tag{5-84}$$

其中

$$G_1^{\sigma}=\frac{K_1\,[K_4^*(K_{34}K_{55}-K_{35}K_{54})+K_{45}(K_{54}K_1^*-K_{34}K_5^*)+K_{44}(K_{35}K_5^*-K_{55}K_1^*)]}{K_{35}(K_{43}K_{54}-K_{44}K_{53})+K_{34}(K_{45}K_{53}-K_{43}K_{55})+K_{33}(K_{44}K_{55}-K_{45}K_{54})},$$

$$G_2^{\sigma}=\frac{K_2^2}{K_{11}-K_{12}}$$

$$G_3^{\sigma}=\frac{K_3\,(K_3\,(K_{11}-K_{12})K_{16}+K_{26}((\dot{K}_{11}+2\dot{K}_{12})K_2-(K_{11}-K_{12})K_4))}{(K_{11}-K_{12})(K_{26}(\dot{K}_{11}+2\dot{K}_{12})-(K_{11}-K_{12})K_{66})}$$

$$G_4^{D}=\frac{K_4\,(K_4^*(K_{35}K_{53}-K_{33}K_{55})+K_{45}(K_{33}K_5^*-K_{53}K_1^*)+K_{43}(K_{55}K_1^*-K_{35}K_5^*))}{K_{35}(K_{43}K_{54}-K_{44}K_{53})+K_{34}(K_{45}K_{53}-K_{43}K_{55})+K_{33}(K_{44}K_{55}-K_{45}K_{54})}$$

$$G_5^{B}=\frac{K_5\,[K_4^*(K_{33}K_{54}-K_{34}K_{53})+K_{44}(K_{53}K_1^*-K_{33}K_5^*)+K_{43}(K_{34}K_5^*-K_{54}K_1^*)]}{K_{35}(K_{43}K_{54}-K_{44}K_{53})+K_{34}(K_{45}K_{53}-K_{43}K_{55})+K_{33}(K_{44}K_{55}-K_{45}K_{54})}$$

G_i^{σ}, G_4^D 和 G_5^B 分别代表由力载荷、电载荷和磁载荷影响部分。其中符号 $K_1^*=K_1+K_{31}\hat{K}_2$，$K_4^*=K_4+K_{41}\hat{K}_2$， $K_5^*=K_5+K_{51}$。

5.4.7 广义应变能强度函数

根据应变能强度理论[93, 94], 磁电热弹耦合材料内部单元体积 $dV=dx_1dx_2dx_3$ 在广义三维应力作用下的应变能为：

$$\begin{aligned}dW=&(\frac{\sigma_{11}^2+\sigma_{22}^2+\sigma_{33}^2}{2E}-v\frac{\sigma_{11}\sigma_{22}+\sigma_{22}\sigma_{33}+\sigma_{33}\sigma_{11}}{E}+\frac{\sigma_{12}^2+\sigma_{13}^2+\sigma_{23}^2}{2\mu}+\frac{D_{11}^2+D_{22}^2+D_{33}^2}{2E'}-\\&v'\frac{D_{11}D_{22}+D_{22}D_{33}+D_{33}D_{11}}{E'}+\frac{D_{12}^2+D_{13}^2+D_{23}^2}{2\mu'}+\frac{B_{11}^2+B_{22}^2+B_{33}^2}{2E'}-v'\frac{B_{11}B_{22}+B_{22}B_{33}+B_{33}B_{11}}{E'}\\&+\frac{B_{12}^2+B_{13}^2+B_{23}^2}{2\mu'})dV\end{aligned}\tag{5-85}$$

其中 E, v, μ, E', v', μ' 分别表示杨氏模量、泊松比、剪切模量，等价杨氏模量，等价泊松比，等价剪切模量. 把上式带入(5-80)，可得到广义应变能量强度函数，表达式如下：

$$\frac{dW}{dV}=\frac{K_n^2}{2\sqrt{2r}}(\frac{a_{3n}^2}{E}+\frac{a_{1n}^2+a_{2n}^2}{\mu}+\frac{a_{4n}^2+a_{5n}^2}{E'})\tag{5-86}$$

广义应变能强度函数在裂纹前沿有 $\dfrac{1}{\sqrt{r}}$ 奇异性，整理后可得到广义应变能强度因子表达式：

$$S\big|_{\theta}=K_n^2\left(\frac{a_{3n}^2}{E}+\frac{a_{1n}^2+a_{2n}^2}{\mu}+\frac{a_{4n}^2+a_{5n}^2}{E'}\right) \tag{5-87}$$

其中 a_{mn} (m,n=1,2,3,4,5,6) 为和材料性质有关的已知参数，具体表达式见附录 D。

5.5 数值方法

5.5.1 数值方法

应用文献[187, 266]中方法，将超奇异积分方程组(5-73) ～（5-74）进行数值化处理。以矩形裂纹为例，如图5-3所示。

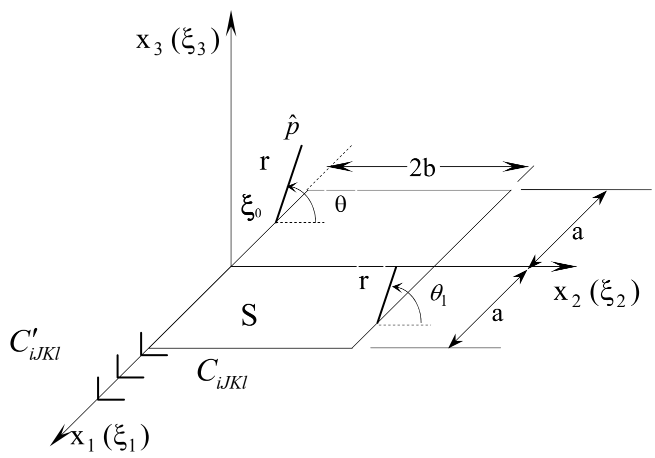


图 5-3 磁压电耦合双材料与界面垂直矩形平片裂纹问题示意图

裂纹前沿广义位移间断函数（5-10）可以简化：

$$\tilde{U}_J(\xi_1,\xi_2)=F_J(\xi_1,\xi_2)\xi_2^{\lambda_J}W(\xi_1,\xi_2) \tag{5-88}$$

其中

$$W=((a^2-\xi_1^2)(2b-\xi_2))^{0.5}$$

$$F_J=\sum_{s=0}^{S}\sum_{h=0}^{H}\ a_{Jsh}\xi_1^s\xi_2^h$$

a_{Jsh} 为未知常数，把方程（5-88）代入(5-73) ～（5-74），朝奇异积分方程组可以转化为一组代数方程组。

$$\sum_{s=0}^{S}\sum_{h=0}^{H}\ a_{Ish}I_{IJsh}=-p_J \tag{5-89}$$

其中

$$I_{12sh} = I_{21sh} = -\int_S 3r^{-3} r_{,1} r_{,2} (K_{11} + K_{12}) S_1 \tag{5-90}$$

$$I_{22sh} = \int_S r^{-3} K_{1\alpha} (1 - 3r_{,\alpha}^2) S_2 \tag{5-91}$$

$$I_{mnsh} = \int_S r^{-3} K_{mn} S_n \tag{5-92}$$

$$I_{11sh} = \int_S r^{-3} (K_{11}(1 - 3r_{,2}^2) + K_{12}(1 - 3r_{,1}^2)) S_1 \tag{5-93}$$

其中

$$S_n = \xi_1^s \xi_2^{\lambda_n + h} W \, d\xi_1 \, d\xi_2$$

$$K_{IJ} = -\delta_{1I}\delta_{1J} c_{44} s_1 / 4\pi + \delta_{1I}\delta_{2J} \sum_{k=1}^{4} D_k \rho_{k2} \hat{\rho}_{k2} + \delta_{mI}\delta_{nJ} \sum_{l=3}^{5} \sum_{k=3}^{5} D_i \rho_{km} \hat{\rho}_{ln}$$

方程(5-90) ～（5-93）为超奇异积分，在数值处理前必须对超奇异积分部分进行处理。应用泰勒展开和极坐标可以得到以下关系式。

$$\xi_1^s (a^2 - x_1^2)^{0.5} = \begin{cases} (a^2 - x_1^2)^{0.5} - x_1 r_1 \cos\theta_1 (a^2 - x_1^2)^{-0.5} - Q_1 r_1^2, & s = 0 \\ x_1^s (a^2 - x_1^2)^{0.5} + x_1^{s-1}(sa^2 - (s+1)x_1^2) r_1 \cos\theta_1 (a^2 - x_1^2)^{-0.5} + Q_2 r_1^2, & s > 0 \end{cases} \tag{5-94}$$

$$\xi_2^{\lambda_J + h}(2b - \xi_2)^{0.5} = x_2^{\lambda_J + h}(2b - x_2)^{0.5} + (x_2^{\lambda_J + h - 1}(4b(\lambda_J + h) - (2\lambda_J + 2h + 1)x_2) r_1 \sin\theta_1 + Q_3^J r_1^2) 0.5 (2b - x_2)^{-0.5} \tag{5-95}$$

其中

$$Q_1 = a^2 (x_1 + \xi_1)((a^2 - x_1^2)^{0.5}((a^2 - x_1^2)^{0.5} + (a^2 - \xi_1^2)^{0.5})(\xi_1 (a^2 - x_1^2)^{0.5} + x_1 (a^2 - \xi_1^2)^{0.5}))^{-1} \cos^2\theta_1 \tag{5-96}$$

$$Q_2 = \begin{cases} (s(s-1)a^4 x_1^{s-2} - (s^2+1)a^2 x_1^s + s(s+1)x_1^{s+2}) 0.5 \cos^2\theta_1 (a^2 - x_1^2)^{-1.5}, & \xi_1 = x_1 \\ r_1^{-2}(\xi_1^s (a^2 - \xi_1^2)^{0.5} - x_1^s (a^2 - x_1^2)^{0.5} - x_1^{s-1}(sa^2 - (s+1)x_1^2)(\xi_1 - x_1)(a^2 - x_1^2)^{-0.5}), & \xi_1 \neq x_1 \end{cases} \tag{5-97}$$

$$Q_3^J = \begin{cases} 2(a^2 - x_1^2)^{-1.5} x_2^{\lambda_J + h - 2} \sin^2\theta_1 (16(\lambda_J + h)(\lambda_J + h - 1)b^2 - 8b(\lambda_J + n)(2\lambda_J + 2h - 1)x_2 + (4(\lambda_J + h)^2 - 1)x_2^2), & \xi_2 = x_2 \\ r_1^{-2}(\xi_2^{\lambda_J + h}(2b - \xi_2)^{0.5} - x_2^{\lambda_J + h}(2b - x_2)^{0.5} - (2b - x_2)^{0.5} x_2^{\lambda_J + h - 1}(2b(\lambda_J + h) - (\lambda_J + h + 0.5)x_2)(\xi_2 - x_2)), & \xi_2 \neq x_2 \end{cases} \tag{5-98}$$

应用式(5-94) ～（5-95），方程(5-90) ～（5-93）中的核函数可表示为：

$$\xi_1^s \xi_2^{\lambda_J + h}((a^2 - \xi_1^2)(2b - \xi_2))^{0.5} = D_0^J + D_1^J r_1 + D_2^J r_1^2 \tag{5-99}$$

其中 D_0^J，D_1^J，D_2^J 为已知函数，可以应用泰勒展开得到。应用有限部积分和关系式（5-99），方程式(5-90) ～（5-93）可简化为：

$$I_{11sh}=\int_0^{2\pi}[K_{11}(1-3\sin^2\theta)+K_{12}(1-3\cos^2\theta)]\dot{S}_1 \tag{5-100}$$

$$I_{12sh}=\int_0^{2\pi}1.5(K_{11}+K_{12})\sin 2\theta\dot{S}_2 \tag{5-101}$$

$$I_{mnsh}=\int_0^{2\pi}K_{mn}\dot{S}_n \tag{5-102}$$

$$I_{21sh}=\int_0^{2\pi}1.5(K_{11}+K_{12})\sin 2\theta\dot{S}_1 \tag{5-103}$$

$$I_{22sh}=\int_0^{2\pi}[K_{11}(1-3\cos^2\theta)+K_{12}(1-3\sin^2\theta)]\dot{S}_2 \tag{5-104}$$

其中

$$\dot{S}_n=(D_1^n\ln R-R^{-1}D_0^n+\int_0^R D_2^n dr_1)d\theta$$

5.5.2 数值结果及讨论

为了检验上述理论及数值方法的正确性，以Li [225]中材料为例，考察含有一矩形裂纹的磁压电耦合双材料弹性体，无限远处作用单位载荷$\tau_{31}^{\infty},\sigma_{33}^{\infty}$。规定①表示材料$BaTiO_3$，②表示材料$C_oFe_2O_4$，③表示材料$BaTiO_3$—$C_oFe_2O_4$。为了便于比较，首先对材料常数进行无量纲化，无量纲公式为：

$$\overline{c}_{ij}=c_{ij}/c_{11},\overline{e}_{ij}=e_{ij}/\sqrt{c_{11}e_{11}},\overline{\in}_{ij}=\in_{ij}/\in_{11} \tag{5-105}$$

$$\overline{d}_{ij}=d_{ij}/\sqrt{c_{11}\mu_{11}},\overline{g}_{ij}=d_{ij}/\sqrt{\in_{11}\mu_{11}},\overline{\mu}_{ij}=\mu_{ij}/\mu_{11} \tag{5-106}$$

把Li [225]中材料常数带入上式，即可得到无量化的材料常数，具体结果如表5-1所示。

表5-1 无量纲化材料参数

	$\overline{c}_{11}$	$\overline{c}_{12}$	$\overline{c}_{13}$	$\overline{c}_{33}$	$\overline{c}_{44}$	$\overline{g}_{11}$	$\overline{g}_{33}$	$\overline{\in}_{11}$	$\overline{\in}_{33}$
①	1	.4639	.4699	.9760	.2590	.338E-2	.139E-2	1	1.125
②	1	.6049	.5944	.9423	.1584	-.230E-4i	-.138E-4i	1	1.625
	$\overline{d}_{15}$	$\overline{d}_{31}$	$\overline{d}_{33}$	$\overline{\mu}_{11}$	$\overline{\mu}_{33}$	$\overline{e}_{15}$	$\overline{e}_{31}$	$\overline{e}_{33}$	
①	0	0	0	1	2	.2690	-.1020	.4314	
②	-.0423i	-.0447i	-.0539i	1	-.2261	0	0	0	

对式(5-81) ～（5-82）进行无量纲，无量纲化广义应力强度因子表达式为：

$$F_{1,\lambda} = K_{1,\lambda} / \sigma_{33}^{\infty} b^{1-\lambda}, F_1 = K_1 / \sigma_{33}^{\infty}\sqrt{b} \tag{5-107}$$

$$F_{2,\lambda} = K_{2,\lambda} / \sigma_{31}^{\infty} b^{1-\lambda}, F_2 = K_2 / \sigma_{31}^{\infty}\sqrt{b} \tag{5-108}$$

$$F_{3,\lambda} = K_{3,\lambda} / \sigma_{32}^{\infty} b^{1-\lambda}, F_3 = K_3 / \sigma_{32}^{\infty}\sqrt{b} \tag{5-109}$$

$$F_{4,\lambda} = K_{4,\lambda} / D_{33}^{\infty} b^{1-\lambda}, F_4 = K_4 / D_{33}^{\infty}\sqrt{b} \tag{5-110}$$

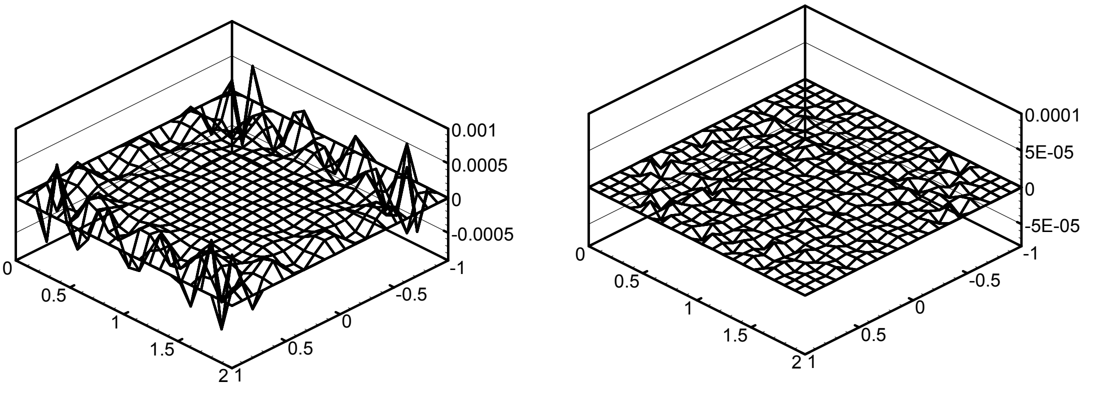
$$F_{5,\lambda} = K_{5,\lambda} / B_{33}^{\infty} b^{1-\lambda}, F_5 = K_5 / B_{33}^{\infty}\sqrt{b} \tag{5-111}$$

同样对式(5-87) 进行无量纲，可得到无量纲化广义应变能强度因子表达式：

$$\bar{S}\Big|_{\lambda,\theta} = (F_{n,\lambda})^2 \left(\frac{a_{3n}^2}{E} + \frac{a_{1n}^2 + a_{2n}^2}{\mu} + \frac{a_{4n}^2 + a_{5n}^2}{E'}\right) \tag{5-112}$$

$$\bar{S}\Big|_{\theta} = (F_n)^2 \left(\frac{a_{3n}^2}{E} + \frac{a_{1n}^2 + a_{2n}^2}{\mu} + \frac{a_{4n}^2 + a_{5n}^2}{E'}\right) \tag{5-113}$$

配置点为 20×20，裂纹表面最小剩余应力随多项式指数变化规律如图 5-4 所示：最小剩余应力 $\sigma_C = (\sigma_{33} / \sigma_{33}^{\infty}) + 1$ 随多项式指数增大而减小，数值结果精度随多项式指数增加而增大。



(a) M=N=9 (b) M=N=13

图 5-4 最小剩余应力随多项式指数变化规律

载荷为 σ_{33}^{∞} 时，配置点为 20×20，多项式指数为 13×13，广义应力强度因子随多项式指数、坐标变化规律如表 5-2 及图 5-5 所示：在 $x_2 = 0, 2b$ 边，应力强度因子 F_I，F_4，F_5 随 x_1 / a 呈对称分布，$x_1 / a = 0$ 取最大值。随多项式指数增加，数值结果是收敛的。结果与其他文献比较显示，本数值方法所得的数值结果是可靠的，有很好的精度。

表5-2 在 $x_2=0,2b$ 边广义应力强度因子 F_1 随多项式指数变化规律(KK=LL=20×20,b/a=1)

x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
M= 13	.7536	.7518	.7464	.7374	.7244	.7067	.6829	.6512	.6092	.5503	.4493
M= 11	.7536	.7518	.7464	.7374	.7245	.7067	.6829	.6512	.6089	.5501	.4520
M= 9	.7536	.7518	.7464	.7374	.7244	.7067	.6829	.6512	.6092	.5503	.4493
M= 7	.7529	.7514	.7466	.7383	.7246	.7069	.6827	.6513	.6071	.5505	.4433
[266]	.7534	.7517	.7465	.7375	.7245	.7065	.6826	.6510	.6086	.5494	.4543

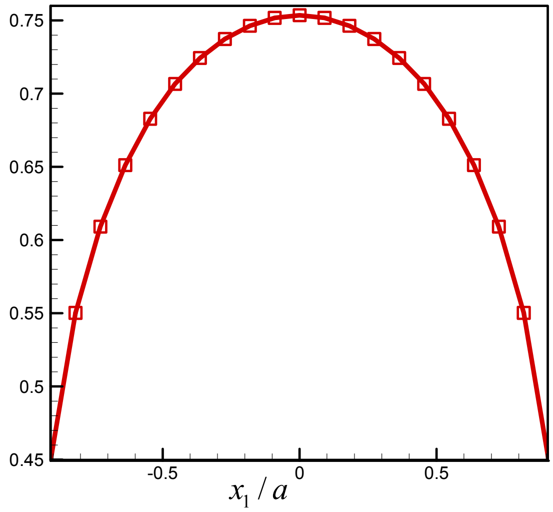


图 5-5 在 $x_2=0,2b$ 无量纲应力强度因子 F_1 随坐标 x_1/a 变化规律(KK=LL=20×20,M=N=13,b/a=1)

表5-3 在 $x_1=\pm a$ 边广义应力强度因子 F_2 随多项式指数变化规律(KK=LL=20×20,b/a=1)

x_2/b		0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
	①	.2666	.3164	.3480	.3705	.3869	.3990	.4081	.4147	.4192	.4218	.4226
M=13	②	.3793	.4496	.4943	.5262	.5492	.5663	.5791	.5885	.5948	.5985	.5996
	③	.2316	.2749	.3023	.3219	.3361	.3466	.3545	.3602	.3641	.3664	.3671
	①	.2667	.3170	.3479	.3704	.3868	.3990	.4080	.4147	.4192	.4217	.4226
M=11	②	.3810	.4505	.4941	5259	5492	5663	5791	5885	5948	5984	5996
	③	.2326	.2754	.3022	.3218	.3361	.3466	.3545	.3602	.3641	.3663	.3671
	①	.2677	.3182	.3479	.3701	.3869	.3991	.4080	.4144	.4190	.4219	.4228
M=9	②	.3810	.4523	.4941	.5255	.5492	.5665	.5790	.5881	.5946	.5986	.6000
	③	.2326	.2765	.3022	.3215	.3361	.3467	.3544	.3600	.3640	.3664	.3673
	①	.2652	.3202	.3490	.3695	.3860	.3990	.4087	.4153	.4192	.4202	.4218
M=7	②	.3776	.5552	.4957	.5246	.5479	.5664	.5801	.5893	.5949	.5977	.5985
	③	.2305	.2782	.3032	.3210	.3353	.3466	.3550	.3607	.3642	.3666	.3674

载荷为 σ_{31}^{∞} 时，配置点为 20×20，多项式指数为 13×13，广义应力强度因子随多项式指数、坐标变化规律如表 5-3、5-4 及图 5-6、5-7 所示：F_2(在 $x_1=\pm a$ 随 x_2/b)及 F_3(在 $x_2=0,2b$ 随 x_1/a)的变化规律与 F_1 (在 $x_2=0,2b$ 随 x_1/a)相似。①与②表示磁压电耦合材料三维裂纹结果。③表示磁压电耦合双材料与界面垂直裂纹结果。

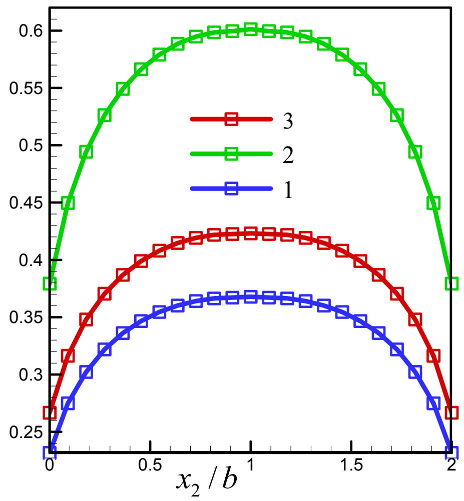


图 5-6 在 $x_1=\pm a$ 无量纲应力强度因子 F_2 随坐标 x_2/b 变化规律(KK=LL=20×20,M=N=13,b/a=1)

表5-4 在 $x_2=0,2b$ 边广义应力强度因子 F_3 随多项式指数变化规律(KK=LL=20×20,b/a=1)

x_1/a		0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
	①	.4763	.4752	.4718	.4659	.4572	.4455	.4302	.4101	.3837	.3482	.2895
M=13	②	.6788	.6772	.6723	.6639	.6516	.6349	.6131	.5844	.5469	.4965	.4131
	③	.4140	.4130	.4100	.4049	.3974	.3872	.3739	.3564	.3335	.3027	.2516
	①	.4764	.4753	.4718	.4659	.4573	.4457	.4303	.4102	.3843	.3491	.2884
M=11	②	.6789	.6773	.6723	.6639	.6517	.6351	.6132	.5846	.5478	.4978	.4116
	③	.4141	.4131	.4100	.4049	.3974	.3873	.3739	.3565	.3340	.3034	.2507
	①	.4764	.4752	.4718	.4659	.4573	.4456	.4304	.4107	.3852	.3493	.2855
M=9	②	.6789	.6772	.6723	.6639	.6516	.6350	.6133	.5854	.5491	4981	.4073
	③	.4140	.4130	.4100	.4049	.3974	.3873	.3740	.3570	.3348	.3036	.2481
	①	.4762	.4751	.4718	.4660	.4575	.4459	.4309	.4115	.3858	.3479	.2789
M=7	②	.6786	.6770	.6723	.6641	.6519	.6355	.6140	.5865	.5500	.4960	.3978
	③	.4138	.4129	.4100	.4050	.3976	.3875	.3744	.3576	.3353	.3023	.2424

配置点为 20×20，多项式指数为 13×13，表 5-5、5-6、5-6 分别表示对应于材料①、②和③应变能强度因子随角度、坐标变化规律；在 $x_2=0,2b$ 边，应变能强度因子随 x_1/a 呈对称分

布，$x_1/a=0$ 取最大值。图 5-8、5-9、5-10 同样反应了对应于材料①、②和③广义应变能强度因子随角度和坐标变化规律；在 180°-360° 的数值结果和 0°-180° 是对称分布的。

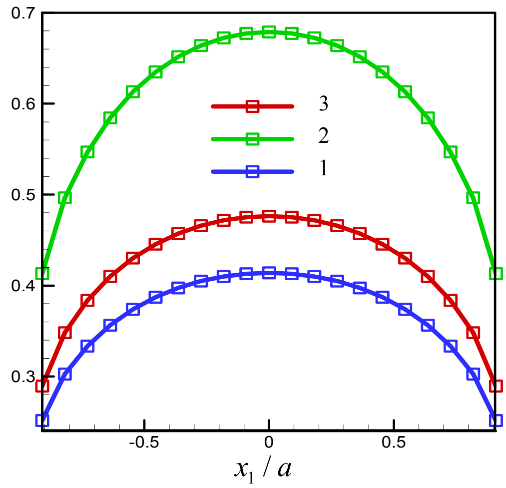


图 5-7 在 $x_2=0,2b$ 无量纲应力强度因子 F_3 随坐标 x_1/a 变化规律(KK=LL=20×20,M=N=13,b/a=1)

表5-5无量纲广义应变能强度因子随 θ，x_1/a 变化规律(KK=LL=20×20,M=N=13,b/a=1,材料①)

θ \ x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
0	0.2329	0.2319	0.2287	0.2233	0.2156	0.2053	0.1920	0.1749	0.1534	0.1259	0.0861
10	0.2398	0.2387	0.2354	0.2299	0.2220	0.2114	0.1977	0.1801	0.1580	0.1296	0.0886
20	0.2428	0.2418	0.2384	0.2328	0.2248	0.2141	0.2002	0.1824	0.1600	0.1313	0.0897
30	0.2411	0.2400	0.2367	0.2311	0.2232	0.2125	0.1987	0.1811	0.1588	0.1303	0.0890
40	0.2341	0.2330	0.2298	0.2244	0.2166	0.2063	0.1929	0.1757	0.1541	0.1264	0.0863
50	0.2220	0.2210	0.2179	0.2128	0.2054	0.1955	0.1828	0.1665	0.1460	0.1198	0.0817
60	0.2055	0.2045	0.2017	0.1969	0.1901	0.1809	0.1691	0.1540	0.1350	0.1107	0.0755
70	0.1859	0.1851	0.1825	0.1781	0.1719	0.1636	0.1529	0.1392	0.1220	0.1000	0.0681
80	0.1649	0.1642	0.1619	0.1580	0.1524	0.1450	0.1355	0.1233	0.1080	0.0886	0.0603
90	0.1445	0.1438	0.1418	0.1384	0.1335	0.1270	0.1186	0.1079	0.0945	0.0774	0.0526
100	0.1265	0.1259	0.1241	0.1211	0.1168	0.1110	0.1037	0.0943	0.0825	0.0677	0.0460
110	0.1126	0.1121	0.1105	0.1078	0.1039	0.0988	0.0922	0.0838	0.0734	0.0602	0.0409
120	0.1042	0.1037	0.1022	0.0997	0.0961	0.0914	0.0853	0.0775	0.0679	0.0557	0.0380
130	0.1019	0.1015	0.1000	0.0976	0.0941	0.0895	0.0835	0.0760	0.0665	0.0546	0.0373
140	0.1061	0.1056	0.1041	0.1016	0.0980	0.0932	0.0870	0.0792	0.0694	0.0570	0.0391
150	0.1161	0.1155	0.1139	0.1112	0.1073	0.1021	0.0954	0.0869	0.0762	0.0626	0.0430
160	0.1308	0.1302	0.1284	0.1254	0.1210	0.1152	0.1077	0.0981	0.0861	0.0707	0.0487
170	0.1488	0.1481	0.1461	0.1427	0.1377	0.1311	0.1227	0.1118	0.0981	0.0806	0.0556
180	0.1682	0.1674	0.1652	0.1613	0.1557	0.1483	0.1387	0.1265	0.1110	0.0913	0.0629

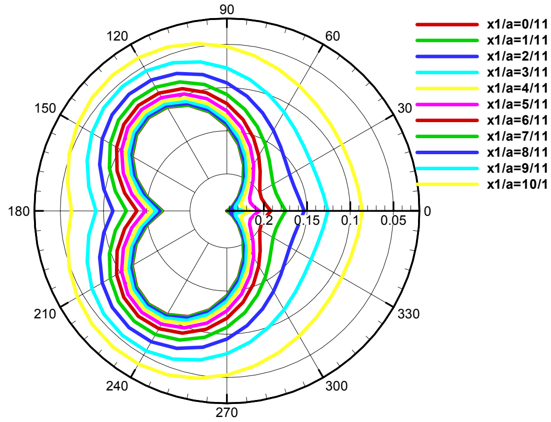


图 5-8 广义无量纲应变能强度因子随 θ 和 x_1/a 变化规律（KK=LL=20×20,M=N=13,b/a=1 材料①）

表5-6无量纲广义应变能强度因子随 θ， x_1/a 变化规律(KK=LL=20×20,M=N=13,b/a=1,材料②）

θ \ x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
0	0.3414	0.3399	0.3352	0.3273	0.3159	0.3008	0.2813	0.2563	0.2248	0.1847	0.1268
10	0.3509	0.3493	0.3445	0.3364	0.3247	0.3092	0.2891	0.2635	0.2311	0.1899	0.1303
20	0.3545	0.3529	0.3480	0.3398	0.3280	0.3123	0.2921	0.2662	0.2335	0.1918	0.1316
30	0.3509	0.3493	0.3445	0.3364	0.3247	0.3091	0.2891	0.2634	0.2310	0.1898	0.1302
40	0.3397	0.3382	0.3335	0.3256	0.3143	0.2992	0.2798	0.2549	0.2236	0.1836	0.1259
50	0.3214	0.3199	0.3155	0.3080	0.2973	0.2830	0.2645	0.2410	0.2113	0.1736	0.1190
60	0.2973	0.2959	0.2918	0.2849	0.2749	0.2616	0.2445	0.2227	0.1952	0.1603	0.1098
70	0.2694	0.2682	0.2644	0.2581	0.2490	0.2369	0.2214	0.2015	0.1767	0.1451	0.0993
80	0.2404	0.2393	0.2359	0.2302	0.2221	0.2112	0.1973	0.1795	0.1573	0.1292	0.0884
90	0.2130	0.2120	0.2090	0.2039	0.1966	0.1870	0.1746	0.1588	0.1391	0.1143	0.0782
100	0.1899	0.1890	0.1863	0.1818	0.1753	0.1666	0.1555	0.1414	0.1238	0.1017	0.0696
110	0.1735	0.1727	0.1702	0.1661	0.1601	0.1521	0.1419	0.1290	0.1130	0.0928	0.0636
120	0.1654	0.1646	0.1623	0.1583	0.1526	0.1450	0.1353	0.1230	0.1077	0.0885	0.0607
130	0.1664	0.1656	0.1632	0.1593	0.1535	0.1459	0.1362	0.1238	0.1085	0.0892	0.0613
140	0.1763	0.1755	0.1730	0.1688	0.1628	0.1548	0.1445	0.1315	0.1152	0.0948	0.0652
150	0.1941	0.1932	0.1905	0.1859	0.1793	0.1706	0.1594	0.1451	0.1272	0.1047	0.0721
160	0.2179	0.2169	0.2139	0.2088	0.2015	0.1917	0.1792	0.1632	0.1431	0.1178	0.0813
170	0.2452	0.2441	0.2407	0.2351	0.2269	0.2159	0.2019	0.1840	0.1614	0.1328	0.0917
180	0.2733	0.2721	0.2684	0.2621	0.2530	0.2409	0.2252	0.2053	0.1802	0.1482	0.1023

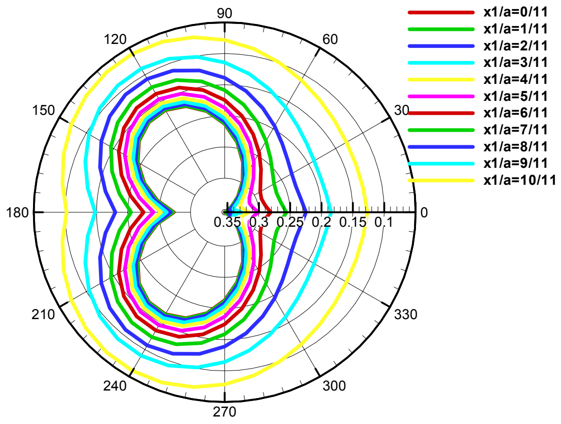


图 5-9 广义无量纲应变能强度因子随 θ 和 x_1/a 变化规律（KK=LL=20×20,M=N=13,b/a=1 材料②）

表5-7无量纲广义应变能强度因子随 θ , x_1/a 变化规律(KK=LL=20×20,M=N=13,b/a=1,材料③）

θ \ x_1/a	0/11	1/11	2/11	3/11	4/11	5/11	6/11	7/11	8/11	9/11	10/11
0	0.2034	0.2025	0.1997	0.1950	0.1883	0.1793	0.1677	0.1528	0.1340	0.1099	0.0750
10	0.2094	0.2085	0.2056	0.2008	0.1939	0.1846	0.1727	0.1573	0.1380	0.1132	0.0773
20	0.2123	0.2114	0.2084	0.2035	0.1965	0.1872	0.1750	0.1595	0.1399	0.1148	0.0783
30	0.2111	0.2102	0.2073	0.2024	0.1954	0.1861	0.1741	0.1586	0.1391	0.1141	0.0778
40	0.2054	0.2045	0.2017	0.1969	0.1901	0.1811	0.1693	0.1542	0.1353	0.1109	0.0756
50	0.1953	0.1944	0.1917	0.1871	0.1807	0.1720	0.1608	0.1465	0.1284	0.1053	0.0718
60	0.1812	0.1803	0.1778	0.1736	0.1676	0.1595	0.1491	0.1358	0.1190	0.0976	0.0664
70	0.1642	0.1634	0.1611	0.1573	0.1518	0.1445	0.1350	0.1229	0.1077	0.0883	0.0600
80	0.1458	0.1451	0.1430	0.1396	0.1347	0.1282	0.1198	0.1090	0.0955	0.0783	0.0531
90	0.1275	0.1269	0.1251	0.1221	0.1178	0.1120	0.1046	0.0952	0.0834	0.0683	0.0463
100	0.1110	0.1105	0.1089	0.1063	0.1025	0.0975	0.0910	0.0828	0.0725	0.0594	0.0403
110	0.0979	0.0975	0.0961	0.0938	0.0904	0.0860	0.0802	0.0730	0.0639	0.0524	0.0355
120	0.0894	0.0890	0.0877	0.0856	0.0825	0.0785	0.0732	0.0666	0.0583	0.0478	0.0325
130	0.0862	0.0858	0.0846	0.0825	0.0796	0.0757	0.0707	0.0643	0.0563	0.0462	0.0315
140	0.0885	0.0881	0.0869	0.0848	0.0818	0.0778	0.0727	0.0661	0.0580	0.0476	0.0326
150	0.0961	0.0956	0.0943	0.0920	0.0888	0.0845	0.0790	0.0719	0.0631	0.0518	0.0356
160	0.1080	0.1075	0.1060	0.1035	0.0999	0.0951	0.0889	0.0811	0.0711	0.0584	0.0402
170	0.1230	0.1225	0.1208	0.1180	0.1139	0.1085	0.1015	0.0925	0.0812	0.0667	0.0460
180	0.1397	0.1391	0.1371	0.1340	0.1294	0.1232	0.1153	0.1051	0.0923	0.0759	0.0522

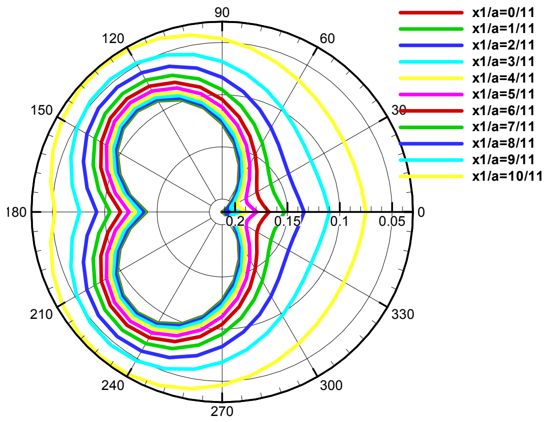


图 5-10 广义无量纲应变能强度因子 S_s 随 θ 和 x_1/a 变化规律（KK=LL=20×20,M=N=13,b/a=1 材料③）

最大广义应力强度因子随裂纹形状比规律如表 5-8、5-9 及图 5-11、5-12、5-13 所示：广义应力强度因子随裂纹形状比增加而增大，当 $b/a \geq 8$ 时，接近二维问题裂纹数值结果。

表5-8 应力强度因子 F_I 随裂纹形状比变化规律(KK=LL=20×20,M=N=13)

b/a	1	2	3	4	5	8	∞
本文结果	.7536	.9183	.9279	.9314	.9379	1.002	1.002
[248]	—	—	—	—	—	—	1.000
[244]	—	—	—	—	—	—	1.005
[243]	—	—	—	—	—	—	1.000

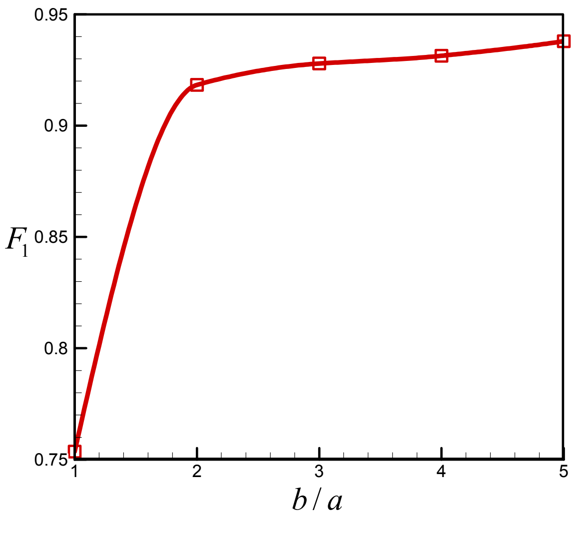


图 5-11 广义无量纲应力强度因子 F_I 随裂纹形状比变化规律（KK=LL=20×20,M=N=13）

表5-9 应力强度因子F_2 ,F_3 随裂纹形状比变化规律(KK=LL=20×20,M=N=13)

b/a		1	2	3	5	6	7	8	∞
	①	.4226	.4731	.4901	.4968	.4973	.4975	.4978	.4980
F_2	②	.5996	.6276	.6510	.6602	.6604	.6605	.6607	.6610
	③	.3671	.4110	.4259	.4316	.4318	.4320	.4322	.4324
	①	.4763	.6162	.6770	.7043	.7198	.7279	.7357	.7463
F_3	②	.6788	.8786	.9693	1.032	1.044	1.051	1.055	1.071
	③	.4140	.5359	.5888	.6261	.6331	.6373	.6400	.6492

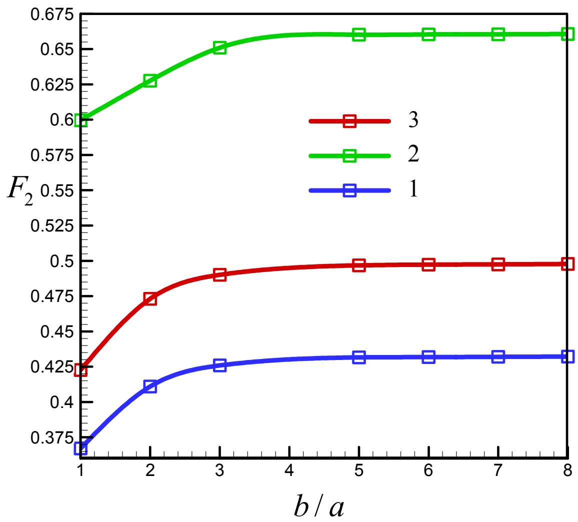


图 5-12 广义无量纲应力强度因子 F_2 随裂纹形状比变化规律（KK=LL=20×20,M=N=13）

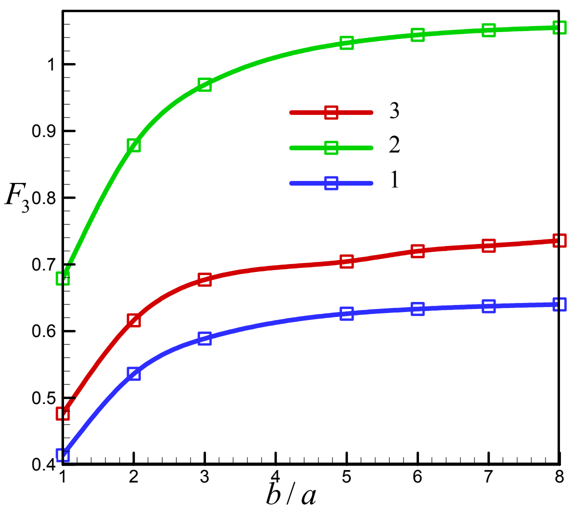


图 5-13 广义无量纲应力强度因子 F_3 随裂纹形状比变化规律（KK=LL=20×20,M=N=13）

表 5-10 表示对应于材料①、②和③最大应变能强度因子随角度和裂纹形状比变化规律；广义应变能强度因子随裂纹形状比增加而增大，当 $b/a \geq 8$ 时，接近二维问题裂纹数值结果。图 5-14、5-15、5-16 同样反应了对应于材料①、②和③应变能强度因子随角度和坐标变化规律；在 180-360 的数值结果和 0-180 是对称分布的。

表5-10无量纲广义应变能强度因子随b/a和θ变化规律(KK=LL=20×20,M=N=13)

b/a	材料	$\theta=0$	$\theta=\pi/6$	$\theta=\pi/3$	$\theta=\pi/2$	$\theta=2\pi/3$	$\theta=5\pi/6$	$\theta=\pi$
	①	.2329	.2411	.2055	.1445	.1042	.1161	.1682
1	②	.3414	.3509	.2973	.2130	.1654	.1941	.2733
	③	.2034	.2111	.1812	.1275	.0894	.0961	.1397
	①	.3422	.3539	.3062	.2224	.1646	.1774	.2470
2	②	.4934	.5066	.4397	.3311	.2656	.2963	.3947
	③	.2977	.3089	.2688	.1950	.1403	.1459	.2039
	①	.3712	.3833	.3331	.2456	.1859	.2005	.2738
3	②	.5475	.5611	.4907	.3771	.3093	.3426	.4463
	③	.3210	.3325	.2903	.2132	.1568	.1638	.2250
	①	.3843	.3965	.3453	.2563	.1960	.2112	.2860
5	②	.5836	.5973	.5256	.4100	.3414	.3757	.4815
	③	.3355	.3471	.3041	.2257	.1687	.1763	.2387
	①	.3924	.4048	.3532	.2635	.2024	.2175	.2928
6	②	.5927	.6060	.5344	.4179	.3486	.3829	.4893
	③	.3400	.3518	.3086	.2295	.1718	.1792	.2420
	①	.4198	.4336	.3793	.2828	.2150	.2279	.3069
7	②	.6247	.6401	.5638	.4387	.3621	.3956	.5078
	③	.3643	.3774	.3320	.2470	.1827	.1874	.2530
	①	.4228	.4366	.3823	.2857	.2178	.2308	.3099
8	②	.6269	.6423	.5660	.4409	.3642	.3977	.5100
	③	.3652	.3784	.3329	.2478	.1836	.1883	.2540
	①	.4268	.4406	.3862	.2896	.2218	.2348	.3139
∞	②	.6355	.6509	.5746	.4494	.3727	.4062	.5186
	③	.3682	.3814	.3360	.2508	.1865	.1913	.2570

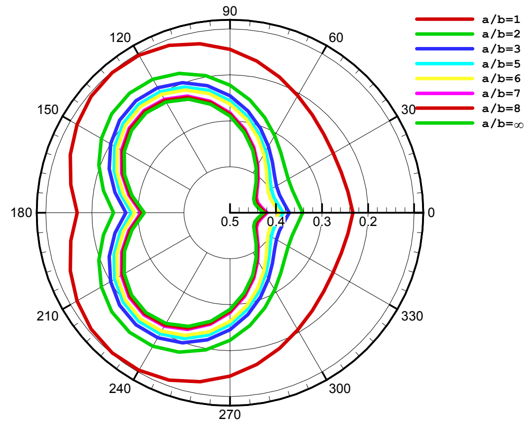


图 5-14 广义无量纲应变能强度因子随 θ 和 b/a 变化规律（KK=LL=20×20,M=N=13,材料①）

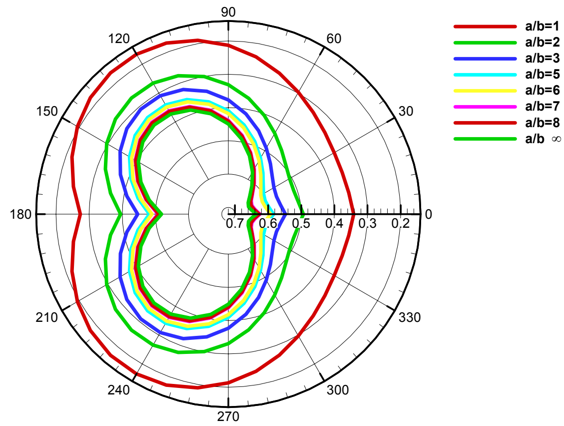


图 5-15 广义无量纲应变能强度因子随 θ 和 b/a 变化规律（KK=LL=20×20,M=N=13,材料②）

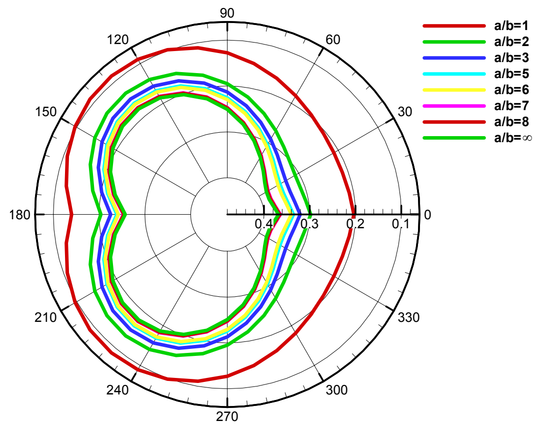


图 5-16 广义无量纲应变能强度因子随 θ 和 b/a 变化规律（KK=LL=20×20,M=N=13,材料③）

5.6 本章小节

1.应用主部分析法和广义位移基本解，将垂直于磁压电耦合双材料界面三维混合型裂纹问题、磁压电耦合材料内含任意形状平片裂纹问题分别转化为求解一组以裂纹位移间断为未知函数的超奇异积分方程问题。采用二维超奇异积分的主部分析法得到了裂纹前沿的广义应力奇异性指数及裂纹前沿光滑点附近奇性应力场的解析表达式。

2.使用有限部积分概念为超奇异积分方程组建立了数值方法，应用 FORTRAN 语言编写了专用程序，计算了部分典型算例，分别得到了磁压电耦合材料内含平片裂纹、垂直于磁压电耦合双材料界面三维裂纹问题广义（应力、应变能）强度因子的数值结果。

3.从数值结果可知，广义（应力、应变能）强度因子数值结果是收敛的，而且有很好的精度。最大（应力、应变能）强度因子随裂纹形状比增加而增大，当 $b/a \geqslant 8$ 时，接近二维平面裂纹问题数值结果。

4.工程中遇到的多场耦合作用下的双材料与界面垂直裂纹问题，可以应用本文提供的裂纹评定准则来分析和研究结构的可靠性。在应力集中区域内，与界面垂直裂纹对材料结构强度和可靠性的影响十分明显，在工程应用中应予以考虑。

第六章 结论与展望

6.1 创新成果

本文在完善和修正超奇异积分法基础上，首次将该方法引入到磁电热弹耦合材料断裂力学问题研究中，丰富和发展了该理论在复合材料断裂力学领域中的应用，具有重要的理论价值和应用前景，对国内外该领域研究有重要的参考价值。本文创新成果主要包括：

1. 引入广义位移间断概念，将磁电热弹耦合材料线弹性任意形状平片裂纹（单裂纹、平行多裂纹、任意多裂纹）问题分别转化为解超奇异积分方程问题；推导出各类问题裂纹前沿广义应力奇性指数和广义奇性应力解析解；定义了广义（应力、应变能）强度因子和能量释放率；将各类问题奇异积分方程转化为代数方程，得到广义（应力、应变能）强度因子数值计算式；对各类问题分别编写了 FORTRAN 专用程序（14000 余条），通过计算，得到了各类裂纹问题的广义（应力、应变能）强度因子变化规律[316-320]。

2. 引入时域超奇异积分、广义位移间断应变率、广义伪体积力和广义伪面力概念；通过将广义热-黏弹塑性对控制微分方程和边界条件的非线性影响作为伪载荷处理，使得塑性项以伪体积力和伪面力形式出现；将磁电热弹耦合材料粘塑性任意形状平片裂纹扩展问题转化为解时域超奇异积分方程问题，推导出广义应力增量场解析表达式，定义了广义 J 积分；通过对超奇异积分过程中遇到的各类奇异积分进行数值处理，为该方程组建立了数值求解方法，编写了专用 FORTRAN 程序（9000 余条），进行了考核性计算，验证了数值方法的收敛性和准确性[301, 302]。

3. 通过对磁压电耦合双材料空间位移基本解进行整理验证，得到适合于超奇异积分方程方法使用的基本解显示表达式；将磁压电耦合材料任意形状平片裂纹、磁压电耦合双材料与界面垂直任意形状平片裂纹问题分别转化为求解超奇异积分方程组问题；推导出裂纹前沿广义应力奇性指数和广义奇性应力场解析解。定义了两类问题广义（应力、应变能）强度因子和广义能量释放率。为两类方程组分别建立了数值求解方法，得到各自广义（应力、应变能）强度因子数值计算式，编写了专用 FORTRAN 程序（7000 余条），进行了考核性计算，得到了两类裂纹问题广义（应力、应变能）强度因子变化规律[210, 321-324]。

6.2 结论

本文在文献[93, 94, 166, 170, 172, 325-330]研究工作基础上，对磁电热弹耦合材料三维断裂力学问题进行了系统深入的理论与数值研究，得出以下主要结论：

1.对已有磁电热弹耦合材料及磁压电耦合双材料空间广义位移基本解资料进行整理、比较和验证。对于磁电热弹耦合材料广义位移基本解，进行整理和验证后，推导出裂纹面上直角坐标系下适合于超奇异积分方程方法中使用的空间广义位移基本解的显示表达式；对于磁压电耦合双材料空间位移基本解进行整理、比较和验证，应用直角坐标系下张量变换规则，经过庞杂的

数学推导，得出了适合于超奇异积分方程方法中应用的直角坐标系下空间位移基本解显示表达式。对于这两类问题广义位移基本解，再根据广义虎克定律、广义边界条件，推导出适合于超奇异积分方程方法中使用的直角坐标系的磁电热弹耦合材料（磁压电耦合双材料）空间问题广义面力基本解和广义应力积分核函数。

2.在广义基本解研究基础上，依次从浅入深，应用边界元法、Somigliana 公式、主部分析法等对磁电热弹耦合材料任意形状平片单裂纹问题、磁电热弹耦合材料任意形状平片多裂纹问题（平行多裂纹、任意多裂纹）、磁电热弹耦合材料粘塑性任意形状平片裂纹扩展问题、磁压电耦合双材料与界面垂直任意形状平片裂纹问题依次进行了系统研究，通过繁杂的数学推导，分别得出了各类裂纹问题的裂纹面坐标系下广义应力场（广义应力场增量）解析表达式，把各类裂纹问题分别转化为以广义位移间断（广义位移间断应变率）为未知函数的超奇异积分方程组问题，这些问题的超奇异积分方程组都属于首次得到。方程组经过退化可以求解磁-电-热-弹两相或多相耦合材料裂纹问题。方程组的得出使超奇异积分方程方法在功能梯度复合材料断裂力学问题中应用向前推进了一大步，丰富和发展了超奇异积分方程方法求解断裂力学理论。

3.在对文献[187, 266]中方法进行拓展和完善的基础上，使用主部分析法对上面得到的各类超奇异积分方程组进行系统研究；分析了裂纹面前沿广义应力奇异性态指数，推导出广义奇性应力（增量）场解析表达式。提出广义位移间断（应变率）概念，引入广义位移间断（应变率）密度函数基本解，对线弹性裂纹问题，定义了以广义位移间断函数表示广义（应力、应变能）强度因子和广义能量释放率；对粘弹塑性裂纹扩展问题定义了广义 J 积分。应用有限部积分方法，把超奇异积分方程组转化为求解代数方程组问题，并对方程组中各奇异积分部分分别进行了专门处理，推导出广义（应力、应变能）强度因子数值计算公式。建立起各类问题超奇异积分方程组数值求解方法。

4.使用 FORTRAN 语言编写了各类裂纹问题广义（应力、应变能）强度因子计算程序（40000 余条），对各类裂纹的典型问题进行了数值计算，考核了本文建立数值方法的收敛性、可靠性、准确性和精度，绝大部分数值结果属于首次得到。在此基础上，分析了广义强度因子随材料剪切模量比、裂纹形状比、多项式指数、配置点数、载荷种类和作用方向等参数变化规律。对于多裂纹问题，分析了裂纹间相互作用规律，广义（应力、应变能）强度因子随裂纹间距变化规律。对于粘塑性裂纹扩展问题，研究了裂纹面应力和位移梯度随时间变化规律，材料参数对于裂纹扩展影响规律。

6.3 展望

综上所述，本文主要目的就是试图在完善和发展超奇异积分方程方法理论基础上，通过把裂纹问题转化为求解超奇异积分方程，把超奇异积分方程方法引入到磁电热弹耦合材料断裂力学问题的分析中来，对以磁-电-热-弹耦合材料为代表的功能梯度复合材料三维断裂力学问题建立起统一而具有一般性的理论和数值求解方法。尽管本文的研究取得了一些积极的成果，为功能梯度复合材料断裂力学问题的研究做出了新贡献，但是限于本文作者的水平原因，还有大量的具体内容没有完成，作者认为可以在以下几个方面进一步开展研究：

1.由于问题的复杂性，未对各种裂纹数值方法做更详细的研究，如椭圆型裂纹问题、裂纹与夹杂相互作用问题等。这些问题可在本文现有程序基础上，改造后获得。由于磁电热弹耦合材料参数较多，本文未能对影响裂纹（应力、应变能）强度因子各个参数分别进行更详细研究，只是给出了一些典型问题结论。这些研究内容作者将在今后的工作中继续完成。

2.磁电热弹耦合材料的破坏涉及到材料宏观—微观—纳观裂纹，晶体结构方向上位错等跨尺度断裂力学问题（多尺度模型与物性跨尺度耦合）。工程结构失效起始于底层，即从材料中的孤立的空洞成核开始，形成微裂纹，发展为宏观裂纹，直至整个结构破坏。虽然近几年目前纳观本构关系和一些基本物理模型已经有了一个相对清晰的理论框架，但是物理学和力学上仍然存在一定问题。如何解决这类问题，可作为今后研究工作一个重点内容。

3.由于问题复杂性和相关理论不成熟，对于磁压电耦合双材料裂纹问题，没有对与界面平行裂纹、界面裂纹、与界面相交任意角度裂纹进行研究。这些问题期望在相关方面研究取得进展的基础上得到深入研究。

4.与准静态断裂问题相比，动态断裂问题在理论和实验上都困难得多。对于弹塑性材料，由于引入了材料的非线性，则要更为困难。由于材料的惯性，载荷是以应力波的形式传播的而裂纹扩展在数学上是个运动边界问题。这种未知的运动边界使问题变得高度非线性，同时载荷与运动边界的相互作用，再加上材料的非线性，使的问题在理论上实际非常复杂。对于这类问题（复杂工作环境，如温度、动态载荷问题），也可由本文方法求解，辅之以实验方法，此类裂纹问题可以得到很好的解决。

5.超奇异积分方法作为解决三维裂纹问题的一个重要理论，经过几十年的发展，在众多学者的不断完善下，已经成为解决断裂力学问题的一个相对完整的理论体系。如何将该理论进一步完善，以期能更好的用来分析断裂力学问题，需要更多人继续的努力。

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作者：朱伯靖

2007 年 9 月

附录 A

$$
\begin{aligned}
a_i ={}& c_{44}l_{24}\delta_{i1}+[c_{11}l_{24}+(c_{44}^2-l_{21}^2)l_{24}+(c_{44}\in_{11}+l_{32}^2)l_{34}+(c_{44}\mu_{11}+l_{32}^2)l_{44}+2(c_{44}e_{15}-l_{21}l_{31})l_{34}+2(c_{44}d_{15}-l_{21}l_{32})l_{41}+\\
& 2(c_{44}g_{11}+l_{31}l_{32})l_{43}]\delta_{i2}+c_{11}l_{24}\delta_{i5}+[c_{11}(c_{44}l_{24}+e_{15}l_{34}+g_{15}l_{41})+c_{44}(c_{33}l_{25}+e_{33}l_{35}+g_{33}l_{45})+(c_{11}c_{33}+c_{44}^2-l_{21}^2)l_{11}\\
& +l_{31}^2l_{22}+l_{32}^2l_{33}+(c_{11}e_{33}+c_{44}e_{15}-2l_{21}l_{32})l_{12}+(c_{11}d_{33}+c_{44}d_{15}-2l_{21}l_{32})l_{13}+2l_{31}l_{32}l_{23}]\delta_{i3}+[c_{44}l_{14}+(c_{11}c_{33}-l_{21}^2)l_{24}+\\
& (c_{1133}+l_{31}^2)l_{42}+(c_{11}\mu_{33}+l_{32}^2)l_{44}+2[(c_{11}e_{33}-l_{21}l_{31})l_{12}+(c_{11}d_{33}-l_{21}l_{32})l_{34}+(c_{11}d_{33}-l_{21}l_{31})l_{41}+(c_{11}g_{33}-l_{31}l_{32})l_{43}]]\delta_{i4}
\end{aligned}
$$

$$
\begin{aligned}
n_{4l} ={}& [-\iota_1l_{14}+\iota_3l_{21}l_{24}+(\iota_3l_{31}-\varsigma_3l_{21})l_{24}+(\iota_3l_{32}-\lambda_3l_{21})l_{41}+\varsigma_3l_{31}l_{42}+(\varsigma_3l_{32}+\lambda_3l_{31})l_{52}+\lambda_3l_{32}l_{53}]\delta_{l1}+[-\iota_1(c_{44}l_{24}\\
& +e_{15}l_{34}+d_{15}l_{31})-(\iota_1c_{33}-\iota_3l_{34})l_{11}-(\iota_1e_{33}-\iota_3l_{31}+d_3l_{21})l_{12}-(\iota_1d_{33}-\iota_3l_{32}+\lambda_3l_{21})l_{13}+(\varsigma_3l_{32}+\lambda_3l_{31})l_{23}+\\
& \varsigma_3l_{31}l_{22}+\lambda_3l_{32}l_{33}]\delta_{l2}+[-\iota_1(c_{44}l_{11}+e_{15}l_{12}+d_{15}l_{13})-(\iota_1c_{33}-\iota_3l_{21})l_{25}-(\iota_1e_{33}-\iota\beta_3l_{31}+\varsigma_3l_{21})l_{35}-(\iota_1d_{33}-\iota_3l_{32}\\
& +\lambda_3l_{21})l_{45}+(\varsigma_3l_{32}+\lambda_3l_{31})l_{52}+\varsigma_3l_{31}l_{51}+\lambda_3l_{32}l_{33}]\delta_{l3}-\iota_1l_{15}\delta_{l4}
\end{aligned}
$$

$$
\begin{aligned}
n_{5l} ={}& c_{44}(\iota_3l_{24}-\varsigma_3l_{34}-\lambda_3l_{41})\delta_{1l}+[(\varsigma_3c_{11}-\iota_1l_{21})l_{41}+c_{44}(\iota_3l_{11}-\varsigma_3l_{12}-\lambda_3l_{13})+(\iota_3l_{31}+\varsigma_3l_{21})(\mu_{33}l_{31}-g_{33}l_{32})-\\
& (\iota_3l_{32}+\lambda_3l_{21})(g_{33}l_{32}-\in_{33}l_{32})+(\varsigma_3l_{32}-\lambda_3l_{31})(d_{33}l_{32}-e_{33}l_{12})]\delta_{2l}+[(\iota_3c_{11}-\iota_1l_{21})l_{11}-c_{44}(\varsigma_3c_{11}+\iota_1l_{32})l_{12}\\
& -(\lambda_3c_{11}+\iota_1l_{32})l_{13}-c_{44}(\varsigma_3l_{35}-\iota_3l_{25}-\lambda_3l_{45})+(\iota_3l_{32}+\varsigma_3l_{21})({}_{11}l_{32}-g_{11}l_{32})-(\iota_3l_{32}+\lambda_3l_{21})(g_{11}l_{32}-\in_{11}l_{32})\\
& +(\varsigma_3l_{32}-\lambda_3l_{31})(d_{15}l_{32}-e_{15}l_{12})]\delta_{3l}+[(\iota_3c_{11}-\iota_1l_{21})l_{25}-(\varsigma_3c_{11}+\iota_1l_{32})l_{35}-(\lambda_3c_{11}+\iota_1l_{32})l_{45}]\delta_{4l}
\end{aligned}
$$

$$
\begin{aligned}
n_{6l} ={}& c_{44}(\iota_3l_{34}+\varsigma_3l_{42}+\lambda_3l_{43})\delta_{1l}+[(\iota_3c_{11}-\iota_1l_{21})l_{34}+l_{42}(\iota_3l_{11}-\varsigma_3l_{12}-\lambda_3l_{13})+l_{43}(\lambda_3c_{11}-\iota_1l_{21})-c_{44}(\iota_3l_{12}+\\
& \varsigma_3l_{22}+\lambda_3l_{23})-(\iota_3l_{31}+\varsigma_3l_{21})(\mu_{33}l_{21}-g_{33}l_{32})-(\iota_3l_{32}+\lambda_3l_{21})(g_{33}l_{21}+e_{33}l_{32})+(\varsigma_3l_{32}-\lambda_3l_{31})(d_{33}l_{32}-\\
& e_{33}l_{12})]\delta_{2l}+[(\iota_3c_{11}-\iota_1l_{21})l_{12}-(\varsigma_3c_{11}+\iota_1l_{42})l_{22}-(\lambda_3c_{11}+\iota_1l_{32})l_{23}+c_{44}(\varsigma_3l_{42}+\iota_3l_{35}+\lambda_3l_{43})-(\iota_3l_{32}+\\
& \varsigma_3l_{21})(\mu_{11}l_{21}+d_{15}l_{32})-(\iota_3l_{32}+\lambda_3l_{21})(g_{11}l_{21}+e_{15}l_{32})+(\varsigma_3l_{32}-\lambda_3l_{31})(d_{15}l_{21}-c_{15}l_{32})]\delta_{3l}+[(\iota_3c_{11}-\iota_1l_{21})l_{35}\\
& +(\varsigma_3c_{11}+\iota_1l_{32})l_{42}+(\lambda_3c_{11}+\iota_1l_{32})l_{52}]\delta_{4l}
\end{aligned}
$$

$$
\begin{aligned}
n_{7l} ={}& c_{44}(\iota_3l_{41}+\varsigma_3l_{43}+\lambda_3l_{44})\delta_{1l}+[(\iota_3c_{11}-\iota_1l_{21})l_{41}+l_{43}(\iota_1l_{31}+\varsigma_3c_{11})+l_{44}(\lambda_3c_{11}+\iota_1l_{32})+c_{44}(\iota_3l_{13}+\varsigma_3l_{23}\\
& +\lambda_3l_{33})+(\iota_3l_{31}+\varsigma_3l_{21})(d_{33}l_{21}+g_{33}l_{32})-(\iota_3l_{32}+\lambda_3l_{21})(\in_{33}l_{21}+e_{33}l_{31})+(\varsigma_3l_{32}-\lambda_3l_{31})(e_{33}l_{21}-\\
& e_{33}l_{31})]\delta_{2l}+[(\iota_3c_{11}-\iota_1l_{21})l_{13}-(\varsigma_3c_{11}+\iota_1l_{32})l_{23}+(\lambda_3c_{11}+\iota_1l_{32})l_{23}+c_{44}(\varsigma_3l_{52}+\iota_3l_{45}+\lambda_3l_{53})-\\
& (\iota_3l_{31}+\varsigma_3l_{21})(d_{11}l_{21}+g_{15}l_{31})-(\iota_3l_{32}+\lambda_3l_{21})(\in_{11}l_{21}+e_{15}l_{31})+(\varsigma_3l_{32}-\lambda_3l_{31})(e_{15}l_{21}-c_{44}l_{31})]\delta_{3l}\\
& +[(\iota_3c_{11}-\iota_1l_{21})l_{45}+(\varsigma_3c_{11}+\iota_1l_{31})l_{52}+(\lambda_3c_{11}+\iota_1l_{32})l_{53}]\delta_{4l}
\end{aligned}
$$

$$R_i=((x_1-\xi_1)^2+(x_2-\xi_2)^2+z_i{}^2)^{0.5}$$

$$\dot{R}_i=R_i+z_i,\lambda_i^w=\kappa_{i2}s_i/\kappa_{i1}$$

$$D_0=-1/4\pi c_{44}$$

$$\lambda_i^\phi=\kappa_{i3}s_i/\kappa_{i1}$$

$$\lambda_i^\varphi=\kappa_{i4}s_i/\kappa_{i1}$$

$$\lambda_i^{\vartheta} = -\kappa_{i5}\delta_{i5} / \kappa_{i1}$$

$$D_i = V^{-1}[0 \quad 0 \quad 0 \quad 0 \quad P_x]^T / 4\pi c_{44}$$

$$A_i = [A_i^P \quad A_i^Q \quad A_i^J \quad A_i^{\Upsilon} \quad A_i^5]^T = W^{-1} I_{5\times5}[P_z \quad Q \quad J \quad \Upsilon \quad 0]^T / 4\pi$$

$$\kappa_{il} = n_{4l}s_i^6 - n_{5l}s_i^4 + n_{6l}s_i^2 - n_{7l}$$

$$\kappa_{55} = n_1 s_5^8 - n_2 s_5^6 + n_3 s_5^4 - n_4 s_5^2 + n_5$$

$$w_{i1} = 1\,, w_{i2} = c_{13} + (c_{33}\kappa_{i2} + e_{33}\kappa_{i3} + d_{33}\kappa_{i4})s_i^2 / \kappa_{i1} - \iota_3\kappa_{55}\delta_{i5} / \kappa_{51}$$

$$w_{i3} = e_{31} + (e_{33}\kappa_{i2} - \in_{33} \kappa_{i3} - g_{33}\kappa_{i4})s_i^2 / \kappa_{i1} - \varsigma_3\kappa_{55}\delta_{i5} / \kappa_{51}$$

$$w_{i4} = d_{31} + (d_{33}\kappa_{i2} - g_{33}\kappa_{i3} - \mu_{33}\kappa_{i4})s_i^2 / \kappa_{i1} - \eta_3\kappa_{55}\delta_{i5} / \kappa_{51}$$

$$w_{i5} = 2[c_{11} - c_{66} + (c_{13}\kappa_{i2} + e_{31}\kappa_{i3} + g_{31}\kappa_{i4})s_i^2 / \kappa_{i1} - \iota_3\kappa_{55}\delta_{i5} / \kappa_{51}]$$

$$v_{il} = \kappa_{il}s_i\,\delta_{kl} / \kappa_{i1} + \kappa_{55}\delta_{i5}\delta_{4l} / \kappa_{i1} + s_i\,\delta_{5l}$$

$$\hat{K}_2 = \frac{K_3(K_{11} - K_{12})K_{16} + K_{26}[(\hat{K}_{11} + 2\hat{K}_{12})K_2 - (K_{11} - K_{12})K_6]}{(K_{11} - K_{12})[K_{26}(\hat{K}_{11} + 2\hat{K}_{12}) - (K_{11} - K_{12})K_{66}]}$$

$$K_5^* = K_5 + K_{51}\hat{K}_2 + K_{56}\hat{K}_6$$

$$\hat{K}_6 = \frac{(\dot{K}_{11} + 2\dot{K}_{12})(K_2 + K_3) - (K_{11} - K_{12})K_6}{(K_{11} - K_{12})K_{66} - K_{16}(\dot{K}_{11} + 2\dot{K}_{12})}$$

$$K_4^* = K_4 + K_{41}\hat{K}_2 + K_{46}\hat{K}_6$$

$$K_1^* = K_1 + K_{31}\hat{K}_2 + K_{36}\hat{K}_6$$

其中

$$l_{11} = \in_{11} \mu_{33} + \in_{33} \mu_{11} - 2g_{11}g_{33}$$

$$l_{12} = e_{33}\mu_{11} + e_{15}\mu_{33} - g_{11}d_{33} - g_{33}d_{15}$$

$$l_{13} = d_{33} \in_{11} + d_{15} \in_{33} - g_{11} e_{33} - g_{33} e_{15}$$

$$l_{21} = c_{13} + c_{44}$$

$$l_{22} = c_{44} \mu_{33} + c_{33} \mu_{11} + 2 d_{15} d_{33}$$

$$l_{23} = -c_{44} g_{33} - c_{33} g_{11} - e_{15} d_{33} - e_{33} g_{15}$$

$$l_{31} = e_{15} + e_{31}$$

$$l_{33} = c_{44} \in_{33} + c_{33} \in_{11} + 2 e_{15} e_{33}$$

$$l_{32} = d_{15} + d_{31}$$

$$l_{14} = -2 g_{33} e_{33} d_{33} - c_{33} d_{33}^2 + g_{33}^2 \in_{33} + (e_{33}^2 + c_{33} \in_{33}) \mu_{33}$$

$$l_{42} = c_{33} \mu_{33} + d_{33}^2$$

$$l_{43} = -c_{33} g_{33} - e_{33} d_{33}$$

$$l_{24} = \mu_{33} \in_{33} - g_{33}^2$$

$$l_{34} = d_{33} g_{33} - e_{33} \mu_{33}$$

$$l_{41} = e_{33} d_{33} - g_{33} \in_{33}$$

$$l_{25} = \mu_{11} \in_{11} - g_{11}^2$$

$$l_{35} = d_{15} g_{11} - e_{15} \mu_{11}$$

$$l_{15} = -2 d_{15} e_{15} g_{11} - c_{44} g_{11}^2 + g_{33}^2 \in_{33} + (e_{33}^2 + c_{33} \in_{33}) \mu_{33}$$

$$l_{51} = c_{44} \mu_{11} + g_{15}^2$$

$$l_{52} = -c_{44} g_{11} - d_{15} e_{15}$$

$$l_{44} = e_{33}^2 - c_{33} \in_{33}$$

$$l_{45} = e_{15} g_{11} - d_{15} \in_{11}$$

$$a_{1n} = \delta_{3n}\sqrt{r_0}\cos\theta_0 + \delta_{1n}\left(\sum_{i=1}^{5} A_i^{\Upsilon}\sqrt{r_1}\cos\theta_1\cot 2\theta_i(\iota_{33}\delta_{3n} + \varsigma_{33}\delta_{4n} + \eta_{33}\delta_{5n}) + \frac{\rho_i^3 s_i t_2 \cot\theta_i \cos^{-1}\theta_i}{\sqrt{r_i}}\right)$$

$$a_{2n} = \sum_{i=1}^{6}\frac{1}{\sqrt{r_i}}[\rho_i^2\cos\theta_i(t_i^2\delta_{1n} + 4t_i^3\delta_{2n} + 4t_i^4\delta_{4n} + 4t_i^5\delta_{5n}) + \delta_{6n}(\lambda_{33}(15\lambda_{33}s_i^2\cot 2\theta_i(3\cos\theta_i - 3\sin^2 2\theta_i\cos\theta_i$$
$$-0.25\sin^2 2\theta_i\cos 6\theta_i) - 2\lambda_{33}r_i^{-2}\sin^{-1}2\theta_i + 0.5\lambda_{32}r_i^{-1}\sin^{-1}\theta(2\cos^{-1}2\theta_i - 3\cos 2\theta_{1i} - \cos 10\theta_i)))]$$

$$a_{3n} = \sum_{i=1}^{6}[\rho_i^3 s_i\iota_{33}(A_i^{\Upsilon}\sqrt{r_i}\cos\theta_i + 6\cot\theta_i)(t_i^3\delta_{2n} + t_i^4\delta_{4n} + t_i^5\delta_{5n}) + \delta_{n1}(A_i^{\Upsilon}\sqrt{r_i}\cos\theta_i\cot 2\theta_i(\iota_{33}\delta_{3n} + \varsigma_{33}\delta_{4n} + \eta_{33}\delta_{5n})$$
$$+\frac{\rho_i^3 t_2\cot 2\theta_i\cos^{-1}\theta_i s_i}{\sqrt{r_i}} + \delta_{n6}(\rho_i^3 s_i^2\lambda_i^9\sqrt{r_i}\cot 2\theta_i\cos\theta_i + \frac{A_i^{\Upsilon}s_i^2\lambda_{33}\lambda_i^9\cos\theta_i(\iota_{33}\delta_{3n} + \varsigma_{33}\delta_{4n} + \eta_{33}\delta_{5n})}{\sqrt{r_i}})]$$

$$a_{4n} = \sum_{i=1}^{6}[\rho_i^4 s_i\zeta_{33}(A_i^{\Upsilon}\sqrt{r_i}\cos\theta_i + 6\cot\theta_i)(t_i^3\delta_{2n} + t_i^4\delta_{4n} + t_i^5\delta_{5n}) + \delta_{n1}(A_i^{\Upsilon}\sqrt{r_i}\cos\theta_i\cot 2\theta_i(\iota_{33}\delta_{3n} + \varsigma_{33}\delta_{4n} + \eta_{33}\delta_{5n})$$
$$+\rho_i^4 s_i\frac{1}{\sqrt{r_i}}t_2\cot 2\theta_i\cos^{-1}\theta_i + \delta_{n6}(\rho_i^4 s_i^2\lambda_i^9\sqrt{r_i}\cot 2\theta_i\cos\theta_i + \frac{A_i^{\Upsilon}s_i^2\lambda_{33}\lambda_i^9\cos\theta_i(\iota_{33}\delta_{3n} + \varsigma_{33}\delta_{4n} + \eta_{33}\delta_{5n})}{\sqrt{r_i}})]$$

$$a_{5n} = \sum_{i=1}^{6}[\rho_i^5 s_i\eta_{33}(A_i^{\Upsilon}\sqrt{r_i}\cos\theta_i + 6\cot\theta_i)(t_i^3\delta_{2n} + t_i^4\delta_{4n} + t_i^5\delta_{5n}) + \delta_{n1}(A_i^{\Upsilon}\sqrt{r_i}\cos\theta_i\cot 2\theta_i(\iota_{33}\delta_{3n} + \varsigma_{33}\delta_{4n} + \eta_{33}\delta_{5n})$$
$$+\frac{\rho_i^5 t_2\cot 2\theta_i\cos^{-1}\theta_i s_i}{\sqrt{r_i}} + \delta_{n6}(\rho_i^5 s_i^2\lambda_i^9\sqrt{r_i}\cot 2\theta_i\cos\theta_i + \frac{A_i^{\Upsilon}s_i^2\lambda_{33}\lambda_i^9\cos\theta_i(\iota_{33}\delta_{3n} + \varsigma_{33}\delta_{4n} + \eta_{33}\delta_{5n})}{\sqrt{r_i}})]$$

$$a_{6n} = \sum_{i=1}^{5}A_i^{\Upsilon}[\frac{\cos\theta_i(\lambda_{32} + 4\lambda_{33}s_{2i}^2)(t_i^3\delta_{3n} + t_i^4\delta_{4n} + t_i^5\delta_{5n})}{\sqrt{r_i}} + \delta_{1n}(\frac{(\cos\theta_i(\lambda_{32} + 4\lambda_{33}s_i)t_2}{\sqrt{r_i}} + 4\lambda_{32}t_2 s_i^2\lambda_i^9(2\lambda_{32}\cos\theta_i$$
$$+3\lambda_{33}r_{1i}^{-2}\sin^{-1}\theta_i)/3) + \delta_{6n}45\lambda_{32}s_{1i}^2\lambda_i^9\cot 2\theta_i(\cos\frac{\theta_i}{2} - \cos\theta_i\sin 2\theta_i - \frac{\cos 3\theta_i\sin^2\theta_i}{6})(2s_i^2\lambda_{33} + \lambda_{32}r_i^{-1}\sin 2\theta_i)]$$

附录 B

$$K_{\alpha i}^{\hat{i}*}=\mathrm{H}_{\alpha i}^{\hat{i}}(R_{\hat{i}}^{-3}(c_{44}^{2}D_{0}s_{0}^{2}(\delta_{\bar{\alpha}\bar{\beta}}-3R_{\hat{i},\bar{\alpha}}R_{\hat{i},\bar{\beta}})+(\delta_{\alpha\beta}-3R_{\hat{i},\alpha}R_{\hat{i},\beta})\sum_{i=1}^{5}\rho_{i}^{2}t_{i}^{2}+3h^{2\hat{i}}R_{\hat{i}}^{-2}(s_{0}\delta_{\alpha\beta}-5R_{\hat{i},\alpha}R_{\hat{i},\beta}))\delta_{\beta i}$$
$$+(3R_{\hat{i}}^{-4}R_{\hat{i},\alpha}\sum_{i=1}^{5}\lambda_{33}s_{i}^{2}t_{i}^{1}+3h^{2\hat{i}}R_{\hat{i},\alpha}^{-2}R_{\hat{i}}^{-2}(1-5s_{0}\delta_{\alpha\beta}-5h^{2\hat{i}}R_{\hat{i}}^{-2})\delta_{6i})$$

$$K_{mi}^{\hat{i}*}=\mathrm{H}_{mi}^{\hat{i}}((R_{\hat{i}}^{-2}R_{\hat{i},\alpha}\sum_{i=1}^{5}A_{i}^{\mathrm{r}}t_{i}^{2}\rho_{i}^{1}+3h^{2\hat{i}}R_{\hat{i},\alpha}^{-2}R_{\hat{i}}^{-2}(1-5s_{0}\delta_{\alpha\beta}-5h^{2\hat{i}}R_{\hat{i}}^{-2}))\delta_{\alpha m}+(R_{\hat{i}}^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_{i}^{m}t_{i}^{t}+3h^{2\hat{i}}R_{\hat{i},\alpha}R_{\hat{i}}(1-5R_{\hat{i}}^{2}h_{i}^{2\hat{i}}))\delta_{nm}$$
$$+(3R_{\hat{i}}^{-4}\lambda_{3\alpha}R_{\hat{i},\alpha}\sum_{i=1}^{5}\lambda_{i}^{v}s_{i}^{2}\rho_{i}^{m}+3h^{2\hat{i}}R_{\hat{i},\alpha}^{-2}R_{\hat{i}}^{-2}(1-5s_{0}\delta_{\alpha\beta}-5h^{2\hat{i}}R_{\hat{i}}^{-2})\delta_{6m})$$

$$K_{6i}^{\hat{i}*}=\mathrm{H}_{6i}^{\hat{i}}(R_{\hat{i}}^{-3}(c_{44}^{2}D_{0}s_{0}^{2}(\delta_{\bar{\alpha}\bar{\beta}}-3R_{\hat{i},\bar{\alpha}}R_{\hat{i},\bar{\beta}})+(\delta_{\alpha\beta}-3R_{\hat{i},\alpha}R_{\hat{i},\beta})\sum_{i=1}^{5}\rho_{i}^{2}t_{i}^{2}+3h^{2\hat{i}}R_{\hat{i}}^{-2}(s_{0}\delta_{\alpha\beta}-5R_{\hat{i},\alpha}R_{\hat{i},\beta}))\delta_{\beta i}+$$
$$(3R_{\hat{i}}^{-4}R_{\hat{i},\alpha}\sum_{i=1}^{5}\lambda_{33}s_{i}^{2}t_{i}^{1}+3h^{2\hat{i}}R_{\hat{i},\alpha}^{-2}R_{\hat{i}}^{-2}(1-5s_{0}\delta_{\alpha\beta}-5h^{2\hat{i}}R_{\hat{i}}^{-2})\delta_{6i})$$

$$K_{\alpha i}^{\hat{i}}=R_{\hat{i}}^{-3}(c_{44}^{2}D_{0}s_{0}^{2}(\delta_{\bar{\alpha}\bar{\beta}}-3R_{\hat{i},\bar{\alpha}}R_{\hat{i},\bar{\beta}})+(\delta_{\alpha\beta}-3R_{\hat{i},\alpha}R_{\hat{i},\beta})\sum_{i=1}^{5}\rho_{i}^{2}t_{i}^{2}+3h^{2\hat{i}}R_{\hat{i}}^{-2}(s_{0}\delta_{\alpha\beta}-5R_{\hat{i},\alpha}R_{\hat{i},\beta}))\delta_{\beta i}$$
$$+(3R_{\hat{i}}^{-4}R_{\hat{i},\alpha}\sum_{i=1}^{5}\lambda_{33}s_{i}^{2}t_{i}^{1}+3h^{2\hat{i}}R_{\hat{i},\alpha}^{-2}R_{\hat{i}}^{-2}(1-5s_{0}\delta_{\alpha\beta}-5h^{2\hat{i}}R_{\hat{i}}^{-2})\delta_{6i}$$

$$K_{mi}^{\hat{i}}=(R_{\hat{i}}^{-2}R_{\hat{i},\alpha}\sum_{i=1}^{5}A_{i}^{\mathrm{r}}t_{i}^{2}\rho_{i}^{1}+3h^{2\hat{i}}R_{\hat{i},\alpha}^{-2}R_{\hat{i}}^{-2}(1-5s_{0}\delta_{\alpha\beta}-5h^{2\hat{i}}R_{\hat{i}}^{-2}))\delta_{\alpha m}+(R_{\hat{i}}^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_{i}^{m}t_{i}^{t}+3h^{2\hat{i}}R_{\hat{i},\alpha}R_{\hat{i}}(1-5R_{\hat{i}}^{2}h_{i}^{2\hat{i}}))\delta_{nm}$$
$$+(3R_{\hat{i}}^{-4}\lambda_{3\alpha}R_{\hat{i},\alpha}\sum_{i=1}^{5}\lambda_{i}^{v}s_{i}^{2}\rho_{i}^{m}+3h^{2\hat{i}}R_{\hat{i},\alpha}^{-2}R_{\hat{i}}^{-2}(1-5s_{0}\delta_{\alpha\beta}-5h^{2\hat{i}}R_{\hat{i}}^{-2})\delta_{6m}$$

$$K_{6i}^{\hat{i}}=R_{\hat{i}}^{-3}(c_{44}^{2}D_{0}s_{0}^{2}(\delta_{\bar{\alpha}\bar{\beta}}-3R_{\hat{i},\bar{\alpha}}R_{\hat{i},\bar{\beta}})+(\delta_{\alpha\beta}-3R_{\hat{i},\alpha}R_{\hat{i},\beta})\sum_{i=1}^{5}\rho_{i}^{2}t_{i}^{2}+3h^{2\hat{i}}R_{\hat{i}}^{-2}(s_{0}\delta_{\alpha\beta}-5R_{\hat{i},\alpha}R_{\hat{i},\beta}))\delta_{\beta i}+$$
$$(3R_{\hat{i}}^{-4}R_{\hat{i},\alpha}\sum_{i=1}^{5}\lambda_{33}s_{i}^{2}t_{i}^{1}+3h^{2\hat{i}}R_{\hat{i},\alpha}^{-2}R_{\hat{i}}^{-2}(1-5s_{0}\delta_{\alpha\beta}-5h^{2\hat{i}}R_{\hat{i}}^{-2})\delta_{6i}$$

$$K_{1i}^{1}=R^{-3}(-3c_{44}^{2}D_{0}s_{0}^{2}R_{,2}R_{,1}+(1-3R_{,1}^{2})\sum_{i=1}^{5}\rho_{i}^{2}t_{i}^{2}+3h^{2}R^{-2}(s_{0}-5R_{,1}^{2}))\delta_{1i}+R^{-3}((1-3c_{44}^{2}D_{0}s_{0}^{2}R_{,2}^{2})$$
$$-3R_{,1}R_{,2}\sum_{i=1}^{5}\rho_{i}^{2}t_{i}^{2}-15h^{2}R^{-2}R_{,1}R_{,2})\delta_{2i}+(3R^{-4}R_{,1}\sum_{i=1}^{5}\lambda_{33}s_{i}^{2}t_{i}^{1}+3h^{2}R_{,1}^{-2}R^{-2}(1-5s_{0}\delta_{1\beta}-5h^{2}R^{-2})\delta_{6i}$$

$$K_{2i}^{1}=R^{-3}(c_{44}^{2}D_{0}s_{0}^{2}(1-3R_{,2}^{2})-3R_{,2}R_{,1}\sum_{i=1}^{5}\rho_{i}^{2}t_{i}^{2}-15h^{2}R^{-2}5R_{,2}R_{,1})\delta_{1i}+R^{-3}(-c_{44}^{2}D_{0}s_{0}^{2}3R_{,2}R_{,1}+$$
$$(1-3R_{,2}^{2})\sum_{i=1}^{5}\rho_{i}^{2}t_{i}^{2}+3h^{2}R^{-2}(s_{0}-5R_{,2}^{2}))\delta_{2i}+(3R^{-4}R_{,2}\sum_{i=1}^{5}\lambda_{33}s_{i}^{2}t_{i}^{1}+3h^{2}R_{,2}^{-2}R^{-2}(1-5s_{0}\delta_{22}-5h^{2}R^{-2})\delta_{6i}$$

$$K_{3i}^1=(R^{-2}R_{,1}\sum_{i=1}^{5}A_i^{\gamma}t_i^2\rho_i^1+3h^2R_{,1}^{-2}R^{-2}(1-5s_0-5h^2R^{-2}))\delta_{1i}+(R^{-2}R_{,2}\sum_{i=1}^{5}A_i^{\gamma}t_i^2\rho_i^1+3h^2R_{,2}^{2}R^{-2}(1-5s_0-5h^2R^{-2}))\delta_{2i}$$
$$+(2R^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^3t_i^t+3h^2R\,(R_{,1}+R_{,2})(1-5R^2h^2))\delta_{3i}+(3R^{-4}\lambda_{31}R_{,1}\sum_{i=1}^{5}\lambda_i^{\nu}s_i^2\rho_i^3+3h^2R_{,1}^{-2}R^{-2}(1-5s_0-5h^2R^{-2})$$
$$+3R^{-4}\lambda_{32}R_{,2}\sum_{i=1}^{5}\lambda_i^{\nu}s_i^2\rho_i^3+3h^2R_{,2}^{-2}R^{-2}(1-5s_0-5h^2R^{-2})\delta_{6i}$$

$$K_{4i}^1=(R^{-2}R_{,1}\sum_{i=1}^{5}A_i^{\gamma}t_i^2\rho_i^1+3h^2R_{,1}^{-2}R^{-2}(1-5s_0-5h^2R^{-2}))\delta_{1i}+(R^{-2}R_{,2}\sum_{i=1}^{5}A_i^{\gamma}t_i^2\rho_i^1+3h^2R_{,2}^{-2}R^{-2}(1-5s_0-5h^2R^{-2}))\delta_{2i}$$
$$+(2R^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^4t_i^t+3h^2R\,(R_{,1}+R_{,2})(1-5R^2h^2))\delta_{4i}+(3R^{-4}\lambda_{31}R_{,1}\sum_{i=1}^{5}\lambda_i^{\nu}s_i^2\rho_i^4+3h^2R_{,1}^{-2}R^{-2}(1-5s_0-5h^2R^{-2})$$
$$+3R^{-4}\lambda_{32}R_{,2}\sum_{i=1}^{5}\lambda_i^{\nu}s_i^2\rho_i^4+3h^2R_{,2}^{-2}R^{-2}(1-5s_0-5h^2R^{-2})\delta_{6i}$$

$$K_{5i}^1=(R^{-2}R_{,1}\sum_{i=1}^{5}A_i^{\gamma}t_i^2\rho_i^1+3h^2R_{,1}^{-2}R^{-2}(1-5s_0-5h^2R^{-2}))\delta_{1i}+(R^{-2}R_{,2}\sum_{i=1}^{5}A_i^{\gamma}t_i^2\rho_i^1+3h^2R_{,2}^{2}R^{-2}(1-5s_0-5h^2R^{-2}))\delta_{2i}$$
$$+(2R^{-3}\sum_{n=3}^{5}\sum_{i=1}^{5}\rho_i^5t_i^t+3h^2R(R_{,1}+R_{,2})(1-5R^2h^2))\delta_{5i}+(3R^{-4}\lambda_{31}R_{,1}\sum_{i=1}^{5}\lambda_i^{\nu}s_i^2\rho_i^5+3h^2R_{,1}^{-2}R^{-2}(1-5s_0-5h^2R^{-2})$$
$$+3R^{-4}\lambda_{32}R_{,2}\sum_{i=1}^{5}\lambda_i^{\nu}s_i^2\rho_i^5+3h^2R_{,2}^{-2}R^{-2}(1-5s_0-5h^2R^{-2})\delta_{6i}$$

$$K_{6i}^1=R^{-3}(c_{44}^2D_0s_0^2(1-3R_{,2}^2)+(1-3R_{,1}^2)\sum_{i=1}^{5}\rho_i^2t_i^2+3h^2R^{-2}(s_0-5R_{,\beta}^2))\delta_{1i}+R^{-3}(-3c_{44}^2D_0s_0^2 3R_{,1}R_{,2}-3R_{,2}R_{,1}\sum_{i=1}^{5}\rho_i^2t_i^2$$
$$-15h^2R^{-2}R_{,2}R_{,1})+R^{-3}(c_{44}^2D_0s_0^2(1-3R_{,1}^2)+(1-3R_{,2}R_{,2}^2)\sum_{i=1}^{5}\rho_i^2t_i^2+3h^2R^{-2}(s_0-5R_{,2}^2))\delta_{2i}+(3R^{-4}(R_{,1}+R_{,2})$$
$$\times\sum_{i=1}^{5}\lambda_{33}s_i^2t_i^1+(R_{,1}^{-2}+R_{,2}^{-2})3h^2R^{-2}(1-5s_0-5h^2R^{-2}))\delta_{6i}$$

$$r_{,\alpha}=(\xi_\alpha-x_\alpha)/r$$

$$R_{i,\alpha}=(\xi_\alpha-x_\alpha)/R$$

$$R_i=((\xi_1-x_1)^2+(\xi_2-x_2)^2+h^{2i})^{1/2}$$

$$K_{\alpha\beta}^1=K_{\alpha\beta}^2$$

$$K_{\alpha m}^1=-K_{\alpha m}^2$$

$$K_{\alpha 6}^1=K_{\alpha 6}^2$$

$$K_{6i}^1=K_{6i}^2$$

$$R_i^{\hat{i}} = ((x_1-\xi_1)^2+(x_2-\xi_2)^2+(z_i+\hat{i}h)^2)^{0.5}$$

$$\dot{R}_i = R_i + z_i(1+\hat{i}h)$$

$$\hat{K}_2^{\hat{i}} = (K_3^{\hat{i}}(K_{11}-K_{12})K_{16}+K_{26}((\hat{K}_{11}+2\hat{K}_{12})K_2^{\hat{i}}-(K_{11}-K_{12})K_6^{\hat{i}}))/(K_{11}-K_{12})(K_{26}(\hat{K}_{11}+2\hat{K}_{12})-(K_{11}-K_{12})K_{66})$$

$$K_1^{*\hat{i}} = K_1^{\hat{i}} + K_{31}\hat{K}_2^{\hat{i}} + K_{36}\hat{K}_6^{\hat{i}}$$

$$K_5^{*\hat{i}} = K_5^{\hat{i}} + K_{51}\hat{K}_2^{\hat{i}} + K_{56}\hat{K}_6^{\hat{i}}$$

$$K_4^{*\hat{i}} = K_4^{\hat{i}} + K_{41}\hat{K}_2^{\hat{i}} + K_{46}\hat{K}_6^{\hat{i}}$$

$$\hat{K}_6^{\hat{i}} = ((\dot{K}_{11}+2\dot{K}_{12})(K_2^{\hat{i}}+K_3^{\hat{i}})-(K_{11}-K_{12})K_6^{\hat{i}})/((K_{11}-K_{12})K_{66}-K_{16}(\dot{K}_{11}+2\dot{K}_{12}))$$

$$Q_1 = a^2(x_1+\xi_1)[(a^2-x_1^2)^{0.5}((a^2-x_1^2)^{0.5}+(a^2-\xi_1^2)^{0.5})(\xi_1(a^2-x_1^2)^{0.5}+x_1(a^2-\xi_1^2)^{0.5})]^{-1}\cos^2\theta_1$$

$$Q_2 = \begin{cases} [s(s-1)a^4x_1^{s-2}-(s^2+1)a^2x_1^s+s(s+1)x_1^{s+2}]\cos^2\theta_1/2(a^2-x_1^2)^{3/2}, & \xi_1 = x_1 \\ r_1^{-2}\{\xi_1^s(a^2-\xi_1^2)^{0.5}-x_1^s(a^2-x_1^2)^{0.5}-x_1^{s-1}[sa^2-(s+1)x_1^2](\xi_1-x_1)/(a^2-x_1^2)^{0.5}\}, & \xi_1 \neq x_1 \end{cases}$$

$$Q_3^J = \begin{cases} 2(a^2-x_1^2)^{-1.5}x_2^{\lambda_J+h-2}\sin^2\theta_1\{16(\lambda_J+h)(\lambda_J+h-1)b^2-8b(\lambda+n)(2\lambda_J+2h-1)x_2+[4(\lambda_J+h)^2-1]x_2^2\}, & \xi_2 = x_2 \\ r_1^{-2}\{\xi_2^{\lambda_J+h}(2b-\xi_2)^{0.5}-x_2^{\lambda_J+h}(2b-x_2)^{0.5}-(2b-x_2)^{0.5}x_2^{\lambda_J+h-1}[2b(\lambda_J+h)-(\lambda_J+h+0.5)x_2](\xi_2-x_2)\}, & \xi_2 \neq x_2 \end{cases}$$

$$K_{11} = c_{44}^2 D_0 s_0^2$$

$$K_{12} = \sum_{i=1}^{5}\rho_i^2 t_i^2$$

$$K_{16} = \sum_{i=1}^{5}\lambda_{33}s_i^2 t_i^1$$

$$\dot{K}_{\alpha\beta} = \sum_{i=1}^{5}A_i^{\Upsilon}\lambda_{3\beta}t_i^2$$

$$K_{mt} = \sum_{i=1}^{5}\rho_i^m t_i^t$$

$$K_{m\alpha} = \sum_{i=1}^{5}A_i^{\Upsilon}t_i^2\rho_i^1$$

$$\dot{K}_{m6} = \sum_{i=1}^{5}v_i^2\lambda_i^{\vartheta}\rho_i^m$$

$$K_{m6}=\sum_{i=1}^{5}A_i^{\Upsilon}v_i^2\lambda_i^{\vartheta}\lambda_{33}\rho_i^6$$

$$a_{\hat{i}1n}=\delta_{3n}r_{\hat{i}0}^{0.5}\cos\theta_{\hat{i}0}+\delta_{1n}(\sum_{i=1}^{5}A_i^{\Upsilon}r_i^{0.5}\cos\theta_{\hat{i}i}\cot 2\theta_{\hat{i}i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})+\rho_i^3 s_{\hat{i}i}r_{\hat{i}i}^{-0.5}t_2\cot\theta_{\hat{i}i}\cos^{-1}\theta_{\hat{i}i})$$

$$a_{\hat{i}2n}=\sum_{i=1}^{5}r_{\hat{i}i}^{-0.5}[\rho_i^2\cos\theta_{\hat{i}i}(t_i^2\delta_{1n}+4t_i^3\delta_{2n}+4t_i^4\delta_{4n}+4t_i^5\delta_{5n})+\delta_{6n}(\lambda_{33}(15\lambda_{33}s_{\hat{i}i}^2\cot 2\theta_{\hat{i}i}(3\cos\theta_{\hat{i}i}-3\sin^2 2\theta_{\hat{i}i}\cos\theta_{\hat{i}i}$$
$$-\frac{1}{4}\sin^2 2\theta_{\hat{i}i}\cos 6\theta_{\hat{i}i})-2\lambda_{33}r_{\hat{i}i}^{-2}\sin^{-1}2\theta_{\hat{i}i}^{i}+0.5\lambda_{32}r_{\hat{i}i}^{-1}\sin^{-1}\theta(2\cos^{-1}2\theta_{\hat{i}i}-3\cos 2\theta_{\hat{i}i}-\cos 10\theta_{\hat{i}i}))]$$

$$a_{\hat{i}3n}=\sum_{i=1}^{5}[\rho_i^3 s_{\hat{i}i}\iota_{33}(A_i^{\Upsilon}r_{\hat{i}i}^{0.5}\cos\theta_{\hat{i}i}+6\cot\theta_{\hat{i}i})(t_i^3\delta_{2n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})+\delta_{n1}(A_i^{\Upsilon}r_{\hat{i}i}^{0.5}\cos\theta_{\hat{i}i}\cot 2\theta_{\hat{i}i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})$$
$$+\rho_i^3 s_{\hat{i}i}r_{\hat{i}i}^{-0.5}t_2\cot 2\theta_{\hat{i}i}\cos^{-1}\theta_{\hat{i}i}+\delta_{n6}(\rho_i^3 s_{\hat{i}i}^2\lambda_i^{\vartheta}r_{\hat{i}i}^{0.5}\cot 2\theta_{\hat{i}i}\cos\theta_{\hat{i}i}+A_i^{\Upsilon}s_{\hat{i}i}^2\lambda_{33}\lambda_i^{\vartheta}r_{\hat{i}i}^{-0.5}\cos\theta_{\hat{i}i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n}))]$$

$$a_{\hat{i}4n}=\sum_{i=1}^{5}[\rho_i^4 s_{\hat{i}i}\zeta_{33}(A_i^{\Upsilon}r_{\hat{i}i}^{0.5}\cos\theta_{\hat{i}i}+6\cot\theta_{\hat{i}i})(t_i^3\delta_{2n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})+\delta_{n1}(A_i^{\Upsilon}r_{\hat{i}i}^{0.5}\cos\theta_{\hat{i}i}\cot 2\theta_{\hat{i}i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})$$
$$+\rho_i^4 s_{\hat{i}i}r_{\hat{i}i}^{-0.5}t_2\cot 2\theta_{\hat{i}i}\cos^{-1}\theta_{\hat{i}i}+\delta_{n6}(\rho_i^4 s_{\hat{i}i}^2\lambda_i^{\vartheta}r_{\hat{i}i}^{0.5}\cot 2\theta_{\hat{i}i}\cos\theta_{\hat{i}i}+A_i^{\Upsilon}s_{\hat{i}i}^2\lambda_{33}\lambda_i^{\vartheta}r_{\hat{i}i}^{-0.5}\cos\theta_{\hat{i}i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n}))]$$

$$a_{\hat{i}5n}=\sum_{i=1}^{5}[\rho_i^5\eta_{33}s_{\hat{i}i}(A_i^{\Upsilon}r_{\hat{i}i}^{0.5}\cos\theta_{\hat{i}i}+6\cot\theta_{\hat{i}i})(t_i^3\delta_{2n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})+\delta_{n1}(A_i^{\Upsilon}r_{\hat{i}i}^{0.5}\cos\theta_{\hat{i}i}\cot 2\theta_{\hat{i}i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})$$
$$+\rho_i^5 s_{\hat{i}i}r_{\hat{i}i}^{-0.5}t_2\cot 2\theta_{\hat{i}i}\cos^{-1}\theta_{\hat{i}i}+\delta_{n6}(\rho_i^5 s_{\hat{i}i}^2\lambda_i^{\vartheta}r_{\hat{i}i}^{0.5}\cot 2\theta_{\hat{i}i}\cos\theta_{\hat{i}i}+A_i^{\Upsilon}s_{\hat{i}i}^2\lambda_{33}\lambda_i^{\vartheta}r_{\hat{i}i}^{-0.5}\cos\theta_{\hat{i}i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n}))]$$

$$a_{\hat{i}6n}=\sum_{i=1}^{5}A_i^{\Upsilon}[r_{\hat{i}i}^{-0.5}\cos\theta_{\hat{i}i}(\lambda_{32}+4\lambda_{33}s_{\hat{i}i}^2)(t_i^3\delta_{3n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})+\delta_{1n}(r_{\hat{i}i}^{-0.5}(\cos\theta_{\hat{i}i}(\lambda_{32}+4\lambda_{33}s_{\hat{i}i}^2)t_2+4\lambda_{32}t_2 s_{\hat{i}i}\lambda_i^{\vartheta}(2\lambda_{32}\cos\theta_{\hat{i}i}$$
$$+3\lambda_{33}r_{\hat{i}i}^{-2}\sin^{-1}\theta_{\hat{i}i})/3)+\delta_{6n}45\lambda_{32}s_i^2\lambda_i^{\vartheta}\cot 2\theta_{\hat{i}i}(\cos\frac{\theta_{\hat{i}i}}{2}-\cos\theta_{\hat{i}i}\sin 2\theta_{\hat{i}i}-\frac{\sin^2\theta_{\hat{i}i}\cos 3\theta_{\hat{i}i}}{6})(2s_{\hat{i}i}^2\lambda_{33}+\lambda_{32}r_{\hat{i}i}^{-1}\sin 2\theta_{\hat{i}i})]$$

$$a_{11n}=\delta_{3n}\sqrt{r_{10}}\cos\theta_{10}+\delta_{1n}(\sum_{i=1}^{5}A_i^{\Upsilon}\sqrt{r_{1i}}\cos\theta_{1i}\cot 2\theta_{1i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})+\frac{\rho_i^3 s_{1i}t_2\cot\theta_{1i}\cos^{-1}\theta_{1i}}{\sqrt{r_{1i}}})$$

$$a_{12n}=\sum_{i=1}^{5}\frac{1}{\sqrt{r_{1i}}}[\rho_i^2\cos\theta_{1i}(t_i^2\delta_{1n}+4t_i^3\delta_{2n}+4t_i^4\delta_{4n}+4t_i^5\delta_{5n})+\delta_{6n}(\lambda_{33}(15\lambda_{33}s_{1i}^2\cot 2\theta_{1i}(3\cos\theta_{1i}-3\sin^2 2\theta_{1i}\cos\theta_{1i}$$
$$-0.25\sin^2 2\theta_{1i}\cos 6\theta_{1i})-2\lambda_{33}r_{1i}^{-2}\sin^{-1}2\theta_{1i}+0.5\lambda_{32}r_{1i}^{-1}\sin^{-1}\theta(2\cos^{-1}2\theta_{1i}-3\cos 2\theta_{1i}-\cos 10\theta_{1i}))]$$

$$a_{13n}=\sum_{i=1}^{5}[\rho_i^3 s_{1i}\iota_{33}(A_i^{\Upsilon}\sqrt{r_{1i}}\cos\theta_{1i}+6\cot\theta_{1i})(t_i^3\delta_{2n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})+\delta_{n1}(A_i^{\Upsilon}\sqrt{r_{1i}}\cos\theta_{1i}\cot 2\theta_{1i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})$$
$$+\frac{\rho_i^3 t_2\cot 2\theta_{1i}\cos^{-1}\theta_{1i}s_{1i}}{\sqrt{r_{1i}}}+\delta_{n6}(\rho_i^3 s_{1i}^2\lambda_i^{\vartheta}\sqrt{r_{1i}}\cot 2\theta_{1i}\cos\theta_{1i}+\frac{A_i^{\Upsilon}s_{1i}^2\lambda_{33}\lambda_i^{\vartheta}\cos\theta_{1i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})}{\sqrt{r_{1i}}})]$$

$$a_{14n}=\sum_{i=1}^{5}[\rho_i^4 s_{1i}\zeta_{33}(A_i^{\Upsilon}\sqrt{r_{1i}}\cos\theta_{1i}+6\cot\theta_{1i})(t_i^3\delta_{2n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})+\delta_{n1}(A_i^{\Upsilon}\sqrt{r_{1i}}\cos\theta_{1i}\cot 2\theta_{1i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})$$
$$+\rho_i^4 s_{1i}\frac{1}{\sqrt{r_{1i}}}t_2\cot 2\theta_{1i}\cos^{-1}\theta_{1i}+\delta_{n6}(\rho_i^4 s_{1i}^2\lambda_i^{\vartheta}\sqrt{r_{1i}}\cot 2\theta_{1i}\cos\theta_{1i}+\frac{A_i^{\Upsilon}s_{1i}^2\lambda_{33}\lambda_i^{\vartheta}\cos\theta_{1i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})}{\sqrt{r_{1i}}})]$$

$$a_{15n}=\sum_{i=1}^{5}[\rho_i^5 s_{1i}\eta_{33}(A_i^{\Upsilon}\sqrt{r_{1i}}\cos\theta_{1i}+6\cot\theta_{1i})(t_i^3\delta_{2n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})+\delta_{n1}(A_i^{\Upsilon}\sqrt{r_{1i}}\cos\theta_{1i}\cot 2\theta_{1i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})$$
$$+\frac{\rho_i^5 t_2\cot 2\theta_{1i}\cos^{-1}\theta_{1i}s_{2i}}{\sqrt{r_{1i}}}+\delta_{n6}(\rho_i^5 s_{1i}^2\lambda_i^9\sqrt{r_{1i}}\cot 2\theta_{1i}\cos\theta_{1i}+\frac{A_i^{\Upsilon}s_{1i}^2\lambda_{33}\lambda_i^9\cos\theta_{1i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})}{\sqrt{r_{1i}}})]$$

$$a_{16n}=\sum_{i=1}^{5}A_i^{\Upsilon}[\frac{\cos\theta_{1i}(\lambda_{32}+4\lambda_{33}s_{2i}^2)(t_i^3\delta_{3n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})}{\sqrt{r_{1i}}}+\delta_{1n}(\frac{(\cos\theta_{1i}(\lambda_{32}+4\lambda_{33}s_{2i})t_2}{\sqrt{r_{1i}}}+4\lambda_{32}t_2 s_{1i}^2\lambda_i^9(2\lambda_{32}\cos\theta_{1i}$$
$$+3\lambda_{33}r_{1i}^{-2}\sin^{-1}\theta_{1i})/3)+\delta_{6n}45\lambda_{32}s_{1i}^2\lambda_i^9\cot 2\theta_{1i}(\cos\frac{\theta_{1i}}{2}-\cos\theta_{1i}\sin 2\theta_{1i}-\frac{\cos 3\theta_{1i}\sin^2\theta_{1i}}{6})(2s_{1i}^2\lambda_{33}+\lambda_{32}r_{2i}^{-1}\sin 2\theta_{1i})]$$

$$a_{21n}=\delta_{3n}\sqrt{r_{20}}\cos\theta_{20}+\delta_{1n}(\sum_{i=1}^{5}A_i^{\Upsilon}\sqrt{r_{2i}}\cos\theta_{2i}\cot 2\theta_{2i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})+\frac{\rho_i^3 s_{2i}t_2\cot\theta_{2i}\cos^{-1}\theta_{2i}}{\sqrt{r_{2i}}})$$

$$a_{22n}=\sum_{i=1}^{5}\frac{1}{\sqrt{r_{2i}}}[\rho_i^2\cos\theta_{2i}(t_i^2\delta_{1n}+4t_i^3\delta_{2n}+4t_i^4\delta_{4n}+4t_i^5\delta_{5n})+\delta_{6n}(\lambda_{33}(15\lambda_{33}s_{2i}^2\cot 2\theta_{2i}(3\cos\theta_{2i}-3\sin^2 2\theta_{2i}\cos\theta_{2i}$$
$$-0.25\sin^2 2\theta_{2i}\cos 6\theta_{2i})-2\lambda_{33}r_{2i}^{-2}\sin^{-1}2\theta_{2i}+0.5\lambda_{32}r_{2i}^{-1}\sin^{-1}\theta(2\cos^{-1}2\theta_{2i}-3\cos 2\theta_{2i}-\cos 10\theta_{2i}))]$$

$$a_{23n}=\sum_{i=1}^{5}[\rho_i^3 s_{2i}\iota_{33}(A_i^{\Upsilon}\sqrt{r_{2i}}\cos\theta_{2i}+6\cot\theta_{2i})(t_i^3\delta_{2n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})+\delta_{n1}(A_i^{\Upsilon}\sqrt{r_{2i}}\cos\theta_{2i}\cot 2\theta_{2i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})$$
$$+\frac{\rho_i^3 t_2\cot 2\theta_{2i}\cos^{-1}\theta_{2i}s_{2i}}{\sqrt{r_{2i}}}+\delta_{n6}(\rho_i^3 s_{2i}^2\lambda_i^9\sqrt{r_{2i}}\cot 2\theta_{2i}\cos\theta_{2i}+\frac{A_i^{\Upsilon}s_{2i}^2\lambda_{33}\lambda_i^9\cos\theta_{2i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})}{\sqrt{r_{2i}}})]$$

$$a_{24n}=\sum_{i=1}^{5}[\rho_i^4 s_{2i}\zeta_{33}(A_i^{\Upsilon}\sqrt{r_{2i}}\cos\theta_{2i}+6\cot\theta_{2i})(t_i^3\delta_{2n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})+\delta_{n1}(A_i^{\Upsilon}\sqrt{r_{2i}}\cos\theta_{2i}\cot 2\theta_{2i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})$$
$$+\rho_i^4 s_{2i}\frac{1}{\sqrt{r_{2i}}}t_2\cot 2\theta_{2i}\cos^{-1}\theta_{2i}+\delta_{n6}(\rho_i^4 s_{2i}^2\lambda_i^9\sqrt{r_{2i}}\cot 2\theta_{2i}\cos\theta_{2i}+\frac{A_i^{\Upsilon}s_{2i}^2\lambda_{33}\lambda_i^9\cos\theta_i^1(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})}{\sqrt{r_{2i}}})]$$

$$a_{25n}=\sum_{i=1}^{5}[\rho_i^5 s_{2i}\eta_{33}(A_i^{\Upsilon}\sqrt{r_{2i}}\cos\theta_{2i}+6\cot\theta_{2i})(t_i^3\delta_{2n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})+\delta_{n1}(A_i^{\Upsilon}\sqrt{r_{2i}}\cos\theta_{2i}\cot 2\theta_{2i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})$$
$$+\frac{\rho_i^5 t_2\cot 2\theta_{2i}\cos^{-1}\theta_{2i}s_{2i}}{\sqrt{r_{2i}}}+\delta_{n6}(\rho_i^5 s_{2i}^2\lambda_i^9\sqrt{r_{2i}}\cot 2\theta_{2i}\cos\theta_{2i}+\frac{A_i^{\Upsilon}s_{2i}^2\lambda_{33}\lambda_i^9\cos\theta_{2i}(\iota_{33}\delta_{3n}+\varsigma_{33}\delta_{4n}+\eta_{33}\delta_{5n})}{\sqrt{r_{2i}}})]$$

$$a_{26n}=\sum_{i=1}^{5}A_i^{\Upsilon}[\frac{\cos\theta_{2i}(\lambda_{32}+4\lambda_{33}s_{2i}^2)(t_i^3\delta_{3n}+t_i^4\delta_{4n}+t_i^5\delta_{5n})}{\sqrt{r_{2i}}}+\delta_{1n}(\frac{(\cos\theta_{2i}(\lambda_{32}+4\lambda_{33}s_{2i})t_2}{\sqrt{r_{2i}}}+4\lambda_{32}t_2 s_{2i}^2\lambda_i^9(2\lambda_{32}\cos\theta_{2i}$$
$$+3\lambda_{33}r_{2i}^{-2}\sin^{-1}\theta_{2i})/3)+\delta_{6n}45\lambda_{32}s_{2i}^2\lambda_i^9\cot 2\theta_{2i}(\cos\frac{\theta_{2i}}{2}-\cos\theta_{2i}\sin 2\theta_{2i}-\frac{\cos 3\theta_2\sin^2\theta_{2i}}{6})(2s_{2i}^2\lambda_{33}+\lambda_{32}r_{2i}^{-1}\sin 2\theta_{2i})]$$

附录 C

$$n_i=c_{44}l_{24}\delta_{i1}+(c_{11}l_{24}+(c_{44}^2-l_{21}^2)l_{24}+(c_{44}\in_{11}+l_{32}^2)l_{34}+(c_{44}\mu_{11}+l_{32}^2)l_{44}+2(c_{44}e_{15}-l_{21}l_{31})l_{34}+2(c_{44}d_{15}-l_{21}l_{32})l_{41}+2(c_{44}g_{11}$$
$$+l_{31}l_{32})l_{43})\delta_{i2}+c_{11}l_{24}\delta_{i5}+(c_{11}(c_{44}l_{24}+e_{15}l_{34}+g_{15}l_{41})+c_{44}(c_{33}l_{25}+e_{33}l_{35}+g_{33}l_{45})+(c_{11}c_{33}+c_{44}^2-l_{21}^2)l_{11}+l_{31}^2l_{22}+l_{32}^2l_{33}+$$
$$(c_{11}e_{33}+c_{44}e_{15}-2l_{21}l_{32})l_{12}+(c_{11}d_{33}+c_{44}d_{15}-2l_{21}l_{32})l_{13}+2l_{31}l_{32}l_{23})\delta_{i3}+(c_{44}l_{14}+(c_{11}c_{33}-l_{21}^2)l_{24}+(c_{1133}+l_{31}^2)l_{42}+(c_{11}\mu_{33}$$
$$+l_{32}^2)l_{44}+2((c_{11}e_{33}-l_{21}l_{31})l_{12}+(c_{11}d_{33}-l_{21}l_{32})l_{34}+(c_{11}d_{33}-l_{21}l_{31})l_{41}+(c_{11}g_{33}-l_{31}l_{32})l_{43}))\delta_{i4}$$

$$n_{4l}=(-\iota_1l_{14}+\iota_3l_{21}l_{24}+(\iota_3l_{31}-\varsigma_3l_{21})l_{24}+(\iota_3l_{32}-\lambda_3l_{21})l_{41}+\varsigma_3l_{31}l_{42}+(\varsigma_3l_{32}+\lambda_3l_{31})l_{52}+\lambda_3l_{32}l_{53})\delta_{l1}+(-\iota_1(c_{44}l_{24}+e_{15}l_{34}$$
$$+d_{15}l_{31})-(\iota_1c_{33}-\iota_3l_{34})l_{11}-(\iota_1e_{33}-\iota_3l_{31}+d_3l_{21})l_{12}-(\iota_1d_{33}-\iota_3l_{32}+\lambda_3l_{21})l_{13}+(\varsigma_3l_{32}+\lambda_3l_{31})l_{23}+\varsigma_3l_{31}l_{22}+\lambda_3l_{32}l_{33})\delta_{l2}$$
$$+(-\iota_1(c_{44}l_{11}+e_{15}l_{12}+d_{15}l_{13})-(\iota_1c_{33}-\iota_3l_{21})l_{25}-(\iota_1e_{33}-\iota\beta_3l_{31}+\varsigma_3l_{21})l_{35}-(\iota_1d_{33}-\iota_3l_{32}+\lambda_3l_{21})l_{45}+(\varsigma_3l_{32}+\lambda_3l_{31})l_{52}$$
$$+\varsigma_3l_{31}l_{51}+\lambda_3l_{32}l_{33})\delta_{l3}-\iota_1l_{15}\delta_{l4}$$

$$n_{5l}=c_{44}(\iota_3l_{24}-\varsigma_3l_{34}-\lambda_3l_{41})\delta_{1l}+((\varsigma_3c_{11}-\iota_1l_{21})l_{41}+c_{44}(\iota_3l_{11}-\varsigma_3l_{12}-\lambda_3l_{13})+(\iota_3l_{31}+\varsigma_3l_{21})(\mu_{33}l_{31}-g_{33}l_{32})-(\iota_3l_{32}+$$
$$\lambda_3l_{21})(g_{33}l_{32}-\in_{33}l_{32})+(\varsigma_3l_{32}-\lambda_3l_{31})(d_{33}l_{32}-e_{33}l_{12}))\delta_{2l}+((\iota_3c_{11}-\iota_1l_{21})l_{11}-c_{44}(\varsigma_3c_{11}+\iota_1l_{32})l_{12}-(\lambda_3c_{11}+\iota_1l_{32})l_{13}$$
$$-c_{44}(\varsigma_3l_{35}-\iota_3l_{25}-\lambda_3l_{45})+(\iota_3l_{32}+\varsigma_3l_{21})(_{11}l_{32}-g_{11}l_{32})-(\iota_3l_{32}+\lambda_3l_{21})(g_{11}l_{32}-\in_{11}l_{32})+(\varsigma_3l_{32}-\lambda_3l_{31})(d_{15}l_{32}-$$
$$e_{15}l_{12}))\delta_{3l}+((\iota_3c_{11}-\iota_1l_{21})l_{25}-(\varsigma_3c_{11}+\iota_1l_{32})l_{35}-(\lambda_3c_{11}+\iota_1l_{32})l_{45})\delta_{4l}$$

$$n_{6l}=c_{44}(\iota_3l_{34}+\varsigma_3l_{42}+\lambda_3l_{43})\delta_{1l}+((\iota_3c_{11}-\iota_1l_{21})l_{34}+l_{42}(\iota_3l_{11}-\varsigma_3l_{12}-\lambda_3l_{13})+l_{43}(\lambda_3c_{11}-\iota_1l_{21})-c_{44}(\iota_3l_{12}+\varsigma_3l_{22}+$$
$$\lambda_3l_{23})-(\iota_3l_{31}+\varsigma_3l_{21})(\mu_{33}l_{21}-g_{33}l_{32})-(\iota_3l_{32}+\lambda_3l_{21})(g_{33}l_{21}+e_{33}l_{32})+(\varsigma_3l_{32}-\lambda_3l_{31})(d_{33}l_{32}-e_{33}l_{12}))\delta_{2l}+((\iota_3c_{11}$$
$$-\iota_1l_{21})l_{12}-(\varsigma_3c_{11}+\iota_1l_{42})l_{22}-(\lambda_3c_{11}+\iota_1l_{32})l_{23}+c_{44}(\varsigma_3l_{42}+\iota_3l_{35}+\lambda_3l_{43})-(\iota_3l_{32}+\varsigma_3l_{21})(\mu_{11}l_{21}+d_{15}l_{32})-(\iota_3l_{32}$$
$$+\lambda_3l_{21})(g_{11}l_{21}+e_{15}l_{32})+(\varsigma_3l_{32}-\lambda_3l_{31})(d_{15}l_{21}-c_{15}l_{32}))\delta_{3l}+((\iota_3c_{11}-\iota_1l_{21})l_{35}+(\varsigma_3c_{11}+\iota_1l_{32})l_{42}+(\lambda_3c_{11}+\iota_1l_{32})l_{52})\delta_{4l}$$

$$n_{7l}=c_{44}(\iota_3l_{41}+\varsigma_3l_{43}+\lambda_3l_{44})\delta_{1l}+((\iota_3c_{11}-\iota_1l_{21})l_{41}+l_{43}(\iota_1l_{31}+\varsigma_3c_{11})+l_{44}(\lambda_3c_{11}+\iota_1l_{32})+c_{44}(\iota_3l_{13}+\varsigma_3l_{23}+\lambda_3l_{33})+$$
$$(\iota_3l_{31}+\varsigma_3l_{21})(d_{33}l_{21}+g_{33}l_{32})-(\iota_3l_{32}+\lambda_3l_{21})(\in_{33}l_{21}+e_{33}l_{31})+(\varsigma_3l_{32}-\lambda_3l_{31})(e_{33}l_{21}-e_{33}l_{31}))\delta_{2l}+((\iota_3c_{11}-\iota_1l_{21})l_{13}$$
$$-(\varsigma_3c_{11}+\iota_1l_{32})l_{23}+(\lambda_3c_{11}+\iota_1l_{32})l_{23}+c_{44}(\varsigma_3l_{52}+\iota_3l_{45}+\lambda_3l_{53})-(\iota_3l_{31}+\varsigma_3l_{21})(d_{11}l_{21}+g_{15}l_{31})-(\iota_3l_{32}+\lambda_3l_{21})(\in_{11}$$
$$l_{21}+e_{15}l_{31})+(\varsigma_3l_{32}-\lambda_3l_{31})(e_{15}l_{21}-c_{44}l_{31}))\delta_{3l}+((\iota_3c_{11}-\iota_1l_{21})l_{45}+(\varsigma_3c_{11}+\iota_1l_{31})l_{52}+(\lambda_3c_{11}+\iota_1l_{32})l_{53})\delta_{4l}$$

$$l_{IJ}=\delta_{11}(\in_{11}\mu_{33}+\in_{33}\mu_{11}-2g_{11}g_{33})+\delta_{12}(e_{33}\mu_{11}+e_{15}\mu_{33}-g_{11}d_{33}-g_{33}d_{15})+\delta_{13}(d_{33}\in_{11}+d_{15}\in_{33}-g_{11}e_{33}-g_{33}e_{15})$$
$$+\delta_{21}(c_{13}+c_{44})+\delta_{22}(c_{44}\mu_{33}+c_{33}\mu_{11}+2d_{15}d_{33})-\delta_{23}(c_{44}g_{33}+c_{33}g_{11}+e_{15}d_{33}+e_{33}g_{15})+\delta_{31}(e_{15}+e_{31})+\delta_{33}(c_{44}$$
$$\in_{33}+c_{33}\in_{11}+2e_{15}e_{33})+\delta_{32}(d_{15}+d_{31})+\delta_{14}(g_{33}^2\in_{33}+(e_{33}^2+c_{33}\in_{33})\mu_{33}-2g_{33}e_{33}d_{33}-c_{33}d_{33}^2)+\delta_{42}(c_{33}\mu_{33}+d_{33}^2)$$
$$-\delta_{43}(c_{33}g_{33}+e_{33}d_{33})+\delta_{24}(\mu_{33}\in_{33}-g_{33}^2)+\delta_{34}(d_{33}g_{33}-e_{33}\mu_{33})+\delta_{41}(e_{33}d_{33}-g_{33}\in_{33})+\delta_{25}(\mu_{11}\in_{11}-g_{11}^2)+\delta_{35}$$
$$(d_{15}g_{11}-e_{15}\mu_{11})+\delta_{15}(g_{33}^2\in_{33}+(e_{33}^2+c_{33}\in_{33})\mu_{33}-2d_{15}e_{15}g_{11}-c_{44}g_{11}^2)+\delta_{51}(c_{44}\mu_{11}+g_{15}^2)-\delta_{52}(c_{44}g_{11}+d_{15}e_{15})$$
$$+\delta_{44}(e_{33}^2-c_{33}\in_{33})+\delta_{45}(e_{15}g_{11}-d_{15}\in_{11})$$

$$w_{iJ}=\delta_{i1}+\delta_{i2}(c_{13}+(c_{33}\kappa_{i2}+e_{33}\kappa_{i3}+d_{33}\kappa_{i4})v_i^2\kappa_{i1}^{-1}-\iota_3\kappa_{55}\delta_{i5}\kappa_{51}^{-1})+\delta_{i3}(e_{31}+(e_{33}\kappa_{i2}-\in_{33}\kappa_{i3}-g_{33}\kappa_{i4})v_i^2\kappa_{i1}^{-1}-\varsigma_3\kappa_{55}\delta_{i5}$$
$$\kappa_{51}^{-1})+\delta_{i4}(d_{31}+(d_{33}\kappa_{i2}-g_{33}\kappa_{i3}-\mu_{33}\kappa_{i4})v_i^2\kappa_{i1}^{-1}-\eta_3\kappa_{55}\delta_{i5}\kappa_{51}^{-1})+2\delta_{i5}(c_{11}-c_{66}+(c_{13}\kappa_{i2}+e_{31}\kappa_{i3}+g_{31}\kappa_{i4})v_i^2\kappa_{i1}^{-1}-$$
$$\iota_3\kappa_{55}\delta_{i5}\kappa_{51}^{-1}),\quad \kappa_{IJ}=n_{4l}v_i^6-n_{5l}v_i^4+n_{6l}v_i^2-n_{7l}$$

$$K_{IJ} = \sum_{i=1}^{5}(\delta_{I1}\delta_{J2}\rho_{i2}\hat{\rho}_{i2} + \delta_{I1}\delta_{J6}\lambda_{33}v_i^2\hat{\rho}_{i1} + \delta_{mI}(\delta_{Jt}\rho_{im}\hat{\rho}_{i1} + \delta_{\alpha J}A_{i4}\hat{\rho}_{i2}\rho_{i1} + \delta_{6J}A_{i4}v_i^3\kappa_{i5}\kappa_{i1}^{-1}\lambda_{33}\rho_{i6}))$$
$$-\delta_{I1}\delta_{J1}c_{44}v_0/4\pi$$

$$\hat{K}_{IJ} = \sum_{i=1}^{5}(\delta_{I\alpha}\delta_{J\beta}A_{i4}\lambda_{3\beta}\hat{\rho}_{i2} + \delta_{mI}\delta_{6J}v_i^3\kappa_{i5}\kappa_{i1}^{-1}\rho_{im})$$

$$\hat{\rho}_{iJ} = \delta_{i1}(c_{44}^2 D v_i + c_{44}A_{i1} + e_{15}A_{i2} + d_{15}A_{i3}) + \delta_{i2}v_i(c_{44} + \kappa_{i1}^{-1}(c_{44}\kappa_{i2} + e_{15}\kappa_{i3} + d_{15}\kappa_{i4})) - \delta_{i3}c_{13} - \delta_{i4}e_{31} - \delta_{i5}d_{31} + v_i^2\kappa_{i1}^{-1}(\delta_{i3}$$
$$(c_{33}\kappa_{i2} + e_{33}\kappa_{i3} + d_{33}\kappa_{i4} + \iota_{33}\kappa_{i5})) + \delta_{i4}(e_{33}\kappa_{i2i} - \in_{33}\kappa_{i3} - g_{33}\kappa_{i4} + \varsigma_{33}\kappa_{i5}) + \delta_{i5}((d_{33}\kappa_{i2} - g_{33}\kappa_{i3} - \mu_{33}\kappa_{i4} + \eta_{33}\kappa_{i5})))$$

$$\rho_{iJ} = \delta_{i1}(\delta_{3m}\iota_{33} + \delta_{4m}\varsigma_{33} + \delta_{5m}\eta_{33}) + \delta_{i2}(c_{44}D_i v_i + c_{44}A_{i1} + e_{15}A_{i2} + d_{15}A_{i3}) + \delta_{i3}(v_i(c_{33}A_{i1} + e_{33}A_{i2} + d_{33}A_{i3}) - D_i c_{13}) + \delta_{i4}(v_i(e_{33}A_{i1} -$$
$$\in_{33}A_{i2} - g_{33}A_{i3}) - D_i e_{31}) + \delta_{i5}(v_i(d_{33}A_{i1} - g_{33}A_{i2} - \mu_{33}A_{i3}) - D_i d_{31}) + \delta_{i6}(\delta_{3m}\iota_{33} + \delta_{4m}\varsigma_{33} + \delta_{5m}\eta_{33} + \delta_{6m})$$

$$\breve{w}_J = 0.5A_{i1}v_i(\delta_{\alpha J}(-2c_{44} + (c_{44}(1+\kappa_{i2}\kappa_{i1}^{-1}) + e_{15}(1+\kappa_{i3}\kappa_{i1}^{-1}) + d_{15}(1+\kappa_{i5}\kappa_{i1}^{-1})) + \delta_{3J}c_{13}(1+\kappa_{i2}\kappa_{i1}^{-1}) + \delta_{4J}e_{31}(1$$
$$+\kappa_{i2}\kappa_{i1}^{-1}) + \delta_{5J}d_{31}(1+\kappa_{i2}\kappa_{i1}^{-1}))$$

$$\hat{w}_J = 1.5A_{i1}\kappa_{i1}^{-1}v_i(\delta_{3J}(e_{33}(\kappa_{i2}s_i^2 + \kappa_{i3}) + d_{33}(\kappa_{i2}s_i^2 + \kappa_{i4}) + 2c_{33}\kappa_{i2}) + \delta_{4J}v_i^2(e_{33}v_i^2(\kappa_{i2}v_i^2 + \kappa_{i3}) - (g_{33}(\kappa_{i2}v_i^2 + \kappa_{i4})$$
$$-2\in_{33}\kappa_{i5}v_i\kappa_{i2}\kappa_{i1}^{-1} - \zeta_{33}\kappa_{i4})) + \delta_{5J}v_i^2(d_{33}(\kappa_{i2}v_i^2 + \kappa_{i3}) - \mu_{33}(\kappa_{i2}v_i^2 + \kappa_{i4}) - g_{33}\kappa_{i5}\kappa_{i2}\kappa_{i1}^{-1}v_i - \eta_{33}\kappa_{i5}))$$

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$$a_i = c_{44}(c_{33}l_{31} + e_{33}l_{36} - d_{33}l_{26})\delta_{i1} + (c_{11}c_{33}c_{44} + c_{44}(c_{44}l_{31} + c_{33}l_{32} + e_{33}l_{23} + d_{33}l_{22}) - l_{34}(l_{34}l_{31} + l_{35}l_{26} - l_{37}l_{36}) + l_{35}(l_{37}l_{39} - l_{34}l_{36} - l_{35}l_{38}) - l_{37}(l_{35}l_{39} - l_{34}l_{26} - l_{37}l_{28}))\delta_{i2} + (c_{11}(c_{44}l_{31} + c_{33}l_{32} + e_{33}l_{23} + d_{33}l_{22}) + c_{44}(c_{44}l_{32} + c_{33}l_{12} + d_{15}l_{24} + e_{15}l_{25}) - l_{34}(l_{34}l_{32} + l_{35}l_{13} - l_{37}l_{14}) - l_{35}(l_{32}l_{13} - l_{35}l_{15} + l_{37}l_{16}) - l_{37}(l_{35}l_{16} - l_{32}l_{14} + l_{37}l_{17}))\delta_{i3} + (c_{11}(c_{44}l_{32} + c_{33}l_{12} - 2e_{15}l_{11} + d_{15}l_{24} + e_{15}w_{25}) + c_{44}(c_{44}l_{12} + l_{27}) - l_{34}(l_{34}l_{12} + l_{35}l_{18} + l_{37}l_{19}) - l_{35}(l_{34}l_{18} - l_{3}l_{29} + l_{37}l_{22}) - l_{37}(l_{35}l_{22} - l_{34}l_{19} - l_{37}l_{21}))\delta_{i4} + c_{11}(c_{11}l_{12} + e_{15}l_{18} - d_{15}l_{19})\delta_{i5}$$

$$l_{in} = \delta_{3i}(\delta_{1n}(\in_{33}\mu_{11} - 2g_{11}^2) + \delta_{2n}(\in_{11}\mu_{33} + \in_{33}\mu_{11} - 2g_{11}g_{33}) + \delta_{3n}(g_{11}d_{33} + d_{15}e_{33}) + \delta_{4n}(c_{13} + c_{44}) + \delta_{5n}(e_{15} + e_{31}) + \delta_{6n}(e_{33}\mu_{33} - d_{33}g_{33}) + \delta_{7n}(d_{15} + d_{31}) + \delta_{8n}(c_{33}\mu_{33} + d_{33}^2) + \delta_{9n}(c_{33}g_{33} + d_{33}e_{33})) + \delta_{1i}(\delta_{1n}(g_{11}d_{33} + g_{33}d_{15}) + \delta_{2n}(c_{11}\mu_{11} - g_{11}^2) + \delta_{3n}(e_{15}\mu_{33} + e_{33}\mu_{11} - d_{15}g_{33} - d_{33}g_{11}) + \delta_{6n}(c_{44}g_{33} + c_{33}g_{11} + d_{15}e_{33} + d_{33}e_{15}) + \delta_{4n}(e_{15}g_{33} + e_{33}g_{11} - d_{15}\in_{33} - d_{33}\in_{11}) + \delta_{5n}(c_{44}\mu_{33} + c_{33}\mu_{11} + 2d_{15}d_{33}) + \delta_{7n}(c_{44}\in_{33} + c_{33}\in_{11} + 2e_{15}e_{33}) + \delta_{8n}(e_{15}\mu_{11} - d_{15}g_{11}) + l_{9n}(e_{15}g_{11} - d_{15}\in_{11})) + \delta_{2i}(\delta_{1n}(c_{44}\in_{11} + e_{15}^2) + \delta_{2n}(d_{33}\in_{11} + 2d_{15}\in_{33}) + \delta_{3n}(e_{33}\mu_{11} + 2\mu_{33}e_{15}) + \delta_{4n}(2\in_{11}d_{33} + d_{15}\in_{33} - 2e_{33}g_{11}) + l_{5n}(2\mu_{11}e_{33} + e_{33}\mu_{33}) + \delta_{6n}(e_{33}g_{33} - d_{33}\in_{33}) + \delta_{7n}(\mu_{11}e_{15}^2 + \in_{11}d_{15}^2 - 2e_{15}g_{11}d_{15}) + \delta_{8n}(c_{33}\in_{33} + e_{33}^2) + \delta_{9n}(c_{44}\mu_{11} + d_{15}^2))$$

$$n_{mi} = (c_{11}l_{12}\delta_{i1} + c_{11}l_{18}\delta_{i2} - c_{11}l_{19}\delta_{i3})\delta_{4m} + \{(c_{11}l_{32} + c_{44}l_{12} + \mu_{11}l_5^2 + \in_{11}l_{37}^2 - 2g_{11}l_{35}l_{37})\delta_{1i} + [c_{11}l_{13} + c_{44}l_{18} - l_{35}(\mu_{11}l_{34} + d_{15}l_{37}) + l_{37}(g_{11}l_{34} + e_{15}l_{37})]\delta_{2i} + [-c_{11}l_{14} - c_{44}l_{19} + l_{35}(g_{11}l_{34} + d_{15}l_{35}) - l_{37}(\in_{11}l_{34} + e_{15}l_{35})]\delta_{3i}\}\delta_{5m} + \{(c_{11}l_{31} + c_{44}l_{32} + \mu_{33}l_{35}^2 + \in_{33}l_{37}^2 - 2g_{33}l_{35}l_{37})\delta_{1i} + [c_{11}l_{26} + c_{44}l_{13} - l_{35}(\mu_{33}l_{34} + d_{33}l_{37}) + l_{37}(g_{33}l_{34} + e_{33}l_{37})]\delta_{2i} + [-c_{11}l_{26} + c_{44}l_{35} + l_{35}(g_{33}l_{34} + d_{33}l_{35}) - l_{37}(\in_{33}l_{34} + e_{33}l_{35})]\delta_{3i}\}\delta_{6m} + (c_{44}l_{31}\delta_{1i} + c_{44}l_{36}\delta_{2i} - c_{44}l_{26}\delta_{3i})\delta_{7m}$$

$$n_i = (l_{34}l_{12} + l_{35}l_{18} - l_{37}l_{18})\delta_{1i} + (l_{34}l_{32} + l_{35}l_{13} - l_{34}l_{14})\delta_{2i} + (l_{34}l_{31} + l_{35}l_{26})\delta_{3i}$$

$$\dot{\kappa}_i = -n_3 s_i^4 + n_2 s_i^2 - n_1$$

$$\kappa_{mi} = n_{7m}s_i^6 - n_{6m}s_i^4 + n_{5m}s_i^2 - n_{4m}$$

$$\dot{v}_{ij} = s_j^{-1}\delta_{4i} + \sum\nolimits_{k=1}^{3}\kappa_{kj}\dot{\kappa}_i^{-1}s_j^{-1}$$

$a'_i, n'_i, n'_{mi}, l'_{in}, \dot{\kappa}'_i, \kappa'_{mj}$ 与 $a_i, n_i, n_{mi}, l_{in}, \dot{\kappa}_i, \kappa_{mj}$ 具有相同结构

$$v_{mn} = (\dot{\kappa}_n^{-1}s_n^{-1}\kappa_{(m-1)n}\sum\nolimits_{k=2}^{4}\delta_{nk} + s_n w_{(m-3)n}\delta_{m5} + w_{(m-4)n}\sum\nolimits_{k=6}^{8}\delta_{mk} - \delta_{m1})\sum\nolimits_{k=1}^{4}\delta_{kn} + (\delta_{m1} + \dot{\kappa}_n'^{-1}s_n^{-1}\kappa'_{(m-1)n}\sum\nolimits_{k=2}^{4}\delta_{mk} + s_n w'_{(m-3)n}\delta_{m5} + w'_{(m-4)n}\sum\nolimits_{k=6}^{8}\delta_{mk})\sum\nolimits_{k=5}^{8}\delta_{kn}$$

$$w_{in} = \delta_{n1} + \delta_{n3}(\dot{\kappa}_i(e_{33}\kappa_{1i} - \in_{33}\kappa_{2i} - g_{33}\kappa_{3i}) - e_{31}) + \delta_{n2}(\dot{\kappa}_i(c_{33}\kappa_{1i} + e_{33}\kappa_{2i} + d_{33}\kappa_{3i}) - c_{13}) + \delta_{4n}(\dot{\kappa}_i(d_{33}\kappa_{1i} - g_{33}\kappa_{2i} - \mu_{33}\kappa_{3i}) - d_{31})$$

$$b_n = -\delta_{n1} - \sum\nolimits_{i=1}^{4}(\kappa_{(n-1)i}\dot{\kappa}_i^{-1}(\delta_{n2} + \delta_{n3} + \delta_{n4}) + s_i w_{(n-3)i} + w_{(n-4)i}\sum\nolimits_{k=6}^{8}\delta_{nk})$$

$$\hat{\rho}_{iJ} = \delta_{J2}(c_{44}s_i^{-1} + \dot{\kappa}_i^{-1}(c_{44}\kappa_{1i} + e_{15}\kappa_{2i} + d_{15}\kappa_{3i})) + \delta_{J3}(\dot{\kappa}_i^{-1}s_i(c_{33}\kappa_{1i} + e_{33}\kappa_{2i} + d_{33}\kappa_{3i}) - c_{13})$$
$$+\delta_{J4}(\dot{\kappa}_i^{-1}s_i(e_{33}\kappa_{1i} - \in_{33}\kappa_{2i}g_{33}\kappa_{3i}) - e_{31}) + \delta_{J5}(\dot{\kappa}_i^{-1}s_i(d_{33}\kappa_{1i} - g_{33}\kappa_{2i} - \mu_{33}\kappa_{3i}) - d_{31})$$

$$\rho_{iJ} = \delta_{2J}(c_{44}v_i^{-1} + \dot{\kappa}_i^{-1}(c_{44}D_i s_i + c_{44}A_{i1} + e_{15}A_{i2} + d_{15}A_{i3}) + \delta_{3J}(s_i(c_{33}A_{i1} + e_{33}A_{i2} + d_{33}A_{i2}) - D_i c_{13}) +$$
$$\delta_{4J}(s_i(e_{33}A_{i1} - \in_{33}A_{i2} - g_{33}A_{i3}) - D_i e_{31}) - e_{31}) + \delta_{5J}(s_i(d_{33}A_{i1} - g_{33}A_{i2} - \mu_{33}A_{i3}) - D_i d_{31})$$

$$K_{\alpha\beta} = ((D_0 + \sum\nolimits_{i=1}^{5} D_i) r_{21}^{-3} + (D_0 + \sum\nolimits_{i=1}^{5} x_2\xi_2 + D_0(x_2+\xi_2)^2) r_{21}^{-5} - ((x_2+\xi_2)^2 D_1 \sum\nolimits_{i=1}^{5} \kappa_{1i}) r_{21}^{-7} +$$
$$(\sum\nolimits_{i=1}^{5}\sum\nolimits_{k=1}^{3} \kappa_{ki} D_i \delta_{ki}) r_{21}^{-1}\dot{r}_1^{-2})\delta_{1\alpha}\delta_{1\beta} + ((x_1-\xi_1)(\sum\nolimits_{i=1}^{5} D_i(x_2+\xi_2) r_{21}^{-5} + D_2\sum\nolimits_{i=1}^{5}\kappa_{2i}x_2\xi_2$$
$$(x_2+\xi_2) r_{21}^{-7} + \dot{\kappa}_2^{-1}(r_{21}^{-3}\dot{r}_1^{-1} + r_{21}^{-2}\dot{r}_1^{-2})))\delta_{1\alpha}\delta_{2\beta} + ((x_1-\xi_1)(\sum\nolimits_{i=1}^{5} D_i(x_2+\xi_2) r_{22}^{-5} +$$
$$D_1\sum\nolimits_{i=1}^{5}\kappa_{1i}x_2\xi_2(x_2+\xi_2) r_{22}^{-7} + \dot{\kappa}_1^{-1}(r_{22}^{-3}\dot{r}_2^{-1} + r_{22}^{-2}\dot{r}_2^{-2})))\delta_{2\alpha}\delta_{1\beta} + ((D_0 + \sum\nolimits_{i=1}^{5} D_i) r_{22}^{-3} +$$
$$(D_0 + \sum\nolimits_{i=1}^{5} D_i x_2\xi_2 + D_0(x_2+\xi_2)^2) r_{22}^{-5} - r_{22}^{-7}(x_2+\xi_2)^2\dot{\kappa}_2^{-1} D_2\sum\nolimits_{i=1}^{5}\kappa_{2i})\delta_{2\alpha}\delta_{2\beta}$$

$$K_{\mathrm{mn}} = A_{\mathrm{m}} r_{2m}^{-3} + (A_m x_2\xi_2 + \sum\nolimits_{i=1}^{5}\dot{\kappa}_i^{-1}s_i^{-1}\kappa_{ni}(x_2+\xi_2)^2) r_{2\mathrm{m}}^{-5} + (\sum\nolimits_{n=1}^{5}\dot{\kappa}_i^{-1}s_i^{-1}\kappa_{ji} - A_m) r_{2m}^{-1}\dot{r}_m^{-2}$$

$$a_{1n} = \delta_{3n}\sqrt{r_0}\cos\theta_0 + \delta_{1n}(\sum\nolimits_{i=1}^{5}\frac{\rho_i^3 s_i t_2 \cot\theta_i \cos^{-1}\theta_i}{\sqrt{r_i}})$$

$$a_{2n} = \sum\nolimits_{i=1}^{5}\frac{1}{\sqrt{r_i}}\rho_i^2\cos\theta_i(t_i^2\delta_{1n} + 4t_i^3\delta_{2n} + 4t_i^4\delta_{4n} + 4t_i^5\delta_{5n})$$

$$a_{3n} = \sum\nolimits_{i=1}^{5}\frac{\rho_i^3 t_2 \cot 2\theta_i \cos^{-1}\theta_i s_i}{\sqrt{r_i}}$$

$$a_{4n} = \sum\nolimits_{i=1}^{5}\rho_i^4 s_i\frac{1}{\sqrt{r_i}} t_2\cot 2\theta_i\cos^{-1}\theta_i$$

$$a_{5n} = \sum\nolimits_{i=1}^{5}\frac{\rho_i^5 t_2\cot 2\theta_i\cos^{-1}\theta_i s_i}{\sqrt{r_i}}$$

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