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# B4.6. NOLL's Classification. Isotropic, Fluid, and Solid Materials

If the material properties<sup>1</sup> at a particle  $X$  of a body are unaltered by rotations of the reference configuration, we say that the material at  $X$  is isotropic. This gives content to the traditional loose identification of isotropy with the material properties being independent of direction; the rotation of the reference configuration corresponds to the examination of properties in different directions. To be precise, we have

Definition B4.6.1. (NOLL, 1958)<sup>2</sup> A simple body  $B$  is isotropic at a particle  $X \in B$  if  $\exists$  a reference configuration  $\mathcal{K}$  of  $B$ , called undistorted,  $\exists$

$$\text{Orth } V_3 \subset \mathcal{L}_n^X.$$

The other familiar kinds of material symmetry, such as transverse isotropy, orthotropy, and the various crystal classes, are defined by replacing the orthogonal group  $\text{Orth } V_3$  in the above definition with certain of its subgroups.

The undistorted reference configuration  $\mathcal{K}$  in the definition of an isotropic particle is, indeed, special. The material symmetry group relative to any other reference configuration  $\mathcal{K}$

<sup>1</sup> Mass density and stress response in continuum mechanical theories.

<sup>2</sup> There exist alternative approaches; CAUCHY was probably the first to consider isotropy.



is given by Theorem B4.5.4 as

$$\mathcal{Q}_{\mu}^X = \mathcal{Q}(X) \mathcal{Q}_\kappa^X \mathcal{G}^{-1}(X) \quad (\mathcal{Q} = D(\mu \circ \kappa^{-1})),$$

and it is not clear that this set contains any orthogonal tensors, let alone all of them. However, by this same machinery, we do have the intuitively expected

Theorem B4.6.1. Let a simple body  $B$  be isotropic  
at a particle  $X$  with  $\kappa$  an undistorted reference  
configuration. Then any reference configuration  $\mu$  obtained  
by a rigid deformation  $\#$  of  $\kappa(\mathcal{Q})$ , i.e.,  $\mu = \# \circ \kappa$ ),  
is also undistorted.

Proof. Exercise B4.6.1.  $\square$

Thus, while undistorted reference configurations for isotropic particles are special, they certainly are not unique.

Imagine next that we were investigating the mechanical properties of a "fluid." A typical laboratory scenario might involve pouring off a beaker of the fluid from a storage tank, carrying the beaker to a lab bench, and then pouring the fluid into some testing device. This is a rather severe deformation of the fluid, yet we have little doubt that the

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<sup>1</sup> Cf. Axiom (B2) of Definition B1.7.1.

fluid in the testing device has the same material properties as that in the storage tank. In general, if the stress response at a particle  $X$  is unaltered by any density preserving change of reference configuration, we say that the material at  $X$  is fluid; i.e., a fluid particle has the maximum amount of material symmetry. More precisely, we have

Definition B4.6.2. (NOLL, 1958) A simple body  $\mathcal{B}$  is fluid at a particle  $X \in \mathcal{B}$  if  $\exists$  a reference configuration  $\mathcal{R}$  of  $\mathcal{B}$   $\ni$

$$\mathcal{D}_n^X = \text{Unim } \mathcal{V}_3.$$

The next theorem shows that the reference configuration used in the definition of a fluid particle is of no special significance, which, of course, is why we did not give it a special name. Contrast this with the undistorted reference configuration in the definition of an isotropic particle.

Theorem B4.6.2. Let a simple body  $\mathcal{B}$  be fluid at a particle  $X$ . Then for any configuration  $\mu$  of  $\mathcal{B}$ ,  $\mathcal{D}_\mu^X = \text{Unim } \mathcal{V}_3$ .

Proof. By Definition B4.6.2,  $\exists$  a reference configuration  $\mathcal{R}$  of  $\mathcal{B} \ni \mathcal{D}_n^X = \text{Unim } \mathcal{V}_3$ . The configuration  $\mu$  is connected to  $\mathcal{R}$  by the deformation  $\mathbb{F} = \mu \circ \mathcal{R}^{-1}$ .<sup>1</sup> Write  $\mathbb{G} = D\mathbb{F}$  and  $\mathbb{X} = \mathcal{R}(X)$ .

Suppose  $H \in \text{Unim } \mathcal{V}_3$ . Then by Theorems A3.8.4 and A3.9.9,

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<sup>1</sup> Cf. Axiom (B1) of Definition B1.7.1.

$$\hat{H} := \underline{G}^{-1}(X) \underline{H} \underline{G}(X) \in \text{Unim } V_3 = \mathcal{D}_n^X.$$

By Theorem B4.5.4, the corresponding element of  $\mathcal{D}_\mu^X$  is

$$\begin{aligned} \underline{G}(X) \hat{H} \underline{G}^{-1}(X) &= \underline{G}(X) \underline{G}^{-1}(X) \underline{H} \underline{G}(X) \underline{G}^{-1}(X) \quad (\text{substitution}) \\ &= \underline{H} \quad (\text{Theorem A3.9.3}), \end{aligned}$$

i.e.,  $\underline{H} \in \mathcal{D}_\mu^X$ . Hence,  $\text{Unim } V_3 \subset \mathcal{D}_\mu^X$ . By definition,  $\mathcal{D}_\mu^X \subset \text{Unim } V_3$ ; and  $\therefore \mathcal{D}_\mu^X = \text{Unim } V_3$ .  $\square$

Theorem B4.6.2 can be interpreted as saying that a fluid particle has no preferred reference configurations. We shall see later that this is not sufficient to characterize a fluid.

Suppose that a simple body  $\mathcal{B}$  is fluid at a particle  $X$ . Then, as we have just seen,  $\mathcal{D}_n^X = \text{Unim } V_3$  for any reference configuration  $\mathcal{K}$  of  $\mathcal{B}$ ; and since  $\text{Orth } V_3 \subset \text{Unim } V_3$ ,  $\text{Orth } V_3 \subset \mathcal{D}_n^X$ . Thus, we have

Theorem B4.6.3. Fluid particles of simple bodies are isotropic with all reference configurations being undistorted.

We would expect the stress response of a "solid" to be altered by a nonrigid deformation of its reference configuration, but

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<sup>1</sup> Cf. Theorem B4.5.2.



rigid deformations may or may not alter its response to subsequent deformations. To be precise, we make

Definition B4.6.3, (NOLL, 1958) A simple body  $\mathcal{B}$  is solid at a particle  $X \in \mathcal{B}$  if  $\exists$  a reference configuration  $\kappa$  of  $\mathcal{B}$ , called undistorted,  $\ni$

$$\mathcal{Q}_\kappa^X \subset \text{Orth } \mathcal{V}_3.$$

As expected, the undistorted reference configuration  $\kappa$  in the definition of a solid particle is of special significance. If  $\mu$  is some other reference configuration, then

$$\mathcal{Q}_\mu^X = \underline{G}(\mathbb{X}) \mathcal{Q}_\kappa^X \underline{G}'(\mathbb{X}) \quad (G = D(\mu \circ \kappa^{-1}))$$

by Theorem B4.5.4. For  $\underline{Q} \in \text{Orth } \mathcal{V}_3$ , generally

$$\underline{G}(\mathbb{X}) \underline{Q} \underline{G}'(\mathbb{X}) \notin \text{Orth } \mathcal{V}_3;$$

so generally  $\mathcal{Q}_\mu^X \notin \text{Orth } \mathcal{V}_3$ . However, we do have

Theorem B4.6.4, Let a simple body  $\mathcal{B}$  be solid at a particle  $X$  with  $\kappa$  an undistorted reference configuration. Then any reference configuration  $\mu$  obtained by a rigid deformation  $\mathbb{F}$  of  $\kappa$  of  $\mathcal{B}$ , i.e.,  $\mu = \mathbb{F} \circ \kappa$ <sup>1</sup>, is also undistorted.

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<sup>1</sup> Cf. Axiom (B2) of Definition B1.7.1.



Proof. Exercise B4.6.2.  $\square$

Suppose that a simple body  $B$  is solid at a particle  $x$ . Then  $\exists$  a reference configuration  $\kappa \ni \mathcal{Q}_\kappa^x \subset \text{Orth } V_3$ . If the particle  $x$  is also fluid, then it follows from Theorem B4.6.2 that  $\mathcal{Q}_\kappa^x = \text{Unim } V_3$ . This is impossible since

$$\text{Unim } V_3 \not\subset \text{Orth } V_3. ^1$$

Hence, the properties of solid and fluid are mutually exclusive; in other words, we have

Theorem B4.6.5. A particle of a simple body cannot be both solid and fluid.

However, it does turn out that a particle can be both solid and isotropic. To see this, we need the following group theoretic result.

Theorem B4.6.6. (Maximality of the Orthogonal Group in the Unimodular Group) If  $G \subset \text{Lin } V_3$  is a group w.r.t. tensor multiplication  $\ni$

$$\text{Orth } V_3 \subset G \subset \text{Unim } V_3,$$

then either  $G = \text{Orth } V_3$  or  $G = \text{Unim } V_3$ .

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<sup>1</sup>G. Theorem B4.5.2.

In other words, there is no group in between the orthogonal group and the unimodular group. The proof of this result is quite long and involved, and we postpone it until the end of the section so as to not interrupt the present development.

To get back to the task at hand, suppose that a simple body  $\mathcal{B}$  is solid at a particle  $X$ . Then  $\exists$  a reference configuration  $\hat{\mathcal{U}} \ni \mathcal{D}_n^X \subset \text{Orth } \mathcal{V}_3$ . If the particle  $X \in \mathcal{B}$  is also isotropic, then  $\exists$  a reference configuration  $\hat{\mathcal{U}} \ni$

$$\text{Orth } \mathcal{V}_3 \subset \mathcal{D}_{\hat{\mathcal{U}}}^X \subset \text{Unim } \mathcal{V}_3.$$

Consequently, by the maximality of  $\text{Orth } \mathcal{V}_3$  in  $\text{Unim } \mathcal{V}_3$ , either

$$\mathcal{D}_{\hat{\mathcal{U}}}^X = \text{Orth } \mathcal{V}_3 \text{ or } \mathcal{D}_{\hat{\mathcal{U}}}^X = \text{Unim } \mathcal{V}_3.$$

the latter possibility is ruled out because its truth would mean that  $X$  is a fluid particle. However, by hypothesis it is a solid particle, and it cannot be both fluid and solid (Theorem B4.6.5). Hence,

$$\mathcal{D}_{\hat{\mathcal{U}}}^X = \text{Orth } \mathcal{V}_3.$$

Next we use Theorem B4.5.4 to relate  $\mathcal{D}_n^X$  to  $\mathcal{D}_{\hat{\mathcal{U}}}^X$ ; i.e.,

$$\hat{\mathcal{Q}} \in \mathcal{D}_{\hat{\mathcal{U}}}^X = \text{Orth } \mathcal{V}_3 \iff \mathcal{Q} = \mathcal{P} \hat{\mathcal{Q}} \mathcal{P}^{-1} \in \mathcal{D}_n^X \subset \text{Orth } \mathcal{V}_3, \quad (*)$$

where  $\underline{P} = D(\mathcal{N} \circ \hat{\mathcal{N}}^{-1})(\hat{\mathcal{N}}(X))$  is the gradient of the deformation that connects the configurations  $\mathcal{N}$  and  $\hat{\mathcal{N}}$ . By Theorem A3.6.3 (iv), (\*) and the orthogonality of  $\underline{Q}$  imply

$$\underline{Q}^T \underline{Q} = [(\underline{P}^{-1})^T \hat{\underline{Q}}^T \underline{P}^T] [\underline{P} \hat{\underline{Q}} \underline{P}^T] = \underline{I} . )^1$$

Then using Theorem A3.9.12 and the associativity of tensor multiplication (Theorem A3.5.1(i)), we obtain

$$\hat{\underline{Q}}^T (\underline{P}^T \underline{P}) \hat{\underline{Q}} = \underline{P}^T \underline{P} .$$

Since this must hold  $\forall \hat{\underline{Q}} \in \text{Orth } \mathcal{V}_3$ , the tensor  $\underline{P}^T \underline{P}$  is isotropic; and by Theorem A8.1.2, it admits the representation

$$\underline{P}^T \underline{P} = \alpha \underline{I} , \quad \alpha \in \mathbb{R} .$$

We saw in the proof of the Polar Decomposition Theorem (Theorem A3.15.4) that  $\underline{P}^T \underline{P}$  is positive definite symmetric. Therefore,  $\alpha > 0$  (Exercise B4.6.3); and using Theorems A3.5.1 (iii) and A3.6.3 (iii), we can express the above result in the form

$$(\frac{1}{\sqrt{\alpha}} \underline{P})^T (\frac{1}{\sqrt{\alpha}} \underline{P}) = \underline{I} .$$

Thus, by Theorem A3.10.4,

$$\underline{\hat{R}} := \frac{1}{\sqrt{\alpha}} \underline{P} \in \text{Orth } \mathcal{V}_3 ;$$

and with manipulations similar to those above, (\*) implies

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<sup>1</sup> Cf. Theorem A3.10.4.

$$\hat{\underline{Q}} = \underline{\hat{R}}^T \underline{Q} \underline{\hat{R}} \quad \text{and} \quad \underline{Q} = \underline{\hat{R}} \hat{\underline{Q}} \underline{\hat{R}}^T. \quad (**)'$$

Recall that  $\underline{Q}$  and  $\hat{\underline{Q}}$  are typical elements of  $\mathcal{Q}_n^x$  and  $\mathcal{Q}_{\hat{n}}^x$ , respectively.

Suppose  $\tilde{\underline{Q}} \in \text{Orth } \mathcal{V}_3$ . Then since  $\text{Orth } \mathcal{V}_3$  is a group,  $\underline{\hat{R}}^T \tilde{\underline{Q}} \underline{\hat{R}} \in \text{Orth } \mathcal{V}_3 = \mathcal{Q}_{\hat{n}}^x$ . Then by (\*\*),

$$\underline{\hat{R}} (\underline{\hat{R}}^T \tilde{\underline{Q}} \underline{\hat{R}}) \underline{\hat{R}}^T = \tilde{\underline{Q}} \in \mathcal{Q}_n^x.$$

Thus,  $\text{Orth } \mathcal{V}_3 \subset \mathcal{Q}_n^x$ . But by hypothesis,  $\mathcal{Q}_n^x \subset \text{Orth } \mathcal{V}_3$ ; hence

$$\mathcal{Q}_n^x = \text{Orth } \mathcal{V}_3.$$

In summary, we have

Theorem B4.6.7. Suppose that a particle  $x$  of a simple body  $B$  is both solid, meaning  $\exists$  a reference configuration

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<sup>1</sup> This derivation of (\*\*) from (\*) was shown to the author by Dr. RANDALL S. MARLOW in 1988.



$\mathcal{M} \ni \mathcal{D}_n^x \subset \text{Orth } \mathcal{V}_3$ , and isotropic, meaning  $\exists$  a reference configuration  $\hat{n} \ni \text{Orth } \mathcal{V}_3 \subset \mathcal{D}_{\hat{n}}^x$ . Then

$$\mathcal{D}_n^x = \mathcal{D}_{\hat{n}}^x = \text{Orth } \mathcal{V}_3.$$

Thus, for an isotropic solid particle, the material symmetry groups relative to the special reference configurations employed in the definitions of solid and isotropic (both termed undistorted) each coincide with the orthogonal group.

We have seen that it is conceivable that an isotropic particle of a simple body also be fluid (Theorem B4.6.3) or also be solid (Theorem B4.6.7). A spectacular consequence of the Maximality of the Orthogonal Group in the Unimodular Group is that these are the only two possibilities for an isotropic particle.

Theorem B4.6.8, An isotropic particle of a simple body is either solid or fluid.

Proof. Suppose that a simple body  $B$  is isotropic at a particle  $X$ . Then  $\exists$  a reference configuration  $\mathcal{M} \ni$

$$\text{Orth } \mathcal{V}_3 \subset \mathcal{D}_n^x \subset \text{Unim } \mathcal{V}_3.$$

By the maximality of  $\text{Orth } \mathcal{V}_3$  in  $\text{Unim } \mathcal{V}_3$ , either

While the undistorted reference configurations  $\mathfrak{x}_0$  and  $\hat{\mathfrak{x}}_0$  above need not be the same, we do have the following corollary of Theorem B4.6.7.

Theorem B4.6.7 a. Let the particle  $X$  of a simple body be both solid and isotropic. Then any undistorted reference configuration  $\mathfrak{x}_0$  associated with the solidity of  $X$  is also an undistorted reference configuration associated with the isotropy of  $X$ . Similarly, any undistorted reference configuration  $\hat{\mathfrak{x}}_0$  associated with the isotropy of  $X$  is also an undistorted reference configuration associated with the solidity of  $X$ .

Proof. By Theorem B4.6.7,

$$\mathcal{D}_{\mathfrak{x}_0}^X = \mathcal{D}_{\hat{\mathfrak{x}}_0}^X = \text{Orth } \mathcal{V}_3.$$

Thus,  $\text{Orth } \mathcal{V}_3 \subset \mathcal{D}_{\mathfrak{x}_0}^X$ , and  $X$  is isotropic with  $\mathfrak{x}_0$  undistorted. Similarly,  $\mathcal{D}_{\hat{\mathfrak{x}}_0}^X \subset \text{Orth } \mathcal{V}_3$ , and  $X$  is solid with  $\hat{\mathfrak{x}}_0$  undistorted.  $\square$

In other words, while the terminology undistorted reference configuration has been used in two generally distinct contexts, when the contexts overlap (i.e., the particle is both solid and isotropic), then the two undistorted reference configurations may be taken as one and the same.

$$\mathcal{D}_n^X = \text{Orth } V_3,$$

in which case  $\mathcal{D}_n^X \subset \text{Orth } V_3$  and  $X$  is a solid particle,  
or

$$\mathcal{D}_n^X = \text{Unim } V_3,$$

in which case  $X$  is a fluid particle.  $\square$

We noted earlier that a fluid particle of a simple body has the maximum amount of material symmetry. It is natural ask what is the minimum amount of symmetry that a particle could have; i.e., what is the smallest possible material symmetry group  $\mathcal{D}_n^X$ ? The answer is easy to construct. By definition, we must have

$$\underline{I} \in \mathcal{D}_n^X.$$

Since  $\underline{I}^{-1} = \underline{I}$  (Theorem A3.9.7 (iv)), the inverse of  $\underline{I}$  is already included. By Theorem B4.5.6,

$$-\underline{I} \in \mathcal{D}_n^X.$$

Since  $(-\underline{I})^{-1} = -\underline{I}$  (Theorem A3.9.7 (ii) and (iv)), the inverse of  $-\underline{I}$  is already included. The products  $\underline{I}\underline{I}$ ,  $(-\underline{I})(-\underline{I})$ ,  $(\underline{I})(-\underline{I})$ , and  $(-\underline{I})(\underline{I})$ , which must be included by closure, lead to nothing new. Thus, the smallest possible material symmetry group is



$$\mathcal{D}_\mu^X = \{ \underline{I}, -\underline{I} \}.$$

Since  $\{ \underline{I}, -\underline{I} \} \subset \text{Orth } V_3$ , we are dealing with a solid particle. Formally, we make

Definition B4.6.4. A simple body  $\mathcal{B}$  is a tridinic solid at a particle  $X \in \mathcal{B}$  if  $\exists$  a reference configuration  $\mu$  of  $\mathcal{B}$ , called undistorted,  $\ni$

$$\mathcal{D}_\mu^X = \{ \underline{I}, -\underline{I} \}.$$

We have just seen that a tridinic solid particle has <sup>the</sup> least possible amount of material symmetry. Another easy result is

Theorem B4.6.9. Let a simple body  $\mathcal{B}$  be a tridinic solid at a particle  $X$ . Then for any reference configuration  $\mu$  of  $\mathcal{B}$ ,  $\mathcal{D}_\mu^X = \{ \underline{I}, -\underline{I} \}.$

Proof. Exercise B4.6.4.  $\square$

Hence, like a fluid particle <sup>1</sup>, a tridinic solid particle has no preferred reference configurations. However, the reason is different; a fluid has the largest possible amount of material symmetry, while a tridinic solid has no nontrivial material symmetry. We then see by this example that having no preferred reference configurations is insufficient to characterize a fluid particle.

<sup>1</sup> Cf. Theorem B4.6.2.



NOLL's definitions of isotropic, fluid, and solid particles of simple bodies in terms of the material symmetry group do not exhaust all of the possibilities of subgroups of the unimodular group. First of all, we have seen that isotropic particles are either solid or fluid and that a particle cannot be both solid and fluid. Since fluids are identified with the full unimodular group, the simplest way to complete the classification is to label all of those particles which are not solid.

Definition B4.6.5. (TRUEDEL, 1977) A particle  $X$  of a simple body  $B$  is a fluid crystal<sup>1</sup> if it is not solid,

I.e.,  $X \in B$  is a fluid crystal if  $\nexists$  a reference configuration  $\mathcal{R} \ni \mathcal{D}_n^X \subset \text{Orth } V_3$ , equivalently, if

$$\mathcal{D}_n^X \not\subset \text{Orth } V_3 \quad \forall \text{ configurations } \mathcal{R}.$$

Thus, for a fluid crystal,  $\mathcal{D}_n^X$  always contains some nonorthogonal elements.

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<sup>1</sup> First studied and labeled liquid crystal by B.D. COLEMAN, "Simple liquid crystals", Archive for Rational Mechanics and Analysis 20 (1965) 41-58, COLEMAN ruled out  $\mathcal{D}_n^X = \text{Unim } V_3$  so that his liquid crystals could not be fluid. The term liquid crystal is now generally reserved for the "oriented" material used in electronic displays. The continuum theory of these nonsimple materials has been developed by ERICKSEN and his associates; (cont.)

While a fluid crystal cannot be solid, it could be fluid or isotropic. In fact, we have

Theorem B4.6.10. A fluid crystal particle of a simple body is fluid iff it is isotropic.

Proof. Exercise B4.6.5.  $\square$

Thus, an anisotropic fluid crystal is neither fluid nor solid.

Following COLEMAN)<sup>1</sup>, we show by example that the notion of fluid crystal is nonempty in that  $\exists$  (at least mathematically) anisotropic fluid crystals. First, we need a proper subgroup of  $\mathcal{D}_n^+$  that contains some nonorthogonal elements. Let  $\underline{e} \in \mathcal{V}_3$  have unit length and consider

$$\mathcal{G}(\underline{e}) := \{ \underline{H} \in \text{Lin } \mathcal{V}_3 : \underline{H}\underline{e} = \pm \underline{e} \}.$$

It is easy to see (Exercise B4.6.6) that  $\mathcal{G}(\underline{e})$  is a group w.r.t. tensor multiplication. To investigate the elements of  $\mathcal{G}(\underline{e})$ , we extend  $\{\underline{e}\}$  to an orthonormal basis for  $\mathcal{V}_3$ <sup>2</sup>.

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(cont.) see, e.g., J.L. ERICKSEN, "Equilibrium theory of liquid crystals", Advances in Liquid Crystals, Vol. 2, ed. G. BROWN, Academic Press, New York (1976).

<sup>1</sup> See previous footnote.

<sup>2</sup> Cf. Exercise A2.2.1, Theorem A2.2.9, and the proof of Theorem A2.3.15.

$\{\underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}, \underline{e}_{\langle 3 \rangle}\}$  with  $\underline{e}_{\langle 1 \rangle} = \underline{e}$ . Then in terms of components<sup>1</sup>,  $H\underline{e} = \pm \underline{e}$  is equivalent to

$$\begin{bmatrix} H^{\langle 11 \rangle} & H^{\langle 21 \rangle} & H^{\langle 31 \rangle} \\ H^{\langle 21 \rangle} & H^{\langle 22 \rangle} & H^{\langle 23 \rangle} \\ H^{\langle 31 \rangle} & H^{\langle 32 \rangle} & H^{\langle 33 \rangle} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} H^{\langle 11 \rangle} \\ H^{\langle 21 \rangle} \\ H^{\langle 31 \rangle} \end{bmatrix} = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix}.$$

We see that

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

defines a unimodular tensor which is not in  $\mathcal{H}(\underline{e})$ , so  $\mathcal{H}(\underline{e}) \neq \text{Unim } \mathcal{V}_3$ . Similarly,

$$[H] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

defines an element of  $\mathcal{H}(\underline{e})$ , but it is nonorthogonal since

$$[H^T H] = [H]^T [H] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 \end{bmatrix} \neq [\underline{I}] \quad )^2$$

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<sup>1</sup>G. Theorem A3.2.9.

<sup>2</sup>G. Theorems A3.6.4, A3.5.4, A3.2.7, A3.10.4.



Thus,  $\mathcal{G}(\underline{e})$  is a group  $\ni$

$$\mathcal{G}(\underline{e}) \neq \text{Idem } \mathcal{V}_3 \text{ and } \mathcal{G}(\underline{e}) \not\subseteq \text{Orth } \mathcal{V}_3.$$

We should also ask if there is any simple body some particle of which has  $\mathcal{G}(\underline{e})$  for its material symmetry group. Consider the simple body  $\mathcal{B}$  (without memory) defined by the constitutive equation

$$\begin{aligned} \underline{T}_{mm}(X, t) &= \mathcal{G}_n^X(F_n^t(\mathcal{M}(X), \cdot)) \\ &= \alpha [F_n(\mathcal{M}(X), t)\underline{e}] \otimes [F_n(\mathcal{M}(X), t)\underline{e}], \end{aligned}$$

where  $\alpha \neq 0$  is a constant and  $\mathcal{M}$  is a reference configuration of  $\mathcal{B}$ . We shall see in Chapter C2 that such a material falls into the class of elastic bodies. We leave it for the reader to show that the above constitutive equation satisfies the Principle of Material Indifference as Exercise B4.6.7.

Since  $\alpha$  is constant, this material is homogeneous<sup>1</sup>, and by Definition B4.5.2, its material symmetry group at any particle  $X$  is given by

$$\mathcal{G}_n^X = \{ \underline{H} \in \text{Idem } \mathcal{V}_3 : (\underline{F}\underline{H}\underline{e}) \otimes (\underline{F}\underline{H}\underline{e}) = (\underline{F}\underline{e}) \otimes (\underline{F}\underline{e}) \forall \underline{F} \in \text{Inv } \mathcal{V}_3 \}$$

We will show that

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<sup>1</sup> Cf. Definition B4.1.3.

$$\mathcal{Q}_n^X = \mathcal{Y}(\underline{e}).$$

First, suppose that  $\underline{H} \in \mathcal{Y}(\underline{e})$ . Then  $\underline{H} \in \text{Inv } V_3$ , and

$$\underline{H}\underline{e} = \pm \underline{e}.$$

Consequently, by substitution and the homogeneity of the tensor product (Theorem A3.1.7 (i)),

$$(\underline{H}\underline{e}) \otimes (\underline{H}\underline{e}) = \underline{e} \otimes \underline{e},$$

which can be written as

$$\underline{H}(\underline{e} \otimes \underline{e})\underline{H}^T = \underline{e} \otimes \underline{e}$$

by Exercise A3.6.6. Then multiplication on the left by  $\underline{F}$  and on the right by  $\underline{F}^T$  yields

$$\underline{F}\underline{H}(\underline{e} \otimes \underline{e})\underline{H}^T\underline{F}^T = \underline{F}(\underline{e} \otimes \underline{e})\underline{F}^T$$

or

$$(\underline{F}\underline{H})(\underline{e} \otimes \underline{e})(\underline{F}\underline{H})^T = \underline{F}(\underline{e} \otimes \underline{e})\underline{F}^T$$

by Theorem A3.6.3 (iv). Then application of Exercise A3.6.6 in the opposite direction results in

$$(\underline{F}\underline{H}\underline{e}) \otimes (\underline{F}\underline{H}\underline{e}) = (\underline{F}\underline{e}) \otimes (\underline{F}\underline{e}) \quad \forall \underline{F} \in \text{Inv } V_3;$$

i.e.,  $\underline{H} \in \mathcal{Q}_n^X$  or  $\mathcal{Y}(\underline{e}) \subset \mathcal{Q}_n^X$ .

Next, suppose  $\underline{H} \in \mathcal{Q}_n^X$ . Then  $\underline{H} \in \text{Unim } \mathcal{V}_3$ , and

$$(\underline{F} \underline{H} \underline{e}) \otimes (\underline{F} \underline{H} \underline{e}) = (\underline{F} \underline{e}) \otimes (\underline{F} \underline{e}) \quad \forall \underline{F} \in \text{Inv } \mathcal{V}_3.$$

By reversing the steps of the previous demonstration this becomes

$$\underline{F} \underline{H} (\underline{e} \otimes \underline{e}) \underline{H}^T \underline{F}^T = \underline{F} (\underline{e} \otimes \underline{e}) \underline{F}^T,$$

which by the invertibility of  $\underline{F}$  implies <sup>1</sup>

$$\underline{H} (\underline{e} \otimes \underline{e}) \underline{H}^T = \underline{e} \otimes \underline{e}$$

or

$$(\underline{H} \underline{e}) \otimes (\underline{H} \underline{e}) = \underline{e} \otimes \underline{e}.$$

Taking the trace, <sup>2</sup> we get

$$(\underline{H} \underline{e}) \cdot (\underline{H} \underline{e}) = \underline{e} \cdot \underline{e};$$

i.e.,

$$|\underline{H} \underline{e}| = |\underline{e}|.$$

In fact,

$$\underline{H} \underline{e} = \pm \underline{e}.$$

<sup>1</sup> Cf. Theorems A3.9.3 and A3.9.12.

<sup>2</sup> Cf. Definition A3.7.1.



For suppose not. Then  $\underline{H}\underline{e} \neq \pm \underline{e}$ . Since we have just seen that  $\underline{H}\underline{e}$  and  $\underline{e}$  have the same length, they must have different directions.<sup>1</sup> Consequently, they are not parallel, and the set  $\{\underline{e}, \underline{H}\underline{e}\}$  is linearly independent.<sup>2</sup>  $\therefore$  we can extend it to a basis for  $\mathcal{V}_2$ ,  $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$  with  $\underline{e}_1 = \underline{e}$ ,  $\underline{e}_2 = \underline{H}\underline{e}$ . By Theorems A3.8.8 and A3.9.9, invertible tensors preserve bases, so  $\{\underline{F}\underline{e}_1, \underline{F}\underline{e}_2, \underline{F}\underline{e}_3\}$  is also basis for  $\mathcal{V}_3$ . By Theorem A3.2.3,  $\{(\underline{F}\underline{e}_i) \otimes (\underline{F}\underline{e}_j) : i, j \in \mathbb{I}_3\}$  is a basis for  $\text{Lin } \mathcal{V}_3$ . But by the definition of  $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$  and the starting hypothesis that  $(\underline{F}\underline{H}\underline{e}) \otimes (\underline{F}\underline{H}\underline{e}) = (\underline{F}\underline{e}) \otimes (\underline{F}\underline{e})$ , we have

$$(\underline{F}\underline{e}_1) \otimes (\underline{F}\underline{e}_1) = (\underline{F}\underline{e}) \otimes (\underline{F}\underline{e}) = (\underline{F}\underline{H}\underline{e}) \otimes (\underline{F}\underline{H}\underline{e}) = (\underline{F}\underline{e}_2) \otimes (\underline{F}\underline{e}_2),$$

which contradicts the linear independence of the basis  $\{(\underline{F}\underline{e}_i) \otimes (\underline{F}\underline{e}_j) : i, j \in \mathbb{I}_3\}$ .  $\therefore \underline{H}\underline{e} = \pm \underline{e}$ ; i.e.,  $\underline{H} \in \mathcal{G}(\underline{e})$  or  $\mathcal{L}_n^x \subset \mathcal{G}(\underline{e})$ .

Thus,  $\mathcal{L}_n^x = \mathcal{G}(\underline{e})$ , and we have a theoretical example of an anisotropic fluid crystal.

In summary, we have NOLL's Classification of Simple Materials:<sup>3</sup> A particle  $X$  of a simple body  $B$  is

<sup>2</sup> Cf. Definition A2.3.6. | <sup>1</sup> Cf. Theorem A2.3.13.

<sup>3</sup> As noted earlier, the classification was completed by TRUESDELL's definition of a fluid crystal, and some aspects of NOLL's definition of isotropy can be traced back to CAUCHY.

isotropic ~~material~~ if  $\text{Orth } V_3 \subset \mathcal{Q}_n^x$ , is undistorted;

solid ~~material~~ if  $\mathcal{Q}_n^x \subset \text{Orth } V_3$ , is undistorted;

a fluid crystal ~~material~~ if  $\mathcal{Q}_n^x \not\subset \text{Orth } V_3 \quad \forall$  configurations  $\mathcal{M}$ ;

fluid ~~material~~ if  $\mathcal{Q}_n^x = \text{Unim } V_3 \quad \forall$  configurations  $\mathcal{M}$ .

We note that while this classification is for simple materials only; the same kind of thinking can be applied to other classes of constitutive equations.

Finally, we turn to the proof of the Maximality of the Orthogonal Group in the Unimodular Group (Theorem B4.6.6). Our demonstration, which is an expanded version of NOLL's proof<sup>1</sup>, is built up from a long sequence of lemmas.

Lemma B4.6.1. Let  $S_0 \in \text{Psym } V_3 \cap \text{Unim } V_3$ . Define

$$\mathcal{H} = \{ \tilde{T} = \tilde{T}_1 \tilde{T}_2 \cdots \tilde{T}_n : \tilde{T}_i \in \text{Orth } V_3 \text{ or } \tilde{T}_i = S_0^m \text{ for } m \in \mathbb{Z}, i \in \mathbb{I}_n \}.$$

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<sup>1</sup>W. NOLL, "Proof of the maximality of the orthogonal group in the unimodular group", Archive for Rational Mechanics and Analysis 18 (1965) 100-102.

<sup>2</sup> Here,  $\mathbb{Z} = \{ \dots, -2, -1, 0, 1, 2, \dots \}$  denotes the set of all integers;  $S_0^0 := \mathbb{I}$ ; and for  $m$  a positive integer,  $S_0^{-m} = (S_0^{-1})^m$ .

Then  $\mathcal{H}$  is a group <sup>1</sup> w.r.t. tensor multiplication.

Proof. Exercise B4.6.8.  $\square$

Lemma B4.6.2. Let  $\underline{S}_0$  and  $\mathcal{H}$  be as in Lemma B4.6.1 and, in addition, let  $\underline{S}_0$  have at least two distinct eigenvalues, say  $s$  and  $t$  with  $s > t$ . Then

$$\text{Psym } \mathcal{V}_3 \cap \text{Unim } \mathcal{V}_3 \subset \mathcal{H}.$$

The proof of Lemma B4.6.2 is based on

Lemma B4.6.3. Let  $\underline{S}_0$  be as in Lemma B4.6.2. Then for every  $\xi \in [(\frac{t}{s})^2, (\frac{s}{t})^2] \exists \underline{R} \in \text{Orth } \mathcal{V}_3 \ni$  the symmetric and unimodular tensor

$$\underline{I} := \underline{S}_0^{-1} \underline{R} \underline{S}_0^2 \underline{R}^{-1} \underline{S}_0^{-1}$$

has the spectrum  $\{\xi, \xi^{-1}, 1\}$ .

Proof of Lemma B4.6.3. Let  $\underline{s}$  and  $\underline{t}$  be normalized eigenvectors of  $\underline{S}_0$  corresponding to  $s$  and  $t$ , respectively. Since  $s > t$ ,  $\underline{s}$  and  $\underline{t}$  are orthogonal.<sup>3</sup> Let  $\underline{u}$  be a unit vector perpendicular to the plane of  $\underline{s}$  and  $\underline{t}$ . By Theorem A3.13.2,

<sup>1</sup> In the language of group theory,  $\mathcal{H}$  is the group generated by  $\underline{S}_0$  and  $\text{Orth } \mathcal{V}_3$ .

<sup>2</sup> Since  $\underline{S}_0$  is positive definite,  $s, t > 0$  cf. Theorem A3.15.1.

<sup>3</sup> See Theorem A3.11.6.

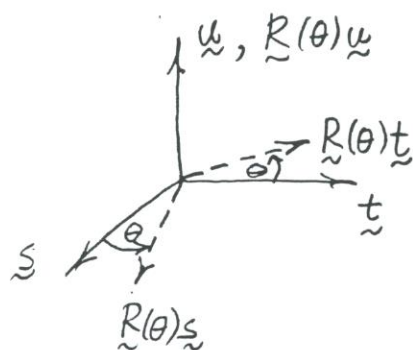


$\{\underline{s}, \underline{t}, \underline{u}\}$  is an orthonormal basis (for  $V_3$ ) of eigenvectors of  $\underline{S}_0$ . By the Spectral Theorem (Theorem A3.13.1),  $\underline{S}_0$  has the spectral resolution

$$\underline{S}_0 = s \underline{s} \otimes \underline{s} + t \underline{t} \otimes \underline{t} + u \underline{u} \otimes \underline{u}.$$

Of course,  $s, t, u > 0$  by the positive definiteness of  $\underline{S}_0$ .

Now, let  $\underline{R}(\theta)$  be the proper orthogonal tensor which corresponds to the "right-handed" rotation about the positive  $\underline{u}$ -axis through the angle  $\theta$  depicted below.)<sup>1</sup>



In terms of its action on the basis  $\{\underline{s}, \underline{t}, \underline{u}\}$ ,  $\underline{R}(\theta)$  is defined)<sup>2</sup> by

$$\underline{R}(\theta) \underline{s} = \cos \theta \underline{s} + \sin \theta \underline{t},$$

$$\underline{R}(\theta) \underline{t} = -\sin \theta \underline{s} + \cos \theta \underline{t},$$

---

<sup>1</sup> Cf. Theorem B1.3.1.

<sup>2</sup> Cf. Theorem A3.2.1.

$$\underline{R}(\theta) \underline{u} = \underline{u}.$$

We leave it for the reader as Exercise B4.6.9 to confirm analytically that  $\underline{R}(\theta)$  is, indeed, proper orthogonal and also that

$$\underline{R}^{-1}(\theta) := [\underline{R}(\theta)]^{-1} = \underline{R}(-\theta).$$

Next, consider

$$\underline{I}(\theta) := \underline{S}_0^{-1} \underline{R}(\theta) \underline{S}_0^2 \underline{R}^{-1}(\theta) \underline{S}_0^{-1},$$

That  $\underline{I}(\theta)$  is symmetric and unimodular follows from straight forward computations. We can easily calculate  $\underline{I}(\theta) \underline{u}$  stepwise:

$$\underline{S}_0^{-1} \underline{u} = \left( \frac{1}{3} \underline{s} \otimes \underline{s} + \frac{1}{6} \underline{t} \otimes \underline{t} + \frac{1}{6} \underline{u} \otimes \underline{u} \right) \underline{u} \quad (\text{Theorem A3.14.2})$$

$$= \frac{1}{3} (\underline{s} \cdot \underline{u}) \underline{s} + \frac{1}{6} (\underline{t} \cdot \underline{u}) \underline{t} + \frac{1}{6} (\underline{u} \cdot \underline{u}) \underline{u} \quad (\text{Theorem A3.1.3 and Definition A3.1.5})$$

$$= \frac{1}{6} \underline{u} \quad (\text{orthonormality of } \{\underline{s}, \underline{t}, \underline{u}\});$$

$$\underline{R}^{-1}(\theta) \underline{S}_0^{-1} \underline{u} = \underline{R}^{-1}(\theta) \left( \frac{1}{6} \underline{u} \right) \quad (\text{substitution})$$

$$= \frac{1}{6} \underline{R}^{-1}(\theta) \underline{u} \quad (\text{linearity of } \underline{R}^{-1}(\theta))$$

$$= \frac{1}{6} \underline{R}(-\theta) \underline{u} \quad (\text{Exercise B4.6.9})$$

$$= \frac{1}{6} \underline{u} \quad (\text{definition of } \underline{R}(\theta));$$

$$\begin{aligned}
S_0^2 R^{-1}(\theta) \tilde{S}_0^{-1} \underline{u} &= S_0^2 \left( \frac{1}{u} \underline{u} \right) \quad (\text{substitution}) \\
&= \frac{1}{u} S_0^2 \underline{u} \quad (\text{linearity of } S_0^2) \\
&= \frac{1}{u} (s^2 \underline{s} \otimes \underline{s} + t^2 \underline{t} \otimes \underline{t} + u^2 \underline{u} \otimes \underline{u}) \underline{u} \quad (\text{Theorem A3.14.1}) \\
&= u \underline{u} \quad (\text{as above});
\end{aligned}$$

$$\begin{aligned}
R(\theta) S_0^2 R^{-1}(\theta) \tilde{S}_0^{-1} \underline{u} &= R(\theta) (u \underline{u}) \quad (\text{substitution}) \\
&= u \underline{u} \quad (\text{definition of } R(\theta));
\end{aligned}$$

$$\begin{aligned}
\tilde{S}_0^{-1} R(\theta) S_0^2 R^{-1}(\theta) \tilde{S}_0^{-1} \underline{u} &= \tilde{S}_0^{-1} (u \underline{u}) \quad (\text{substitution}) \\
&= u \tilde{S}_0^{-1} \underline{u} \quad (\text{linearity of } \tilde{S}_0^{-1}) \\
&= u \left( \frac{1}{u} \underline{u} \right) \quad (\text{substitution}) \\
&= \underline{u} \quad \left( \frac{u}{u} = 1 \right),
\end{aligned}$$

Hence,

$$\tilde{T}(\theta) \underline{u} = \underline{u};$$

i.e.,  $(1, \underline{u})$  is an eigenpair of  $\tilde{T}(\theta) \forall \theta$ . Similar calculations (Exercise B4.6.11) show that

$$\tilde{T}(\pi/2) \underline{s} = \left( \frac{t}{s} \right)^2 \underline{s} \quad \text{and} \quad \tilde{T}(\pi/2) \underline{t} = \left( \frac{s}{t} \right)^2 \underline{t}.$$

Thus, the ordered spectrum of  $\tilde{T}(\pi/2)$  is  $\left\{ \left( \frac{t}{s} \right)^2, 1, \left( \frac{s}{t} \right)^2 \right\}$ .



It is even easier to see that  $\underline{T}(0) = \underline{I}$ , so the spectrum of  $\underline{T}(0)$  is  $\{1, 1, 1\}$ .<sup>1</sup>

In general, the eigenvalues of  $\underline{T}(\theta)$  are the roots of the characteristic polynomial

$$-\lambda^3 + \iota_1(\underline{T}(\theta))\lambda^2 - \iota_2(\underline{T}(\theta))\lambda + \iota_3(\underline{T}(\theta)). \quad )^2$$

From the discussion above, 1 is always a root; and  $\therefore$  the polynomial must factor into the form

$$-(\lambda - 1)(\lambda^2 + b(\theta)\lambda + c(\theta)).$$

Multiplying this out and comparing the result with the general form above, we see that

$$b(\theta) = 1 - \iota_1(\underline{T}(\theta)) = \iota_2(\underline{T}(\theta)) - \iota_3(\underline{T}(\theta)), \quad )^3$$

$$c(\theta) = \iota_3(\underline{T}(\theta)).$$

The other two eigenvalues are then the roots of

$$\lambda^2 + b(\theta)\lambda + c(\theta).$$

<sup>1</sup> Cf. Theorem A3.13.2.

<sup>2</sup> Cf. Theorem A3.11.B.

<sup>3</sup> This special relation between the invariants is no cause for concern; after all,  $\underline{T}(\theta)$  is rather special.

By the quadratic formula, they are

$$\lambda_1(\theta) = -\frac{1}{2}b(\theta) - \frac{1}{2}\sqrt{b^2(\theta) - 4c(\theta)}, \quad \lambda_2(\theta) = -\frac{1}{2}b(\theta) + \frac{1}{2}\sqrt{b^2(\theta) - 4c(\theta)}. \quad )^1$$

By direct calculation and the fact that  $\underline{T}(\theta)$  is unimodular,

$$\lambda_1(\theta)\lambda_2(\theta) = c(\theta) = \tau_3(\underline{T}(\theta)) = 1.$$

Thus, the two eigenvalues of  $\underline{T}(\theta)$  which are not always 1 are reciprocals. <sup>2</sup> We also see from the above formulas for them that they are continuous functions of

$$b(\theta) = 1 - \tau_1(\underline{T}(\theta)) = 1 - \text{tr } \underline{T}(\theta).$$

Now the trace function is a continuous function since it is differentiable, <sup>3</sup> and  $\underline{T}$  is a continuous function of  $\theta$  by its construction. Hence, the eigenvalues  $\lambda_1$  and  $\lambda_2$  are continuous functions of  $\theta$ . <sup>4</sup> It follows from our earlier observations that

<sup>1</sup> Since  $\underline{T}(\theta)$  is symmetric, its eigenvalues must be real; and  $\therefore$  there is no possibility of  $b^2(\theta) - 4c(\theta) < 0$ .

<sup>2</sup> Actually, this follows more directly from  $\det \underline{T}(\theta) = \lambda_1(\theta) \cdot \lambda_2(\theta) \cdot 1$ .

<sup>3</sup> Cf. Theorems A5.2.4 and A5.5.4 (v).

<sup>4</sup> The study of the continuity properties of the eigenvalues of general tensor functions is quite difficult; see T. KATO's A Short Introduction to Perturbation Theory for Linear Operators, Springer (1982).

the smaller of these two eigenvalues has the property that

$$\lambda_1(0) = 1, \quad \lambda_1(\pi/2) = \left(\frac{t}{s}\right)^2,$$

while for the larger one

$$\lambda_2(0) = 1, \quad \lambda_2(\pi/2) = \left(\frac{t}{s}\right)^{-2}.$$

By continuity, all of the values in between are assumed as  $\theta$  varies between 0 and  $\pi/2$ .<sup>1</sup> Thus, given any

$$\xi \in \left[ \left(\frac{t}{s}\right)^2, \left(\frac{t}{s}\right)^{-2} \right],$$

$\exists \theta \in [0, \pi/2]$  for which the lemma holds,  $\square$

Now we are in position for the

Proof of Lemma B4.6.2. Let  $\underline{H} \in \text{Psym } \mathcal{V}_3 \cap \text{Unim } \mathcal{V}_3$ .  
Then  $\underline{H}$  has the spectral resolution

$$\underline{H} = h_1 \underline{e}_{\langle 1 \rangle} \otimes \underline{e}_{\langle 1 \rangle} + h_2 \underline{e}_{\langle 2 \rangle} \otimes \underline{e}_{\langle 2 \rangle} + h_1^{-1} h_2^{-1} \underline{e}_{\langle 3 \rangle} \otimes \underline{e}_{\langle 3 \rangle}. \quad )^2$$

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<sup>1</sup> This is just the Intermediate Value Theorem of elementary calculus. See, e.g., COURANT's Differential and Integral Calculus, Volume I, Interscience (1934), p. 67.

<sup>2</sup> Cf. Theorems A3.13.1, A3.15.1, A3.13.4.



Define

$$H_1 = h_1 e_{\sim\langle 1 \rangle} \otimes e_{\sim\langle 1 \rangle} + h_1^{-1} e_{\sim\langle 2 \rangle} \otimes e_{\sim\langle 2 \rangle} + 1 e_{\sim\langle 3 \rangle} \otimes e_{\sim\langle 3 \rangle}$$

and

$$H_2 = 1 e_{\sim\langle 1 \rangle} \otimes e_{\sim\langle 1 \rangle} + h_1 h_2 e_{\sim\langle 2 \rangle} \otimes e_{\sim\langle 2 \rangle} + (h_1 h_2)^{-1} e_{\sim\langle 3 \rangle} \otimes e_{\sim\langle 3 \rangle}.$$

Then by direct calculation,

$$H_{\sim} = H_1 H_2.$$

Next, choose  $m_1, m_2 \in \mathbb{Z} \ni$

$$\xi_1 := (h_1)^{1/m_1} \quad \text{and} \quad \xi_2 := (h_1 h_2)^{1/m_2}$$

meet

$$\left(\frac{t}{s}\right)^2 \leq \xi_i \leq \left(\frac{s}{t}\right)^2,$$

(Show that this is possible as Exercise B4.6.11. Hint: Take logarithms.) Let  $I_1$  and  $I_2$  be the corresponding symmetric, unimodular tensors given by Lemma B4.6.3. Then  $I_1$  has the spectral resolution

$$I_1 = \xi_1 \bar{e}_{\sim\langle 1 \rangle} \otimes \bar{e}_{\sim\langle 1 \rangle} + \xi_1^{-1} \bar{e}_{\sim\langle 2 \rangle} \otimes \bar{e}_{\sim\langle 2 \rangle} + 1 \bar{e}_{\sim\langle 3 \rangle} \otimes \bar{e}_{\sim\langle 3 \rangle}.$$

and

$$\begin{aligned} I_1^{m_1} &= \xi_1^{m_1} \bar{e}_{\sim\langle 1 \rangle} \otimes \bar{e}_{\sim\langle 1 \rangle} + \xi_1^{-m_1} \bar{e}_{\sim\langle 2 \rangle} \otimes \bar{e}_{\sim\langle 2 \rangle} + \bar{e}_{\sim\langle 3 \rangle} \otimes \bar{e}_{\sim\langle 3 \rangle} \\ &= h_1 \bar{e}_{\sim\langle 1 \rangle} \otimes \bar{e}_{\sim\langle 1 \rangle} + h_1^{-1} \bar{e}_{\sim\langle 2 \rangle} \otimes \bar{e}_{\sim\langle 2 \rangle} + \bar{e}_{\sim\langle 3 \rangle} \otimes \bar{e}_{\sim\langle 3 \rangle}. \end{aligned}$$

By Theorem A3.10.8,  $\exists Q_1 \in \text{Orth } V_3 \ni$

$$\bar{e}_{\langle i \rangle} = Q_1 e_{\langle i \rangle}, \quad i \in \mathbb{I}_3.$$

Making this substitution, we see by Exercise A3.6.6, the distributivity and homogeneity of tensor multiplication (Theorem A3.5.1), and  $Q_1^T = Q_1^{-1}$  (Theorem A3.10.3) that

$$\begin{aligned} T_1^{m_1} &= Q_1 (h_1 e_{\langle 1 \rangle} \otimes e_{\langle 1 \rangle} + h_1^{-1} e_{\langle 2 \rangle} \otimes e_{\langle 2 \rangle} + e_{\langle 3 \rangle} \otimes e_{\langle 3 \rangle}) Q_1^{-1} \\ &= Q_1 H_1 Q_1^{-1}. \end{aligned}$$

$\therefore$

$$H_1 = Q_1^{-1} T_1^{m_1} Q_1. \quad (\text{Theorem A3.9.3}).$$

In the same way  $\exists Q_2 \in \text{Orth } V_3 \ni$

$$H_2 = Q_2^{-1} T_2^{m_2} Q_2 \quad (\text{Exercise B4.6.12}),$$

Whence,

$$H = H_1 H_2 = Q_1^{-1} (\bar{S}_0^{-1} R_1 \bar{S}_0^{-2} R_1^{-1} \bar{S}_0^{-1})^{m_1} Q_1 Q_2^{-1} (\bar{S}_0^{-1} R_2 \bar{S}_0^{-2} R_2^{-1} \bar{S}_0^{-1})^{m_2} Q_2,$$

which obviously is an element of  $\mathcal{H}$ .  $\therefore$

$$\text{Psym } V_3 \cap \text{Unim } V_3 \subset \mathcal{H}, \quad \square$$

Lemma B4.6.4. Let  $\bar{S}_0$  and  $\mathcal{H}$  be as in Lemma B4.6.2.

Then

$$\text{Unim } V_3 \subset \mathcal{H}.$$

Proof. Let  $H \in \text{Unim } V_3$ . Then by the Polar Decomposition Theorem (Theorem A3.15.4)  $\exists$  unique  $Q \in \text{Orth } V_3$  and  $S \in \text{Psym } V_3 \ni$

$$H = QS.$$

Since  $H \in \text{Unim } V_3$ ,  $S \in \text{Unim } V_3$  (Exercise B4.6.13); so, in fact,  $S \in \text{Psym } V_3 \cap \text{Unim } V_3$ .

By Lemma B4.6.2,  $S \in \mathcal{H}$ ; and by definition,  $Q \in \mathcal{H}$ . Since  $\mathcal{H}$  is a group w.r.t. tensor multiplication,

$$H = QS \in \mathcal{H}.$$

$\therefore \text{Unim } V_3 \subset \mathcal{H}$ .  $\square$

Finally, we are in position to prove

Theorem B4.6.6. (Maximality of the Orthogonal Group in the Unimodular Group) If  $\mathcal{G} \subset \text{Lin } V_3$  is a group w.r.t. tensor multiplication  $\ni$

$$\text{Orth } V_3 \subset \mathcal{G} \subset \text{Unim } V_3,$$

then either  $\mathcal{G} = \text{Orth } V_3$  or  $\mathcal{G} = \text{Unim } V_3$ .

Proof. There are two possibilities:



(i)  $\mathcal{G}$  contains a positive definite symmetric tensor with at least two distinct eigenvalues,

(ii)  $\mathcal{G}$  does not contain a positive definite tensor with two distinct eigenvalues.

Case (i):  $\exists \underline{S}_0 \in \mathcal{G} \ni \underline{S}_0 \in \text{Psym } V_3 \cap \text{Unim } V_3$  has at least two distinct eigenvalues, say  $s > t$ . Let  $\mathcal{H}$  be the group generated by  $\underline{S}_0$  and  $\text{Orth } V_3$  in the sense of Lemma B4.6.1. By construction, each element of  $\mathcal{H}$  is a product of powers of  $\underline{S}_0$  and orthogonal tensors. But  $\text{Orth } V_3 \subset \mathcal{G}$  and  $\underline{S}_0 \in \mathcal{G}$ , and  $\therefore$  since  $\mathcal{G}$  is a group  $\mathcal{H} \subset \mathcal{G}$ . Then by Lemma B4.6.4,  $\text{Unim } V_3 \subset \mathcal{G}$ . But by hypothesis  $\mathcal{G} \subset \text{Unim } V_3$ , so in this case

$$\mathcal{G} = \text{Unim } V_3.$$

Case (ii): Consider any  $\underline{H} \in \mathcal{G}$ . By the polar decomposition theorem,

$$\underline{H} = \underline{Q} \underline{S} \text{ with } \underline{Q} \in \text{Orth } V_3 \text{ and } \underline{S} \in \text{Psym } V_3 \cap \text{Unim } V_3.$$

Since  $\mathcal{G}$  is a group and since  $\underline{Q}^{-1} \in \text{Orth } V_3 \subset \mathcal{G}$

$$\underline{S} = \underline{Q}^{-1} \underline{H} \in \mathcal{G}.$$

$\therefore \underline{S}$  has only one distinct eigenvalue, <sup>1</sup> apply the Spectral Theorem

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<sup>1</sup>Of course, the case of three distinct eigenvalues includes the case of two distinct eigenvalues and is thereby ruled out here.

(Theorem A3.13.1),  $\underline{S}$  is spherical; i.e.,

$$\underline{S} = \lambda \underline{I}.$$

Since  $\underline{S} \in \text{Psym } V_3$ ,  $\lambda > 0$ ; and since  $\underline{S} \in \text{Umin } V_3$ ,  $\lambda = 1$  )'.

$\therefore \underline{H} = \underline{Q} \in \text{Orth } V_3$ , i.e.,  $\mathcal{Q} \subset \text{Orth } V_3$ . But by hypothesis  $\text{Orth } V_3 \subset \mathcal{Q}$ , so in this case

$$\mathcal{Q} = \text{Orth } V_3, \square$$

Supplementary Reading

Same as for § B4.1

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<sup>1</sup> $\mathcal{Q}$ . Theorem A3.8.2 (iv) and (iii).

# B4.7. Material Symmetry in the Relative Description

As was intimated in § B1.16, the relative description is particularly useful in the constitutive theory of fluids with long-range memory. But we just saw in the previous section that the notion of a particle being a fluid is defined in terms of the particle's material symmetry group. Thus, it is of some interest to investigate the material symmetry of a simple material when its constitutive equation is formulated in terms of the relative description.

We start with (SM-6) of Theorem B4.4.1; i.e.,

$$\underline{T}_m(x, t) = \underline{R}_n(x, t) \hat{H}_n^x ( \underline{R}_n^T(x, t) \underline{C}_{(t)}^+(x, \cdot) \underline{R}_n(x, t); \underline{C}_n(x, t) ) \underline{R}_n^T(x, t).$$

By Definition B4.5.2,  $\underline{H} \in \text{Unim } V_3$  is an element of the material symmetry group  $\mathcal{S}_n^x$  if replacement of  $\underline{F}_n$  by  $\underline{F}_n \underline{H}$  in the constitutive equation results in the same stress. To see how  $\underline{C}_{(t)}^+$  transforms when

$$\underline{F}_n(x, \tau) \rightarrow \underline{F}_n(x, \tau) \underline{H},$$

we use Theorem B1.16.1:

$$\begin{aligned} \underline{F}_{(t)}(x, \tau) &= \underline{F}_n(x, \tau) \underline{F}_n^{-1}(x, t) \rightarrow [\underline{F}_n(x, \tau) \underline{H}] [\underline{F}_n(x, t) \underline{H}]^{-1} \\ &= \underline{F}_n(x, \tau) \underline{F}_n^{-1}(x, t) \left( \begin{array}{l} (\underline{A} \underline{B})^{-1} = \underline{B}^{-1} \underline{A}^{-1}, \\ \text{assoc. of tensor mult., and} \\ \underline{H} \underline{H}^{-1} = \underline{I} \end{array} \right) \end{aligned}$$

$$= \underline{F}_{\sim(t)}(\mathbb{X}, \tau) \quad (\text{Theorem B1.16.1}).$$

Therefore,

$$\underline{C}_{\sim(t)}(\mathbb{X}, \tau) \rightarrow \underline{C}_{\sim(t)}(\mathbb{X}, \tau) \text{ and } \underline{C}_{\sim(t)}^t(\mathbb{X}, s) \rightarrow \underline{C}_{\sim(t)}^t(\mathbb{X}, s).$$

Of course, the transformation  $\underline{F}_{\sim n}(\mathbb{X}, \tau) \rightarrow \underline{F}_{\sim n}(\mathbb{X}, \tau) \underline{H}$  represents a change of the fixed reference configuration  $\mathbb{X}_0$ , and this should not affect the relative deformation nor the fields derived from it.

We also need to know how  $R_n$  transforms, and here we restrict our attention to orthogonal  $\underline{H}$ 's. For  $\underline{H} = \underline{Q} \in \text{Orth } V_3$ ,

$$\underline{F}_{\sim n}(\mathbb{X}, \tau) \rightarrow \underline{F}_{\sim n}(\mathbb{X}, \tau) \underline{Q} = \underline{V}_{\sim n}(\mathbb{X}, \tau) \underline{R}_{\sim n}(\mathbb{X}, \tau) \underline{Q},$$

where we have made the left polar decomposition of  $\underline{F}_{\sim n}(\mathbb{X}, \tau)$ .<sup>1</sup> By Theorem A3.10.6,  $\underline{R}_{\sim n}(\mathbb{X}, \tau) \underline{Q}$  is orthogonal; and, therefore, by the uniqueness of the polar decomposition (Theorem A3.15.4),

$$\underline{R}_{\sim n}(\mathbb{X}, \tau) \rightarrow \underline{R}_{\sim n}(\mathbb{X}, \tau) \underline{Q} \text{ and } \underline{V}_{\sim n}(\mathbb{X}, \tau) \rightarrow \underline{V}_{\sim n}(\mathbb{X}, \tau).$$

Similarly,

$$\underline{C}_{\sim n}(\mathbb{X}, \tau) = \underline{F}_{\sim n}^T(\mathbb{X}, \tau) \underline{F}_{\sim n}(\mathbb{X}, \tau) \rightarrow [\underline{F}_{\sim n}(\mathbb{X}, \tau) \underline{Q}]^T [\underline{F}_{\sim n}(\mathbb{X}, \tau) \underline{Q}]$$

$$= \underline{Q}^T [\underline{F}_{\sim n}^T(\mathbb{X}, \tau) \underline{F}_{\sim n}(\mathbb{X}, \tau)] \underline{Q} \left( \begin{array}{l} (\underline{AB})^T = \underline{B}^T \underline{A}^T \\ \text{and} \\ \text{assoc. of tensor mult.} \end{array} \right)$$

$$= \underline{Q}^T \underline{C}_{\sim n}(\mathbb{X}, \tau) \underline{Q} \quad (\text{defn. of } \underline{C}_{\sim n}).$$

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<sup>1</sup>cf. Definition B1.13.5.



Hence, if  $\underline{Q} \in \text{Orth } \mathcal{V}_3$  and  $\underline{Q} \in \mathcal{D}_n^X$ , then

$$\underline{T}_m(\underline{x}, t) = \underline{R}_n(\underline{x}, t) \underline{Q} \hat{\mathcal{H}}_n^X(\underline{Q}^T \underline{R}_n^T(\underline{x}, t) \underline{C}_{\sim(t)}^t(\underline{x}, \cdot); \underline{R}_n(\underline{x}, t) \underline{Q}; \underline{Q}^T \underline{C}_{\sim(t)}(\underline{x}, t) \underline{Q}) \underline{Q}^T \underline{R}_n^T(\underline{x}, t).$$

If the particle  $X$  is isotropic with  $\mathfrak{x}_0$  an undistorted reference configuration,<sup>1</sup> then this must hold  $\forall \underline{Q} \in \text{Orth } \mathcal{V}_3$ . In particular, for given  $X$  and  $t$ , we can choose  $\underline{Q} = \underline{R}_n^T(\underline{x}, t)$  to get

$$\underline{T}_m(\underline{x}, t) = \hat{\mathcal{H}}_n^X(\underline{C}_{\sim(t)}^t(\underline{x}, \cdot); \underline{R}_n(\underline{x}, t) \underline{C}_{\sim(t)}(\underline{x}, t) \underline{R}_n^T(\underline{x}, t)).$$

By Theorem B1.5.1 (iii),  $\underline{R}_n \underline{C}_{\sim(t)} \underline{R}_n^T$  is the left deformation tensor  $\underline{B}_n$ ; and thus we are led to

Theorem B4.7.1. If a particle  $X$  of a simple body  $\mathcal{B}$  is isotropic with the reference configuration  $\mathfrak{x}_0$  undistorted, then its constitutive equation can be written as

$$\underline{T}_m(\underline{x}, t) = \hat{\mathcal{H}}_n^X(\underline{C}_{\sim(t)}^t(\underline{x}, \cdot); \underline{B}_n(\underline{x}, t)).$$

Conversely, a particle  $X$  (of a simple body  $\mathcal{B}$ ) with a constitutive equation of the above form is isotropic with the reference configuration  $\mathfrak{x}_0$  undistorted.

Proof. Only the converse assertion remains to be proven, and this is left as Exercise B4.7.1.  $\square$

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<sup>1</sup>See Definition B4.6.1.

Even though we obtained Theorem B4.7.1 by starting with the reduced constitutive equation (SM-6), the constitutive equation of Theorem B4.7.1 appears to be considerably different from any of the reduced forms of Theorem B4.4.1. Indeed, in adjusting the form of (SM-6) so as to get an isotropic material, we somehow have lost the automatic satisfaction of the Principle of Material Indifference.

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Exercise B4.7.2. Determine the restrictions placed on the response functional of Theorem B4.7.1 by the Principle of Material Indifference.

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We are especially interested in the additional simplifications that occur when the material is fluid.<sup>1</sup> Since fluids are necessarily isotropic (Theorem B4.6.3), we can start with the constitutive equation of Theorem B4.7.1; i.e.,

$$T(\underline{x}, t) = \hat{H}_{\underline{n}}^X (C_{(t)}^t(\underline{x}, \cdot); \underline{B}_{\underline{n}}(\underline{x}, t)).$$

We have already seen that when  $\underline{F}_{\underline{n}}(\underline{x}, \tau) \rightarrow \underline{F}_{\underline{n}}(\underline{x}, \tau) \underline{H}$ ,  $C_{(t)}^t(\underline{x}, s) \rightarrow C_{(t)}^t(\underline{x}, s)$ . Obviously,

$$\underline{B}_{\underline{n}}(\underline{x}, \tau) = \underline{F}_{\underline{n}}(\underline{x}, \tau) \underline{F}_{\underline{n}}^T(\underline{x}, \tau) \rightarrow \underline{F}_{\underline{n}}(\underline{x}, \tau) \underline{H} \underline{H}^T \underline{F}_{\underline{n}}^T(\underline{x}, \tau).$$

Therefore, a fluid particle  $X$  must satisfy

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<sup>1</sup>See Definition B4.6.2.



$$\underline{T}_m(X, t) = \hat{\mathbb{M}}_m^X(\underline{C}_{(t)}^t(x, \cdot); \underline{F}_n(\mathbb{X}, t) \underline{H} \underline{H}^T \underline{F}_n^T(\mathbb{X}, t)) \quad \forall \underline{H} \in \text{Linim } \mathcal{V}_3. \quad (*)$$

For given  $X$  and  $t$ , the choice  $\underline{H} = |\det \underline{F}_n(\mathbb{X}, t)|^{1/3} \underline{F}_n^{-1}(\mathbb{X}, t)$  yields

$$\underline{T}_m(X, t) = \hat{\mathbb{M}}_m^X(\underline{C}_{(t)}^t(x, \cdot); |\det \underline{F}_n(\mathbb{X}, t)|^{2/3} \underline{I}).$$

Since  $|\det \underline{F}_n(\mathbb{X}, t)|$  determines the mass density at  $x$  in the configuration  $\mathbb{X}(\cdot, t)$  (i.e.,  $\rho(x, t)$ ) through the Referential Density Equation (Theorem B2.1.5),

$$\rho(x, t) = \frac{\rho_n(\mathbb{X})}{|\det \underline{F}_n(\mathbb{X}, t)|},$$

and knowledge of the mass density at  $\mathbb{X}$  in the reference configuration  $\mathbb{X}$  (i.e.,  $\rho_n(\mathbb{X}) = \rho_n(\mathbb{X}(X))$ ),<sup>1</sup> we can write the above constitutive equation as

$$\underline{T}_m(X, t) = \hat{\mathbb{M}}_m^X(\underline{C}_{(t)}^t(x, \cdot); \rho(x, t)). \quad (**)$$

Of course, it is important ask (and answer) if  $(**)$  automatically defines a fluid for all response functions  $\hat{\mathbb{M}}_m^X$ ; which is tantamount to asking if we could get more information out of  $(*)$  by making additional choices of  $\underline{H}$ . We saw earlier that  $\underline{C}_{(t)}^t(x, \cdot)$  does not change when the reference configuration  $\mathbb{X}$  is changed. Similarly, the density  $\rho(x, t)$  has nothing to do with the reference configuration. Therefore, if the reference configuration is changed from  $\mathbb{X}$  to  $\mathbb{P}$ , say,

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<sup>1</sup>We take this knowledge as part of the prescription of the reference configuration

then

$$\underline{T}_m(x, t) = \underline{f}_{\tilde{x}}^x(\underline{C}_{\tilde{x}(t)}^t(x, \cdot); \rho(x, t)) = \underline{f}_{\tilde{\mu}}^x(\underline{C}_{\tilde{\mu}(t)}^t(x, \cdot); \rho(x, t)).$$

Therefore,  $\underline{f}_{\tilde{x}}^x = \underline{f}_{\tilde{\mu}}^x \quad \forall$  reference configurations  $\tilde{x}$  and  $\tilde{\mu}$ ; i.e., the response function is independent of the choice of reference configuration. Accordingly, we write  $(**)$  as

$$\underline{T}_m(x, t) = \underline{f}_{\tilde{x}}^x(\underline{C}_{\tilde{x}(t)}^t(x, \cdot); \rho(x, t)); \quad (\#)$$

and this clearly defines a fluid.

In view of Exercise B4.7.2 and the remark which preceded it, we must question the status of  $(\#)$  w.r.t. the Principle of Material Indifference. Following the developments in §B4.2, we consider the transformation

$$\underline{F}_n(x, \tau) \rightarrow \underline{F}_n^*(x, \tau) = \underline{Q}(\tau) \underline{F}_n(x, \tau)$$

for arbitrary  $\underline{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } \mathcal{V}_3)$ .<sup>1</sup> By Theorem B1.16.1,

$$\begin{aligned} \underline{F}_{\tilde{x}(t)}(x, \tau) &= \underline{F}_{\tilde{x}}(x, \tau) \underline{F}_{\tilde{x}}^{-1}(x, t) \rightarrow \underline{F}_{\tilde{x}(t)}^*(x, \tau) = [\underline{Q}(\tau) \underline{F}_{\tilde{x}}(x, \tau)] [\underline{Q}(t) \underline{F}_{\tilde{x}}(x, t)]^{-1} \\ &= \underline{Q}(\tau) [\underline{F}_{\tilde{x}}(x, \tau) \underline{F}_{\tilde{x}}^{-1}(x, t)] \underline{Q}^T(t) \left( \begin{array}{l} (\underline{A}\underline{B})^{-1} = \underline{B}^{-1}\underline{A}^{-1}, \\ \text{assoc. of tensor mult.,} \\ \text{and } \underline{Q}^{-1} = \underline{Q}^T \end{array} \right) \\ &= \underline{Q}(\tau) \underline{F}_{\tilde{x}(t)}(x, \tau) \underline{Q}^T(t) \quad (\text{substitution}). \end{aligned}$$

Then

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<sup>1</sup>  $N \geq 2$  is the smoothness index of (M3).



$$\begin{aligned}
\underset{\sim}{C}_{(t)}(\underline{x}, \tau) &= \underset{\sim}{F}_{(t)}^T(\underline{x}, \tau) \underset{\sim}{F}_{(t)}(\underline{x}, \tau) \rightarrow \underset{\sim}{C}_{(t)}^*(\underline{x}^*, \tau) = \\
&= [\underset{\sim}{Q}(\tau) \underset{\sim}{F}_{(t)}(\underline{x}, \tau) \underset{\sim}{Q}^T(\tau)]^T [\underset{\sim}{Q}(\tau) \underset{\sim}{F}_{(t)}(\underline{x}, \tau) \underset{\sim}{Q}^T(\tau)] \\
&= \underset{\sim}{Q}(\tau) \underset{\sim}{F}_{(t)}^T(\underline{x}, \tau) [\underset{\sim}{Q}^T(\tau) \underset{\sim}{Q}(\tau)] \underset{\sim}{F}_{(t)}(\underline{x}, \tau) \underset{\sim}{Q}^T(\tau) \left( \begin{array}{l} (\underline{A}\underline{B})^T = \underline{B}^T \underline{A}^T, \\ (\underline{A}^T)^T = \underline{A}, \text{ and} \\ \text{assoc. of tensor mult.} \end{array} \right) \\
&= \underset{\sim}{Q}(\tau) \underset{\sim}{C}_{(t)}(\underline{x}, \tau) \underset{\sim}{Q}^T(\tau) \left( \begin{array}{l} \underset{\sim}{Q}^T \underset{\sim}{Q} = \underline{I}, \text{ assoc. of tensor mult.}, \\ \text{and defn. of } \underset{\sim}{C}_{(t)} \end{array} \right),
\end{aligned}$$

and the substitution  $\tau = t - s$  yields

$$\underset{\sim}{C}_{(t)}^t(\underline{x}, s) \rightarrow \underset{\sim}{Q}(t) \underset{\sim}{C}_{(t)}^t(\underline{x}, s) \underset{\sim}{Q}^T(t).$$

On p. B4.2.22 (cf. also the remark on p. B4.2.26), we saw that

$$\rho(\underline{x}, t) \rightarrow \rho(\underline{x}, t).$$

Consequently, the Principle of Material Indifference applied to the constitutive equation (#) requires that<sup>1</sup>

$$\underset{\sim}{Q}(t) \underset{\sim}{F}^x(\underset{\sim}{C}_{(t)}^t(\underline{x}, \cdot); \rho(\underline{x}, t)) \underset{\sim}{Q}^T(t) = \underset{\sim}{F}^x(\underset{\sim}{Q}(t) \underset{\sim}{C}_{(t)}^t(\underline{x}, \cdot) \underset{\sim}{Q}^T(t); \rho(\underline{x}, t))$$

$\forall \underset{\sim}{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } \mathcal{V}_3)$ . The choice of constant  $\underset{\sim}{Q}$  results in the requirement that  $\underset{\sim}{F}^x(\cdot, \rho)$  be an isotropic function.<sup>2</sup> Conversely, if  $\underset{\sim}{F}^x$  is isotropic, then Material Indifference is

<sup>1</sup> See p. B4.2.12 and the preceding discussion.

<sup>2</sup> Cf. Definition A8.5.1.

seen to be automatically satisfied.

We gather these results together as

Theorem B4.7.2. Let the particle  $X$  of a simple body  $\mathcal{B}$  be a fluid particle. Then its constitutive equation can be written as

$$(SF) \quad \underline{T}_m(x, t) = \underline{F}^x(\underline{C}_{(t)}^t(x, \cdot); \rho(x, t)),$$

where the response function  $\underline{F}^x: C^{N-1}([0, \infty), \text{Psym } V_3) \times \mathbb{R}^+ \rightarrow \text{Sym } V_3$   
is isotropic; i.e.,

$$\underline{F}^x(\underline{Q} \underline{C} \underline{Q}^T; \rho) = \underline{Q} \underline{F}^x(\underline{C}; \rho) \underline{Q}^T \quad \forall \underline{Q} \in \text{Orth } V_3$$

and  $\forall \underline{C} \in C^{N-1}([0, \infty), \text{Psym } V_3)$  and  $\forall \rho \in \mathbb{R}^+$ .

Conversely, a particle with a constitutive equation of the above form is a simple fluid particle.

Moreover, the above constitutive equation satisfies the Principle of Material Indifference for all response functions  $\underline{F}^x$

Suppose that a simple fluid body<sup>1</sup> has been at rest for all times prior to and including the present time  $t$ . Thus, the

<sup>1</sup>  $N \geq 2$  is the smoothness index of Axiom (M3).

<sup>2</sup> I.e., a simple body every particle of which is a fluid.

body has been sitting forever in the configuration  $\mathbb{X}(\cdot; t)$ , the relative deformation  $\mathbb{X}_{(t)}(\mathbb{X}, \tau) = \mathbb{X}$ , and  $\mathbb{C}_{(t)}^t(\mathbb{X}, \varepsilon) = \mathbb{I}$ . Then for any  $\underline{Q} \in \text{Orth}_3$ ,

$$\begin{aligned} \underline{T}_m(\underline{X}, t) &= \underline{\mathbb{T}}^x(\underline{\mathbb{I}}; \rho(\mathbb{X}, t)) && (SF) \\ &= \underline{\mathbb{T}}^x(\underline{Q}\underline{\mathbb{I}}\underline{Q}^T; \rho(\mathbb{X}, t)) && \left( \begin{array}{l} \text{assoc. of tensor mult.,} \\ \underline{A}\underline{\mathbb{I}} = \underline{A}, \text{ and } \underline{Q}\underline{Q}^T = \underline{\mathbb{I}} \end{array} \right) \\ &= \underline{Q}\underline{\mathbb{T}}^x(\underline{\mathbb{I}}; \rho(\mathbb{X}, t))\underline{Q}^T && (\text{isotropy of } \underline{\mathbb{T}}^x) \\ &= \underline{Q}\underline{T}_m(\underline{X}, t)\underline{Q}^T. \end{aligned}$$

Thus, the stress  $\underline{T}_m(\underline{X}, t)$  is isotropic.<sup>1</sup>

Next, we investigate the density. By Exercise B2.1.3a,

$$\rho(\mathbb{X}_{(t)}(\mathbb{X}, \tau), \tau) = \frac{\rho(\mathbb{X}, t)}{\underline{J}_{(t)}(\mathbb{X}, \tau)},$$

which in the present circumstances reduces to

$$\rho(\mathbb{X}, \tau) = \rho(\mathbb{X}, t).$$

Thus,  $\rho(\mathbb{X}, t)$  does not depend upon  $t$ ; and, consequently, neither does

$$\underline{T}_m(\underline{X}, t) = \underline{\mathbb{T}}^x(\underline{\mathbb{I}}; \rho(\mathbb{X}, t)).$$

Hence,  $\underline{T}_s(\underline{X}, t) = \underline{T}_m(\underline{X}, t)$  is a constant (time-independent

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<sup>1</sup>See Definition AB.1.2.



isotropic tensor. Therefore, by Theorem A3.1.2, it has the hydrostatic form

$$\underline{T}_s(\underline{x}, t) = \sigma(\underline{x}) \underline{I};$$

and we have proven

Theorem B4.7.3. In a simple fluid body  $\mathcal{B}$  which has always been at rest, the stress field is constant and hydrostatic; i.e.,

$$\underline{T}_s(\underline{x}, t) = \sigma(\underline{x}) \underline{I}, \quad \underline{x} \in \mathcal{B}_t.$$

Theorem B4.7.3 is the mathematical version, in our framework, of the common assertion A fluid at rest cannot support a shear stress, or, alternatively, If a fluid at rest is subjected to a shear stress, then it must flow. Some solids also flow in this sense, and, therefore, these statements do not suffice to characterize a fluid (if we accept NOLL's classification).

It follows from Theorem A6.9.3 that the First Stress Equation of Motion (Theorem B3.4.3) reduces, in the case of a simple fluid body which has always been at rest, to

$$\nabla \sigma(\underline{x}) + \rho(\underline{x}) \underline{b}(\underline{x}) = \underline{0}, \quad \underline{x} \in \mathcal{B}_t,$$

where  $\rho$  is the constant density and  $\underline{b}$  is the body force field, which is necessarily time independent in the present circumstances and such that

$$\text{curl}(\rho \underline{b}) = \underline{0} \text{ on } \mathcal{B}_t.$$

Thus, if  $\underline{b} \equiv \underline{0}$  on  $\mathcal{B}_t$ ,  $\sigma$  is uniform (independent of position) on  $\mathcal{B}_t$ .<sup>1</sup> More generally,

$$\sigma(\underline{x}) = \sigma(\underline{x}_0) - \int_{\underline{x}_0}^{\underline{x}} \rho(\underline{s}) \underline{b}(\underline{s}) d\underline{s}, \quad )^2$$

provided  $\mathcal{B}_t$  is simply connected, where  $\underline{x}_0$  is an arbitrary point in  $\mathcal{B}_t$  and the line integral is over any continuous curve in  $\mathcal{B}_t$  from  $\underline{x}_0$  to  $\underline{x}$ .

Supplementary Reading

Same as for  $\S$  B4.1.

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<sup>1</sup> See Theorems A5.9.3 and A5.4.1.

<sup>2</sup> See, e.g., Theorem 10-37 of APOSTOL's Mathematical Analysis.

## B4.B. Internal Constraints

Some bodies are internally constrained in the sense that their constitution does not permit them to undergo arbitrary motions. The most familiar example is a rigid body, which is idealized as being able to undergo only rigid motions. A less degenerate example is provided by an incompressible body, which can sustain only volume preserving motions. In practice, bodies also are often externally constrained; e.g., a portion of the surface might be clamped or forced to move in a prescribed way. External constraints are incorporated into the model through the boundary conditions in particular problems rather than through the constitutive theory; they are not our concern in the present section.

Definition B4.B.1. A simple body  $\mathcal{B}$  is simply constrained at the particle  $X \in \mathcal{B}$  w.r.t. the reference configuration  $\kappa_0$  if  $\exists$  a function

$$\gamma_n^X: \text{Inv } \mathcal{V}_3 \rightarrow \mathbb{R}$$

$\exists$  the deformation gradient  $F_n$  corresponding to any motion  $\{\chi(\cdot, t): t \in \mathbb{R}\}$  of  $\mathcal{B}$  is required to satisfy

$$(SC) \quad \gamma_n^X(F_n(\kappa(X), t)) = 0 \quad \forall t \in \mathbb{R}.$$

The function  $\gamma_n^X$  is called the constraint function, and (SC) is called the constraint equation.



Generally, a constrained body is constrained at each of its particles; Theorem B4.8.1 will show that the constraints at the various particles may be viewed as being w.r.t. the same reference configuration. Moreover, bodies usually are constrained in the same way at each of their particles. Then the constraint function  $\delta_x^X$  does not depend on  $X$ , so we call it  $\delta_x$ ; and the constraint equation (SC) becomes an identity in  $\mathbb{X} = \mathbb{X}(X) \in \mathcal{X}(\mathcal{B})$  as well as in  $t \in \mathbb{R}$ .

The most important example is that of an incompressible body, which is internally constrained in such a way that it can undergo only isochoric motions. It follows from Theorem B1.14.5 that for an incompressible body (SC) has the form

$$\delta_x(F(\mathbb{X}, t)) := |\det \underline{F}(\mathbb{X}, t)| - \hat{J}_x(\mathbb{X}) = 0 \quad \forall (\mathbb{X}, t) \in \mathcal{X}(\mathcal{B}) \times \mathbb{R}.$$

Similarly, a rigid body is internally constrained in such a way that it can undergo only rigid motions. By Theorem B1.14.2, a motion is rigid iff

$$\underline{C}_x(\mathbb{X}, t) = \hat{\underline{C}}_x(\mathbb{X}) \quad \forall (\mathbb{X}, t) \in \mathcal{X}(\mathcal{B}) \times \mathbb{R},$$

and by the positive definiteness of the magnitude of a tensor,<sup>1</sup>

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<sup>1</sup> Cf. Definition A2.3.2, Theorem A2.3.6, and the discussion on p. A3.7.5.

this is equivalent to

$$|\underline{C}_{\sim n}(\mathbb{X}, t) - \hat{\underline{C}}_{\sim n}(\mathbb{X})| = 0 \quad \forall (\mathbb{X}, t) \in \mathcal{M}(\mathcal{B}) \times \mathbb{R}.$$

But  $\underline{C}_{\sim n} = \underline{F}_{\sim n}^T \underline{F}_{\sim n}$ ,<sup>1</sup> and  $\therefore$  for a rigid body (SC) has the form

$$\delta_{\sim n}(\underline{F}_{\sim n}(\mathbb{X}, t)) := |\underline{F}_{\sim n}^T(\mathbb{X}, t) \underline{F}_{\sim n}(\mathbb{X}, t) - \hat{\underline{C}}_{\sim n}(\mathbb{X})|^2 = 0 \quad \forall (\mathbb{X}, t) \in \mathcal{M}(\mathcal{B}) \times \mathbb{R}.$$

We shall examine the constraints of incompressibility and rigidity more fully in the next section. It will turn out that the above description of rigidity is inappropriate.

The next theorem shows that the choice of reference configuration does not play a crucial role in the definition of a simple constraint.

Theorem B4.8.1. If a simple body  $\mathcal{B}$  is simply constrained at a particle  $X$  w.r.t. one reference configuration  $\mu_0$ , then it is simply constrained at  $X$  w.r.t. any other reference configuration  $\mu$ . In fact, if  $\mu_0$  and  $\mu$  are connected through the diffeomorphism  $\# = \mu \circ \mu_0^{-1}$ ,<sup>2</sup> then for any motion  $\{\mathbb{X}(\cdot, t) : t \in \mathbb{R}\}$  of  $\mathcal{B}$

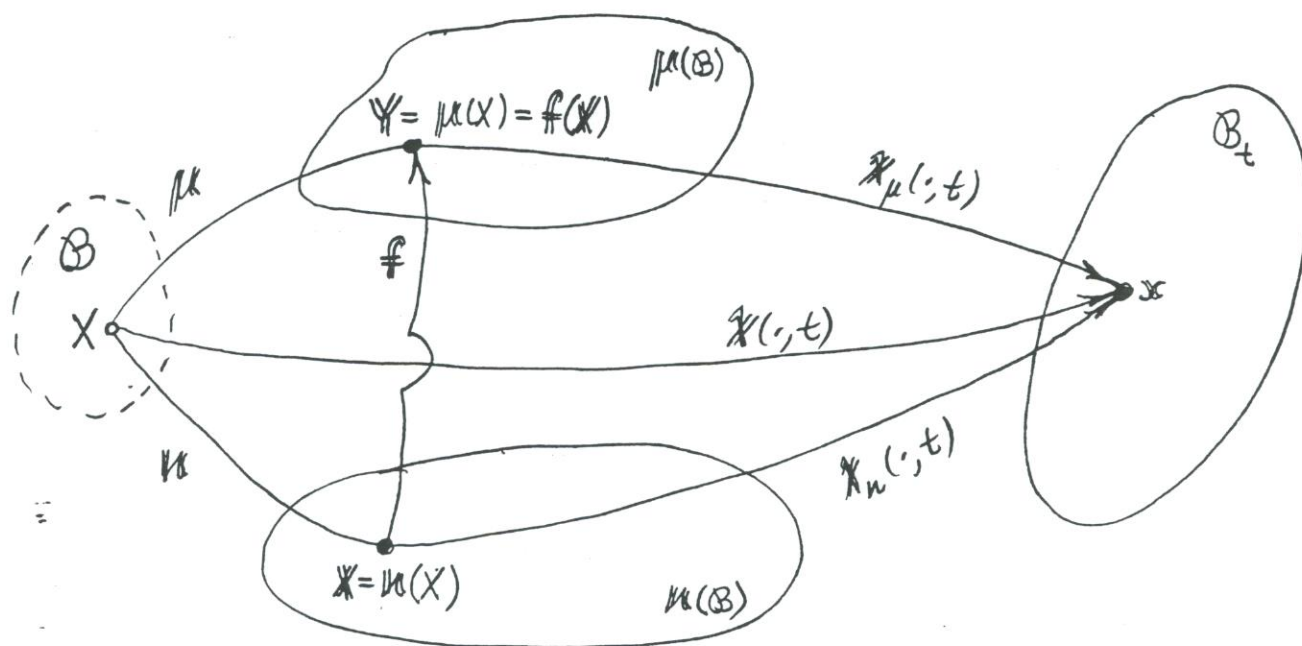
<sup>1</sup> See Definition B1.13.5,

<sup>2</sup> Here, of course,  $\underline{F}_{\sim n}^T(\mathbb{X}, t) := [\underline{F}_{\sim n}(\mathbb{X}, t)]^T$ .

<sup>3</sup> Cf. Axiom (B1) of Definition B1.7.1.

$$\gamma_{\mu}^X(F_{\mu}(\mu(X), t)) := \gamma_n^X(F_{\mu}(\mu(X), t) \underline{G}(\mu(X))) = 0, \forall t \in \mathbb{R},$$

where  $\underline{G} = Df$ .



Proof. By Theorem B1.13.B,

$$\underline{F}_n(X, t) = \underline{F}_\mu(f(X), t) \underline{G}(X).$$

$\therefore$  the constraint equation becomes

$$\begin{aligned} 0 &= \gamma_n^X(\underline{F}_n(\mu(X), t)) = \gamma_n^X(\underline{F}_\mu(\mu(X), t) \underline{G}(\mu(X))) \\ &=: \gamma_\mu^X(\underline{F}_\mu(\mu(X), t)) \quad \forall t \in \mathbb{R}. \quad \square \end{aligned}$$

Even though a simply constrained body is simply constrained w.r.t. all reference configurations, the constraint function depends



on the particular reference configuration being employed. Theorem B4.8.1 provides a transformation rule for finding the new constraint function when the reference configuration is changed.

The constraint equation

$$(SC) \quad \gamma_n^X(\underline{F}_n(\kappa(X), t)) = 0 \quad \forall t \in \mathbb{R}$$

is part of the constitutive description of the material at  $X$ , and  $\therefore$  it is subject to the Principle of Material Indifference. Since the constraint function is scalar-valued, we expect its values to be the same for all equivalent motions. Now as we saw in the development preceding Theorem B4.2.2, the deformation gradients corresponding to equivalent motions are related by

$$\underline{\tilde{F}}_n^*(\kappa, \tilde{t}) = \underline{Q}(t) \underline{F}_n(\kappa, t),$$

where  $\underline{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } \mathcal{V}_3)$ . Thus, here we want

$$\gamma_n^X(\underline{Q}(t) \underline{F}_n(\kappa(X), t)) = \gamma_n^X(\underline{F}_n(\kappa(X), t))$$

$\forall \underline{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } \mathcal{V}_3)$  and  $\forall$  motions  $\{\kappa(\cdot, t) : t \in \mathbb{R}\}$ .

Since, unlike for the stress response of §B4.2, there is no memory or history dependence involved here (Should we have

included memory effects in Definition B4.8.1?), the only thing important (for the purpose at hand) about the functions  $\underline{Q}$  and  $\underline{F}_n(\underline{u}(X), \cdot)$ <sup>1</sup> is that their values at time  $t$  are orthogonal and invertible tensors, respectively. Thus, we are lead to requiring

The Principle of Material Indifference for Simple Constraints, The constraint function  $\gamma_n^x$  must satisfy

$$(MISC) \quad \gamma_n^x(\underline{Q}\underline{F}) = \gamma_n^x(\underline{F}) \quad \forall \underline{Q} \in \text{Orth } \mathcal{V}_3 \text{ and } \forall \underline{F} \in \text{Inv } \mathcal{V}_3.$$

From our experience in § B4.2, we know how to "solve" this equation. We make the polar decomposition  $\underline{F} = \underline{R}\underline{U}$ , employing here and in the sequel the now familiar notation of Theorem A3.15.4 and its proof, to get

$$\gamma_n^x(\underline{Q}\underline{R}\underline{U}) = \gamma_n^x(\underline{F}).$$

The choice  $\underline{Q} = \underline{R}^{-1}$  leads to

$$\gamma_n^x(\underline{F}) = \gamma_n^x(\underline{U}).$$

Then since  $\underline{U} = \sqrt{\underline{C}}$ , where  $\underline{C} = \underline{F}^T \underline{F}$ ,

$$\gamma_n^x(\underline{U}) = \gamma_n^x(\sqrt{\underline{C}}) =: \chi_n^x(\underline{C}).$$

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<sup>1</sup>cf. the discussion pp. B4.1.3-4 concerning the domain of the stress response function  $\gamma_n^{x,t}$ .

Thus, we are lead to

Theorem B4.B.2. If the constraint equation

$$(SC) \quad \gamma_n^x(F_n(\kappa(x), t)) = 0 \quad \forall t \in \mathbb{R}$$

satisfies the Principle of Material Indifference for Simple Constraints in the sense that the constraint function  
 $\gamma_n^x$  meets (MISC), then it can be written in either  
of the following reduced forms:

$$(SC-1) \quad \gamma_n^x(\underline{U}_n(\kappa(x), t)) = 0 \quad \forall t \in \mathbb{R};$$

$$(SC-2) \quad \lambda_n^x(\underline{C}_n(\kappa(x), t)) = 0 \quad \forall t \in \mathbb{R};$$

where  $\lambda_n^x: \text{Psym } \mathcal{V}_3 \rightarrow \mathbb{R}$ .

Conversely, the reduced constraint equations (SC-1) and (SC-2) each satisfy the Principle of Material Indifference for Simple Constraints  $\forall$  choices of the constraint functions  $\gamma_n^x$  and  $\lambda_n^x$ .

Proof. Only the converse assertion remains to be proven.  
 This is left as Exercise B4.B.1.  $\square$

We will generally work with the reduced constraint equation (SC-2) because it is as simple as (SC),



$\underline{C}_n = \underline{F}_n^T \underline{F}_n$  is easy to calculate, and for any  $\lambda_n^x$  we can rest assured that Material Indifference is satisfied.

We will need to know what happens to the reduced constraint equation (SC-2) if the reference configuration is changed. By either applying Theorem B4.B.1 or reasoning as in its proof, we get

Theorem B4.B.3. Let a simple body  $\mathcal{B}$  be simply constrained at a particle  $X \in \mathcal{B}$  w.r.t. the reference configuration  $\mathcal{U}$  by the reduced constraint equation

$$\lambda_n^x(\underline{C}_n(\mathcal{U}(X), t)) = 0 \quad \forall t \in \mathbb{R}.$$

If  $\mu$  is any other reference configuration of  $\mathcal{B}$ , then

$$\lambda_\mu^x(\underline{C}_\mu(\mu(X), t)) := \lambda_n^x(\underline{G}^T(\mathcal{U}(X)) \underline{C}_n(\mu(X), t) \underline{G}(\mathcal{U}(X))) = 0 \quad \forall t \in \mathbb{R},$$

where  $\underline{G} = D(\mu \circ \mathcal{U}^{-1})^1$ .

Proof. Exercise B4.B.2.  $\square$

Next we exploit the fact that the constraint equation (SC-2) is an identity in time by differentiating it w.r.t. time.

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<sup>1</sup> Cf. Axiom (B1) of Definition B1.7.1.

Suppose then that  $\lambda_n^x: \text{Psym } \mathcal{V}_3 \rightarrow \mathbb{R}$  is differentiable; this makes sense since  $\text{Psym } \mathcal{V}_3$  is an open subset of the linear space  $\text{Sym } \mathcal{V}_3$ .<sup>1</sup> Then (SC-2)  $\Rightarrow$

$$0 = D\lambda_n^x(\underline{C}_n(\mathbb{X}, t)) [D_2 \underline{C}_{n,2}(\mathbb{X}, t) [\alpha]] \quad (\text{Theorem A5.7.1})$$

$$= D\lambda_n^x(\underline{C}_n(\mathbb{X}, t)) [\alpha \underline{C}_{n,2}(\mathbb{X}, t)] \quad (\text{Theorem A5.3.1})$$

$$= \alpha D\lambda_n^x(\underline{C}_n(\mathbb{X}, t)) [\underline{C}_{n,2}(\mathbb{X}, t)] \quad (\text{linearity of the derivative})$$

$$= \alpha \nabla \lambda_n^x(\underline{C}_n(\mathbb{X}, t)) \cdot \underline{C}_{n,2}(\mathbb{X}, t) \quad (\text{Theorem A5.4.1}),$$

and by the arbitrariness of  $\alpha$

$$\nabla \lambda_n^x(\underline{C}_n(\mathbb{X}, t)) \cdot \underline{C}_{n,2}(\mathbb{X}, t) = 0.$$

Now, by Theorem B1.13.10

$$\underline{C}_{n,2} = 2 \underline{F}_n^T \underline{D}_n \underline{F}_n;$$

whence

$$\nabla \lambda_n^x(\underline{C}_n) \cdot (\underline{F}_n^T \underline{D}_n \underline{F}_n) = 0.$$

But

$$\underline{A} \cdot \underline{B} = \text{tr}(\underline{A} \underline{B}^T), \quad )^2$$

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<sup>1</sup>See Theorem A5.11.4.

<sup>2</sup>See Theorem A3.7.3.

so by Theorem A3.6.3 and the symmetry of  $\underline{D}_n$ ,

$$\text{tr} [\nabla \lambda_n^x(C_n) \underline{E}_n^T \underline{D}_n \underline{E}_n] = 0.$$

By Theorem A3.7.2 (ii), this can be written as

$$\text{tr} [\underline{E}_n \nabla \lambda_n^x(C_n) \underline{E}_n^T \underline{D}_n] = [\underline{E}_n \nabla \lambda_n^x(C_n) \underline{E}_n^T] \cdot \underline{D}_n = 0.$$

Hence, we are lead to

Theorem B4.8.4. Let  $\lambda_n^x : \text{Psym } \mathcal{V}_3 \rightarrow \mathbb{R}$  be  
differentiable. Then the constraint equation

$$(SC-2) \quad \lambda_n^x(C_n(\mathbb{X}, t)) = 0 \quad \forall t \in \mathbb{R}$$

$$\Rightarrow [\underline{E}_n(\mathbb{X}, t) \nabla \lambda_n^x(C_n(\mathbb{X}, t)) \underline{E}_n^T(\mathbb{X}, t)] \cdot \underline{L}_n(\mathbb{X}, t) = 0 \quad \forall t \in \mathbb{R}.$$

"Conversely", this last identity  $\Rightarrow$

$$\lambda_n^x(C_n(\mathbb{X}, t)) = \lambda_n^x(C_n(\mathbb{X}, t_0)) \quad \forall t \in \mathbb{R},$$

where  $t_0 \in \mathbb{R}$  is arbitrary; i.e.,  $\lambda_n^x(C_n(\mathbb{X}, t))$  is  
constant.

Proof. From the preceding analysis, (SC-2)  $\Rightarrow$

$$[\underline{E}_n \nabla \lambda_n^x(C_n) \underline{E}_n^T] \cdot \underline{D}_n = 0.$$



Since the domain of  $\lambda_n^x$  is  $\text{Psym } \mathcal{V}_3 \subset \text{Sym } \mathcal{V}_3$ , it follows from Theorem A5.4.1 that

$$\nabla \lambda_n^x(\underline{C}_n) \in \text{Sym } \mathcal{V}_3;$$

which in turn  $\Rightarrow$

$$F_n \nabla \lambda_n^x(\underline{C}_n) F_n^T \in \text{Sym } \mathcal{V}_3.$$

$\therefore$  by Theorem A3.7.4,

$$[F_n \nabla \lambda_n^x(\underline{C}_n) F_n^T] \cdot D_n = [F_n \nabla \lambda_n^x(\underline{C}_n) F_n^T] \cdot L_n$$

To establish the "converse" assertion, we merely note from the above result and the analysis preceding the statement of the theorem that the identity  $\Rightarrow$

$$D\Delta(t) = 0 \quad \forall t \in \mathbb{R},$$

where, for fixed  $X$ ,  $\Delta(t) := \lambda_n^x(\underline{C}_n(X, t))$ . The assertion then follows from Theorem A5.9.3 and the connectivity of  $\mathbb{R}$ .  $\square$

The fact that the "converse" assertion in the above theorem is not a strict converse raises a subtle but important issue that has been largely overlooked in the continuum mechanics literature. The point is that our subsequent developments will be based on

$$[F_{\sim n} \nabla \lambda_n^x(C_{\sim n}) E_{\sim n}^T] \cdot L_{\sim n} = 0$$

rather than on the constraint equation

$$\lambda_n^x(C_{\sim n}) = 0.$$

Consequently, we want the former condition to be equivalent to the latter one. The present form of the "converse" assertion suggests the adjustments we need to make to obtain the desired equivalence.

First, we choose the reference configuration  $\kappa$  to be the one that the body occupies at some reference time  $t_0 \in \mathbb{R}$ ; i.e.,

$$\kappa = \kappa_0 := \mathbb{X}(\cdot, t_0).$$

Then the deformation  $\mathbb{X}_{\kappa_0}(\cdot; t_0)$  is the identity map defined by

$$\mathbb{X}_{\kappa_0}(\mathbb{X}, t_0) = \mathbb{X}, \quad \mathbb{X} \in \kappa_0(B) = B_{t_0},$$

and

$$F_{\sim \kappa_0}(\mathbb{X}, t_0) = C_{\sim \kappa_0}(\mathbb{X}, t_0) = \underline{I}, \quad \mathbb{X} \in B_{t_0}.$$

Now by the "converse" assertion of Theorem B4.B.4,

$$[F_{\sim \kappa_0} \nabla \lambda_{\kappa_0}^x(C_{\sim \kappa_0}) E_{\kappa_0}^T] \cdot L_{\sim \kappa_0} = 0$$

$\Rightarrow$

$$\lambda_{\sim \kappa_0}^x(C_{\sim \kappa}) = \lambda_{\kappa_0}^x(\underline{I}).$$

Next, we note that since

$$C_{x_0}(\mathbb{X}, t_0) = \underline{I}, \quad \mathbb{X} \in \mathcal{B}_{x_0}$$

holds  $\forall$  motions, it necessarily holds for motions that satisfy the constraint defined by

$$\lambda_{x_0}^x(C_{x_0}(\mathbb{X}, t)) = 0 \quad \forall t \in \mathbb{R}.$$

Then by considering  $t = t_0$ , we get

Theorem B4.8.5. Let  $x_0$  be the configuration  
occupied by the body at time  $t_0$ . Then the reduced  
constraint function  $\lambda_{x_0}^x$  must have the property that

$$\lambda_{x_0}^x(\underline{I}) = 0.$$

Combining this with Theorem B4.8.4 and the analysis following its proof yields

Theorem B4.8.6. Let  $x_0$  be the configuration occupied  
by the body at time  $t_0$ , and let the reduced constraint  
function  $\lambda_{x_0}^x$  be differentiable. Then the constraint equation

$$(SC-2) \quad \lambda_{x_0}^x(C_{x_0}(\mathbb{X}, t)) = 0 \quad \forall t \in \mathbb{R}$$

is equivalent to the condition



$$[F_{\sim x_0}(\mathbb{X}, t) \nabla \lambda_{x_0}^x(C_{\sim x}(\mathbb{X}, t)) F_{\sim x_0}^T(\mathbb{X}, t)] \cdot L_{\sim x_0}(\mathbb{X}, t) = 0 \quad \forall t \in \mathbb{R}.$$

For the remainder of this section then, we shall work with the reference configuration  $\mathcal{X}_0$  occupied by the body at time  $t_0$  in every allowable motion. We recall from Theorem B1.13.11 that another consequence of this, in addition to Theorems B4.8.5 and 6, is that

$$\det F_{\sim x_0}(\mathbb{X}, t) > 0 \quad \forall (\mathbb{X}, t) \in \mathcal{X}_0(\mathcal{B}) \times \mathbb{R}.$$

To avoid theorem statements that are excessively cumbersome, we shall refer to  $\mathcal{X}_0$  simply as an occupied reference configuration.

In order to proceed, we need to alter our basic constitutive equation (SM)<sup>1</sup> so as to recognize the presence of the internal constraint (SC-2). To motivate this, we recall some of our experience with constrained systems in

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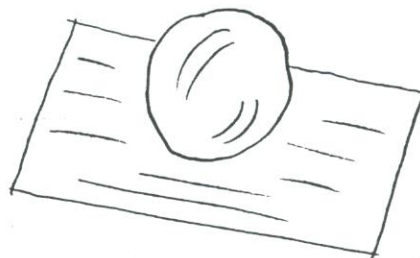
<sup>1</sup>cf. Definition B4.1.2 and Theorem B4.2.3.

simpler contexts: e.g.,

in particle mechanics — a bead sliding on a smooth wire;



in rigid body mechanics — a sphere rolling (without slip) on a plane;



in structural mechanics — a beam with pinned ends.



In such cases, the constraints are maintained by unknown (at least a priori) forces — called reactions. Moreover, the reactions have zero power in any motion that satisfies the constraint.

In continuum mechanics, we deal directly with stresses rather than forces. The Energy Transfer Theorem (Theorem B3.4.5) makes it clear that the power (per unit of volume at time  $t$ ) of a stress tensor  $\underline{T}_s$  (not necessarily symmetric) in a motion with spatial velocity gradient  $\underline{L}_s$  is the specific stress power  $\underline{T}_s \cdot \underline{L}_s$ . Thus, we are led to assume

The Principle of Determinism for an Internally Constrained Simple Body. (TRUEDELL and NOLL, 1965) Let a simple body  $\mathcal{B}$  be simply constrained w.r.t. <sup>(occupied)</sup> a reference configuration  $\kappa_0$  in the sense that for each particle  $X \in \mathcal{B} \exists$  a differentiable function

$$\lambda_{\kappa_0}^X: \text{Psym } \mathcal{V}_3 \rightarrow \mathbb{R}$$

$\ni$  the right deformation tensor  $\underline{C}_{\kappa_0}$  corresponding to any motion  $\{\chi(\cdot, t): t \in \mathbb{R}\}$  of  $\mathcal{B}$  is required to satisfy

$$(SC-2) \quad \lambda_{\kappa_0}^X(\underline{C}_{\kappa_0}(\kappa_0(X), t)) = 0 \quad \forall t \in \mathbb{R}.$$

Then the constitutive equation for  $\mathcal{B}$  is

$$(CSM) \quad \underline{T}_{\kappa_0}(X, t) = \underline{N}(X, t) + \underline{Y}_{\kappa_0}^X(\underline{F}_{\kappa_0}^t(\kappa_0(X), \cdot)), \quad (X, t) \in \mathcal{B} \times \mathbb{R},$$

where

$$\underline{N}(X, t) \cdot \underline{L}_{\kappa_0}(X, t) = 0$$

$\forall$  motions satisfying (SC-2). The domain of the response function  $\underline{Y}_{\kappa_0}^X$  is

$$\underline{Y}_{\kappa_0}^X = \left\{ \underline{F} \in C^{N-1}([0, \infty), \text{Inv } \mathcal{V}_3) : \lambda_{\kappa_0}^X(\underline{F}^T(t) \underline{F}(t)) = 0 \quad \forall t \in [0, \infty) \right\}.$$

The part of the stress determined by the history of the

<sup>1</sup> In contrast to Definition B4.1.2, because of the presence of  $\underline{N}$ , we cannot say here that the codomain must be  $\text{Sym } \mathcal{V}_3$ .



of the deformation gradient, i.e.,

B4.B.16

$$\underline{T}_m^D(X, t) = \underline{g}_{n_0}^X(\underline{F}_{n_0}^T(n_0(X), \cdot)),$$

is called the determinate stress)<sup>1</sup>. The indeterminate part of the stress  $\underline{N}(X, t)$  is called the constraint stress.<sup>2</sup>  
 $N \geq 2$  is the smoothness index of Axiom (N3).

While the constraint stress  $\underline{N}$  is not determined by the history of the deformation gradient through a constitutive equation, there are still some things to be said about it. We have the requirement that

$$\underline{N}(X, t) \cdot \underline{L}_m(X, t) = 0$$

$\forall$  motions meeting (SC-2). But by Theorem B4.B.6,  $\forall$  such motions,  $\underline{L}_{n_0}$  must satisfy

$$[\underline{F}_{n_0}(\mathbb{X}, t) \nabla \underline{g}_{n_0}^X(\underline{C}_{n_0}(\mathbb{X}, t)) \underline{F}_{n_0}^T(\mathbb{X}, t)] \cdot \underline{L}_{n_0}(\mathbb{X}, t) = 0 \quad \forall t \in \mathbb{R}.$$

Now the functions  $\underline{L}_m$  and  $\underline{L}_{n_0}$  are different, but their values agree at corresponding arguments; thus,

$$\underline{L}_m(X, t) = \underline{L}_{n_0}(\mathbb{X}, t). \quad )^3$$

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<sup>1</sup> Often, extra stress; sometimes, active stress.

<sup>2</sup> Often, reaction stress; sometimes, reactive stress.

<sup>3</sup> Cf. Definition B1.13.2.

$\therefore$  in geometric language and abridged notation,  $\underline{N}$  is orthogonal to all  $\underline{L}$  which are orthogonal  $\underline{E}_{\underline{n}_0} \nabla \lambda_{\underline{n}_0}^X(C_{\underline{n}_0}) \underline{E}_{\underline{n}_0}^T$ ; so by Theorem A2.3.12,  $\underline{N}$  is parallel to  $\underline{E}_{\underline{n}_0} \nabla \lambda_{\underline{n}_0}^X(C_{\underline{n}_0}) \underline{E}_{\underline{n}_0}^T$ . Hence, we have

Theorem B4.B.7. The constraint stress of a simply constrained simple body  $\mathcal{B}$  has the form

$$\underline{N}_{\underline{n}_0}(\underline{x}, t) = q(\underline{x}, t) \underline{E}_{\underline{n}_0}(\underline{x}, t) \nabla \lambda_{\underline{n}_0}^X(C_{\underline{n}_0}(\underline{x}, t)) \underline{E}_{\underline{n}_0}^T(\underline{x}, t), (\underline{x}, t) \in \mathcal{M}_0(\mathcal{B}) \times \mathbb{R}.$$

It is important to realize that  $q$  is not a function which characterizes the material of the body  $\mathcal{B}$ . It will vary from situation to situation (problem to problem), and in the end it is determined only after the balance principles and the boundary and initial conditions appropriate to the problem under consideration have been imposed.

An important corollary to Theorem B4.B.7 is

Theorem B4.B.8. The constraint stress  $\underline{N}(\underline{x}, t)$  and the values of the response function  $\lambda_{\underline{n}_0}^X$  of a simply constrained simple body are symmetric tensors.

Proof. As was noted in the proof of Theorem B4.B.4,  $\underline{E}_{\underline{n}_0} \nabla \lambda_{\underline{n}_0}^X(C_{\underline{n}_0}) \underline{E}_{\underline{n}_0}^T$  is symmetric; consequently, so is  $\underline{N}$ . The symmetry of  $\lambda_{\underline{n}_0}^X(\underline{E}_{\underline{n}_0}^T(\mathcal{M}_0(\underline{x}), \cdot))$  then follows from (SCM) and the symmetry of the total stress  $\underline{T}$ .  $\square$

Clearly, the symmetry of  $\underline{N}$  is primarily a consequence of our insistence that the constraint equation (SC) satisfy the Principle of Material Indifference so that it can be reduced to (SC-2). In a neat demonstration in his text, WANG starts with

$$(SC) \quad \gamma_{n_0}^x(\underline{F}_{n_0}(\underline{x}, t)) = 0 \quad \forall t \in \mathbb{R},$$

derives the constraint stress

$$\underline{N}_{n_0}(\underline{x}, t) = q(\underline{x}, t) \nabla \gamma_{n_0}^x(\underline{F}_{n_0}(\underline{x}, t)) \underline{F}_{n_0}^T(\underline{x}, t)$$

from the condition that  $\underline{N}_{n_0} \cdot \underline{L}_{n_0} = 0 \quad \forall$  motions meeting (SC),<sup>1</sup> and finally shows that  $\underline{N}_{n_0}(\underline{x}, t)$  is symmetric only if the constraint equation (SC) meets Material Indifference.

Next, we investigate the Principle of Material Indifference as it applies to the full constitutive equation (CSM), which with the aid of Theorem B4.B.7 we write concisely as

$$(CSM) \quad \underline{T} = q \underline{F}_{n_0} \nabla \lambda_{n_0}^x(\underline{C}_{n_0}) \underline{F}_{n_0}^T + \underline{T}^D.$$

As in § B4.2, we consider an equivalent motion  $\{\underline{x}^*(\cdot, t) : t \in \mathbb{R}\}$  for which (CSM) delivers the stress

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<sup>1</sup>Work this out as Exercise B4.B.3.



$$\tilde{\underline{\underline{T}}}^* = q \tilde{\underline{\underline{F}}}_{n_0}^* \nabla \lambda_{n_0}^x(\tilde{\underline{\underline{C}}}_{n_0}^*) \tilde{\underline{\underline{F}}}_{n_0}^{*T} + \tilde{\underline{\underline{T}}}^D.$$

But

$$\tilde{\underline{\underline{F}}}_{n_0}^* = \underline{\underline{Q}} \tilde{\underline{\underline{F}}}_{n_0} )^1 \text{ and } \tilde{\underline{\underline{C}}}_{n_0}^* = \underline{\underline{C}}_{n_0} )^2,$$

so

$$\tilde{\underline{\underline{T}}}^* = q \underline{\underline{Q}} \tilde{\underline{\underline{F}}}_{n_0} \nabla \lambda_{n_0}^x(\underline{\underline{C}}_{n_0}) \tilde{\underline{\underline{F}}}_{n_0}^T \underline{\underline{Q}}^T + \tilde{\underline{\underline{T}}}^D.$$

Now, Material Indifference requires that

$$\tilde{\underline{\underline{T}}}^* = \underline{\underline{Q}} \tilde{\underline{\underline{T}}} \underline{\underline{Q}}^T = q \underline{\underline{Q}} \tilde{\underline{\underline{F}}}_{n_0} \nabla \lambda_{n_0}^x(\underline{\underline{C}}_{n_0}) \tilde{\underline{\underline{F}}}_{n_0}^T \underline{\underline{Q}}^T + \underline{\underline{Q}} \tilde{\underline{\underline{T}}}^D \underline{\underline{Q}}^T,$$

and  $\therefore$

$$\tilde{\underline{\underline{T}}}^D = \underline{\underline{Q}} \tilde{\underline{\underline{T}}}^D \underline{\underline{Q}}^T.$$

Thus, Material Indifference places no restrictions on the constraint stress

$$\underline{\underline{N}}_{n_0} = q \tilde{\underline{\underline{F}}}_{n_0} \nabla \lambda_{n_0}^x(\underline{\underline{C}}_{n_0}) \tilde{\underline{\underline{F}}}_{n_0}^T,$$

which is not surprising since we have already incorporated Material Indifference by working with the reduced constraint equation (SC-2); however, the determinate stress is required to meet Material Indifference by itself. Since the constitutive equation for the determinate stress,

<sup>1</sup>See p. B4.2.11.

<sup>2</sup>See p. B4.2.17.

$$\underline{T}^D(X, t) = \underline{g}_{\underline{n}_0}^X(F_{\underline{n}_0}^t(\underline{n}_0(X), \cdot)),$$

is of the same form as (SM) for an unconstrained body and since the reduced constraint equation (SC-2) meets Material Indifference so that

$$\lambda_{\underline{n}_0}^X(\underline{C}_{\underline{n}_0}) = 0 \iff \lambda_{\underline{n}_0}^X(\underline{C}_{\underline{n}_0}^*) = 0,$$

the analysis of §B4.2 applies; and we can read off

Theorem B4.8.9. The constitutive equation

$$(CSM) \left\{ \begin{array}{l} \underline{T}_{\underline{m}}(X, t) = \underline{g}_{\underline{m}}(X, t) F_{\underline{n}_0}(X, t) \nabla \lambda_{\underline{n}_0}^X(\underline{C}_{\underline{n}_0}(X, t)) F_{\underline{n}_0}^T(X, t) + \underline{T}_{\underline{m}}^D(X, t), \\ \underline{T}_{\underline{m}}^D(X, t) = \underline{g}_{\underline{n}_0}^X(F_{\underline{n}_0}^t(\underline{n}_0(X), \cdot)), \\ (X, t) \in \mathcal{B} \times \mathbb{R} \end{array} \right.$$

for a simple body  $\mathcal{B}$  constrained at  $X$  by

$$(SC-2) \quad \lambda_{\underline{n}_0}^X(\underline{C}_{\underline{n}_0}(\underline{n}_0(X), t)) = 0 \quad \forall t \in \mathbb{R}$$

satisfies the Principle of Material Indifference iff the constitutive equation for the determinate stress  $\underline{T}^D$  has one of the reduced forms:

$$(CSM-1) \quad \underline{T}_{\underline{m}}^D(X, t) = \underline{R}_{\underline{n}_0}(\underline{n}_0(X), t) \underline{g}_{\underline{n}_0}^X(\underline{U}_{\underline{n}_0}^t(\underline{n}_0(X), \cdot)) \underline{R}_{\underline{n}_0}^T(\underline{n}_0(X), t), \quad (X, t) \in \mathcal{B} \times \mathbb{R};$$

$$(CSM-2) \quad T_{\sim m}^D(X, t) = F_{\sim n_0}(u_0(X), t) \hat{Y}_{\sim n_0}^X(U_{\sim n_0}^T(u_0(X), \cdot)) F_{\sim n_0}^T(u_0(X), t), (X, t) \in \mathcal{B} \times \mathbb{R};$$

$$(CSM-3) \quad T_{\sim m}^D(X, t) = F_{\sim n_0}(u_0(X), t) \bar{Y}_{\sim n_0}^X(C_{\sim n_0}^T(u_0(X), \cdot)) F_{\sim n_0}^T(u_0(X), t), (X, t) \in \mathcal{B} \times \mathbb{R};$$

where the response functions  $\hat{Y}_{\sim n_0}^X$  and  $\bar{Y}_{\sim n_0}^X$  are on

$$\mathcal{U}_{\sim n_0}^X = \{U \in C^{N-1}([0, \infty), \text{Sym } \mathcal{V}_3) : \lambda_{\sim n_0}^X(U^2(t)) = 0 \quad \forall t \in [0, \infty)\}$$

to  $\text{Sym } \mathcal{V}_3$ , while the response function  $\bar{Y}_{\sim n_0}^X$  is on

$$\mathcal{C}_{\sim n_0}^X = \{C \in C^{N-1}([0, \infty), \text{Sym } \mathcal{V}_3) : \lambda_{\sim n_0}^X(C(t)) = 0 \quad \forall t \in [0, \infty)\}$$

to  $\text{Sym } \mathcal{V}_3$ .

INSERT material on p. B4.B.21a

If there are several simple constraints at the particle  $X$ , then the single reduced constraint equation (SC-2) is replaced by

$$\alpha \lambda_{\sim n_0}^X(C_{\sim n_0}(X, t)) = 0 \quad \forall t \in \mathbb{R}, \alpha \in \mathbb{I}_m.$$

Of course, Theorem B4.B.6 applies to each constraint so that we have the equivalent conditions:

$$[F_{\sim n_0} \nabla^{\alpha} \lambda_{\sim n_0}^X(C_{\sim n_0}) F_{\sim n_0}^T] \cdot \underline{L}_{\sim n_0} = 0, \alpha \in \mathbb{I}_m.$$

Now, thinking geometrically, the linear span

$$\bar{J} := \text{Lsp} \{ F_{\sim n_0} \nabla^{\alpha} \lambda_{\sim n_0}^X(C_{\sim n_0}) F_{\sim n_0}^T : \alpha \in \mathbb{I}_m \}$$



Because of the arbitrary scalar factor  $q$  in the constraint stress in our constitutive equation

$$(CSM) \quad \underline{T} = q \underline{F}_{\sim n_0} \nabla \lambda_{\sim n_0}^x(\underline{C}_{\sim n_0}) \underline{F}_{\sim n_0}^T + \underline{T}^D,$$

any component of the determinate stress  $\underline{T}^D$  parallel to the constraint stress can be absorbed into the constraint stress; in fact, this feature would prevent us from being able to fully find the response function for the determinate stress if some restriction were not put on its range. To take care of this, we note that  $\underline{T} \in \text{Sym } \mathcal{V}_3$  and

$$q \underline{F}_{\sim n_0} \nabla \lambda_{\sim n_0}^x(\underline{C}_{\sim n_0}) \underline{F}_{\sim n_0}^T \in \mathcal{S} := \text{span} \{ \underline{F}_{\sim n_0} \nabla \lambda_{\sim n_0}^x(\underline{C}_{\sim n_0}) \underline{F}_{\sim n_0}^T \}$$

suggest the orthogonal decomposition<sup>1</sup>

$$\text{Sym } \mathcal{V}_3 = \mathcal{S} \oplus \mathcal{S}^\perp$$

Thus, we require the normalization

$$\underline{T}^D \in \mathcal{S}^\perp;$$

i.e., for each  $\underline{F} \in \mathcal{F}_{\sim n_0}^x$ ,

$$\mathcal{L}_{\sim n_0}^x(\underline{F}) \cdot [\underline{F}(t) \nabla \lambda_{\sim n_0}^x(\underline{F}^T(t) \underline{F}(t)) \underline{F}^T(t)] = 0,$$

with similar statements in terms of reduced forms of Theorem B4.B.9.

<sup>1</sup>See Theorem A3.12.5 (the Projection Theorem).

is a linear subspace of the linear space  $\text{Lin } \mathcal{V}_3$ ; <sup>1</sup> and in terms of the orthogonal direct sum decomposition

$$\text{Lin } \mathcal{V}_3 = \bar{\mathcal{S}} \oplus \bar{\mathcal{S}}^\perp,$$

the above conditions are equivalent to

$$\underline{L}_{x_0} \in \bar{\mathcal{S}}^\perp, \quad )^2$$

Of course, in accord with the Principle of Determinism for a simple body subject to a single simple constraint, we assume here that the stress is determined by the history of the deformation gradient to within an additive powerless stress that maintains the constraint. I.e., we make the constitutive assumption that

$$\underline{T}(X, t) = \underline{N}(X, t) + \mathcal{L}_{\underline{x}_0}^X(F_{\underline{x}_0}^t(x_0(X), \cdot)),$$

where

$$\underline{N}(X, t) \cdot \underline{L}(X, t) = 0$$

$\forall$  motions that satisfy the multiple constraint equations

$$\alpha \lambda_{\underline{x}_0}^X(C_{\underline{x}_0}(X, t)) = 0, \quad \alpha \in \Pi_M \quad \forall t \in \mathbb{R}.$$

---

<sup>1</sup> See Theorem A2.1.12.

<sup>2</sup> Actually, since  $\underline{F}_{x_0} \nabla \alpha \lambda_{\underline{x}_0}^X(C_{\underline{x}_0}) \underline{F}_{x_0}^T \in \text{Sym } \mathcal{V}_3$ ,  $\bar{\mathcal{S}}$  is a linear subspace of  $\text{Sym } \mathcal{V}_3$ . Here, we consider the orthogonal complement of  $\bar{\mathcal{S}}$  in  $\text{Lin } \mathcal{V}_3$  because  $\underline{L}_{x_0} \notin \text{Sym } \mathcal{V}_3$ .

Thus,

$$N_{\tilde{x}_0} \cdot \tilde{L}_{x_0} = 0$$

for every  $\tilde{L}_{x_0}$  corresponding to a motion that satisfies the multiple constraint equations; but all and only such  $\tilde{L}_{x_0}$  belong to  $\bar{S}^\perp$ , where  $\text{Lin } \mathcal{V}_3 = \bar{S} \oplus \bar{S}^\perp$ . In other words,

$$N_{\tilde{x}_0} \cdot \tilde{L}_{x_0} = 0 \quad \forall \quad \tilde{L}_{x_0} \in \bar{S}^\perp ;$$

i.e.,

$$N_{\tilde{x}_0} \in (\bar{S}^\perp)^\perp = \bar{S}. \quad )'$$

Consequently,

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<sup>1</sup>Speaking abstractly, let  $S$  be a linear subspace of a finite-dimensional inner product space  $\mathcal{V}$ ; and show, as Exercise B4.B.4, that  $(S^\perp)^\perp = S$ .



$$\underset{\sim}{N}_{n_0} = \sum_{\alpha=1}^M \alpha q_{\sim n_0} F_{\sim n_0} \nabla^{\alpha} \lambda_{n_0}^x(C_{\sim n_0}) F_{\sim n_0}^T.$$

As before, the constraint stress  $\underset{\sim}{N}$ , and consequently the determinate stress  $\underset{\sim}{I}^D$ , come out to be symmetric. And since each of the terms

$$\alpha q_{\sim n_0} F_{\sim n_0} \nabla^{\alpha} \lambda_{n_0}^x(C_{\sim n_0}) F_{\sim n_0}^T$$

is objective, the constitutive equation for the determinate stress,

$$\underset{\sim}{I}_m^D(X, t) = \underset{\sim}{g}_{\sim n_0}^x(F_{\sim n_0}^t(n_0(X), \cdot))$$

must satisfy the Principle of Material Indifference. Of course, in the case of multiple constraints, the domain of  $\underset{\sim}{g}_{\sim n_0}^x$  is

$$\{ F \in C^{N-1}([0, \infty), \text{Inv}^+ V_3) : \alpha \lambda_{n_0}^x(F(t)) F(t) = 0 \quad \forall t, \alpha \in \mathbb{I}_M \},$$

with corresponding adjustments to the domains of the associated reduced response functions.

The normalization<sup>1</sup> in this case is accomplished by requiring  $\underset{\sim}{I}_D \in \bar{\mathcal{S}}^{\perp}$ , where

$$\bar{\mathcal{S}} := \text{Lsp} \{ F_{\sim n_0} \nabla^{\alpha} \lambda_{n_0}^x(C_{\sim n_0}) F_{\sim n_0}^T : \alpha \in \mathbb{I}_M \},$$

---

<sup>1</sup> Cf. the discussion on p. B4.8.21 a.

and

$$\text{Sym } \mathcal{V}_3 = \bar{\mathcal{S}} \oplus \bar{\mathcal{S}}^\perp.$$

Here, in contrast to the analysis on p. B4.8.22, we take the orthogonal complement of  $\bar{\mathcal{S}}$  ( $= \bar{\mathcal{S}}$ ) in  $\text{Sym } \mathcal{V}_3$  rather than  $\text{Lin } \mathcal{V}_3$  because we know at this stage that the determinate stress  $\underline{T}_D \in \text{Sym } \mathcal{V}_3$ .

Of course, we want the multiple constraints

$$\alpha \lambda_{x_0}^x(\underline{C}_{x_0}(t)) = 0 \quad \forall t \in \mathbb{R}, \alpha \in \mathbb{I}_M$$

to be "independent" in some appropriate sense. It is left for the reader, as Exercise B4.8.5, to establish that  $\bar{\mathcal{S}}^\perp$ , the space where the determinate stress lives, is a  $(6-M)$ -dimensional linear subspace of  $\text{Sym } \mathcal{V}_3$  iff the set

$$\{ \nabla^\alpha \lambda_{x_0}^x(\underline{C}_{x_0}(x, t)) : \alpha \in \mathbb{I}_M \}$$

is linearly independent. In fact, in this case, it can be shown<sup>1</sup> that

$$\{ \underline{C} \in \text{Psym } \mathcal{V}_3 : \alpha \lambda_{x_0}^x(\underline{C}) = 0, \alpha \in \mathbb{I}_M \}$$

is a  $(6-M)$ -dimensional submanifold of  $\text{Sym } \mathcal{V}_3$ .

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<sup>1</sup>See the paper by MARLOW listed in the Supplementary Reading.

Supplementary ReadingCHADWICK, Continuum MechanicsTRUESDELL, A First Course in Rational Continuum Mechanics, Vol. 1TRUESDELL, The Elements of Continuum MechanicsTRUESDELL and NOLL, "The non-linear field theories of mechanics", Handbuch der Physik, vol III/3 (edit. FLÜGGE) 1965WANG, Mathematical Principles of Mechanics and Electromagnetism,  
Part A: Analytical and Continuum Mechanics{ MARLOW, "On the stress in an internally constrained elastic material", J. Elasticity (in press).



# B4.9. Incompressible, Inextensible, and Rigid Bodies

As alluded to in the previous section, the idea of incompressibility is that the motion is constrained to be volume preserving. According to Theorem B1.14.5, a motion  $\{\chi(\cdot, t) : t \in \mathbb{R}\}$  of a body  $\mathcal{B}$  is volume preserving iff  $\exists$  a function  $\hat{j}_n : \mathcal{N}(\mathcal{B}) \rightarrow \mathbb{R}^+ \ni$

$$|\det \underline{F}_n(\mathbb{X}, t)| = \hat{j}_n(\mathbb{X}) \quad \forall (\mathbb{X}, t) \in \mathcal{N}(\mathcal{B}) \times \mathbb{R}.$$

This preserves the volume of  $\mathcal{B}$  and each of its arbitrary parts. It is natural then to make the following

Definition B4.9.1. Let  $\kappa$  be a reference configuration of a body  $\mathcal{B}$ . Then a motion  $\{\chi(\cdot, t) : t \in \mathbb{R}\}$  of  $\mathcal{B}$  is locally volume preserving at the particle  $X = \kappa^{-1}(\mathbb{X})$  if  $\exists$  a time-independent number  $j_n(\mathbb{X}) \ni$

$$|\det \underline{F}_n(\mathbb{X}, t)| = j_n(\mathbb{X}) \quad \forall t \in \mathbb{R}.$$

Then, of course, an incompressible particle is one which can sustain only locally volume preserving motions. Formally,

Definition B4.9.2. A particle  $X$  of a body  $\mathcal{B}$  is incompressible if it is simply constrained w.r.t. all reference configurations  $\kappa$  by

$$\chi_n^X(\underline{F}_n(\mathbb{X}, t)) = |\det \underline{F}_n(\mathbb{X}, t)| - j_n(\mathbb{X}) = 0 \quad \forall t \in \mathbb{R}.$$

It is expected and easy to see that if this condition is satisfied w.r.t. one reference configuration, then it is satisfied w.r.t. all reference configurations. The demonstration is left as Exercise B4.9.1.

Of course, if the whole body  $B$  is incompressible, then the above constraint equation is required to be an identity in  $\mathbb{X} \in \mathcal{M}(B)$  as well as in  $t \in \mathbb{R}$ . This is the usual situation in the practice of continuum mechanics.

In view of Theorem B2.2.1, incompressibility at a particle or for the whole body can easily be stated in terms of the mass density. Often, this is the approach in fluid mechanics.

To get the reduced form (SC-2) of the incompressibility constraint, we note that

$$\begin{aligned}
 & |\det \underline{F}_n(\mathbb{X}, t)| = j_n(\mathbb{X}) \\
 \Rightarrow & \\
 & j_n^2 = (\det \underline{F}_n)^2 \quad (\text{definition of 1.1}) \\
 & = (\det \underline{F}_n)(\det \underline{F}_n^T) \quad (\text{Theorem A3.8.6}) \\
 & = \det (\underline{F}_n^T \underline{F}_n) \quad (\text{Theorem A3.8.4}) \\
 & = \det \underline{C}_n \quad (\text{definition of } \underline{C}_n).
 \end{aligned}$$

Since these steps are reversible, we have

Theorem B4.9.1. A particle  $X$  of a body  $B$  is incompressible iff it is constrained w.r.t. all reference configurations  $\kappa_0$  by

$$\lambda_n^X(\underline{C}_n(\mathbb{X}, t)) = \det \underline{C}_n(\mathbb{X}, t) - [j_n(\mathbb{X})]^2 = 0 \quad \forall t \in \mathbb{R},$$

In view of Theorem B4.8.2, this form of the incompressibility constraint automatically meets the Principle of Material Indifference.

To incorporate the results of the previous section, we need to work with a reference configuration  $\kappa_0$  that the body actually occupies at some time  $t_0$  in every motion. Then, in view of Theorem B1.14.3, the function  $j_n$  above becomes  $j_{\kappa_0} \equiv 1$ .

By Theorem B4.8.7, the constraint stress has the form

$$N_{\kappa_0} = q \underline{F}_{\kappa_0} \nabla \lambda_{\kappa_0}^X(\underline{C}_{\kappa_0}) \underline{F}_{\kappa_0}^T,$$

In the present case ( $\lambda_{\kappa_0}^X(\underline{C}_{\kappa_0}) = \det \underline{C}_{\kappa_0} - 1$ ), it follows from Theorem A5.8.1 that

$$\nabla \lambda_{\kappa_0}^X(\underline{C}_{\kappa_0}) = (\det \underline{C}_{\kappa_0}) \underline{C}_{\kappa_0}^{-1},$$

where we have used both the invertibility and the symmetry of  $\underline{C}_{\kappa_0}$ .

Since  $\underline{C}_{\kappa_0} = \underline{F}_{\kappa_0}^T \underline{F}_{\kappa_0}$ ,

$$\underline{C}_{\kappa_0}^{-1} = \underline{F}_{\kappa_0}^{-1} (\underline{F}_{\kappa_0}^T)^{-1}; \quad (\text{Theorem A3.9.7 (iii)})$$



and  $\therefore$

$$\underline{N}_{x_0} = q(\det \underline{C}_{x_0}) \underline{I}.$$

But since all motions must satisfy the constraint,

$$\underline{N}_{x_0}(\mathbb{X}, t) = q(\mathbb{X}, t) \underline{I}.$$

(calling

$$q(\mathbb{X}, t) = -p_{x_0}(\mathbb{X}, t),$$

we have

Theorem B4.9.2. The constraint stress at an incompressible particle  $X$  of a body  $B$  is a hydrostatic pressure; i.e.,

$$\underline{N}_m(X, t) = -p_m(X, t) \underline{I}, \quad t \in \mathbb{R}.$$

The minus sign is put in by convention so that a positive pressure  $p$  corresponds to compressive stress. The hydrostatic form for the constraint stress of an incompressible particle agrees with long standing experience with various fluids (say water) and solids (say rubber) that are usually modeled as being incompressible.

INSERT material on p. B4.9.4a

Next, we turn to the notion of inextensibility. Let  $\underline{C}_n \in C^1([0, 1], \mathcal{N}(B))$  define a smooth curve  $\underline{C}_n$  in the region of space that a body  $B$  occupies in the reference configuration  $\mathcal{N}$ ; i.e.,

$$\underline{C}_n = \{ \mathbb{X} = \underline{C}_n(\tau) : \tau \in [0, 1] \}.$$

The image at time  $t$  of  $\underline{C}_n$  under the motion  $\{ \mathbb{X}(\cdot, t) : t \in \mathbb{R} \}$  of  $B$

INSERT for p. B4.9.4

The normalization in the incompressible case is particularly easy. Referring to the discussion on p. B4.8.21 a and the above calculation, we have

$$\mathcal{S} = \text{Lsp} \{ \underline{\mathbb{I}} \} ;$$

and, therefore,  $\underline{T}^D \in \mathcal{S}^\perp$  iff  $\underline{T}^D \cdot \underline{\mathbb{I}} = 0$  or to  $\underline{T}^D = 0$ .<sup>10</sup>  
Thus, we have

Theorem B4.9.3. The normalization of the determinate stress for an incompressible particle  $X$  is

$$\text{to } \underline{T}_m^D(X, t) = 0, \quad t \in \mathbb{R}.$$

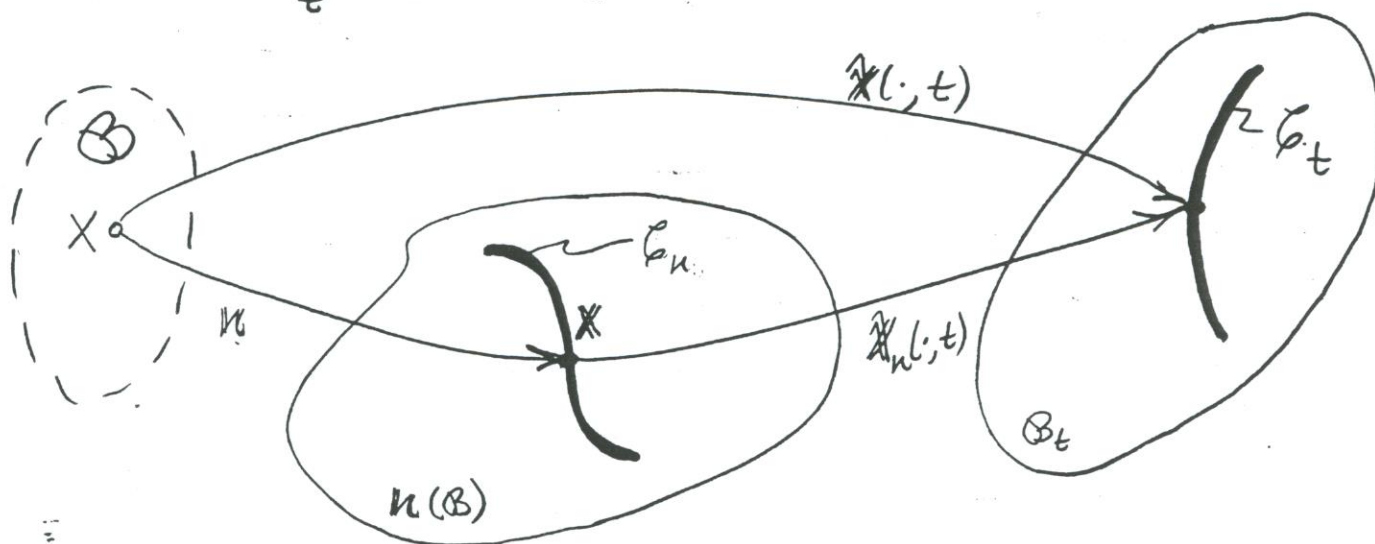
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<sup>1</sup>See Theorem A3.7.3.

return to p. B4.9.4

is the smooth curve

$$\mathcal{C}_t = \{x = \chi_n(\mathcal{C}_n(\tau), t) : \tau \in [0, 1]\}.$$



Of course, here we are speaking of a material curve which always consists of the same particles of  $B$ .

The length of  $\mathcal{C}_n$  is

$$\int_0^1 |\mathcal{C}'_n(\tau)| d\tau ;$$

while by Theorem B1.5.2, the length of  $\mathcal{C}_t$  is

$$\int_0^1 \sqrt{\mathcal{C}'_n(\tau) \cdot \chi_n(\mathcal{C}_n(\tau), t) \mathcal{C}'_n(\tau)} d\tau ,$$

This leads us to



Definition B4.9.3. A motion  $\{\mathbb{X}(\cdot, t) : t \in \mathbb{R}\}$  of a body  $\mathcal{B}$  is length preserving along the smooth material curve defined in the reference configuration  $\mathcal{C}_0$  by  
 $\mathcal{C}_n = \{\mathbb{X} = \mathcal{C}_n(\tau) : \tau \in [0, 1]\}$  if

$$\int_{\tau_1}^{\tau_2} \sqrt{\mathcal{C}_n'(\tau) \cdot \tilde{\mathcal{C}}_n(\mathcal{C}_n(\tau), t) \mathcal{C}_n'(\tau)} d\tau = \int_{\tau_1}^{\tau_2} \sqrt{\mathcal{C}_n'(\tau) \cdot \tilde{\mathcal{C}}_n(\mathcal{C}_n(\tau), s) \mathcal{C}_n'(\tau)} d\tau$$

$$\forall [\tau_1, \tau_2] \subset [0, 1] \text{ and } \forall t, s \in \mathbb{R}.$$

There are two aspects of this definition that warrant additional comment. First, we are preserving the length of the material curve and each of its arbitrary "parts," which is why we speak of the motion being length preserving along the material curve. Second, since we do not require the body to ever occupy the reference configuration  $\mathcal{C}_0$ , there is no reason for the length of  $\mathcal{C}_n$  to be the same as the length of  $\mathcal{C}_t$ ; and we do not require

$$\int_{\tau_1}^{\tau_2} \sqrt{\mathcal{C}_n'(\tau) \cdot \tilde{\mathcal{C}}_n(\mathcal{C}_n(\tau), t) \mathcal{C}_n'(\tau)} d\tau = \int_{\tau_1}^{\tau_2} |\mathcal{C}_n'(\tau)| d\tau.$$

By the one-dimensional version of the localization theorem (Theorem B1.6.2), we see that a motion is length preserving along a curve iff

$$\sqrt{c'_n(\tau) \cdot \tilde{C}_n(c_n(\tau), t) c'_n(\tau)} = \sqrt{c'_n(\tau) \cdot \tilde{C}_n(c_n(\tau), s) c'_n(\tau)}$$

$\forall \tau \in [0, 1]$  and  $\forall t, s \in \mathbb{R}$ , which is to say that for each  $\tau \in [0, 1]$

$$\sqrt{c'_n(\tau) \cdot \tilde{C}_n(c_n(\tau), t) c'_n(\tau)}$$

is independent of the time  $t$ . Thus, we are lead to

Theorem B4.9.4. Let  $\{\mathbb{X}(\cdot; t) : t \in \mathbb{R}\}$  and  $v_0$  be a motion  
and a reference configuration of a body  $\mathcal{B}$ . Then the following  
statements are equivalent:

(i) the motion is length preserving along the smooth  
material curve defined in the reference configuration  $v_0$  by  
 $C_n = \{\mathbb{X} = c_n(\tau) : \tau \in [0, 1]\}$ ;

(ii)  $\exists$  a function  $c_n : [0, 1] \rightarrow \mathbb{R} \ni$

$$c'_n(\tau) \cdot \tilde{C}_n(c_n(\tau), t) c'_n(\tau) = c_n(\tau) \quad \forall \tau \in [0, 1] \text{ and } \forall t \in \mathbb{R}$$

(iii) for each  $t_0 \in \mathbb{R}$ ,

$$c'_{n_0}(\tau) \cdot \tilde{C}_{n_0}(c_{n_0}(\tau), t) c'_{n_0}(\tau) = c'_{n_0}(\tau) \cdot c'_{n_0}(\tau) \quad \forall \tau \in [0, 1] \text{ and } \forall t \in \mathbb{R}$$

where  $n_0 := \mathbb{X}(\cdot; t_0)$ ;

$$(iv) \quad c'_n(\tau) \cdot \tilde{C}_{n_{32}}(c_n(\tau), t) c'_n(\tau) = 0 \quad \forall \tau \in [0, 1] \text{ and } \forall t \in \mathbb{R};$$

$$(v) [\tilde{F}_n(c_n(\tau), t) \dot{c}_n(\tau)] \cdot \tilde{D}_n(c_n(\tau), t) \tilde{F}_n(c_n(\tau), t) \dot{c}_n(\tau) = 0$$

$$\forall \tau \in [0, 1] \text{ and } \forall t \in \mathbb{R}.$$

Proof. We have already shown that  $(i) \Leftrightarrow (ii)$ ; the rest of the demonstration is left as Exercise B4.9.2.  $\square$

Now, the vector  $\dot{c}_n(\tau)$  is tangent to the curve  $\mathcal{C}_n$  at the point  $c_n(\tau)$ , and this leads us to

Definition B4.9.4. A motion  $\{x(\cdot, t) : t \in \mathbb{R}\}$  of a body  $\mathcal{B}$  is locally length preserving at the particle  $x$  in the material direction defined in the reference configuration  $x_0$  by the unit vector  $\underline{e}_n(x)$  if  $\exists$  a time independent number  $\hat{C}_n(x) \ni$

$$\underline{e}_n(x) \cdot \tilde{C}_n(x, t) \underline{e}_n(x) = \hat{C}_n(x) \quad \forall t \in \mathbb{R},$$

where  $x = x(x)$ .

Of course, for a reference configuration  $x_0 = x(\cdot, t_0)$  occupied at the time  $t_0$ ,  $\tilde{C}_{n_0}(\cdot, t_0) = \underline{I}$ ; and

$$\hat{C}_{n_0}(x) = 1.$$

A particle is inextensible in a given material direction if it can sustain only those motions which are locally length preserving in that direction. Formally, we make



Definition B4.9.5. A particle  $X$  of a body  $B$  is inextensible in the material direction defined in the reference configuration  $\kappa$  by the unit vector  $\underline{e}_n(X)$  if it is simply constrained w.r.t.  $\kappa$  by

$$\lambda_n^X(C_n(X, t)) = \underline{e}_n(X) \cdot \underline{c}_n(X, t) \underline{e}_n(X, t) - \hat{c}_n(X) = 0 \quad \forall t \in \mathbb{R}.$$

Clearly, a body could be inextensible at different particles and in different material directions at each of them. In particular, it might be inextensible along a smooth material curve. This provides a means to model composite materials which consist of a matrix reinforced by relatively stiff (compared to the matrix) fibers.

It is amusing to see what happens to the above inextensibility constraint if we change the reference configuration from  $\kappa$  to  $\mu = \# \circ \kappa$  by the deformation  $\#$ .<sup>1</sup> Then by Theorem B4.B.3,

$$\begin{aligned} \lambda_\mu^X(C_\mu(Y, t)) &= \lambda_n^X(\underline{G}^T(X) \underline{c}_\mu(Y, t) \underline{G}(X)) = 0 \\ &= \underline{e}_n(X) \cdot \underline{G}^T(X) \underline{c}_\mu(Y, t) \underline{G}(X) \underline{e}_n(X) - \hat{c}_n(X) \\ &= [\underline{G}(X) \underline{e}_n(X)] \cdot \underline{c}_\mu(Y, t) \underline{G}(X) \underline{e}_n(X) - \hat{c}_n(X) \quad (\text{defn. of transpose}) \end{aligned}$$

where  $\underline{G} = D\#$ .

Now, since  $\underline{e}_n(X)$  defines a material direction, we interpret

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<sup>1</sup>cf. Axiom (B2) of Definition B1.7.1.

it is a unit tangent at  $\mathbb{X}$  to a smooth material curve defined in  $\mathcal{M}$  by

$$\mathcal{C}_n = \{ \mathbb{X} = \mathbb{C}_n(\tau) : \tau \in [0, 1] \}.$$

The corresponding curve in the configuration  $\mu$  is

$$\mathcal{C}_\mu = \{ \mathbb{Y} = \mathbb{C}_\mu(\tau) := \mathbb{F}(\mathbb{C}_n(\tau)) : \tau \in [0, 1] \},$$

and as we saw on p. B1.5.7, a unit tangent to  $\mathcal{C}_\mu$  at the corresponding point  $\mathbb{Y} = \mathbb{F}(\mathbb{X})$  is given by

$$\underline{e}_\mu(\mathbb{Y}) = \frac{\underline{G}(\mathbb{X}) \underline{e}_n(\mathbb{X})}{|\underline{G}(\mathbb{X}) \underline{e}_n(\mathbb{X})|}.$$

Thus, we can write the constraint equation as

$$\begin{aligned} \underline{e}_\mu(\mathbb{Y}) \cdot \underline{C}_\mu(\mathbb{Y}, t) \underline{e}_\mu(\mathbb{Y}) &= \frac{\hat{C}_n(\mathbb{X})}{|\underline{G}(\mathbb{X}) \underline{e}_n(\mathbb{X})|^2} \\ &= \frac{\hat{C}_n(\mathbb{F}^{-1}(\mathbb{Y}))}{|\underline{G}(\mathbb{F}^{-1}(\mathbb{Y})) \underline{e}_n(\mathbb{F}^{-1}(\mathbb{Y}))|} \\ &=: \hat{C}_\mu(\mathbb{Y}) \quad \forall t \in \mathbb{R}. \end{aligned}$$

Since  $\underline{e}_\mu(\mathbb{Y})$  defines the same material direction at  $\mathbb{Y} = \mu(x)$  as does  $\underline{e}_n(\mathbb{X})$  at  $\mathbb{X} = \kappa(x)$ , this is as it should be.

Since we chose to formulate the inextensibility constraint in terms of the right deformation tensor  $\underline{C}_n$  rather than the right stretch tensor  $\underline{U}_n$ ,<sup>1</sup> it has come out in a form that automatically satisfies the Principle of Material Indifference.<sup>2</sup>

By Theorem B4.8.7, the constraint stress has the form

$$\underline{N}_{n_0} = \gamma \underline{F}_{n_0} \nabla \lambda_{n_0}^x(\underline{C}_{n_0}) \underline{F}_{n_0}^T.$$

for any occupied reference configuration  $\kappa_0$ . For inextensibility,

$$\lambda_{n_0}^x(\underline{C}_{n_0}) = \underline{e}_{n_0} \cdot \underline{C}_{n_0} \underline{e}_{n_0} - 1.$$

The first term is linear in  $\underline{C}_{n_0}$ , while the second term is independent of  $\underline{C}_{n_0}$ ; hence, it follows from Theorems A5.5.1-3 that

$$\nabla \lambda_{n_0}^x(\underline{C}_{n_0})[\underline{U}] = \underline{e}_{n_0} \cdot \underline{U} \underline{e}_{n_0}$$

$$= (\underline{e}_{n_0} \otimes \underline{e}_{n_0}) \cdot \underline{U} \quad (\text{Theorem A3.7.4 (iii)}).$$

$\therefore$  by Theorem A5.4.1,

$$\nabla \lambda_{n_0}^x(\underline{C}_{n_0}) = \underline{e}_{n_0} \otimes \underline{e}_{n_0}.$$

Then

<sup>1</sup> cf. Theorem B1.5.2.

<sup>2</sup> cf. Theorem B4.8.2.



$$\begin{aligned} \underline{N}_{n_0} &= \rho_{n_0} F(\underline{e}_{n_0} \otimes \underline{e}_{n_0}) F^T \\ &= \rho_{n_0} (F \underline{e}_{n_0}) \otimes (F \underline{e}_{n_0}) \quad (\text{Exercise A3.6.6}). \end{aligned}$$

What is the nature of this stress? It will be helpful to be a bit more precise in our notation and write

$$\underline{N}_{n_0}(\mathbb{X}, t) = \rho_{n_0}(\mathbb{X}, t) [F(\mathbb{X}, t) \underline{e}_{n_0}(\mathbb{X})] \otimes [F(\mathbb{X}, t) \underline{e}_{n_0}(\mathbb{X})].$$

As we just saw in our investigation of the effect of changing the reference configuration from  $n_0$  to  $n$ , the vector  $F(\mathbb{X}, t) \underline{e}_{n_0}(\mathbb{X})$  defines the same material direction at  $\mathbf{x} = \mathbf{x}_{n_0}(\mathbb{X}, t) = \mathbf{x}(\mathbb{X}, t)$  as does  $\underline{e}_{n_0}(\mathbb{X})$  at  $\mathbb{X} = n_0(\mathbf{x})$ .<sup>1</sup> For brevity let us write

$$\underline{e}(\mathbf{x}, t) = F(\mathbb{X}, t) \underline{e}_{n_0}(\mathbb{X}).$$

Now,  $\underline{N}_{n_0}(\mathbb{X}, t)$  is a stress tensor acting at the point  $\mathbf{x} = \mathbf{x}_{n_0}(\mathbb{X}, t)$ ; and according to our Second Notational Agreement (p. B1.13.6),

$$\underline{N}_{n_0}(\mathbb{X}, t) = \underline{N}_{\mathbf{x}}(\mathbf{x}, t),$$

Similarly,

$$\rho_n(\mathbb{X}, t) = \rho_{\mathbf{x}}(\mathbf{x}, t).$$

Then

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<sup>1</sup>Is  $F(\mathbb{X}, t) \underline{e}_{n_0}(\mathbb{X}, t)$  a unit vector? — Exercise B.4.9.3.

$$\underline{N}_{\underline{a}}(\underline{x}, t) = q_{\underline{a}}(\underline{x}, t) \underline{e}(\underline{x}, t) \otimes \underline{e}(\underline{x}, t).$$

A stress tensor of this form is called a uniaxial tension<sup>1</sup> in the direction  $\underline{e}$ , since for any  $\underline{n}$

$$\underline{N}_{\underline{a}} \underline{n} = q_{\underline{a}}(\underline{e} \cdot \underline{n}) \underline{e}$$

by the definition of the tensor product (Theorem A3.1.5).

To gain additional insight into this stress state of uniaxial tension, extend  $\{\underline{e}(\underline{x}, t)\}$  to an orthonormal basis for  $V_3$ , say  $\{\underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}, \underline{e}_{\langle 3 \rangle}\}$ , with

$$\underline{e}_{\langle 1 \rangle} \parallel \underline{e}(\underline{x}, t). \quad )^2$$

Then

$$\underline{N}_{\underline{a}}(\underline{x}, t) = \hat{q}_{\underline{a}}(\underline{x}, t) \underline{e}_{\langle 1 \rangle} \otimes \underline{e}_{\langle 1 \rangle}, \quad \hat{q}_{\underline{a}} = |\underline{e}|^2 q_{\underline{a}},$$

and the matrix of components of  $\underline{N}_{\underline{a}}(\underline{x}, t)$  relative to this basis is

$$[N^{\langle ij \rangle}(\underline{x}, t)] = \begin{bmatrix} \hat{q}_{\underline{a}}(\underline{x}, t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad )^3$$

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<sup>1</sup> Sometimes, pure tension.

<sup>2</sup> Cf. Exercise A2.2.1, Theorem A2.2.9, and the proof of Theorem A2.3.16.

<sup>3</sup> Cf. Theorem A3.2.5 (iii).

In summary, we have

Theorem B4.9.5. The constraint stress at a particle  
 $X$  of a body  $B$  which is inextensible in the material  
direction defined in the <sup>(occupied)</sup> reference configuration  $\kappa_0$  by the unit  
vector  $\underline{e}_{\kappa_0}(X)$  is a uniaxial tensor in the same material  
direction; i.e.,

$$\underline{N}_{\kappa_0}(X, t) = q_{\kappa_0}(X, t) [\underline{F}_{\kappa_0}(X, t) \underline{e}_{\kappa_0}(X)] \otimes [\underline{F}_{\kappa_0}(X, t) \underline{e}_{\kappa_0}(X)], \quad t \in \mathbb{R}.$$

INSERT material in p. B4.9.14a

Finally, we consider rigidity. According to Theorem B1.14.2, a motion  $\{X(\cdot, t) : t \in \mathbb{R}\}$  of a body  $B$  is rigid iff  $\exists$  a function  $\hat{\underline{C}}_n : \mathcal{K}(B) \rightarrow \text{Psym } \mathcal{V}_3 \ni$

$$\underline{C}_n(X, t) = \hat{\underline{C}}_n(X) \quad \forall (X, t) \in \mathcal{K}(B) \times \mathbb{R},$$

This preserves the distance between every pair of particles of  $B$ . In analogy with Definitions B4.9.1 and 4, we make

Definition B4.9.6. Let  $\kappa$  be a reference configuration of  
a body  $B$ . Then a motion  $\{X(\cdot, t) : t \in \mathbb{R}\}$  of  $B$   
is locally rigid at the particle  $X = \kappa^{-1}(X)$  if  $\exists$  a  
time-independent positive definite symmetric tensor  $\hat{\underline{C}}_n(X)$   
 $\ni$

$$\underline{C}_n(X, t) = \hat{\underline{C}}_n(X) \quad \forall t \in \mathbb{R}.$$



INSERT for p. B4.9.14

For the normalization in the inextensible case, we see from the discussion on p. B4.8.21a and the above calculation that

$$\mathcal{S} = \text{Lsp} \{ \underline{F} \underline{e}_{\underline{x}_0} \otimes \underline{F} \underline{e}_{\underline{x}_0} \};$$

and, therefore,  $\underline{T}^D \in \mathcal{S}^\perp$  iff  $\underline{T}^D \cdot (\underline{F} \underline{e}_{\underline{x}_0} \otimes \underline{F} \underline{e}_{\underline{x}_0}) = 0$ , which perhaps is more conveniently stated as  $\underline{e}_{\underline{x}_0} \cdot (\underline{F}^T \underline{T}^D \underline{F}) \underline{e}_{\underline{x}_0} = 0$ .<sup>1</sup>  
Thus, we have

Theorem B4.9.6, The normalization of the determinate  
stress for a particle  $X$  that is inextensible in the material  
direction defined in the <sup>(occupied)</sup> reference configuration  $\mathcal{V}_0$  by the unit  
vector  $\underline{e}_{\underline{x}_0}(X)$  is

$$\underline{e}_{\underline{x}_0}(\mathcal{X}_0(X)) \cdot \left[ \underline{F}^T(t) \underline{T}^D_{\text{mm}}(X, t) \underline{F}(t) \right] \underline{e}_{\underline{x}_0}(\mathcal{X}_0(X)) = 0 \quad \forall \underline{F} \in \mathcal{J}_{\underline{x}_0}^X, t \in \mathbb{R}.$$

<sup>1</sup>State and prove the identity used here as Exercise B.9.9.4.

return to p. B4.9.14

As was observed on pp. B4.8.2 and 2a, this is equivalent to

$$|\underline{C}_{\sim x}(\mathbb{X}, t) - \hat{\underline{C}}_{\sim x}(\mathbb{X})| = 0 \quad \forall t \in \mathbb{R};$$

and we are lead to describe rigidity of the particle  $X$  by the reduced simple constraint equation

$$\lambda_{\sim x}^x(\underline{C}_{\sim x}(\mathbb{X}, t)) = |\underline{C}_{\sim x}(\mathbb{X}, t) - \hat{\underline{C}}_{\sim x}(\mathbb{X})|^2 = 0 \quad \forall t \in \mathbb{R}.$$

Of course, for a reference configuration  $x_0 = \mathbb{X}(\cdot; t_0)$  occupied by the body at some time  $t_0$ ,  $\underline{C}_{\sim x_0}(\cdot, t_0) = \underline{I}$ ; and

$$\hat{\underline{C}}_{\sim x_0}(\mathbb{X}) = \underline{I}. )^1$$

In order to utilize the results of § B4.8, we need to work with such an occupied reference configuration.

Consider then  $\lambda_{\sim x_0}^x: \text{Psym } \mathcal{V}_3 \rightarrow \mathbb{R}$  defined by

$$\lambda_{\sim x_0}^x(\underline{C}) = |\underline{C} - \underline{I}|^2.$$

By writing  $|\underline{C} - \underline{I}|^2 = (\underline{C} - \underline{I}) \cdot (\underline{C} - \underline{I})$ , it is easy to calculate that

$$\nabla \lambda_{\sim x_0}^x(\underline{C}) = 2(\underline{C} - \underline{I}).$$

Thus, whenever the constraint is satisfied,

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<sup>1</sup>cf. (iv) of Theorem B1.14.1.

$$\nabla \lambda_{x_0}^x (C_{\sim n}(\mathbb{X}, t)) = 0.$$

But then by Theorem B4.8.7, this implies that the constraint stress

$$N_{\sim x_0}(\mathbb{X}, t) = 0$$

— a result which is assuredly incorrect! The problem here is that we have described the constraint set, i.e. the set of all right deformation tensors that satisfy the constraint, as the zero set of the constraint function  $\lambda_{x_0}^x$ . In order for the constraint set to be a (6-1)-dimensional manifold,  $\lambda_{x_0}^x$  must be such that  $\nabla \lambda_{x_0}^x(C) \neq 0$  for all  $C$  that satisfy the constraint. More generally, when there are  $M$  constraint equations

$$\alpha \lambda_{x_0}^x (C_{\sim n_0}(\mathbb{X}, t)) = 0, \alpha \in \mathbb{I}_M,$$

the set

$$\{ \nabla^\alpha \lambda_{x_0}^x (C_{\sim n_0}(\mathbb{X}, t)) : \alpha \in \mathbb{I}_M \}$$

must be linearly independent for all  $C$ 's that satisfy the constraints in order to have a proper manifold description. This point is elucidated in the paper by MARLOW listed in the Supplementary Reading; see also Exercise 4.8.5.

The "tensorial" constraint of local rigidity can be fit easily into our scheme by viewing it as a multiple "scalar" constraint. To do this, we let  $\{\underline{E}_1, \underline{E}_2, \dots, \underline{E}_6\}$  be a



basis for  $\text{Sym } \mathcal{V}_3$  and note from Theorems A2.2.12 and A2.5.8 that the constraint of local rigidity at the particle  $X$  is equivalent to

$$\underline{E}_\alpha \cdot \underline{C}_{\sim n}(\mathbb{X}, t) = \underline{E}_\alpha \cdot \hat{\underline{C}}_{\sim n}(\mathbb{X}) \quad \forall t \in \mathbb{R}, \alpha \in \Pi_6.$$

Of course, a rigid particle is one which can sustain only locally rigid motions. Formally, we make

Definition B4.9.7, A particle  $X$  of a body  $B$  is rigid  
if it is simply constrained w.r.t. all reference configurations  $\mathbb{K}$  by the multiple constraints

$${}^\alpha \lambda_{\sim n}^X(\underline{C}_{\sim n}(\mathbb{X}, t)) = \underline{E}_\alpha \cdot \underline{C}_{\sim n}(\mathbb{X}, t) - \underline{E}_\alpha \cdot \hat{\underline{C}}_{\sim n}(\mathbb{X}) = 0 \quad \forall t \in \mathbb{R}, \alpha \in \Pi_6,$$

where  $\{\underline{E}_1, \underline{E}_2, \dots, \underline{E}_6\}$  is a basis for  $\text{Sym } \mathcal{V}_3$ .

Of course, a body is rigid if all of its particles are rigid. Then the above constraints must be identities in  $\mathbb{X} \in \mathbb{K}(B)$  as well as in  $t \in \mathbb{R}$ .

and on p. B4.9.10 for the inextensible case

As we saw in Exercise B4.9.1 for the incompressible case, if the rigidity constraints hold w.r.t. one reference configuration  $\mathbb{K}$ , then they hold w.r.t. all reference configurations. The details of this are left as Exercise B4.9.5.

The six constraints of Definition B4.9.7 are each already in the reduced form (SC-2), so that the Principle of Material Indifference is automatically met.

INSERT material on p. B4.9.18a

For the constraint stress, we need  $\nabla^\alpha \lambda_{n_0}^\alpha(C_{\sim n_0})$ .  
Clearly, <sup>1</sup>

$$D^\alpha \lambda_{n_0}^\alpha(C_{\sim n_0})[\underline{U}] = \underline{E}_\alpha \cdot \underline{U},$$

and  $\therefore$

$$\nabla^\alpha \lambda_{n_0}^\alpha(C_{\sim n_0}) = \underline{E}_\alpha.$$

Hence,

$$\underline{N}_{n_0} = \sum_{\alpha=1}^6 \alpha q \underline{F}_{\sim n} \nabla^\alpha \lambda_{n_0}^\alpha(C_{\sim n_0}) \underline{F}_{\sim n_0}^T \quad \left( \begin{array}{l} \text{discussion on} \\ \text{pp. B4.8, 21-24} \end{array} \right)$$

$$= \sum_{\alpha=1}^6 \alpha q \underline{F}_{\sim n_0} \underline{E}_\alpha \underline{F}_{\sim n_0}^T \quad (\text{substitution})$$

$$= \underline{F}_{\sim n_0} \left( \sum_{\alpha=1}^6 \alpha q \underline{E}_\alpha \right) \underline{F}_{\sim n_0}^T \quad \left( \begin{array}{l} \text{distributivity and homogeneity} \\ \text{of tensor multiplication} \end{array} \right).$$

Now,  $\{\underline{E}_1, \underline{E}_2, \dots, \underline{E}_6\}$  spans  $\text{Sym } \mathcal{V}_3$ , and the  $\alpha q$  are arbitrary; so

$$\sum_{\alpha=1}^6 \alpha q \underline{E}_\alpha$$

is an arbitrary symmetric tensor. Since  $\underline{F}_{\sim n}$  is invertible,  $\underline{N}_{n_0}$  also is arbitrary; and then the determinate stress can be absorbed by  $\underline{N}_{n_0}$ . Thus, there is no point in bothering with a constitutive equation for  $\underline{T}^D$ . It is not surprising then that the concept of stress has played no role in determining the motion of rigid bodies. This is not to

<sup>1</sup> q. the analysis on p. B4.9.11.

INSERT for p. B4.9.18

B4.9.18a

As noted earlier, for an occupied reference configuration  $\psi_0$ ,  $\hat{C}_{\sim\psi_0}(\mathbb{X}) = \underline{I}$ ;

and  $\therefore$

$${}^{\alpha}\chi_{\psi_0}^{\mathbb{X}}(\underline{C}_{\sim\psi}) = \underline{E}_{\sim\alpha} \cdot \underline{C}_{\sim\psi}(\mathbb{X}, t) - \underline{E}_{\sim\alpha} \cdot \underline{I}$$

return to p. B4.9.18



# B4.10. Material Symmetry of Constrained Bodies

It is conceivable that an internally constrained body might possess some material symmetry; indeed, water is usually modeled as an incompressible fluid, while rubber is generally taken to be an incompressible, isotropic solid.

We start with an internally constrained simple body  $B$ . It is mechanically characterized w.r.t. a reference configuration  $\kappa$  by its density

$$\rho_\kappa: \mathcal{M}(B) \rightarrow \mathbb{R}^+,$$

a reduced simple constraint equation for the particle  $X$

$$\lambda_\kappa^X(C_{\sim\kappa}(X, t)) = 0 \quad \forall t \in \mathbb{R},$$

and a constitutive equation

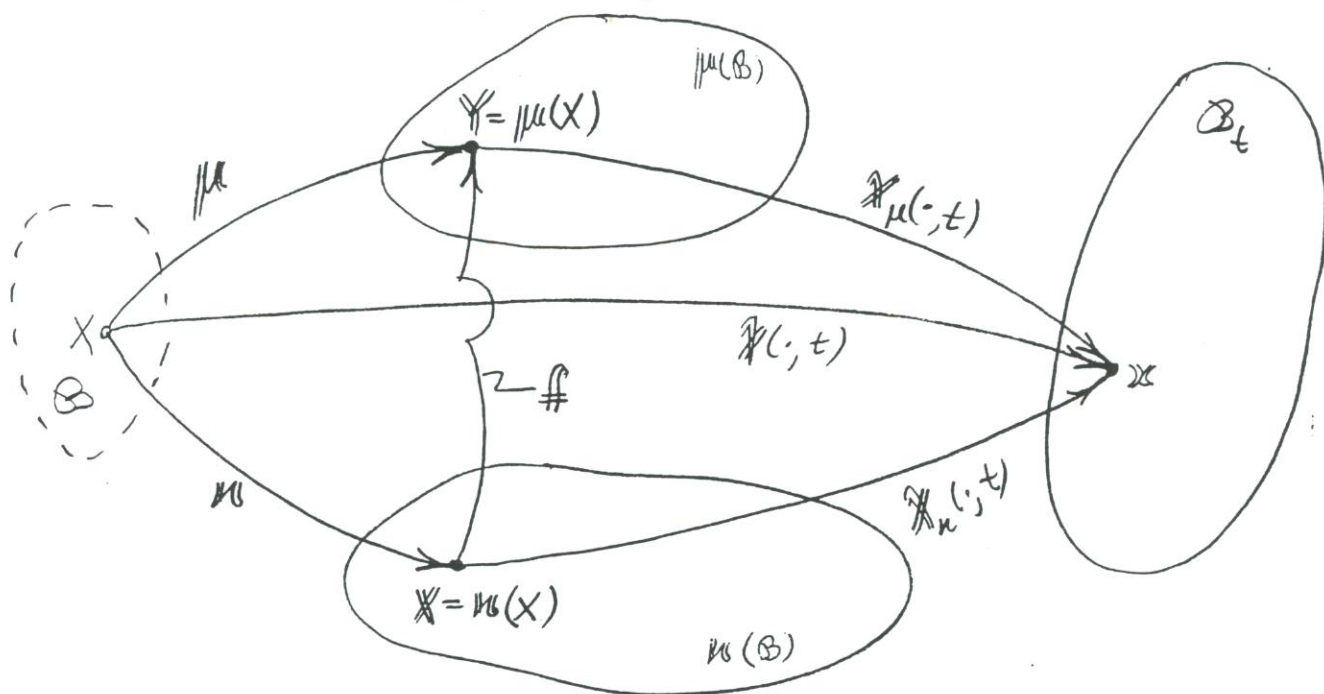
$$\underline{T} = \underline{N} + \underline{T}^D,$$

where the constraint stress has the form

$$\underline{N}_{\sim\kappa}(X, t) = q_\kappa(X, t) \underline{F}_\kappa(X, t) \nabla \lambda_\kappa^X(C_{\sim\kappa}(X, t)) \underline{F}_\kappa^T(X, t)$$

with  $q_\kappa$  arbitrary at this stage, and the determinate stress is given by the reduced constitutive equation

With regard to material symmetry, as in §B4.5, we are interested in other reference configurations which are mechanically indistinguishable from  $\kappa$ . By Axiom (B1) of Definition B1.7.1, any other reference configuration  $\mu$  is connected to  $\kappa$  by the deformation  $\sharp = \mu \circ \kappa^{-1}$ .



Throughout, we shall write

$$\mathbb{X} = \kappa(X), \quad \mathbb{Y} = \mu(X), \quad \underline{G}(\mathbb{X}) = D\sharp(\mathbb{X})$$

as before.

Since the body must actually occupy both configurations  $\kappa$  and  $\mu$ , the connecting deformation  $\#$  must be the end result of some motion which actually carries the body from the configuration  $\kappa$  to the configuration  $\mu$ . Consequently,

$$\det \underline{G}(\underline{X}) > 0 \quad \forall \underline{X} \in \kappa(B); \quad )^1$$

moreover,  $\#$  must satisfy the constraint at  $X$ :

$$\lambda_n^X (\underline{G}^T(\underline{X}) \underline{G}(\underline{X})) = 0.$$

Now we are in position to consider the invariance of the mechanical properties of the body under the change of reference configuration from  $\kappa$  to  $\mu$ .

Just as on p. 4.5.4, for the mass density at  $Y = \mu(X)$  in the configuration  $\mu$  of  $B$  to be the same as the mass density at the corresponding point  $\underline{X} = \kappa(X)$  in the configuration  $\kappa$ , it is necessary and sufficient that

$$|\det \underline{G}(\underline{X})| = 1.$$

Here, however, the absolute value is pointless.

Combining the last two observations, we have

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<sup>1</sup>cf. Theorem B1.13.11.



Theorem B4.10.1. The deformation  $\mathbb{F}$  which carries the  
body  $\mathcal{B}$  from the reference configuration  $\mu$  to the new  
reference configuration  $\mu$  satisfies the constraint and  
preserves the mass density at the particle  $X$  iff

$$D\mathbb{F}(X) \in \mathcal{H}_n^X := \{ \underline{H} \in \text{Unim}^+ \mathcal{V}_3 : \lambda_n^X(\underline{H}^T \underline{H}) = 0 \}.$$

Next we consider the constraint equation

$$\lambda_n^X(\underline{C}_{\sim n}(X, t)) = 0 \quad \forall t \in \mathbb{R},$$

INSERT material on p. B4.10.5a

For the change of reference configuration  $\mu \rightarrow \mu$  defined  
 by the deformation  $\mathbb{F}$  to be a symmetry transformation  
 for the constraint set,  $\mathcal{C}_\mu^X = \mathcal{C}_n^X$ ; i.e.,

$$\lambda_n^X(\underline{C}) = 0 \iff \lambda_\mu^X(\underline{C}) = 0, \quad \underline{C} \in \text{Psym} \mathcal{V}_3.$$

By Theorem B4.8.3, the constraint function  $\lambda_\mu^X$  is defined  
 in terms of  $\lambda_n^X$  by

$$\lambda_\mu^X(\underline{C}) = \lambda_n^X(\underline{G}^T(X) \underline{C} \underline{G}(X)),$$

and thus we are led to

Definition B4.10.2. The symmetry group for the  
mass density and the constraint equation of the particle  $X$   
of the simply constrained body  $\mathcal{B}$  relative to the reference

INSERT for p. B4.10.5

which defines the class of allowable motions. To this end the following definition will be convenient.

Definition B4.10.1. The constraint set relative to the  
reference configuration  $\kappa$  is the set

$$\mathcal{C}_{\kappa}^{\times} = \{ \underline{C} \in \text{Psym } V_3 : \lambda_{\kappa}^{\times}(\underline{C}) = 0 \}.$$

Thus, the constraint set is the set of all admissible right deformation tensors relative to the reference configuration  $\kappa$ .

return to p. B4.10.5

configuration  $n$  is the set

$$\mathcal{Q}_n^x(p, \lambda) = \{ \underline{H} \in \mathcal{H}_n^x : \lambda_n^x(\underline{H}^T \underline{C} \underline{H}) = 0 \Leftrightarrow \lambda_n^x(\underline{C}) = 0, \underline{C} \in \text{Psym } \mathcal{V}_3 \}$$

Of course, our cumbersome terminology reflects the fact that we have incorporated invariance of both the density and the constraint equation. Note that we have not required invariance of the constraint function itself. What is physically important is the constraint set; i.e., the "submanifold" in  $\text{Lin } \mathcal{V}_3$  defined by the constraint equation

$$\lambda_n^x(\underline{C}) = 0, \underline{C} \in \text{Psym } \mathcal{V}_3.$$

In fact, the constraint function is not uniquely determined by the given submanifold of possible rigid deformation tensors. E.g., the constraint functions

$$\lambda_n^x, \quad \alpha \lambda_n^x \quad (\alpha \neq 0), \quad \text{and} \quad (\lambda_n^x)^2$$

all define the same submanifold. Recall, also, the different ways we had of describing the constraint of rigidity in the previous section.

Theorem B4.10.2. The symmetry group  $\mathcal{Q}_n^x(p, \lambda)$  is, indeed, a group w.r.t. tensor multiplication.

Proof. Closure: Let  $\underline{H}_1, \underline{H}_2 \in \mathcal{Q}_n^x(p, \lambda)$ . Then by the properties of the transpose and the associativity of tensor multiplication,



$$\lambda_n^x((H_{\sim 1} H_{\sim 2})^T (H_{\sim 1} H_{\sim 2})) = \lambda_n^x(H_{\sim 2}^T (H_{\sim 1}^T H_{\sim 1}) H_{\sim 2})$$

Now, we saw in the proof of the polar decomposition theorem (Theorem A3.15.4) that

$$H_{\sim 1}^T H_{\sim 1} \in \text{Psym } \mathcal{V}_3.$$

We also have

$$\lambda_n^x(H_{\sim 1}^T H_{\sim 1}) = 0$$

since  $H_{\sim 1} \in \mathcal{D}_n^x(\rho, \lambda) \subset \mathcal{H}_n^x$ .<sup>1</sup> Then since  $H_{\sim 2} \in \mathcal{D}_n^x(\rho, \lambda)$ ,

$$\lambda_n^x(H_{\sim 1}^T H_{\sim 1}) = 0 \implies \lambda_n^x(H_{\sim 2}^T (H_{\sim 1}^T H_{\sim 1}) H_{\sim 2}) = 0.$$

$\therefore$

$$\lambda_n^x((H_{\sim 1} H_{\sim 2})^T (H_{\sim 1} H_{\sim 2})) = 0,$$

so

$$(H_{\sim 1} H_{\sim 2}) \in \mathcal{H}_n^x; \quad )^2$$

and the product is a candidate for membership in  $\mathcal{D}_n^x(\rho, \lambda)$ .

Next, let  $\underline{C} \in \text{Psym } \mathcal{V}_3$  and consider

<sup>1</sup> Note the important role played here by the requirement that deformations defining changes reference configurations satisfy the constraint — a requirement whose justification resides in our use of occupied reference configurations. <sup>2</sup> Note how we use  $\mathcal{D}_n^x(\rho, \lambda)$  to get  $H_{\sim 1}, H_{\sim 2} \in \mathcal{H}_n^x$ . Evidently,  $\mathcal{H}_n^x$  itself is not a group.

$$0 = \lambda_n^x((H_1 H_2)^T C (H_1 H_2))$$

$$= \lambda_n^x(H_2^T (H_1^T C H_1) H_2) \quad ((AB)^T = B^T A^T \text{ and associativity}),$$

which is equivalent to

$$\lambda_n^x(H_1^T C H_1) = 0$$

since  $H_2 \in \mathcal{D}_n^x(\rho, \lambda)$  and  $H_1^T C H_1 \in \text{Psym } \mathcal{V}_3$  (Establish this last fact as Exercise B4.10.1.), which in turn is equivalent to

$$\lambda_n^x(C) = 0$$

since  $H_1 \in \mathcal{D}_n^x(\rho, \lambda)$ . Hence,  $H_1 H_2 \in \mathcal{D}_n^x(\rho, \lambda)$ .

Associativity: Theorem A3.5.1.

Identity Element<sup>†</sup>: By Theorem B4.8.5,

$$\lambda_n^x(I) = 0 \Rightarrow I \in \mathcal{H}_n^x. )^1$$

Then, trivially  $I \in \mathcal{D}_n^x(\rho, \lambda)$ .

Inverse Elements: Let  $H \in \mathcal{D}_n^x(\rho, \lambda) \Rightarrow H \in \mathcal{H}_n^x \Rightarrow H \in \text{Unim}^+ \mathcal{V}_3 \Rightarrow H^{-1} \in \text{Unim}^+ \mathcal{V}_3$  since  $\text{Unim}^+ \mathcal{V}_3$  is a

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<sup>†</sup>Again, an important consequence of the use of occupied reference configurations.

group (Theorem B4.5.1). Next, we use Theorem B4.8.5 again to write

$$\begin{aligned}
 0 &= \lambda_n^x(\underline{\mathbb{I}}) \\
 &= \lambda_n^x(\underline{H}^T(\underline{H}^{-T}\underline{H}^{-1})\underline{H}) \quad (\text{associativity and property of inverse}) \\
 \Rightarrow \lambda_n^x(\underline{H}^{-T}\underline{H}^{-1}) &= 0 \quad (\underline{H}^{-T}\underline{H}^{-1} \in \text{Psym } \mathcal{V}_3 \text{ and } \underline{H} \in \mathcal{D}_n^x(\rho, \lambda)). \\
 \therefore \underline{H}^{-1} &\in \mathcal{H}_n^x. \quad )^1
 \end{aligned}$$

Next, let  $\underline{C} \in \text{Psym } \mathcal{V}_3$  and consider

$$\begin{aligned}
 0 &= \lambda_n^x(\underline{C}) \\
 &= \lambda_n^x(\underline{H}^T(\underline{H}^{-T}\underline{C}\underline{H}^{-1})\underline{H}) \quad (\text{associativity and property of inverse}) \\
 \Leftrightarrow \lambda_n^x(\underline{H}^{-T}\underline{C}\underline{H}^{-1}) &= 0 \quad (\underline{H}^{-T}\underline{C}\underline{H}^{-1} \in \text{Psym } \mathcal{V}_3 \text{ and } \underline{H} \in \mathcal{D}_n^x(\rho, \lambda)) \\
 \therefore \underline{H}^{-1} &\in \mathcal{D}_n^x(\rho, \lambda). \quad \square
 \end{aligned}$$

Next, we turn to the invariance of the constraint stress response. The constraint stress associated with the place  $\mathbb{X} = \mathcal{W}(\mathbf{X})$  when the reference configuration is  $\mathcal{W}$  is

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<sup>1</sup> Again, an important consequence of the use of occupied reference configurations.



$$\underline{N}_{\sim n}(\mathbb{X}, t) = \underline{q}_n(\mathbb{X}, t) \underline{F}_{\sim n}(\mathbb{X}, t) \nabla \lambda_n^x(\underline{C}_{\sim n}(\mathbb{X}, t)) \underline{F}_{\sim n}^T(\mathbb{X}, t),$$

and for the corresponding place  $\mathbb{Y} = \mu(X)$  when the reference configuration is  $\mu$ ,

$$\underline{N}_{\sim \mu}(\mathbb{Y}, t) = \underline{q}_{\mu}(\mathbb{Y}, t) \underline{F}_{\sim \mu}(\mathbb{Y}, t) \nabla \lambda_{\mu}^x(\underline{C}_{\sim \mu}(\mathbb{Y}, t)) \underline{F}_{\sim \mu}^T(\mathbb{Y}, t).$$

Because of the presence of the unknown multipliers  $\underline{q}_n$  and  $\underline{q}_{\mu}$ , the only important characteristic of these constraint stresses is their directions. Therefore, for the constraint stress response at the particle  $X$  out of the reference configuration  $\mu$  to be indistinguishable from the response out of  $n$ , we must have

$$\underline{F} \nabla \lambda_{\mu}^x(\underline{F}^T \underline{F}) \underline{F}^T \parallel \underline{F} \nabla \lambda_n^x(\underline{F}^T \underline{F}) \underline{F}^T \quad \forall \underline{F} \in \mathcal{f}_n^x,$$

where

$$\mathcal{f}_n^x := \{ \underline{F} \in \text{Inv } \mathcal{V}_3 : \det \underline{F} > 0, \lambda_n^x(\underline{F}^T \underline{F}) = 0 \}$$

is the set of all admissible<sup>1</sup> deformation gradients relative to the reference configuration  $n$ . We note that since

We restrict our consideration to those reference configurations  $\mu$  that are connected to  $n$  by a deformation  $\mathbb{F}$  with gradient  $\underline{G}(\mathbb{X}) \in \mathcal{D}_n^x(\rho, \lambda)$  so that the mass density and the constraint equation are the same in  $n$  and  $\mu$  at  $X$ ,

<sup>1</sup>Admissible in the sense that the constraint is satisfied.

$$\underline{G}(\mathbb{X}) \in \mathcal{D}_n^X(p, \lambda),$$

$$\underline{F} \in \underline{f}_n^X \Rightarrow \lambda_n^X(\underline{F}^T \underline{F}) = 0 \Leftrightarrow \lambda_n^X(\underline{G}^T(\mathbb{X})(\underline{F}^T \underline{F})\underline{G}(\mathbb{X})) = 0;$$

but by Theorem B4.8.3,

$$\lambda_n^X(\underline{G}^T(\mathbb{X})(\underline{F}^T \underline{F})\underline{G}(\mathbb{X})) = \lambda_\mu^X(\underline{F}^T \underline{F}),$$

so that

$$\underline{F} \in \underline{f}_n^X \Rightarrow \lambda_\mu^X(\underline{F}^T \underline{F}) = 0.$$

Thus,

$$\underline{F} \in \underline{f}_n^X \Rightarrow \underline{F} \in \underline{f}_\mu^X;$$

and  $\underline{F} \nabla \lambda_\mu^X(\underline{F}^T \underline{F}) \underline{F}^T$  has the interpretation that we want here.

Now the tensors  $\underline{F} \nabla \lambda_\mu^X(\underline{F}^T \underline{F}) \underline{F}^T$  and  $\underline{F} \nabla \lambda_n^X(\underline{F}^T \underline{F}) \underline{F}^T$  are parallel iff they are linearly dependent (Definition A2.3.6). Since  $\underline{F}$  is invertible, this makes it clear that they are parallel iff

$$\nabla \lambda_\mu^X(\underline{C}) \parallel \nabla \lambda_n^X(\underline{C}) \quad \forall \underline{C} \in \mathcal{C}_n^X,$$

where

$$\mathcal{C}_n^X := \{ \underline{C} \in \text{Psym } \mathcal{V}_3 : \lambda_n^X(\underline{C}) = 0 \}$$

is the constraint set introduced in Definition B4.10.1. But this condition is automatically

met because  $\nabla \lambda_\mu^x(\underline{C})$  is a normal to the submanifold  $\mathcal{C}_\mu^x$  at  $\underline{C}$ <sup>1</sup> and  $\nabla \lambda_n^x(\underline{C})$  is a normal to  $\mathcal{C}_n^x$  at  $\underline{C}$  and  $\mathcal{C}_\mu^x = \mathcal{C}_n^x$  since  $\underline{C} \in \mathcal{D}_n^x(\rho, \lambda)$ . INSERT p. B4.10.12a That symmetry of the constraint equation carries over to the constraint stress response is not surprising in view of the fact that both are formulated in terms of the constraint function.

The last step is to consider invariance of the determinate stress response under the change of reference configuration  $\kappa \rightarrow \mu$  defined by the deformation  $\mathbb{F}$  with gradient  $\underline{G}$ . At the outset, we require  $\underline{G} \in \mathcal{D}_n^x(\rho, \lambda)$  so that the density, constraint equation, and constraint stress response are left unchanged. The reduced constitutive equation for the determinate stress is

$$\underline{T}_m^D(X, t) = \underline{F}_n(\kappa(X), t) \underline{\mathcal{T}}_n^x(\underline{C}_n^t(\kappa(X), \cdot)) \underline{F}_n^T(X, t),$$

where the domain of  $\underline{\mathcal{T}}_n^x$  is

$$\mathcal{C}_n^x = \{ \underline{C} \in C^{N-1}([0, \infty), \text{Psym } \mathcal{V}_3) : \lambda_n^x(\underline{C}(t)) = 0 \ \forall t \in [0, \infty) \},$$

the set of all right deformation tensor histories relative to the reference

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<sup>1</sup>See, e.g., COURANT's Differential and Integral Calculus, Vol. II, p. 130.

<sup>2</sup>G. Theorem B4.8.9.



INSERT for p. B4.10.12

To see this, we note that since  $\underline{G} \in \mathcal{D}_n^X(p, \lambda)$ ,

$$\underline{C} \in \mathcal{C}_n^X \Leftrightarrow \lambda_n^X(\underline{C}) = 0 \Leftrightarrow \lambda_n^X(\underline{G}^T \underline{C} \underline{G}) = 0 = \lambda_\mu^X(\underline{C}) \Leftrightarrow \underline{C} \in \mathcal{C}_\mu^X.$$

Return to p. B4.10.12

configuration  $\mathfrak{u}$  which honor the constraint. By Theorem B1.13.15,

$$\underline{F}_{\mathfrak{u}}(\mathbb{X}, t) = \underline{F}_{\mu}(\#(\mathbb{X}), t) \underline{G}(\mathbb{X})$$

and  $\therefore$

$$\begin{aligned} & \underline{T}_{\mathfrak{u}}^D(X, t) \\ &= \underline{F}_{\mu}(\mu(X), t) \underline{G}(\mathfrak{u}(X)) \underline{\mathcal{J}}_{\mathfrak{u}}^X(\underline{G}^T(\mathfrak{u}(X)) \underline{C}_{\mu}^t(\mu(X), \cdot) \underline{G}(\mathfrak{u}(X))) \underline{G}^T(\mathfrak{u}(X)) \underline{F}_{\mu}^T(\mu(X), t), \end{aligned}$$

which makes it clear that

$$\underline{\mathcal{J}}_{\mu}^X(\underline{C}^t(\cdot)) = \underline{G}(\mathfrak{u}(X)) \underline{\mathcal{J}}_{\mathfrak{u}}^X(\underline{G}^T(\mathfrak{u}(X)) \underline{C}^t(\cdot) \underline{G}(\mathfrak{u}(X))) \underline{G}^T(\mathfrak{u}(X)).$$

For the determinate stress response to be unaltered by this change of reference configuration, we must have

$$\underline{\mathcal{J}}_{\mu}^X = \underline{\mathcal{J}}_{\mathfrak{u}}^X.$$

For this to make sense, the domains of  $\underline{\mathcal{J}}_{\mu}^X$  and  $\underline{\mathcal{J}}_{\mathfrak{u}}^X$  must be the same. Since  $\underline{G} \in \mathcal{J}_{\mathfrak{u}}^X(\rho, \lambda)$ ,  $\underline{C}_{\mu}^X = \underline{C}_{\mathfrak{u}}^X$  as we noted before, and therefore, indeed,

$$\underline{C}_{\mu}^X = \underline{C}_{\mathfrak{u}}^X.$$

There also is the normalization of the response function  $\underline{\mathcal{J}}_{\mathfrak{u}}^X$  to consider. As in the development on p. B4.8.21a, we want

$$\underline{\mathcal{J}}_{\mathfrak{u}}^X(\underline{F}) \cdot [\underline{F}(\underline{t}) \nabla \underline{X}_{\mathfrak{u}}^X(\underline{F}^T(\underline{t}) \underline{F}(\underline{t})) \underline{F}^T(\underline{t})] = 0 \quad \forall \underline{F} \in \mathcal{Y}_{\mathfrak{u}}^X.$$

In our consideration of the constraint stress, we just saw that for  $\underline{G} \in \mathcal{D}_x^x(\rho, \lambda)$

$$\underline{F} \nabla \lambda_x^x (\underline{F}^T \underline{F}) \underline{F}^T \parallel \underline{F} \nabla \lambda_\mu^x (\underline{F}^T \underline{F}) \underline{F}^T;$$

and, of course, for symmetry, we have  $\underline{\mathcal{D}}_x^x = \underline{\mathcal{D}}_\mu^x$ . Thus, the normalization does not interfere with our symmetry considerations, and we finally are in position to make



Definition B4.10.3. The material symmetry group of the particle  $x$  of the simply constrained simple body  $\mathcal{B}$  relative to the reference configuration  $\kappa$  is the set

$$\mathcal{D}_n^x = \{ \underline{H} \in \mathcal{D}_n^x(\rho, \lambda) : \underline{H} \underline{Y}_n^x(\underline{H}^T \underline{C} \underline{H}) \underline{H}^T = \underline{Y}_n^x(\underline{C}) \quad \forall \underline{C} \in \mathcal{C}_n^x \},$$

where

$$\mathcal{C}_n^x = \{ \underline{C} \in C^{N-1}([0, \infty), \text{Psym } \mathcal{V}_3) : \lambda_n^x(\underline{C}(t)) = 0 \quad \forall t \in [0, \infty) \}$$

with  $N$  being the smoothness index of Axian (M3).

As expected from our terminology, we have

Theorem B4.10.3. The material symmetry group  $\mathcal{D}_n^x$  of a constrained simple body is, indeed, a group w.r.t. tensor multiplication.

Proof. Closure: Exercise B4.10.2.

Associativity: Property of tensor multiplication.

Identity Element: Trivially,  $\underline{I} \in \mathcal{D}_n^x$ .

Inverse Element: Let  $\underline{H} \in \mathcal{D}_n^x$ . Then necessarily  $\underline{H} \in \mathcal{D}_n^x(\rho, \lambda) \Rightarrow \underline{H}^{-1} \in \mathcal{D}_n^x(\rho, \lambda)$  since  $\mathcal{D}_n^x(\rho, \lambda)$  is a group. Next let  $\underline{C} \in \mathcal{C}_n^x$  and consider

$$\begin{aligned}\bar{\mathcal{D}}_n^X(\underline{C}) &= \bar{\mathcal{D}}_n^X(\underline{H}^T(\underline{H}^T \underline{C} \underline{H}^{-1})\underline{H}) \quad (\text{tensor algebra}) \\ &= \underline{H}^{-1} \bar{\mathcal{D}}_n^X(\underline{H}^T \underline{C} \underline{H}^{-1}) \underline{H}^{-T} \quad (\underline{H}^T \underline{C} \underline{H}^{-1} \in \mathcal{C}_n^X \text{ and } \underline{H} \in \mathcal{D}_n^X).\end{aligned}$$

$$\therefore \underline{H}^{-1} \in {}_c\mathcal{D}_n^X. \quad \square$$

As expected from §4.5, for orthogonal elements of the material symmetry group of a constrained particle, we get more familiar looking results.

Theorem B4.10.4, Let  $\underline{Q} \in \text{Orth}^+ V_3$ , then  $\underline{Q} \in \mathcal{D}_n^X(\rho, \lambda)$  iff for  $\underline{C} \in \text{Psym } V_3$

$$\lambda_n^X(\underline{Q}^T \underline{C} \underline{Q}) = 0 \iff \lambda_n^X(\underline{C}) = 0.$$

And  $\underline{Q} \in {}_c\mathcal{D}_n^X$  iff  $\underline{Q} \in \mathcal{D}_n^X(\rho, \lambda)$  and

$$\bar{\mathcal{D}}_n^X(\underline{Q}^T \underline{C} \underline{Q}) = \underline{Q}^T \bar{\mathcal{D}}_n^X(\underline{C}) \underline{Q} \quad \forall \underline{C} \in \mathcal{C}_n^X.$$

Proof, Exercise B4.10.3,  $\square$

Here, since all elements of  $\mathcal{D}_n^X(\rho, \lambda)$  must have a strictly positive determinant, we have

$$\underline{Q} \in \mathcal{D}_n^X(\rho, \lambda) \Rightarrow -\underline{Q} \notin \mathcal{D}_n^X(\rho, \lambda),$$

in contrast to Theorem B4.5.6.

Since our formulation of material symmetry for a constrained particle has employed an underlying reference configuration  $\kappa$ , we need to investigate the effect on the theory of using a different reference configuration, say  $\mu$ , as we did for an unconstrained particle in Theorem B4.5.4,

Theorem B4.10.5. Let the reference configurations  $\kappa$  and  $\mu$  of a simply constrained simple body  $\mathcal{B}$  be connected by the deformation  $\# = \mu \circ \kappa^{-1}$  <sup>1</sup> and write  $\underline{G} = D\#$ . Then

$$\underline{H} \in \mathcal{D}_n^x(p, \lambda) \iff \underline{G}(\#) \underline{H} \underline{G}^{-1}(\#) \in \mathcal{D}_\mu^x(p, \lambda)$$

and

$$\underline{H} \in {}_c\mathcal{D}_n^x \iff \underline{G}(\#) \underline{H} \underline{G}^{-1}(\#) \in {}_c\mathcal{D}_\mu^x.$$

Proof. Exercise B4.10.4,  $\square$

Thus, Noll's rule holds for the groups  $\mathcal{D}_n^x(p, \lambda)$  and  ${}_c\mathcal{D}_n^x$ .

Next, let us consider the particular case of an incompressible particle  $X$ . Since the reference configuration is occupied at some time in all motions of  $\mathcal{B}$ , it follows from Theorem B1.14.3 and the deliberations of §4.9 that the constraint is

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<sup>1</sup>Gf. Axiom (B1) of Definition B1.7.1. Here, of course,  $\#$  must satisfy the constraint; see the discussion on p. B4.10.4.



$$\lambda_n^x(\underline{C}_{\sim n}(\mathbb{X}, t)) = \det \underline{C}_{\sim n}(\mathbb{X}, t) - 1 = 0 \quad \forall t \in \mathbb{R}.$$

Let  $\underline{H} \in \text{Unim}^+ \mathcal{V}_3$ . Then

$$\lambda_n^x(\underline{H}^T \underline{H}) = \det(\underline{H}^T \underline{H}) - 1 = 0$$

by Theorems A3.8.4 and A3.8.6. Thus,  $\underline{H} \in \mathcal{H}_n^x$ . Let  $\underline{C} \in \text{Psym} \mathcal{V}_3$ . Then by the same reasoning

$$\lambda_n^x(\underline{H}^T \underline{C} \underline{H}) = \det(\underline{H}^T \underline{C} \underline{H}) - 1 = \det \underline{C} - 1 = \lambda_n^x(\underline{C}).$$

Thus,  $\underline{H} \in \mathcal{D}_n^x(\rho, \lambda)$ .  $\therefore \text{Unim}^+ \mathcal{V}_3 \subset \mathcal{D}_n^x(\rho, \lambda)$ , which  $\Rightarrow$

$$\mathcal{D}_n^x(\rho, \lambda) = \text{Unim}^+ \mathcal{V}_3,$$

since necessarily  $\mathcal{D}_n^x(\rho, \lambda) \subset \text{Unim}^+ \mathcal{V}_3$ .

Consequently, in this case

$${}_c \mathcal{D}_n^x = \{ \underline{H} \in \text{Unim}^+ \mathcal{V}_3 : \underline{H} \underline{\mathcal{J}}_n^x(\underline{H}^T \underline{C} \underline{H}) \underline{H}^T = \underline{\mathcal{J}}_n^x(\underline{C}) \quad \forall \underline{C} \in \mathcal{C}_n^x \}$$

which is the same as in the unconstrained case)<sup>1</sup> except that here the domain of the response function  $\underline{\mathcal{J}}_n^x$  is restricted to those right deformation tensor histories that satisfy the constraint. Thus, we have

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<sup>1</sup>Cf. Theorem B4.5.7.

Theorem B 4.10.6. The constraint of incompressibility places no restrictions on the material symmetry group in the sense that it is just the symmetry group of the stress response as in the unconstrained case,

This explains why historically no difficulties arose when investigators cavalierly coupled incompressibility and material symmetry considerations such as isotropy with no apparent recognition of the possibility that the two concepts might be incompatible.

What is the most general internal constraint which places no restriction on material symmetry? — apart from the need to use occupied reference configurations and to account for the constraint in the domain of the response function for the determinate stress. As the case of incompressibility makes clear, technically all we need is that  $\mathcal{I}_n^X(\rho, \lambda) = \ell n i m + V_3$ .

Theorem B 4.10.7. If the constraint function  $\lambda_n^X: \text{Psym} V_3 \rightarrow \mathbb{R}$  has the form

$$\lambda_n^X(\underline{C}) = \varphi_n^X(\det \underline{C}) \quad ,$$

where the function  $\varphi_n^X: \mathbb{R}^+ \rightarrow \mathbb{R}$  has the property

$$\varphi_n^X(1) = 0,$$

then

$$\mathcal{I}_n^X(\rho, \lambda) = \ell n i m + V_3.$$

"Conversely", if  $\mathcal{D}_n^x(\rho, \lambda) = \text{Unim}^+ \mathcal{V}_3$ , then the restriction of  $\lambda_n^x$  to the constraint set

$$\mathcal{C}_n^x = \{ \underline{C} \in \text{Psym } \mathcal{V}_3 : \lambda_n^x(\underline{C}) = 0 \}$$

of right deformation tensors that honor the constraint has the above form.

Proof. Suppose  $\lambda_n^x$  has this form. Let  $\underline{H} \in \text{Unim}^+ \mathcal{V}_3$ .  
Then

$$\lambda_n^x(\underline{H}^T \underline{H}) = \varphi_n^x(\det(\underline{H}^T \underline{H})) \quad (\text{substitution})$$

$$= \varphi_n^x(1) \quad (\text{Theorems A3.8.4 and 6})$$

$$= 0 \quad (\text{hypothesis}).$$

$\therefore \underline{H} \in \mathcal{H}_n^x$ . Similarly, for any  $\underline{C} \in \text{Psym } \mathcal{V}_3$ ,

$$\lambda_n^x(\underline{H}^T \underline{C} \underline{H}) = \varphi_n^x(\det(\underline{H}^T \underline{C} \underline{H})) = \varphi_n^x(\det \underline{C}) = \lambda_n^x(\underline{C});$$

and  $\underline{H} \in \mathcal{D}_n^x(\rho, \lambda)$ . Hence,  $\text{Unim}^+ \mathcal{V}_3 \subset \mathcal{D}_n^x(\rho, \lambda)$ , and  $\therefore$

$$\mathcal{D}_n^x(\rho, \lambda) = \text{Unim}^+ \mathcal{V}_3.$$

Conversely, suppose that this last equation holds.  
Then for  $\underline{C} \in \text{Psym}$ ,

$$\lambda_n^x(\underline{H}^T \underline{C} \underline{H}) = 0 \iff \lambda_n^x(\underline{C}) = 0 \quad \forall \underline{H} \in \text{Unim}^+ \mathcal{V}_3.$$



Consequently, for  $\underline{C} \in \mathcal{C}_n^X = \{ \underline{C} \in \text{Psym } \mathcal{V}_3 : \lambda_n^X(\underline{C}) = 0 \}$ ,

$$\lambda_n^X(\underline{H}^T \underline{C} \underline{H}) = \lambda_n^X(\underline{C}) \quad \forall \underline{H} \in \text{Unim}^+ \mathcal{V}_3. \quad (*)$$

In particular, for  $\underline{C} \in \mathcal{C}_n^X$ ,

$$\lambda_n^X(\underline{Q}^T \underline{C} \underline{Q}) = \lambda_n^X(\underline{C}) \quad \forall \underline{Q} \in \text{Orth}^+ \mathcal{V}_3.$$

Thus, the restriction  $\lambda_n^X|_{\mathcal{C}_n^X}$  of  $\lambda_n^X$  to  $\mathcal{C}_n^X$  is isotropic; and by the representation theorem for an isotropic scalar-valued function of one symmetric tensor (Theorem A8.4.4),  $\exists$  a function  $\vartheta_n^X: \mathbb{R}^3 \rightarrow \mathbb{R}$   $\ni$

$$\lambda_n^X|_{\mathcal{C}_n^X}(\underline{C}) = \vartheta_n^X(I_1(\underline{C}), I_2(\underline{C}), I_3(\underline{C})),$$

Then by (\*), for any  $\underline{C} \in \mathcal{C}_n^X$

$$\vartheta_n^X(I_1(\underline{C}), I_2(\underline{C}), I_3(\underline{C})) = \vartheta_n^X(I_1(\underline{H}^T \underline{C} \underline{H}), I_2(\underline{H}^T \underline{C} \underline{H}), I_3(\underline{H}^T \underline{C} \underline{H})) \quad \forall \underline{H} \in \text{Unim}^+ \mathcal{V}_3 \quad (**)$$

In order to solve (\*\*), we first note that since  $\underline{C}$  is positive definite, it has a positive definite square root<sup>1</sup>; i.e.,  $\exists \underline{U} \in \text{Psym } \mathcal{V}_3 \ni \underline{C} = \underline{U}^2$ . Next, we make the special choice

$$\underline{H} = (\det \underline{U})^{1/3} \underline{U}^{-1},$$

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<sup>1</sup> Cf. Theorem A3.15.3.

which belongs to  $\text{Unim}^+ V_3$  by Theorems A3.8.2(iv) and A3.9.10. Then

$$\underline{\underline{H}}^T \underline{\underline{C}} \underline{\underline{H}} = \underline{\underline{H}}^T \underline{\underline{U}}^2 \underline{\underline{H}} = (\det \underline{\underline{U}})^{2/3} \underline{\underline{I}} = (\det \underline{\underline{C}})^{1/3} \underline{\underline{I}} ;$$

and by straight forward calculations <sup>1</sup>,

$$l_1(\underline{\underline{H}}^T \underline{\underline{C}} \underline{\underline{H}}) = 3(\det \underline{\underline{C}})^{1/3},$$

$$l_2(\underline{\underline{H}}^T \underline{\underline{C}} \underline{\underline{H}}) = 3(\det \underline{\underline{C}})^{2/3},$$

$$l_3(\underline{\underline{H}}^T \underline{\underline{C}} \underline{\underline{H}}) = \det \underline{\underline{C}}.$$

Thus, (\*\*) becomes

$$\begin{aligned} \vartheta_n^x(l_1(\underline{\underline{C}}), l_2(\underline{\underline{C}}), l_3(\underline{\underline{C}})) &= \vartheta_n^x(3(\det \underline{\underline{C}})^{1/3}, 3(\det \underline{\underline{C}})^{2/3}, \det \underline{\underline{C}}) \\ &=: \varphi_n^x(\det \underline{\underline{C}}). \end{aligned}$$

Clearly, for any  $\vartheta_n^x$  of this form, (\*\*) is automatically satisfied; so there is nothing more to be learned from (\*\*) by additional special choices of  $\underline{\underline{H}}$ .

Putting these results together, we obtain

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<sup>1</sup> cf. Theorem A3.11.8.

$$\lambda_n^x \Big|_{\mathcal{C}_n^x}(\underline{C}) = \varphi_n^x(\det \underline{C}).$$

By Theorem B4.10.1,

$$\lambda_n^x(\underline{I}) = \lambda_n^x \Big|_{\mathcal{C}_n^x}(\underline{I}) = \varphi_n^x(1) = 0. \quad \square$$

How much more general than incompressibility is the constraint

$$\varphi_n^x(\det \underline{C}) = 0 \quad ?$$

The following result is relevant

Theorem B4.10.8. Let the function  $\varphi_n^x: \mathbb{R}^+ \rightarrow \mathbb{R}$  have the property  $\varphi_n^x(1) = 0$  and also be one-to-one.  
Then for  $\underline{C} \in \text{Psym} \mathcal{V}_3$ ,

$$\varphi_n^x(\det \underline{C}) = 0 \iff \det \underline{C} = 1.$$

Proof. Exercise B4.10.5.  $\square$

Thus, when  $\varphi_n^x$  is one-to-one, the constraint

$$\varphi_n^x(\det \underline{C}) = 0$$

is equivalent to incompressibility. The example



$$\phi_n^x(x) = \sin(\pi x)$$

shows that the one-to-one hypothesis is necessary for this conclusion. For while

$$\det \underline{C} = 1 \Rightarrow \sin(\pi \det \underline{C}) = 0,$$

$$\sin(\pi \det \underline{C}) = 0 \Rightarrow \det \underline{C} = m, m = 1, 2, 3, \dots$$

On the other hand, the set

$$\{ \underline{C} \in \text{Psym } \mathcal{V}_3 : \det \underline{C} = m, m = 1, 2, 3, \dots \}$$

is not connected (Establish this as Exercise B4.10.6), and this suggests that perhaps our definition of a constraint is too general. At this point our constraint is defined by

$$\lambda_n^x(\underline{C}_n(n(x), t)) = 0 \quad \forall t \in \mathbb{R},$$

where  $\lambda_n^x : \text{Psym } \mathcal{V}_3 \rightarrow \mathbb{R}$  is differentiable. We probably should require also that the constraint set

$$\kappa_n^x = \{ \underline{C} \in \text{Psym } \mathcal{V}_3 : \lambda_n^x(\underline{C}) = 0 \}$$

be connected,

Most internal constraints will restrict the material symmetry group. E.g., suppose a particle  $X \in \mathcal{B}$  is inextensible in the material direction defined in the reference configuration  $\mathcal{R}_0$  by the unit vector  $\underline{e}_n(X)$ . Then by Definition B4.9.5 and the remark following Definition B4.9.4, we have the constraint

$$\lambda_n^X(C_{\sim n}(X, t)) = \underline{e}_n(X) \cdot C_{\sim n}(X, t) \underline{e}_n(X) - 1 = 0.$$

In this case, we expect intuitively that all rotations about  $\underline{e}_n(X)$  are included in

$$\mathcal{I}_n^X(\rho, \lambda) = \{H \in H_n^X : \lambda_n^X(H^T C H) = 0 \iff \lambda_n^X(C) = 0\}.$$

To prove this let  $\underline{R}(\theta)$  be a "right-handed" rotation about  $\underline{e}_n(X)$  through the angle  $\theta$ .<sup>1</sup> Then, in somewhat abridged notation,

$$\begin{aligned} \lambda_n^X(\underline{R}^T \underline{R}) &= \underline{e}_n \cdot (\underline{R}^T \underline{R}) \underline{e}_n - 1 \quad (\text{substitution}) \\ &= \underline{e}_n \cdot \underline{R}^T (\underline{R} \underline{e}_n) - 1 \quad (\text{defn. of tensor multiplication}) \\ &= \underline{R} \underline{e}_n \cdot \underline{R} \underline{e}_n - 1 \quad (\text{defn. of transpose}) \\ &= \underline{e}_n \cdot \underline{e}_n - 1 \quad (\text{defn. of } \underline{R}) \\ &= 0 \quad (\underline{e}_n \text{ a unit vector}). \end{aligned}$$

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<sup>1</sup>cf. the discussion in pp. B4.6.22-23.

$\therefore \underline{R}(\theta) \in N_n^X$ . Similarly, for any  $\underline{C} \in \text{Psym } V_3$

$$\lambda_n^X(\underline{R}^T \underline{C} \underline{R}) = \lambda_n^X(\underline{C}) \quad (\text{Exercise B4.10.7});$$

and  $\therefore \underline{R}(\theta) \in \mathcal{D}_n^X(\rho, \lambda)$ . It is convenient here to introduce

Definition B4.10.4. Let  $\underline{R}(\theta)$  denote the right-handed rotation about the axis  $\underline{e}$  ( $|\underline{e}|=1$ ) through the angle  $\theta$ .  
Then

$$\text{Orth}^+(\underline{e}, V_3) = \{ \underline{R}(\theta) : \theta \in [0, 2\pi] \}$$

is the group<sup>1</sup> of rotations about  $\underline{e}$ .

What we have just shown is that for the constraint of inextensibility at  $X$  in the direction  $\underline{e}_n(\mathbb{X})$ ,

$$\text{Orth}^+(\underline{e}_n(\mathbb{X}), V_3) \subset \mathcal{D}_n^X(\rho, \lambda)$$

Of course, there is no reason to expect in general that

$$\text{Orth}^+(\underline{e}_n(\mathbb{X}), V_3) \subset_c \mathcal{D}_n^X.$$

We are led to the formal

Definition B4.10.5. A simple body  $\mathcal{B}$  is transversely isotropic w.r.t. the material direction  $\underline{e}$  at the particle  $X \in \mathcal{B}$

<sup>1</sup>cf. Exercise B1.3.3.



if  $\exists$  an occupied reference configuration  $\kappa$  of  $\mathcal{B}$ , called undistorted,  $\ni$

$$\text{Orth}^+(\underline{e}_n(\kappa(X)), \mathcal{V}_3) \subset \mathcal{D}_n^X.$$

Of course, for a constrained particle, we require

$$\text{Orth}^+(\underline{e}_n(\kappa(X)), \mathcal{V}_3) \subset_c \mathcal{D}_n^X.$$

The above considerations show that a particle  $X$  which is inextensible in the material direction defined by  $\underline{e}_n(\kappa)$  has the potential of being transversely isotropic w.r.t.  $\underline{e}_n(\kappa)$ . Whether or not it is depends entirely on the determinate stress response function. It will be transversely isotropic if

$$\underline{\mathcal{Y}}_{\sim n}^X(R^T \underline{\mathcal{C}} R) = R^T \underline{\mathcal{Y}}_{\sim n}^X(\underline{\mathcal{C}}) R$$

$\forall R \in \text{Orth}^+(\underline{e}_n(\kappa(X)), \mathcal{V}_3)$  and  $\forall \underline{\mathcal{C}} \in \mathcal{C}_n^X$ , where

$$\mathcal{C}_n^X = \{ \underline{\mathcal{C}} \in C^{N-1}([0, \infty), \text{Psym} \mathcal{V}_3) : \underline{e}_n(\kappa) \cdot \underline{\mathcal{C}}(t) \underline{e}_n(\kappa) = 1 \quad \forall t \in [0, \infty) \}.$$

Of course, the definitions of isotropic, fluid, solid, and fluid crystal particles can be carried over from the unconstrained case in §B4.6 to the constrained case just as we did above for transverse isotropy. It is important to realize that most constraints will rule out certain symmetries. E.g., one cannot have a fluid that is inextensible in any direction.

Finally, suppose that there are several constraints at the particle  $X$ , say

$${}^{\alpha}\lambda_n^x(\underline{C}_n(\#,t)) = 0 \quad \forall t \in \mathbb{R}, \alpha \in \mathbb{I}_m,$$

as on pp. B4.8.21-25. Then by analogy with Theorem B4.10.1, it is easy to see that the deformation  $\#$  which carries the body  $\mathcal{B}$  from the reference configuration  $\mathcal{V}_0$  to the new reference configuration  $\mathcal{V}_\#$  preserves the mass density at  $X$  and honors all of the constraints iff

$$D\#(\#) \in {}^M\mathcal{H}_n^x := \{ \underline{H} \in \text{Unim}^+\mathcal{V}_3 : {}^{\alpha}\lambda_n^x(\underline{H}^T \underline{H}) = 0, \alpha \in \mathbb{I}_m \}.$$

For  $\#$  to leave the  $M$  constraint equations invariant, we must have

$${}^{\alpha}\lambda_n^x(\underline{C}) = 0 \iff {}^{\alpha}\lambda_n^x(\underline{C}) = 0, \quad \underline{C} \in \text{Psym}\mathcal{V}_3$$

for each  $\alpha \in \mathbb{I}_m$ ; and thus the symmetry group for the mass density and the constraint equations of the particle  $X$  relative to  $\mathcal{V}_0$  is

$${}^M\mathcal{G}_n^x(\rho, \lambda) = \{ \underline{H} \in {}^M\mathcal{H}_n^x : {}^{\alpha}\lambda_n^x(\underline{H}^T \underline{C} \underline{H}) = 0 \iff {}^{\alpha}\lambda_n^x(\underline{C}) = 0, \underline{C} \in \text{Psym}\mathcal{V}_3 \quad \forall \alpha \in \mathbb{I}_m \}$$

We leave it as Exercise B4.10.8 to show that  ${}^M\mathcal{G}_n^x(\rho, \lambda)$  is a group w.r.t. tensor multiplication.

As on pp. B4.10.9-12, the symmetry of the constraint equations

carries over to the constraint stress

$$\underline{N}_{\underline{n}} = \sum_{\alpha=1}^M \alpha q_{\underline{n}} \underline{F}_{\underline{n}} \nabla^{\alpha} \lambda_{\underline{n}}^{\times}(\underline{C}_{\underline{n}}) \underline{F}_{\underline{n}}^T$$

in the sense that if  $D\#(\mathbb{X}) \in {}^M\mathcal{G}_{\underline{n}}^{\times}(\rho, \lambda)$ , then

$$\underline{N}_{\underline{\mu}} = \sum_{\alpha=1}^M \alpha q_{\underline{\mu}} \underline{F}_{\underline{\mu}} \nabla^{\alpha} \lambda_{\underline{\mu}}^{\times}(\underline{C}_{\underline{\mu}}) \underline{F}_{\underline{\mu}}^T$$

with

$$\underline{F} \nabla^{\alpha} \lambda_{\underline{\mu}}^{\times}(\underline{F}^T \underline{F}) \underline{F}^T \parallel \underline{F} \nabla^{\alpha} \lambda_{\underline{n}}^{\times}(\underline{F}^T \underline{F}) \underline{F}^T \quad \forall \underline{F} \in {}^M\mathcal{J}_{\underline{n}}^{\times},$$

where

$${}^M\mathcal{J}_{\underline{n}}^{\times} := \{ \underline{F} \in \text{Inv } \mathcal{V}_3 : \det \underline{F} > 0, {}^{\times}\lambda_{\underline{n}}^{\times}(\underline{F}^T \underline{F}) = 0 \quad \forall \alpha \in \mathbb{I}_m \}.$$

In the multiply constrained case, the determinate stress is given by

$$\underline{T}_{\underline{m}}^D(\mathbb{X}, t) = \underline{F}_{\underline{n}}(\mathbb{X}, t) \underline{\mathcal{G}}_{\underline{n}}^{\times}(\underline{C}_{\underline{n}}^t(\mathbb{X}, \cdot)) \underline{F}_{\underline{n}}^T(\mathbb{X}, t),$$

where the domain of  $\underline{\mathcal{G}}_{\underline{n}}^{\times}$  is

$${}^M\mathcal{C}_{\underline{n}}^{\times} = \{ \underline{C} \in C^{N-1}([0, \omega), \text{Psym } \mathcal{V}_3) : {}^{\times}\lambda_{\underline{n}}^{\times}(\underline{C}(t)) = 0 \quad \forall t \in [0, \omega) \quad \forall \alpha \in \mathbb{I}_m \}.$$

For  $\#$  also to leave the determinate stress response invariant, we must have

$$\underline{\mathcal{G}}_{\underline{\mu}}^{\times} = \underline{\mathcal{G}}_{\underline{n}}^{\times},$$

where



$$\tilde{\mathcal{G}}_{\mu}^X(\underline{C}^t) = \underline{G}(\mathbb{X}) \tilde{\mathcal{G}}_{\mu}^X(\underline{G}^T(\mathbb{X}) \underline{C}^t \underline{G}(\mathbb{X})) \underline{G}^T(\mathbb{X})$$

with  $\underline{G} = D\#$  as on pp. B4.10.12-14. Thus, the material symmetry group of the particle  $\mathbb{X}$  of the multiply constrained body  $\mathcal{B}$  is

$${}^M\mathcal{D}_{\mu}^X = \{ \tilde{H} \in {}^M\mathcal{D}_{\mu}^X(p, \lambda) : \tilde{H} \tilde{\mathcal{G}}_{\mu}^X(\tilde{H}^T \underline{C} \tilde{H}) \tilde{H}^T = \tilde{\mathcal{G}}_{\mu}^X(\underline{C}) \quad \forall \underline{C} \in {}^M\mathcal{B}_{\mu}^X \}.$$

Just as in Theorem B4.10.4,  ${}^M\mathcal{D}_{\mu}^X$  is group w.r.t. tensor multiplication.

We leave it as Exercise B4.10.9 to decide whether or not the groups  ${}^M\mathcal{D}_{\mu}^X(p, \lambda)$  and  ${}^M\mathcal{D}_{\mu}^X$  follow Noll's rule.)<sup>1</sup>

Of course, the definitions of isotropic, fluid, solid, and fluid crystal particles can be carried over from before and given in terms of  ${}^M\mathcal{D}_{\mu}^X$ .

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<sup>1</sup> Cf. Theorem B4.10.4.

# Supplementary Reading

COHEN and WANG, "On the response and symmetry of elastic materials with internal constraints", Arch. Rational Mech. Analysis 91 (1987) 1-36.

COHEN and WANG, "On the stored energy functions of hyperelastic materials with internal constraints", Arch. Rational Mech. Analysis 102 (1988) 119-140.

GURTIN and PODIO-GUIDUGLI, "The thermodynamics of constrained materials", Arch. Rational Mech. Analysis 51 (1973) 192-208.

MARLOW, "On the stress in an internally constrained elastic material", J. Elasticity (in press).

PODIO-GUIDUGLI, "Constrained elasticity", Atti Accad. Naz. Lincei Rend. Mat. (1990).

PODIO-GUIDUGLI and VIANELLO, "Constraint manifolds for isotropic solids", Arch. Rational Mech. Analysis 105 (1989) 105-121.

VIANELLO, "On the active part of the stress for elastic materials with internal constraints", J. Elasticity 24 (1990) 289-294.

VIANELLO, "The representation problem for constrained hyperelastic materials", Arch. Rational Mech. Analysis 111 (1990) 87-98.