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Chapter B2Mass. Momentum. Kinetic Energy

The title concepts are probably familiar to most readers from courses in particle and rigid body mechanics. Here we introduce the variants of these notions appropriate to continuum mechanics.

B2.1. Mass Density. Conservation of Mass

With each part P of a body B , we associate a strictly positive number, called the mass of the part, which is supposed to represent the amount of material contained in the part. Experience indicates that mass should have the following two properties:

- (i) the mass of a body is the sum of the masses of its parts (for any decomposition of the body into parts);
- (ii) the mass of a part of a body is the same for all configurations of the body.

Property (i) suggests that measure theory would be the proper mathematical vehicle for introducing mass.)¹ In this spirit but employing considerably less sophistication, we shall adopt the more customary Riemann integral approach and assume

The Principle of Conservation of Mass. For each configuration $\mathbb{X}$ of a body B , $\exists$ a smooth scalar field $\rho_{\mathbb{X}}: \mathbb{X}(B) \rightarrow \mathbb{R}^+$, called the mass density of the body B in the configuration $\mathbb{X}$, $\exists$ the mass of any part $P \subset B$ is given by

$$m(P) = \int_{\mathbb{X}(P)} \rho_{\mathbb{X}} dV.$$

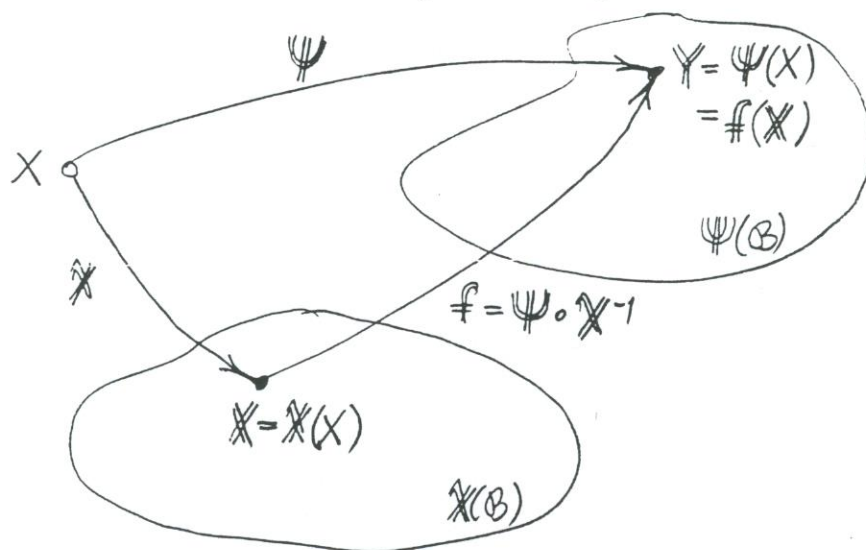
¹See the articles by NOLL listed in the supplementary reading.

Furthermore, mass is conserved in the sense that

$$(D1) \quad m(\mathcal{O}) = \int_{\mathbb{X}(\mathcal{O})} \rho_{\mathbb{X}} dV = \int_{\Psi(\mathcal{O})} \rho_{\Psi} dV \quad \forall \text{ parts } \mathcal{O} \subset \mathcal{B} \text{ and } \forall \text{ configurations } \mathbb{X} \text{ and } \Psi \text{ of } \mathcal{B}.$$

Obviously, it is appropriate to think of the mass density as the mass per unit volume.

By Axiom (B1) of Definition B1.7.1, the configurations $\mathbb{X}$ and Ψ are connected by the deformation $\mathbb{F} = \Psi \circ \mathbb{X}^{-1}$,



and this leads us to the following local form of the principle of conservation of mass.

(The Density Equation)

Theorem B2.1.1. Let the configurations $\mathbb{X}$ and Ψ of a body $\mathcal{B}$ be connected by the deformation $\mathbb{F} = \Psi \circ \mathbb{X}^{-1}$. Then mass is conserved iff $\forall$ configurations $\mathbb{X}$ and Ψ of $\mathcal{B}$

$$\rho_{\Psi}(\mathbb{F}(\mathbb{X})) = \frac{\rho_{\mathbb{X}}(\mathbb{X})}{|\det D\mathbb{F}(\mathbb{X})|}, \quad \mathbb{X} \in \mathbb{X}(\mathcal{B}).$$

Proof. Suppose that mass is conserved. Then by the change of variable theorem for volume integrals (Theorem A6.10.2),

$$\int_{X(P)} \rho_X(X) dV_X = \int_{\Psi(P)} \rho_\Psi(Y) dV_Y = \int_{X(P)} \rho_\Psi(f(X)) |\det Df(X)| dV_X. \quad (*)$$

Since this must hold $\forall$ parts $P \subset B$, it holds $\forall$ regular subregions $X(P) \subset X(B)$. Indeed, given any regular subregion $R \subset X(B)$, $P := X^{-1}(R)$ is a part of $B \ni X(P) = R$. Then we conclude from the Localization Theorem (Theorem B1.6.2) that

$$\rho_X(X) = \rho_\Psi(f(X)) |\det Df(X)| \quad \forall X \in X(B). \quad (**)$$

Since f is a deformation, $\det Df \neq 0$ on $X(B)$ by Definition B1.1.1; and we can perform the division in $(**)$ necessary to reach the density equation.

Conversely, if the density equation holds, then $(**)$ holds. Integration of $(**)$ over $X(P)$ yields $(*)$; i.e., mass is conserved. $\square$

In the case of a motion, there is a mass density for each configuration associated with the motion.

Definition B2.1.1. Let $\{X(\cdot, t) : t \in \mathcal{I}\}$ be a motion of a body B . Then the mass density $\rho_{X(\cdot, t)}$ for the configuration $X(\cdot, t)$, $t \in \mathcal{I}$ is denoted by

$$\rho(\cdot, t) = \rho_{X(\cdot, t)}(\cdot).$$

Thus, $\rho(x, t)$ is the mass density at the place x for the configuration that the body is in at time t in the motion $\{\chi(\cdot, t) : t \in \mathcal{Q}\}$.

For motions the principle of conservation of mass adopts the form given by

Theorem B2.1.2. Let $\{\chi(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body B . If mass is conserved, then

$$(*) \quad \int_{\mathcal{P}_t} \rho(x, t) dV = \int_{\mathcal{P}_s} \rho(x, s) dV \quad \forall \text{ parts } \mathcal{P} \subset B \text{ and } \forall t, s \in \mathcal{Q}.$$

Conversely, if $(*)$ holds, then mass is conserved w.r.t. all configurations in the motion.

Proof. If mass is conserved, then by definition Axiom (D1) must hold $\forall$ configurations χ and ψ of B . In particular, it must hold for

$$\chi = \chi(\cdot, t) \text{ and } \psi = \chi(\cdot, s);$$

and $(*)$ above follows.

Conversely, if $(*)$ holds, then (D1) obtains $\forall$ configurations in the motion. $\square$

take this as the working definition for
B2.1.4: $\rho(x, t)$
& forget $\rho(x)$.

Differentiation of (*) of Theorem B2.1.2 w.r.t. t for fixed s leads to the following balance form of the principle of conservation of mass.

Theorem B2.1.3. Let $\{\chi(\cdot, t): t \in \mathcal{Q}\}$ be a motion
of a body B , If mass is conserved, then

$$(\#) \quad \frac{d}{dt} \int_{\mathcal{Q}_t} \rho(x, t) dV_x = 0 \quad \forall \text{ parts } P \subset B \text{ and } \forall t \in \mathcal{Q},$$

Conversely, if (#) holds, then mass is conserved w.r.t.
all configurations in the motion.

Proof. Only the converse assertion remains to be proved. Integration of (#) from s to t yields (*) of Theorem B2.1.2; i.e., $(\#) \Rightarrow$ conservation of mass w.r.t. all configurations in the motion. $\square$

For motions relative to a reference configuration, the principle of conservation of mass adopts the form given by

Theorem B2.1.4. Let $\{\chi(\cdot, t): t \in \mathcal{Q}\}$ be a motion
and κ a reference configuration of a body B . If mass is
conserved, then

$$(\dagger) \quad \int_{\mathcal{Q}_t} \rho(\chi, t) dV_x = \int_{\kappa(P)} \rho_\kappa(\chi) dV_x \quad \forall \text{ parts } P \subset B \text{ and } \forall t \in \mathcal{Q}.$$

Conversely, if $(\dagger)$ holds, then mass is conserved w.r.t. all configurations in the motion.

Proof. Exercise B2.1.1. $\square$

The local form of the referential version of the principle of conservation of mass is given by

Theorem B2.1.5. (The Referential Density Equation) Let $\{\kappa(\cdot, t) : t \in \mathcal{D}\}$ be a motion and κ a reference configuration of a body $\mathcal{B}$. If mass is conserved, then

$$(-\dagger) \quad \rho(\kappa(\cdot, t), t) = \frac{\rho_{\kappa}(\cdot)}{J_{\kappa}(\cdot, t)} \quad \forall (\cdot, t) \in \kappa(\mathcal{B}) \times \mathcal{D}.$$

Conversely, if $(-\dagger)$ holds, then mass is conserved w.r.t. all configurations in the motion.

Proof. Exercise B2.1.2. $\square$

INSERT material on p. B2.1.6 a

The referential density equation is a rather remarkable result. If we know the mass density in some reference configuration and if we know the motion, then the referential density equation $(-\dagger)$ gives us the mass density at any time t . It also supplies the smoothness properties of ρ that we need to get a local form of conservation of mass in the spatial description.

INSERT for p. B2.1.6 (immediately after the proof of Theorem B2.1.5)

Exercise B2.1.3a, Show that

$$\rho(\mathcal{X}_{(t)}(\mathcal{X}, \tau), \tau) = \frac{\rho(\mathcal{X}, t)}{J_{(t)}(\mathcal{X}, \tau)} \quad \forall (\mathcal{X}, t) \in \mathcal{I}, \tau \in \mathcal{D}$$

— a form of the Referential Density Equation appropriate to the relative description of Ξ B1.1b.

Return to p. B2.1.6

Theorem B2.1.6. Let $\{\chi(\cdot, t) : t \in \mathcal{I}\}$ be a motion
of a body $\mathcal{B}$, and let mass be conserved. Then

$$\rho \in C^1(\mathcal{I}, \mathbb{R}).$$

Proof. Let κ be an arbitrary reference configuration
of $\mathcal{B}$. Then since $\mathbf{x} = \chi_n(\mathbf{x}, t)$, the referential density
equation $\Rightarrow$

$$\rho(\mathbf{x}, t) = \frac{\rho_n(\chi_n^{-1}(\mathbf{x}, t))}{J_n(\chi_n^{-1}(\mathbf{x}, t), t)} \quad \forall (\mathbf{x}, t) \in \mathcal{I}.$$

Then the smoothness of ρ on $\mathcal{I}$ follows from the known
smoothness properties of ρ_n ¹, J_n ², and χ_n^{-1} ³
via the product rule (Theorem A5.6.3), the
chain rule (Theorem A5.7.1), and the Extension Theorem
(Theorem B1.2.2). $\square$

A local form of the principle of conservation of mass
in the spatial description, which is especially useful in
fluid mechanics, is given by

¹g. the Principle of Conservation of Mass.

² Theorem B1.13.12.

³ Theorem B1.13.4.

Theorem B2.1.7. (The Spatial Density Equation)¹ Let
 $\{x(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$. If mass is
conserved, then

$$(\star) \quad \rho_{,2} + \nabla_1 \rho \cdot \underline{v}_x + \rho \operatorname{div}_1 \underline{v}_x = 0 \quad \text{on } \mathcal{I}, \quad)^2$$

or equivalently,

$$(\star\star) \quad \rho_{,2} + \operatorname{div}_1 (\rho \underline{v}_x) = 0 \quad \text{on } \mathcal{I}.$$

Conversely, if $(\star)$ holds, then mass is conserved w.r.t.
all configurations in the motion,

Proof. Suppose that mass is conserved. Then by Theorem
 B2.1.3,

$$\frac{d}{dt} \int_{\mathcal{Q}_t} \rho(x, t) dV_x = 0 \quad \forall \text{ parts } \mathcal{Q} \subset \mathcal{B} \text{ and } \forall t \in \mathcal{Q}.$$

In view of the smoothness of ρ guaranteed by the previous theorem, the Transport Theorem (Theorem B1.13.14) can be

¹ Often, the continuity equation, and then our referential density equation is called the Lagrangian continuity equation. There appears to be no good reason, apart from "continuity" to perpetuate this terminology.

² $\nabla_1 \rho$ is defined by $D_1 \rho(x, t)[u] = \nabla_1 \rho(x, t) \cdot u$, and $\operatorname{div}_1 \underline{v}_x = \operatorname{tr}(D_1 \underline{v}_x)$; see §A6.5, Definition A6.6.1, and §B1.8.

applied with the result that

$$\int_{\mathcal{Q}_t} (\rho_{,2} + \nabla_1 \rho \cdot \underline{v}_x + \rho \operatorname{div}_1 \underline{v}_x) dV_x = 0.$$

Since $\mathcal{Q}$ is an arbitrary part of $\mathcal{B}$, $\mathcal{Q}_t$ is an arbitrary regular subregion of $\mathcal{B}_t$ ¹, and we conclude from the Localization Theorem (Theorem B1.6.2) that $(\star)$ holds.

Conversely, suppose that $(\star)$ holds. Then integration over $\mathcal{Q}_t$ and application of the Transport Theorem lead back to

$$\frac{d}{dt} \int_{\mathcal{Q}_t} \rho dV = 0 \quad \forall \text{ parts } \mathcal{Q} \subset \mathcal{B} \text{ and } \forall t \in \mathcal{I},$$

and mass is conserved w.r.t. all configurations in the motion by Theorem B2.1.3.

The equivalency of $(\star)$ and $(\star\star)$ rests on the identity

$$\operatorname{div}_1 (\rho \underline{v}_x) = \nabla_1 \rho \cdot \underline{v}_x + \rho \operatorname{div}_1 \underline{v}_x,$$

which is the trace of the identity

$$D_1 (\rho \underline{v}_x) = \underline{v}_x \otimes \nabla_1 \rho + \rho D_1 \underline{v}_x,$$

which was established in Exercise A5.6.4. $\square$

¹cf. the proof of Theorem B2.1.1.

Exercise B2.1.3, Establish the first assertion in Theorem B2.1.7 by differentiating the referential density equation

$$\rho_n(\mathbb{X}) = J_n(\mathbb{X}, t) \rho(\mathbb{X}_n(\mathbb{X}, t), t)$$

w.r.t. time.

Our notational agreements with regard to the material, spatial, and referential descriptions do not work perfectly when the mass density is concerned. To see this, recall that the notion of density arose in the context of the referential description with $\rho_n(\mathbb{X})$ being the mass per unit volume at the place $\mathbb{X}$ for the configuration $\mathbb{X}_n$. This leads us to $\rho_n(X, t) = \rho_n(\mathbb{X}(X))$ for the density (at $\mathbb{X} = \mathbb{X}(X)$ for the configuration $\mathbb{X}_n$) of that particle X that is at the place $\mathbb{X}$ in the configuration $\mathbb{X}_n$ and to $\rho_s(\mathbb{X}, t) = \rho_n(\mathbb{X}_n^{-1}(\mathbb{X}, t))$ for the density (at $\mathbb{X} = \mathbb{X}_n^{-1}(\mathbb{X}, t)$ for the configuration $\mathbb{X}_n$) of that particle that is at the place $\mathbb{X}$ at time t . On the other hand, $\rho(\mathbb{X}, t)$ is the mass density at the place $\mathbb{X}$ for the configuration $\mathbb{X}(\cdot, t)$ that the body is in at time t . Thus, in general, $\rho(\mathbb{X}, t) \neq \rho_s(\mathbb{X}, t)$, and to minimize confusion we shall not use the notations ρ_s or ρ_n .

i.e. In the conventional sense, $\rho_s(\mathbb{X}, t) = \rho(\mathbb{X}, t)$ means the density of the particle at $\mathbb{X}$ at time t .

This definition of the mass density, however, gives for $\rho_s(\mathbb{X}, t)$ the density of the particle at $\mathbb{X}$ and time t , the density of the same particle in the reference configuration

From this point forward, we assume without further comment in general that mass is conserved.

Most volume integrals in continuum mechanics have the mass density as a factor in the integrand; therefore, the following two theorems, which are consequences of conservation of mass, are quite useful.

Theorem B2.1.8. Let $\{\mathbb{X}(\cdot, t) : t \in \mathbb{R}\}$ be a motion and $\mathcal{M}$ a reference configuration of a body $\mathcal{B}$. Suppose $f_n \in C^0(\mathcal{M}(\mathcal{B}) \times \mathbb{R}, \mathbb{R})$, equivalently $f_n \in C^0(\mathcal{I}, \mathbb{R})$.¹ Then

$$\int_{\mathcal{M}(\mathcal{B})} f_n(\mathbb{X}, t) \rho_n(\mathbb{X}) dV_{\mathbb{X}} = \int_{\mathcal{P}_t} f_n(\mathbf{x}, t) \rho(\mathbf{x}, t) dV_{\mathbf{x}} \quad \forall \text{ parts } \mathcal{P} \subset \mathcal{B} \text{ and } \forall t \in \mathbb{R}.$$

Proof. The result follows readily from the Change of Variable Theorem for volume integrals and the referential density equation. The details are left as Exercise B2.1.4. $\square$

It should be clear to the reader that Theorem 2.1.8 holds for tensor fields of any order.

Theorem B2.1.9. (The Reduced Transport Theorem) Let $\{\mathbb{X}(\cdot, t) : t \in \mathbb{R}\}$ be a motion of a body $\mathcal{B}$. Suppose $g \in C^1(\mathcal{I}, \mathbb{R})$ and $\underline{g} \in C^1(\mathcal{I}, \mathcal{V}_3)$. For each, part $\mathcal{P} \subset \mathcal{B}$,

¹ Cf. Theorem B1.13.5 a.

define $k : \mathcal{Q} \rightarrow \mathbb{R}$ and $\tilde{k} : \mathcal{Q} \rightarrow \mathbb{V}_3$ by

$$k(t) = \int_{\mathcal{Q}_t} g(x, t) \rho(x, t) dV_x \quad \text{and} \quad \tilde{k}(t) = \int_{\mathcal{Q}_t} \underline{g}(x, t) \rho(x, t) dV_x.$$

Then $k \in C^1(\mathcal{Q}, \mathbb{R})$ and $\tilde{k} \in C^1(\mathcal{Q}, \mathbb{V}_3)$, and their ordinary derivatives are given by

$$\begin{aligned} \dot{k}(t) &= \int_{\mathcal{Q}_t} \{ g_{,2}(x, t) + D_1 g(x, t) [\underline{v}_a(x, t)] \} \rho(x, t) dV_x \\ &= \int_{\mathcal{Q}_t} [g_{,2}(x, t) + \nabla_1 g(x, t) \cdot \underline{v}_a(x, t)] \rho(x, t) dV_x \end{aligned}$$

and

$$\dot{\tilde{k}}(t) = \int_{\mathcal{Q}_t} \{ \underline{g}_{,2}(x, t) + D_1 \underline{g}(x, t) [\underline{v}_a(x, t)] \} \rho(x, t) dV_x.$$

Proof. There are two natural approaches to the reduced transport theorem. We leave it for the reader as Exercise B2.1.5 to establish the above formula for $\dot{k}$ by applying the Transport Theorem to $g\rho$ and then reducing (i.e., simplifying) the result with the spatial density equation. As Exercise B2.1.6, work out the above expression for $\dot{\tilde{k}}$ as in the proof of the transport theorem making use of the referential density equation after the conversion to referential variables. $\square$

Some of the results of this section often are expressed in terms of the material time derivative introduced at the end of §B1.8. E.g., the Spatial Density Equation (Theorem B2.1.7) becomes

$$\dot{\rho} + \rho \operatorname{div}_x \underline{v}_a = 0,$$

and the Reduced Transport Theorem (Theorem B2.1.9) becomes

$$\frac{d}{dt} \int_{\mathcal{Q}_t} g(\underline{x}, t) \rho(\underline{x}, t) dV_x = \int_{\mathcal{Q}_t} \dot{g}(\underline{x}, t) \rho(\underline{x}, t) dV_x,$$

$$\frac{d}{dt} \int_{\tilde{\mathcal{Q}}_t} g(\underline{x}, t) \rho(\underline{x}, t) dV_x = \int_{\tilde{\mathcal{Q}}_t} \dot{g}(\underline{x}, t) \rho(\underline{x}, t) dV_x.$$

Supplementary Reading

Same as for § B1.1 plus

NOLL, "The foundations of classical mechanics in the light of recent advances in continuum mechanics", in The Axiomatic Method (Berkeley Symposium), edit. by Henkin, Suppes, and Tarski (1959).

NOLL, "Lectures on the foundations of continuum mechanics and thermodynamics", Archive for Rational Mechanics and Analysis 52 (1973) 62-92.

B2.2. Mass Conservation in Isochoric and Rigid Motions

The following theorem summarizes the implications of conservation of mass for isochoric motions.

Theorem B2.2.1. Let $\{\mathbb{X}(\cdot, t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$. Let $\mathcal{K}$ be an arbitrary reference configuration of $\mathcal{B}$, and let $\mathcal{K}_0 = \mathbb{X}(\cdot, t_0)$ denote the configuration of $\mathcal{B}$ at the time $t_0 \in \mathcal{Q}$. Then the following statements are equivalent:

(i) the motion is isochoric;

(ii) $\dot{\rho} = \rho_{,2} + \nabla_1 \rho \cdot \underline{v}_a = 0$ on $\mathcal{I}$;

(iii) $\rho(\mathbb{X}_n(\mathbb{X}, t), t) = \rho_{\mathcal{K}_0}(\mathbb{X}) \quad \forall (\mathbb{X}, t) \in \mathcal{K}_0(\mathcal{B}) \times \mathcal{Q}$;

(iv) $\exists$ a function $\hat{\rho}_n : \mathcal{K}(\mathcal{B}) \rightarrow \mathbb{R}^+ \ni$

$$\rho(\mathbb{X}_n(\mathbb{X}, t), t) = \hat{\rho}_n(\mathbb{X}) \quad \forall (\mathbb{X}, t) \in \mathcal{K}(\mathcal{B}) \times \mathcal{Q};$$

(v) $\exists$ a function $\hat{\rho} : \mathcal{B} \rightarrow \mathbb{R}^+ \ni$

$$\rho(\mathbb{X}(x, t), t) = \hat{\rho}(x) \quad \forall (x, t) \in \mathcal{B} \times \mathcal{Q}.$$

Proof. (i) $\Leftrightarrow$ (ii): Suppose the motion is isochoric. Then by Theorem B1.14.4, $\text{div}_1 \underline{v}_a = 0$ on $\mathcal{I}$; and the spatial density equation (Theorem B2.1.7) reduces to

$$\rho_2 + \nabla_1 \rho \cdot \underline{v}_a = 0 \text{ on } \mathcal{I},$$

or by Exercise B1.8.6,

$$\dot{\rho} = 0 \text{ on } \mathcal{I},$$

Thus, (i) $\Rightarrow$ (ii).

Conversely, suppose (ii) holds. In this case, the spatial density equation reduces to $\rho \operatorname{div}_1 \underline{v}_a = 0$ on $\mathcal{I}$. Since $\rho \neq 0$, we conclude that $\operatorname{div}_1 \underline{v}_a = 0$ on $\mathcal{I}$, and $\therefore$ the motion is isochoric by Theorem B1.14.4. Thus, (ii) $\Rightarrow$ (i).

(i) $\Leftrightarrow$ (iii): Suppose the motion is isochoric. Then by Theorem B1.14.3, $J_{n_0} = 1$ on $\mathcal{N}_0(\mathcal{B}) \times \mathcal{Q}$; and the referential density equation (Theorem B2.1.5) reduces to

$$\rho(\mathbb{X}_{n_0}(\mathbb{X}, t), t) = \rho_{n_0}(\mathbb{X}), \quad (\mathbb{X}, t) \in \mathcal{N}_0(\mathcal{B}) \times \mathcal{Q}.$$

Thus, (i) $\Rightarrow$ (iii).

Conversely, suppose (iii) holds. In this case, the referential density equation reduces to $J_{n_0} = 1$ on $\mathcal{N}_0(\mathcal{B}) \times \mathcal{Q}$; and the motion is isochoric by Theorem B1.14.3. Thus, (iii) $\Rightarrow$ (i).

(i) $\Leftrightarrow$ (iv): Suppose the motion is isochoric. Then by Theorem B1.14.5, $\exists$ a function $\hat{J}_n: \mathcal{N}(\mathcal{B}) \times \mathcal{Q} \rightarrow \mathbb{R}^+ \ni J_n(\mathbb{X}, t) = \hat{J}_n(\mathbb{X}) \quad \forall (\mathbb{X}, t) \in \mathcal{N}(\mathcal{B}) \times \mathcal{Q}$. In this case, the referential density equation (Theorem B2.1.5) reduces to

$$\rho(\mathbb{X}_n(\mathbb{X}, t), t) = \frac{\rho_n(\mathbb{X})}{\hat{J}_n(\mathbb{X})} =: \hat{\rho}_n(\mathbb{X}), \quad (\mathbb{X}, t) \in \mathcal{N}(\mathcal{B}) \times \mathcal{Q}.$$

Thus, (i) $\Rightarrow$ (iv).

Conversely, suppose (iv) holds. Then the referential density equation reduces to

$$J_n(\mathbb{X}, t) = \frac{\rho_n(\mathbb{X})}{\hat{\rho}_n(\mathbb{X})} =: \hat{J}_n(\mathbb{X}), \quad (\mathbb{X}, t) \in \mathcal{U}(\mathcal{B}) \times \mathcal{Q},$$

and the motion is isochoric by Theorem B1.14.5. Thus, (iv) $\Rightarrow$ (i).

(iv) $\Leftrightarrow$ (v): Suppose (iv) holds; i.e.,

$$\rho(\mathbb{X}_n(\mathbb{X}, t), t) = \hat{\rho}_n(\mathbb{X}), \quad (\mathbb{X}, t) \in \mathcal{U}(\mathcal{B}) \times \mathcal{Q},$$

To get (v), we simply transform to the material description. On the left, we note that

$$\mathbb{X}_n(\mathbb{X}, t) = \mathbf{x} = \mathbb{X}(X, t);$$

and on the right,

$$\mathbb{X} = \mathcal{U}(X).$$

Hence,

$$\rho(\mathbb{X}(X, t), t) = \hat{\rho}_n(\mathcal{U}(X)) =: \hat{\rho}(X), \quad (X, t) \in \mathcal{B} \times \mathcal{Q}.$$

Thus, (iv) $\Rightarrow$ (v).

Conversely, suppose (v) holds. Then reversing the steps immediately above, i.e. transforming back to the referential description, leads us back to (iv), $\square$

Let us examine (i) $\Leftrightarrow$ (iii) above. With the understanding that $\mathbf{x} = \mathbf{x}_n(\mathbf{X}, t)$, we have

the motion is isochoric iff $\rho(\mathbf{x}, t) = \rho_n(\mathbf{X})$, $(\mathbf{X}, t) \in \mathcal{K}(\mathcal{B}) \times \mathcal{I}$.

In other words, the motion is isochoric iff the mass density at $\mathbf{x}$ in the configuration the body is in at time t is the same as the mass density at the corresponding place $\mathbf{X} = \mathbf{x}_n^{-1}(\mathbf{x}, t)$ in the reference configuration $\mathcal{K}$. In this sense,

the motion is isochoric iff the mass density is constant.

Of course, the technical meaning of the statement "the mass density is constant" is expressed in (iii) or (iv) or (v) of Theorem B2.2.1.

Since rigid motions are necessarily isochoric,¹ statements (ii) - (v) of Theorem B2.2.1 must hold for rigid motions. However, the satisfaction of any of them does not guarantee that the motion is rigid. Formally, we have

¹ The demonstration of this fact is left for the reader as Exercise B2.2.1.

Theorem B2.2.2. Let $\{\chi(\cdot, t) : t \in \mathcal{Q}\}$ be a rigid
motion of a body $\mathcal{B}$. Let $\mathcal{K}$ be an arbitrary reference
configuration of $\mathcal{B}$, and let $\mathcal{K}_0 = \chi(\cdot, t_0)$ denote the
configuration of $\mathcal{B}$ at the time $t_0 \in \mathcal{Q}$. Then the mass
density of $\mathcal{B}$ is constant in the sense that (iii),
(iv), and (v) of Theorem B2.2.1 are satisfied.
Furthermore, the spatial density equation reduces to (ii)
of Theorem B2.2.1.

Supplementary Reading

Same as for § B2.1

B2.4. Momentum. Kinetic Energy

In this section, we introduce several quantities which will appear later in the basic laws and main theorems of continuum mechanics.

Definition B2.4.1. Let $\{x(\cdot; t) : t \in \mathcal{Q}\}$ be a motion of a body $\mathcal{B}$. Then the momentum¹ of part $P \subset \mathcal{B}$ at time $t \in \mathcal{Q}$ is

$$\tilde{g}(P, t) = \int_{P_t} \tilde{v}_x(x, t) \rho(x, t) dV_x ;$$

the moment of momentum² about point $y \in E_3$ ³ of part $P \subset \mathcal{B}$ at time $t \in \mathcal{Q}$ is

$$\tilde{h}(P, t, y) = \int_{P_t} (x - y) \times \tilde{v}_x(x, t) \rho(x, t) dV_x ; \quad)^4$$

and the kinetic energy of part $P \subset \mathcal{B}$ at time $t \in \mathcal{Q}$ is

¹Often, linear momentum.

²Often, angular momentum.

³Point y may be fixed or moving; it need not be identified with any particle of $\mathcal{B}$.

⁴We assume here and in the sequel that the translation space $\mathcal{V}_3$ has been oriented.

$$T(P, t) = \frac{1}{2} \int_{P_t} |\underline{v}_s(\underline{x}, t)|^2 \rho(\underline{x}, t) dV_{\underline{x}}.$$

It is of interest to know what happens if the point about which the moment of momentum is computed is changed.

Theorem B2.4.1. Let $\{\underline{x}(\cdot, t): t \in \mathcal{Q}\}$ be a motion of
a body $\mathcal{B}$. Then for any part $P \subset \mathcal{B}$ and any points $\underline{y}, \underline{z} \in E_3$,

$$\underline{h}(P, t, \underline{z}) = \underline{h}(P, t, \underline{y}) + (\underline{y} - \underline{z}) \times \underline{g}(P, t).$$

Proof. Exercise B2.4.1. $\square$

Formulas for the time rates of change (denoted here by a superposed dot) of the momentum, moment of momentum, and kinetic energy follow readily from the Reduced Transport Theorem (Theorem B2.1.9).

Theorem B2.4.2. Let $\{\underline{x}(\cdot, t): t \in \mathcal{Q}\}$ be a motion of
a body $\mathcal{B}$. Then for each part $P \subset \mathcal{B}$, $\underline{g}(P, \cdot) \in C^1(\mathcal{Q}, \mathcal{V}_3)$,
 $\underline{h}(P, \cdot, \underline{y}(\cdot)) \in C^1(\mathcal{Q}, \mathcal{V}_3)$ provided $\underline{y} \in C^1(\mathcal{Q}, E_3)$, and
 $T(P, \cdot) \in C^1(\mathcal{Q}, \mathbb{R})$. Moreover,

$$\dot{\underline{g}}(P, t) = \int_{P_t} \underline{a}_s(\underline{x}, t) \rho(\underline{x}, t) dV_{\underline{x}},$$

$$\dot{\tilde{h}}(\mathcal{O}, t, y(t)) = \int_{\mathcal{O}_t} [x - y(t)] \times \underline{a}_s(x, t) \rho(x, t) dV_x - \dot{y}(t) \times \underline{g}(\mathcal{O}, t),$$

and

$$\dot{T}(\mathcal{O}, t) = \int_{\mathcal{O}_t} v_s(x, t) \cdot \underline{a}_s(x, t) \rho(x, t) dV_x.$$

Proof. Exercises B2.4.2, B2.4.3, and B2.4.4. $\square$

Supplementary Reading

Same as for $\S$ B2.1

Chapter B3

Force. Stress, Momentum Balance

Deformations and motions of bodies are caused by forces. In this chapter, we introduce the notion of force and the basic balance laws which relate force and motion.

B3.1. The Stress Principle and Balance of Momentum

Using measure theory, one can introduce the concept of force in continuum mechanics in a very natural and intuitive way.¹ However, as with an treatment of mass, we shall be more traditional and start out by expressing forces in terms of Riemann integrals. Forces are usually thought of as the "agents" that cause the motion of a body. Introductory texts often give the definition: "Force is the 'action' of one body on another body." Of course, the abstraction of force must embody the "pushes" and "pulls" of everyday mechanical experience.

Consider a motion $\{\chi(\cdot, t) : t \in \mathcal{I}\}$ of a body $\mathcal{B}$. Let $\mathcal{P}$ be an arbitrary part of $\mathcal{B}$.² We envisage that at each instant $t \in \mathcal{I}$ the part $\mathcal{P}$ is subjected to two types of forces:

(i) Surface Force.³ Force distributed over the boundary $\partial \mathcal{P}_t$ due to part $\mathcal{P}$ being in contact at $\partial \mathcal{P}_t$ with the rest of the body. In the case $\mathcal{P} = \mathcal{B}$, then the force on $\partial \mathcal{B}_t$ is due to contact with other bodies (i.e., the environment of $\mathcal{B}$); a similar remark holds when a subset of $\partial \mathcal{P}_t$ is in $\partial \mathcal{B}_t$.

¹ See the articles by NOLL or the text by TRUESDELL listed in the Supplementary Reading.

² As always, we may have $\mathcal{P} = \mathcal{B}$.

³ Often, surface traction.

(ii) External Body Force.¹ Force distributed throughout the region $\mathcal{Q}_t$ due to bodies other than $\mathcal{B}$, i.e. due to the environment of $\mathcal{B}$. E.g., the weight force of an airplane due to the presence of the earth.

We will not consider

(iii) Internal Body Force.² Force distributed throughout the region $\mathcal{Q}_t$ due to the body $\mathcal{B}$ itself.

As with mass, experience indicates that the above force distributions are additive. E.g., if $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ and $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \emptyset$ so that part $\mathcal{Q}$ is comprised of two separate subparts, then the total external body force on $\mathcal{Q}$ is the sum of the total external body forces on $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Of course, integrals are additive in this sense, and we are lead to adopt CAUCHY'S

Stress Principle. Let $\{\mathbb{X}(\cdot, t) : t \in \mathcal{I}\}$ be a motion of a body $\mathcal{B}$. Then for each unit vector $\underline{n} \in \mathcal{V}_3$, $\exists$ a function $\underline{t}_{\underline{n}} : \mathcal{I} \rightarrow \mathcal{V}_3$ $\ni$

$$(F1) \quad \int_{\partial \mathcal{Q}_t} \underline{t}_{\underline{n}(\underline{x}, t)}(\underline{x}, t) dA_x,$$

¹ Often, extrinsic body force.

² Often, intrinsic or mutual body force.

where $\underline{n}(\underline{x}, t)$ is the outward unit normal to ∂P_t at $\underline{x}$, is the total surface force on part $P \subset B$ at time $t \in \mathcal{I}$. For each $\underline{n}$ and t , the field $\underline{t}(\cdot, t)$ is smooth on B_t .¹ The vector $\underline{t}(\underline{x}, t)$ is called the "stress vector"² at the point $\underline{x}$ on "the" surface through $\underline{x}$ with unit normal $\underline{n}$ at $\underline{x}$ at time t .

Of course, there are many surfaces through $\underline{x}$ with normal $\underline{n}$. NOLL's measure theoretic development shows that the stress vector can depend on no characteristics of the surface (e.g., the curvature) other than its normal. In following CAUCHY's approach, we must assume this. Obviously, it is appropriate to think of the stress vector as the surface force per unit area; i.e., the stress vector is the surface force density.

In order to handle the external body force, we assume

Let $\{\underline{x}(\cdot, t) : t \in \mathcal{I}\}$ be a motion of a body B . Then $\exists$ a function $\underline{b} : \mathcal{I} \rightarrow \mathcal{V}_3 \ni$

$$(F2) \quad \int_{P_t} \underline{b}(\underline{x}, t) f(\underline{x}, t) dV_{\underline{x}}$$

is the total external body force on part $P \subset B$ at time $t \in \mathcal{I}$.

¹ This is more than sufficient to guarantee the existence of the integral which defines the total surface force.

² often, traction vector.

For each t , the field $\underline{b}(\cdot, t)$ is continuous on $\mathcal{Q}_t$.¹ The vector $\underline{b}(\underline{x}, t)$ is called the body force density per unit mass at the point $\underline{x}$ at time t .

It is sometimes convenient to decompose the stress vector $\underline{t}_{\underline{n}}(\underline{x}, t)$ into components parallel and perpendicular to $\underline{n}$.

Theorem B3.1.1. Given the stress vector $\underline{t}_{\underline{n}}(\underline{x}, t)$ corresponding to $(\underline{x}, t) \in \mathcal{I}$ and the unit vector $\underline{n}$, $\exists$ unique $N(\underline{x}, t, \underline{n}) \in \mathbb{R}$ and $\underline{s}(\underline{x}, t, \underline{n}) \in \mathbb{V}_3 \ni$

$$\underline{t}_{\underline{n}}(\underline{x}, t) = N(\underline{x}, t, \underline{n})\underline{n} + \underline{s}(\underline{x}, t, \underline{n}), \quad \underline{s}(\underline{x}, t, \underline{n}) \cdot \underline{n} = 0.$$

The vectors $N(\underline{x}, t, \underline{n})\underline{n}$ and $\underline{s}(\underline{x}, t, \underline{n})$ are called the normal stress vector and the shear stress vector, respectively.

Proof. Exercise B3.1.1. $\square$

Definition B3.1.2. Referring to Axioms (F1) and (F2), we call the pair $(\underline{t}_{\underline{n}}, \underline{b})$ a system of forces corresponding to the motion $\{\underline{x}(\cdot, t): t \in \mathcal{I}\}$ of body $\mathcal{B}$. The total force on part $\mathcal{Q}$ of $\mathcal{B}$ at time t is

$$\underline{f}(\mathcal{Q}, t) = \int_{\partial \mathcal{Q}_t} \underline{t}_{\underline{n}(\underline{x}, t)}(\underline{x}, t) dA_{\underline{x}} + \int_{\mathcal{Q}_t} \underline{b}(\underline{x}, t) \rho(\underline{x}, t) dV_{\underline{x}},$$

¹This ensures the existence of the integral which defines the total external body force.

and the total moment¹ about point y ² on part P of B at time t is

$$m(P, t, y) = \int_{\partial P_t} (x - y) \times \underline{\underline{t}}_{n(x, t)}(x, t) dA_x + \int_{P_t} (x - y) \times \underline{\underline{b}}(x, t) \rho(x, t) dV_x. \quad)^3$$

The following theorem demonstrates the dependence of the total moment on the point about which it is computed.

Theorem B3.1.2. Let $(\underline{\underline{t}}_n, \underline{\underline{b}})$ be a system of forces corresponding to a motion $\{X(\cdot, t) : t \in \mathcal{D}\}$ of a body B . Then

$$m(P, t, z) = m(P, t, y) + (y - z) \times \underline{\underline{f}}(P, t)$$

for any part $P \subset B$ and any points $y, z \in E_3$.

Proof. Exercise B3.1.2. $\square$

We assume that force and motion are related through the following two axioms.

¹Often, total torque.

²As in Definition B2.4.1, the arbitrary point $y \in E_3$ may be fixed or moving; it need not be identified with any particle of B .

³See Footnote 4 on p. B2.4.1.

Principle of Balance of Momentum.¹ Let $(\underline{t}_n, \underline{b})$ be a system of forces corresponding to a motion $\{\underline{x}(\cdot, t); t \in \mathcal{I}\}$ of a body $\mathcal{B}$.
Then

$$(MB1) \quad \dot{\underline{g}}(\underline{P}, t) = \underline{f}(\underline{P}, t) \quad \forall \text{ parts } \underline{P} \subset \mathcal{B} \text{ and } \forall t \in \mathcal{I}.$$

Principle of Balance of Moment of Momentum.² Let $(\underline{t}_n, \underline{b})$ be a system of forces corresponding to a motion $\{\underline{x}(\cdot, t); t \in \mathcal{I}\}$ of a body $\mathcal{B}$. Then for each fixed point $\underline{x} \in \mathbb{E}_3$,

$$(MB2) \quad \dot{\underline{h}}(\underline{P}, t, \underline{x}) = \underline{m}(\underline{P}, t, \underline{x}) \quad \forall \text{ parts } \underline{P} \subset \mathcal{B} \text{ and } \forall t \in \mathcal{I}.$$

In words these axioms assert that the total force equals the time rate of change of momentum and that the total moment about any fixed point $\underline{x}$ equals the time rate of change of the moment of momentum about $\underline{x}$. When (MB1) is satisfied, one says that the momentum is balanced; and when (MB2) holds, the moment of momentum about $\underline{x}$ is balanced. Axiom (MB1) is the continuum mechanical version of NEWTON's Second Law, $\underline{f} = m\underline{a}$, of particle mechanics. Axiom (MB2) is the continuum mechanical analog of the theorem in particle mechanics concerning balance of moment of momentum, $\underline{r} \times \underline{f} = (\underline{r} \times m\underline{v})'$.

¹Often, EULER's first law.

²Often, EULER's second law.

Even closer in form to NEWTON's Second Law is

Theorem B3.1.3. Let $(\underline{t}_n, \underline{L})$ be a system of forces
corresponding to a motion $\{\underline{x}(t); t \in \mathcal{I}\}$ of a body $\mathcal{B}$. Then
the momentum is balanced iff

$$\underline{f}(\mathcal{P}, t) = m(\mathcal{P}) \ddot{\underline{c}}_0(\mathcal{P}, t) \quad \forall \text{ parts } \mathcal{P} \subset \mathcal{B} \text{ and } \forall t \in \mathcal{I},$$

In particular, momentum balance $\Rightarrow$

$$\underline{f}(\mathcal{B}, t) = m(\mathcal{B}) \ddot{\underline{c}}_0(t).$$

Proof. Exercise B3.1.3. $\square$

where $\underline{c}_0$ is the mass center of part $\mathcal{P}$ (Definition B2.3.1).

The following interesting result follows from Theorems B3.1.2 and B2.4.1.

Theorem B3.1.4. Let $(\underline{t}_n, \underline{b})$ be a system of forces
corresponding to a motion $\{\mathbb{X}(\cdot, t) : t \in \mathbb{I}\}$ of a body $\mathcal{B}$.
Suppose that the momentum is balanced, then if the
moment of momentum about any one fixed point is
balanced, it is balanced about all fixed points.

Proof. Exercise B3.1.4. $\square$

The next theorem is especially useful in rigid body mechanics, but it holds for all bodies.

Theorem B3.1.5. Let $(\underline{t}_n, \underline{b})$ be a system of forces
corresponding to a motion $\{\mathbb{X}(\cdot, t) : t \in \mathbb{I}\}$ of a body $\mathcal{B}$.
Suppose that the momentum is balanced. Then the moment of
momentum is balanced about all fixed points iff it is
balanced about the mass center; i.e. iff

$$\dot{\underline{h}}(\mathcal{P}, t, \mathcal{C}_\mathcal{P}(t)) = \underline{m}(\mathcal{P}, t, \mathcal{C}_\mathcal{P}(t)) \quad \forall \text{ parts } \mathcal{P} \subset \mathcal{B} \text{ and } \forall t \in \mathbb{I}.$$

Proof. Exercise B3.1.5. $\square$

From this point forward, we assume without further
comment in general that momentum and moment of momentum
are balanced.

Remark. While our treatment of the Principles of Balance of Momentum and Moment of Momentum has been case-free, to physically locate points and measure velocities and accelerations evidently requires the employment of a reference frame¹. For the Principles of Momentum Balance to agree with observations, it is sufficient that the underlying reference frame be in translation with constant velocity relative to the "fixed stars". Such frames are said to be inertial or Newtonian.

For many engineering purposes, satisfactory results are obtained by treating reference frames fixed relative to the earth as inertial. It also should be noted that agreement with observations is lost if the velocities are too large (speed of light) or the bodies too small (subatomic).

Of course, a choice of reference frame amounts to a choice of the Euclidean space E_3 (and its translation space $\mathcal{T}_3$) on which the considerations of Parts B and C are based.

Supplementary Reading

Same as for § B2.1.

¹ Cf. Definition A6.2.3, See TRUESDELL's text for additional discussion.

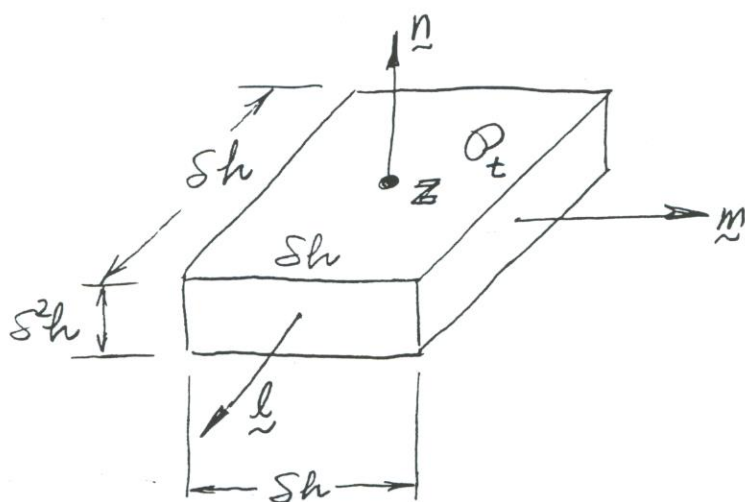
B3.4. The Stress Tensor and the Stress Equations of Motion

In this section, we show that the stress vector $\underline{t}_{\underline{n}}(\underline{x}, t)$ is linear in $\underline{n}$; i.e., we establish the existence of the stress tensor. We then use this to derive local forms of the Principles of Momentum Balance. The following first step in this program generally is referred to as CAUCHY's Lemma. ((Action/Reaction Theorem))

Theorem B3.4.1. Let $(\underline{t}_{\underline{n}}, \underline{b})$ be a system of forces corresponding to a motion $\{\underline{x}(\cdot, t) : t \in \mathcal{Q}\}$ of a body $\mathcal{B}$.
Then for each unit vector $\underline{n}$,

$$\underline{t}_{-\underline{n}}(\underline{x}, t) = -\underline{t}_{\underline{n}}(\underline{x}, t) \quad \forall (\underline{x}, t) \in \mathcal{Q}.$$

Proof. Arbitrarily fix $\underline{n}$ and $t \in \mathcal{Q}$, and let $\underline{z}$ be any interior point of $\mathcal{B}_t$. Choose the part $\mathcal{Q}$ of $\mathcal{B} \ni \mathcal{Q}_t \subset \mathcal{B}_t$ is the rectangular "pillbox" indicated below.



The faces of the pillbox are orthogonal to the unit vectors $\underline{l}$, $\underline{m}$, and $\underline{n}$, point $\underline{x}$ is centered in the top face, h is a length, and δ is dimensionless. We call the top face, whose outward unit normal is $\underline{n}$, $\mathcal{N}^+$ and the bottom face $\mathcal{N}^-$. Similarly, the face with normal $\underline{l}$ is $\mathcal{L}^+$, etc.

Next we apply the Principle of Balance of Momentum (MB1) to part $\mathcal{P}$ and use Theorem B2.4.2 to get

$$\begin{aligned} & \int_{\mathcal{L}^+} \underline{t}_{\underline{l}}(\underline{x}, t) dA_x + \int_{\mathcal{L}^-} \underline{t}_{-\underline{l}}(\underline{x}, t) dA_x \\ & + \int_{\mathcal{M}^+} \underline{t}_{\underline{m}}(\underline{x}, t) dA_x + \int_{\mathcal{M}^-} \underline{t}_{-\underline{m}}(\underline{x}, t) dA_x \\ & + \int_{\mathcal{N}^+} \underline{t}_{\underline{n}}(\underline{x}, t) dA_x + \int_{\mathcal{N}^-} \underline{t}_{-\underline{n}}(\underline{x}, t) dA_x \\ & + \int_{\mathcal{P}} [\underline{b}(\underline{x}, t) - \underline{a}_a(\underline{x}, t)] \rho(\underline{x}, t) dV_x = 0. \end{aligned}$$

By the Mean-Value Theorems for surface and volume integrals (Theorem A6.10.3), the i^{th} component relative to some basis

for $\forall_3$ of this equation can be expressed as

$$\begin{aligned} & \delta^2 h^2 \left\{ t_{\tilde{l}}^i(x_1, t) \delta + t_{-\tilde{l}}^i(x_2, t) \delta \right. \\ & + t_{\tilde{m}}^i(x_3, t) \delta + t_{-\tilde{m}}^i(x_4, t) \delta + t_{\tilde{n}}^i(x_5, t) + t_{-\tilde{n}}^i(x_6, t) \\ & \left. + [b^i(x_7, t) - a^i(x_7, t)] \rho(x_7, t) \delta^2 h \right\} = 0, \end{aligned}$$

for some $x_1 \in \mathcal{L}^+$, $x_2 \in \mathcal{L}^-$, $x_3 \in \mathcal{M}^+$, $x_4 \in \mathcal{M}^-$, $x_5 \in \mathcal{N}^+$, $x_6 \in \mathcal{N}^-$, and $x_7 \in \mathcal{B}_t$. This equation (with $\delta^2 h^2$ divided out) has been derived on the basis that $\delta > 0$, but the l.h.s. of the equation is a continuous function of δ even at $\delta = 0$. $\therefore$ by the one-dimensional version of the Extension Theorem (Theorem B1.2.2), we can let $\delta \rightarrow 0$ (keeping $\mathbb{Z} \in \mathcal{N}^+$) to obtain

$$t_{\tilde{n}}^i(\mathbb{Z}, t) + t_{-\tilde{n}}^i(\mathbb{Z}, t) = 0.$$

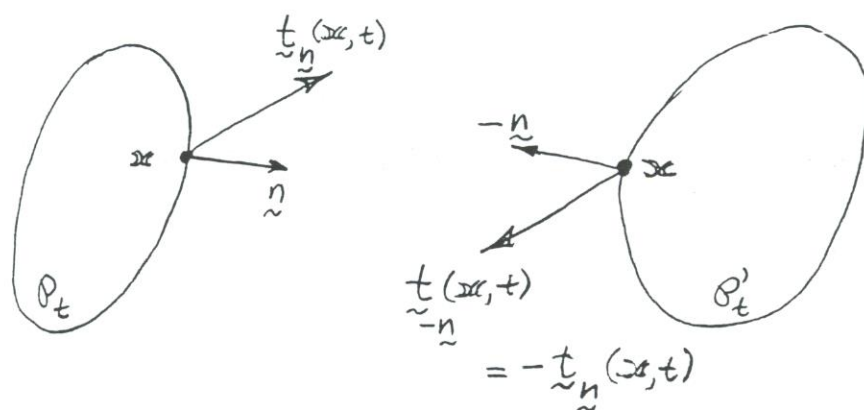
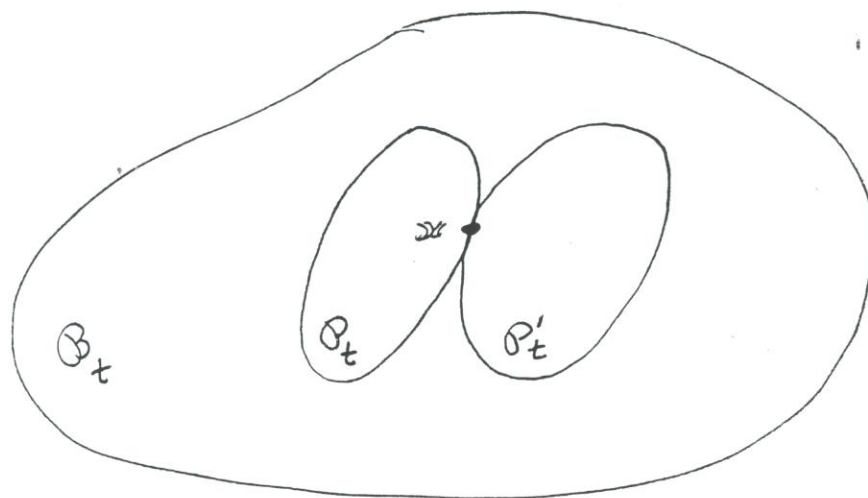
Thus, we have the desired result

$$t_{-\tilde{n}}^i(\mathbb{Z}, t) = -t_{\tilde{n}}^i(\mathbb{Z}, t)$$

for interior points. Since for each $\tilde{n}$ and t , the field $t_{\tilde{n}}^i(\cdot, t)$ is continuous on $\mathcal{B}_t$ by hypothesis, the Extension Theorem $\Rightarrow$

$$t_{-\tilde{n}}^i(x, t) = -t_{\tilde{n}}^i(x, t) \quad \forall x \in \mathcal{B}_t. \quad \square$$

The reason for the "action/reaction" terminology is made clear by the following sketch.



The Action/Reaction Theorem is the continuum mechanical analog of NEWTON's Third Law in particle mechanics. Interestingly, in continuum mechanics it is a theorem, while in particle mechanics it is an axiom — just the opposite of the situation for the Principle of Balance of Moment of Momentum.

In order to proceed, we introduce special notation for the components of the stress vector on rectangular Cartesian coordinate planes.

Definition B3.4.1. Let $(\underline{t}_n, \underline{b})$ be a system of forces corresponding to a motion $\{\chi(\cdot, t) : t \in \mathbb{R}\}$ of a body $\mathcal{B}$. Then for $\{\underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}, \underline{e}_{\langle 3 \rangle}\}$ an orthonormal basis for V_3 ,

$$T^{\langle ij \rangle}(\underline{x}, t) = \underline{e}_{\langle i \rangle} \cdot \underline{t}_{\sim \underline{e}_{\langle j \rangle}}(\underline{x}, t).$$

Thus, $T^{\langle ij \rangle}(\underline{x}, t)$ is the i th component of the stress vector on the surface with unit normal $\underline{e}_{\langle j \rangle}$ at the point $\underline{x}$ at time t .¹ In line with Theorem B3.1.1, the $T^{\langle ii \rangle}$ (no sum) and $T^{\langle ij \rangle}$ ($i \neq j$) are called normal and shear stresses, respectively.

Now we are in position to establish

¹ Most authors interchange the role of the indices, our approach saves a step. In any case, we shall eventually see that $T^{\langle ji \rangle} = T^{\langle ij \rangle}$.

Theorem B3.4.2.¹ (Existence of the Stress Tensor) Let
 $(\underline{t}_n, \underline{b})$ be a system of forces corresponding to a motion
 $\{\underline{x}(\cdot, t) : t \in \mathcal{Q}\}$ of a body $\mathcal{B}$. Then $\exists$ a unique function

$$\underline{T} : \mathcal{I} \rightarrow \text{Lin } \mathcal{V}_3$$

$\Rightarrow$ for each unit vector $\underline{n}$,

$$\underline{t}_{\underline{n}}(\underline{x}, t) = \underline{T}(\underline{x}, t) \underline{n} \quad \forall (\underline{x}, t) \in \mathcal{I}.$$

Moreover, for each $t \in \mathcal{Q}$, the tensor field $\underline{T}(\cdot, t)$ is
smooth on $\mathcal{B}_t$. $\underline{T}(\underline{x}, t)$ is called the stress tensor² at
the place $\underline{x}$ at the time t .

Proof. Let $\{\underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}, \underline{e}_{\langle 3 \rangle}\}$ be an orthonormal
 basis for $\mathcal{V}_3$, and define

$$\underline{T}(\underline{x}, t) = T^{\langle ij \rangle}(\underline{x}, t) \underline{e}_{\langle i \rangle} \otimes \underline{e}_{\langle j \rangle}, \quad (\underline{x}, t) \in \mathcal{I},$$

where the $T^{\langle ij \rangle}$ are given by Definition B3.4.1. We will
 show that this tensor $\underline{T}$ has the desired properties.
 First of all, its smoothness follows from the definition
 of the $T^{\langle ij \rangle}$ and the assumed smoothness of the stress
 vector.

¹ Almost always, CAUCHY's Theorem.

² Often, CAUCHY stress tensor; sometimes, time stress tensor.

Next, consider the case $\underline{n} = \underline{e}_{<1>}$. Then (with $\underline{x}$ and $\underline{t}$ suppressed)

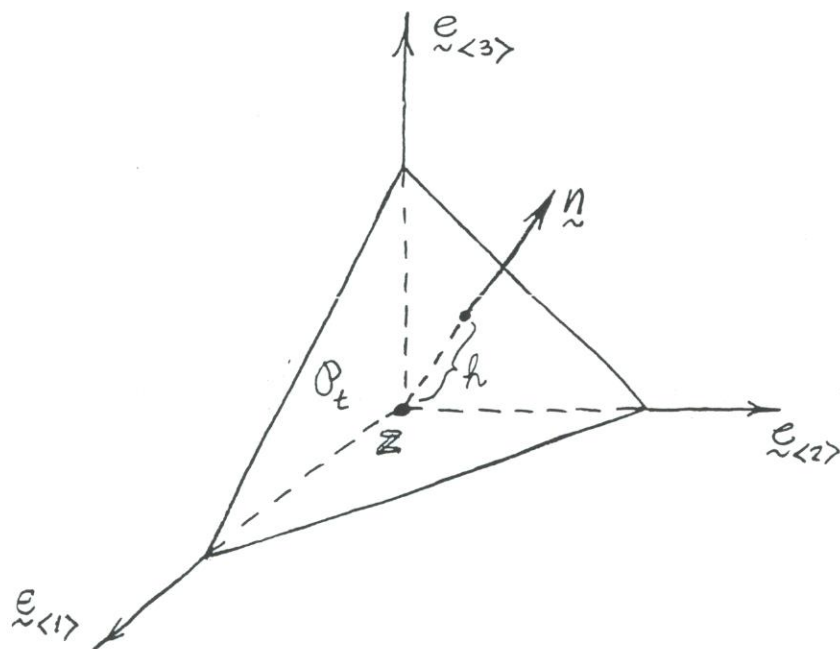
$$\begin{aligned}
 \underline{T} \underline{e}_{<1>} &= (T^{<ij>} \underline{e}_{<i>} \otimes \underline{e}_{<j>}) \underline{e}_{<1>} \quad (\text{definition of } \underline{T}) \\
 &= T^{<ij>} [(\underline{e}_{<i>} \otimes \underline{e}_{<j>}) \underline{e}_{<1>}] \quad (\text{Lin } \mathcal{V}_3 \text{ a linear space}) \\
 &= T^{<ij>} [(\underline{e}_{<j>} \cdot \underline{e}_{<1>}) \underline{e}_{<i>}] \quad (\text{definition of } \underline{u} \otimes \underline{v}) \\
 &= T^{<ij>} [\delta_{j1} \underline{e}_{<i>}] \quad (\{\underline{e}_{<i>}\} \text{ orthonormal}) \\
 &= (T^{<ij>} \delta_{j1}) \underline{e}_{<i>} \quad (\mathcal{V}_3 \text{ is a linear space}) \\
 &= T^{<i1>} \underline{e}_{<i>} \quad (\text{substitution property of } \delta_{ke}) \\
 &= \sum_{i=1}^3 (\underline{e}_{<i>} \cdot \underline{t}_{\underline{e}_{<1>}}) \underline{e}_{<i>} \quad (\text{definition of } T^{<ij>}) \\
 &= \underline{t}_{\underline{e}_{<1>}} \quad (\text{Theorem A2.3.14}).
 \end{aligned}$$

Thus, $\underline{T} \underline{n} = \underline{t}_{\underline{n}}$ whenever $\underline{n} = \underline{e}_{<1>}$ (or $\underline{e}_{<2>}$ or $\underline{e}_{<3>}$).

The case $\underline{n} = -\underline{e}_{<i>}$ is handled by applying the Action/Reaction Theorem to $\underline{t}_{-\underline{e}_{<i>}}$ and then using the above result:

$$\underline{t}_{-\underline{e}_{<i>}} = -\underline{t}_{\underline{e}_{<i>}} = -\underline{T} \underline{e}_{<i>} = \underline{T} (-\underline{e}_{<i>}),$$

Finally, consider the case where $\underline{n}$ is not parallel to any of the basis vectors. For definiteness, suppose that $\underline{n}$ points into the first octant of $\{\underline{e}_{\langle 1 \rangle}, \underline{e}_{\langle 2 \rangle}, \underline{e}_{\langle 3 \rangle}\}$. Arbitrarily fix $t \in \mathcal{Q}$ and $\underline{z} \in \mathcal{B}_t$, and choose the part $\mathcal{P}$ $\Rightarrow \mathcal{P}_t$ is a tetrahedron of altitude h with faces S_i orthogonal to $\underline{e}_{\langle i \rangle}$ ($i \in \mathbb{I}_3$) and meeting at $\underline{z}$ and S orthogonal to $\underline{n}$.



If A denotes the area of S , then the area of S_i is $A(\underline{e}_{\langle i \rangle} \cdot \underline{n})$ and the volume of $\mathcal{P}_t$ is $\frac{1}{3} Ah$.

Now apply the Principle of Balance of Momentum (MB1) to part $\mathcal{P}$ and use Theorem B2.4.2 to get

$$\int_S \underline{t}_{\underline{n}}(\underline{x}, t) dA_x + \sum_{i=1}^3 \int_{S_i} \underline{t}_{\underline{e}_{\langle i \rangle}}(\underline{x}, t) dA_x$$

$$+ \int_{\mathcal{Q}_t} [b(x, t) - a_s(x, t)] \rho(x, t) dV_x = Q. \quad (*)$$

By the Action/Reaction Theorem,

$$\underline{t}_{-\underline{e}_{<i>}} = -\underline{t}_{<\underline{e}_{<i>}};$$

so for the j^{th} component

$$\begin{aligned} \underline{t}_{-\underline{e}_{<i>}}^{<j>} &= \underline{e}_{<j>} \cdot (-\underline{t}_{<\underline{e}_{<i>}}) \quad (\text{Theorem A2.3.14}) \\ &= -\underline{e}_{<j>} \cdot \underline{t}_{<\underline{e}_{<i>}} \quad (\text{bilinearity of inner product}) \\ &= -T^{<j>i>} \quad (\text{Definition B3.4.1}), \end{aligned}$$

With this result and the Mean-Value Theorems for surface and volume integrals, the j^{th} component of (*) can be expressed as
(Theorem A6.10.3)

$$\begin{aligned} A \left\{ \underline{t}_{\underline{n}}^{<j>}(\underline{x}_0, t) - \sum_{i=1}^3 T^{<j>i>}(\underline{x}_i, t) (\underline{e}_{<i>} \cdot \underline{n}) \right. \\ \left. + [b^{<j>}(\underline{x}_4, t) - a_s^{<j>}(\underline{x}_4, t)] \rho(\underline{x}_4, t) \frac{1}{3} h \right\} = 0, \end{aligned}$$

where $\underline{x}_0 \in S$, $\underline{x}_i \in S_i$ ($i \in \mathbb{I}_3$), and $\underline{x}_4 \in \mathcal{Q}_t$. This equation (with A divided out) has been derived on the basis that $h > 0$, but the l.h.s. of the equation, is a continuous function of h even at $h = 0$, $\therefore$ by the Extension Theorem

(Theorem B1.2.2), we can let $h \rightarrow 0$ (keeping $\underline{z}$ fixed) to obtain

$$\begin{aligned}
 t_{\underline{n}}^{<j>}(\underline{z}, t) &= T^{<j^i>}(\underline{z}, t) (\underline{e}_{<i>} \cdot \underline{n}) \\
 \Rightarrow \underline{t}_{\underline{n}}(\underline{z}, t) &= T^{<j^i>}(\underline{z}, t) (\underline{e}_{<i>} \cdot \underline{n}) \underline{e}_{<j>} \quad (\{\underline{e}_{<j>}\} \text{ a basis}) \\
 &= T^{<j^i>}(\underline{z}, t) (\underline{e}_{<j>} \otimes \underline{e}_{<i>}) \underline{n} \quad (\text{definition of } \underline{u} \otimes \underline{v}) \\
 &= \underline{T}(\underline{z}, t) \underline{n} \quad (\text{definition of } T^{<j^i>})
 \end{aligned}$$

Since $\underline{t}_{\underline{n}}(\cdot, t)$ and $\underline{T}(\cdot, t)$ are continuous on $\mathcal{B}_t$. This result must hold on the boundary of $\mathcal{B}_t$ as well as on its interior by the Extension Theorem. The cases where $\underline{n}$ points into the other octants are handled similarly.

All that remains to be established is the uniqueness of $\underline{T}$. Suppose $\underline{T}'$ were another tensor field on $\mathcal{T} \ni$

$$\underline{t}_{\underline{n}}(\underline{x}, t) = \underline{T}'(\underline{x}, t) \underline{n}.$$

But we also have

$$\underline{t}_{\underline{n}}(\underline{x}, t) = \underline{T}(\underline{x}, t) \underline{n},$$

and $\therefore$

$$\underline{T}' \underline{n} = \underline{T} \underline{n} \quad \forall \text{ unit vectors } \underline{n}.$$

Thus, by Theorem A3.1.2,

$$\underline{T}'(\underline{x}, t) = \underline{T}(\underline{x}, t) \quad \forall (\underline{x}, t) \in \mathcal{J}. \quad \square$$

With the knowledge that the stress vector $\underline{t}_{\underline{n}}$ is linear in $\underline{n}$, it is easy to get the local forms of the momentum balance principles; and we turn to this next.

Theorem B3.4.3. (First Stress Equation of Motion)¹ Let $(\underline{t}_{\underline{n}}, \underline{b})$ be a system of forces corresponding to a motion $\{\underline{x}(\cdot, t); t \in \mathcal{I}\}$ of a body $\mathcal{B}$. If the momentum is balanced, then

$$\operatorname{div}_1 \underline{T}(\underline{x}, t) + \rho(\underline{x}, t) \underline{b}(\underline{x}, t) = \rho(\underline{x}, t) \underline{a}_{\underline{x}}(\underline{x}, t) \quad \forall (\underline{x}, t) \in \mathcal{J}.$$

"Conversely", if $\exists \underline{T} \in C^1(\mathcal{J}, \operatorname{Lin} \mathcal{V}_3) \Rightarrow \underline{t}_{\underline{n}} = \underline{T} \underline{n}$ on $\mathcal{J}$ and if

$$\operatorname{div}_1 \underline{T} + \rho \underline{b} = \rho \underline{a}_{\underline{x}} \text{ on } \mathcal{J},$$

then the momentum is balanced.)²

¹Often, CAUCHY's First Equation of Motion.

²This is not a strict converse of the prior assertion because it assumes the existence of a stress tensor, and by the previous theorem this existence rests on the balance of momentum.

Proof. Suppose that the momentum is balanced. Then by Theorem B2.4.2,

$$\int_{\partial P_t} \underline{t}_{\underline{n}(\underline{x},t)} dA_x + \int_{P_t} [\underline{b}(\underline{x},t) - \underline{a}_s(\underline{x},t)] \rho(\underline{x},t) dV_x = \underline{0}, \quad (*)$$

which in terms of the stress tensor (Theorem B3.4.2) becomes

$$\int_{\partial P_t} \underline{T}(\underline{x},t) \underline{n}(\underline{x},t) dA_x + \int_{P_t} [\underline{b}(\underline{x},t) - \underline{a}_s(\underline{x},t)] \rho(\underline{x},t) dV_x = \underline{0}.$$

Converting the surface integral to a volume integral via the Divergence Theorem (Theorem A6.10.1), we get

$$\int_{P_t} [\operatorname{div}_1 \underline{T} + (\underline{b} - \underline{a}_s) \rho] dV_x = \underline{0} \quad (**)$$

Since this must hold $\forall$ parts $P \subset B$, the region P_t is an arbitrary regular subregion of B_t ; ¹ and we conclude from the Localization Theorem (Theorem B1.6.2) that

$$\operatorname{div}_1 \underline{T} + \rho \underline{b} = \rho \underline{a}_s \text{ on } \mathcal{I}. \quad (\#)$$

Conversely, if the first stress equation of motion (#) holds, we can integrate it over P_t to obtain (**). Then the Divergence Theorem and $\underline{t}_{\underline{n}} = \underline{T} \underline{n}$ yield (*); i.e., the

¹cf. the proof of Theorem B2.1.1.

momentum is balanced. $\square$

Theorem B3.4.4, (Second Stress Equation of Motion)¹ Let
 $(\tilde{t}_n, \tilde{b})$ be a system of forces corresponding to a motion
 $\{\tilde{x}(\cdot, t) : t \in \mathcal{Q}\}$ of a body B . Suppose that the momentum
is balanced. Then a necessary and sufficient condition that
the moment of momentum be balanced is that

$$[\tilde{T}(\underline{x}, t)]^T = \tilde{T}(\underline{x}, t) \quad \forall (\underline{x}, t) \in \mathcal{I};$$

i.e., the stress tensor is symmetric.

Proof. By Theorem B2.4.2, the moment of momentum is
 balanced iff $\tilde{Q}(\mathcal{P}, t) = \underline{0} \quad \forall$ parts $\mathcal{P} \subset B$ and $\forall t \in \mathcal{Q}$, where

$$\tilde{Q}(\mathcal{P}, t) := \int_{\partial \mathcal{P}_t} (\underline{x} - \underline{x}_0) \times \tilde{t}_{\tilde{n}(\underline{x}, t)} dA_x + \int_{\mathcal{P}_t} (\underline{x} - \underline{x}_0) \times [\tilde{b}(\underline{x}, t) - \tilde{a}_\rho(\underline{x}, t)] \rho(\underline{x}, t) dV_x \quad (*)$$

for any fixed point $\underline{x}_0 \in \mathcal{E}_3$. In terms of the stress tensor
 (Theorem B3.4.2), the first term in (*) becomes

$$\tilde{Q} := \int_{\partial \mathcal{P}_t} (\underline{x} - \underline{x}_0) \times \tilde{t}_{\tilde{n}(\underline{x}, t)} dA_x = \int_{\partial \mathcal{P}_t} (\underline{x} - \underline{x}_0) \times \tilde{T}(\underline{x}, t) \tilde{n}(\underline{x}, t) dA_x.$$

This expression is readily reduced by resorting to components
 and coordinates. But to keep our considerations basis free,

¹Often, CAUCHY's Second Equation of Motion.

let $\underline{w}$ be an arbitrary uniform vector and consider

$$\underline{w} \cdot \underline{c} = \int_{\partial \mathcal{Q}_t} \underline{w} \cdot [(\underline{x} - \underline{x}_0) \times \underline{T} \underline{n}] dA_x \quad (\text{uniformity of } \underline{w})$$

$$= \int_{\partial \mathcal{Q}_t} [\underline{w} \times (\underline{x} - \underline{x}_0)] \cdot \underline{T} \underline{n} dA_x \quad \left((\underline{u} \times \underline{v}) \cdot \underline{w} = \Delta(\underline{u}, \underline{v}, \underline{w}) \right. \\ \left. \text{plus skew symmetry of } \Delta \right)$$

$$= \int_{\partial \mathcal{Q}_t} [\underline{T}^T (\underline{w} \times (\underline{x} - \underline{x}_0))] \cdot \underline{n} dA_x \quad (\text{definition of transpose})$$

$$= \int_{\mathcal{Q}_t} \text{div}_1 [\underline{T}^T (\underline{w} \times (\underline{x} - \underline{x}_0))] dV_x \quad (\text{Divergence Theorem})$$

$$= \int_{\mathcal{Q}_t} [(\text{div}_1 \underline{T}) \cdot (\underline{w} \times (\underline{x} - \underline{x}_0)) + \underline{T} \cdot D(\underline{w} \times (\underline{x} - \underline{x}_0))] dV_x \quad \left(\begin{array}{l} \text{Theorem} \\ \text{A6.9.6} \end{array} \right).$$

In order to calculate the derivative $D(\underline{w} \times (\underline{x} - \underline{x}_0))$, define the vector field $\underline{d}: \mathcal{Q}_t \rightarrow \mathcal{V}_3$ by

$$\underline{d}(\underline{x}) = \underline{w} \times \underline{e}(\underline{x}),$$

where $\underline{e}: \mathcal{Q}_t \rightarrow \mathcal{V}_3$ is defined by

$$\underline{e}(\underline{x}) = \underline{x} - \underline{x}_0.$$

By Theorems A6.9.1, A5.5.2, and the point version of Theorem

A5.5.4 (ii), $\underline{c}$ is differentiable on $\mathcal{B}_t$ and

$$D\underline{c}(\underline{x}) = \underline{I}.$$

Of course, the uniform field $\underline{w}$ is differentiable with zero derivative, and then by Theorem A5.6.5 $\underline{d}$ is differentiable with

$$D\underline{d}(\underline{x})[\underline{u}] = \underline{w} \times \underline{u} = \underline{W}\underline{u},$$

where $\underline{W}$ is the skew tensor whose axial vector is $\underline{w}$. In components relative to a positive orthonormal basis,

$$W_{\langle ij \rangle} = -\varepsilon_{ijk} w^{\langle k \rangle}. \quad (**)$$

Thus,

$$D\underline{d}(\underline{x}) = \underline{W},$$

and

$$\underline{w} \cdot \underline{c} = \int_{\mathcal{P}_t} [(\operatorname{div}_1 \underline{T}) \cdot (\underline{w} \times (\underline{x} - \underline{x}_0)) + \underline{T} \cdot \underline{w}] dV_{\underline{x}}. \quad (\#)$$

Placing this into $\underline{w} \cdot \underline{a}$, we have

$$\underline{w} \cdot \underline{a} = \int_{\mathcal{P}_t} [(\operatorname{div}_1 \underline{T} + \rho \underline{b} - \rho \underline{a}_0) \cdot (\underline{w} \times (\underline{x} - \underline{x}_0)) + \underline{T} \cdot \underline{w}] dV_{\underline{x}}.$$

Since the linear momentum is balanced by hypothesis, it

¹G. Theorem A3.6.9.

follows from the First Stress Equation of Motion that

$$\underline{w} \cdot \underline{a}(P, t) = \int_{P_t} \underline{T}(x, t) \cdot \underline{w} \, dV_x. \quad (\#\#)$$

Necessity: Suppose that the moment of momentum is balanced. Then $\underline{a}(P, t) = \underline{0} \, \forall$ parts $P \subset B$ and $\forall t \in \mathcal{Q}$, and $(\#\#) \Rightarrow$

$$\int_{P_t} \underline{T}(x, t) \cdot \underline{w} \, dV_x = 0$$

$\forall t \in \mathcal{Q}$, $\forall$ regular subregions $P_t \subset B_t$,¹ and $\forall$ vectors $\underline{w}$. By the arbitrariness of P_t , we conclude from the Localization Theorem (Theorem B1.6.2) that

$$\underline{T} \cdot \underline{w} = 0 \text{ on } \mathcal{I}.$$

Since $\underline{w} \in \text{Skw } \mathcal{V}_3$, it follows from Theorem A3.7.4 that

$$(\text{skw } \underline{T}) \cdot \underline{w} = 0 \text{ on } \mathcal{I}.$$

Now inspection of $(\#\#)$ shows that $\underline{w}$ may be viewed as an arbitrary skew tensor because of the arbitrariness of the vector $\underline{w}$. (Work this out as Exercise B3.4.1.) Thus,

¹ Cf. the proof of Theorem B2.1.1.

everywhere on $\mathcal{I}$, we have

$$(\text{skw } \underline{T}) \cdot \underline{W} = 0 \quad \forall \underline{W} \in \text{Skw } \mathcal{V}_3.$$

Since $\text{Skw } \mathcal{V}_3$ is an inner product space,¹ we conclude from Theorem A2.3.4 that $\text{skw } \underline{T} = \underline{Q}$; i.e.,

$$\underline{T}^T = \underline{T} \text{ on } \mathcal{I}.$$

Sufficiency. Suppose that the stress tensor is symmetric on $\mathcal{I}$. Then by Theorem A3.7.4 and ($\#\#$),

$$\underline{W} \cdot \underline{a}(\mathcal{P}, t) = 0 \quad \forall \underline{W} \in \mathcal{V}_3.$$

Thus, $\underline{a}(\mathcal{P}, t) = \underline{Q} \quad \forall$ parts $\mathcal{P} \subset \mathcal{B}$ and $\forall t \in \mathcal{I}$, and we conclude from the remarks at the beginning of the proof that the moment of momentum is balanced. $\square$

Exercise B3.4.2. Reduce $\underline{W} \cdot \underline{Q}$ to the expression given by ($\#$) by the use of components and coordinates.

As an easy consequence of the Stress Equations of Motion, we have

¹ Cf. Theorems A3.6.7 and A3.7.3.

Theorem B3.4.5. (The Energy Transfer Theorem) Let
 $(\underline{t}_n, \underline{b})$ be a system of forces corresponding to a motion
 $\{\chi(\cdot, t) : t \in \mathfrak{Q}\}$ of a body $\mathcal{B}$. Suppose the momentum is
balanced. Then $\forall$ parts $\mathcal{P} \subset \mathcal{B}$ and $\forall t \in \mathfrak{Q}$,

$$\dot{T}(\mathcal{P}, t) = \int_{\partial \mathcal{P}_t} \underline{t}_n \cdot \underline{v}_s \, dA + \int_{\mathcal{P}_t} \underline{b} \cdot \underline{v}_s \rho \, dV - \int_{\mathcal{P}_t} \underline{T} \cdot \underline{L}_s \, dV. \quad)^1$$

If, in addition, the moment of momentum is balanced, then

$$\dot{T}(\mathcal{P}, t) = \int_{\partial \mathcal{P}_t} \underline{t}_n \cdot \underline{v}_s \, dA + \int_{\mathcal{P}_t} \underline{b} \cdot \underline{v}_s \rho \, dV - \int_{\mathcal{P}_t} \underline{T} \cdot \underline{D}_s \, dV,$$

where $\underline{L}_s$ and $\underline{D}_s = \text{sym } \underline{L}_s$ are the spatial velocity gradient
and the stretching tensor. }^2

Proof. Exercise B3.4.3. $\square$

INSERT material on p. B3.4.18 a

Finally, we remark that the stress tensor $\underline{T}$ arose naturally in terms of the spatial variables, and there was no need to write $\underline{T}(\underline{x}, t) = \underline{T}_s(\underline{x}, t)$. In the theory of constitutive equations, it is more clear to work with the material description; consequently in accordance with our First Notational Agreement (p. B1.8.7), we introduce

¹ $\underline{T} \cdot \underline{L}_s = \text{tr}(\underline{T} \underline{L}_s^T)$; cf. the discussion after Theorem A3.7.3

² See Definition B1.8.6.

INSERT for p. B3.4.18

Obviously, the term

$$\int_{\mathcal{P}_t} \underline{T} \cdot \underline{L}_s dV$$

has the units of power; it is called the stress power of the part $\mathcal{P}$ at the time t . The energy transfer theorem asserts that for each part of a body the time rate of change of kinetic energy is balanced by the power of the forces on the part minus the stress power.

return to p. B3.4.18

$$\underline{T}_m(X, t) = \underline{T}_s(\mathbb{X}(X, t), t).$$

In most problems in solid mechanics, it is more convenient to work with a reference configuration κ ; and in accordance with our Second Notational Agreement (p. B1.13.6), we introduce

$$\underline{T}_n(\mathbb{X}, t) = \underline{T}_m(\kappa^{-1}(\mathbb{X}), t) = \underline{T}_s(\mathbb{X}_n(\mathbb{X}, t), t),$$

In the literature, $\underline{T}_s$, $\underline{T}_m$, and $\underline{T}_n$ are all called (true or Cauchy) stress tensors since they all represent the same physical quantity. For corresponding X , $\mathbb{X}$, and $\mathfrak{x}$; i.e., for

$$\mathfrak{x} = \mathbb{X}(X, t) = \mathbb{X}_n(\mathbb{X}, t), \quad \mathbb{X} = \kappa(X),$$

$$\underline{T}_s(\mathfrak{x}, t) = \underline{T}_m(X, t) = \underline{T}_n(\mathbb{X}, t).$$

Supplementary Reading

Same as for § B2.1

B3.6. Nominal Stress

The CAUCHY stress tensor arose naturally in §B3.4 as a function of the spatial variables (x, t) , but this is an awkward feature for many situations in solid mechanics because x varies in the deformed region $\mathcal{B}_t$, which typically is unknown. To get around this difficulty, it would seem reasonable simply to view the CAUCHY stress as a function of the referential variables (X, t) by writing

$$\underline{T}_x(X, t) = \underline{T}(\underline{X}_x(X, t), t)$$

in accordance with our Second Notational Agreement (p. B1.13.6). Then we have to determine the corresponding transformations of the stress equations of motion¹

$$\operatorname{div}_x \underline{T}(x, t) + f(x, t) \rho(x, t) = \rho(x, t) a_x(x, t)$$

and

$$(\underline{T}(x, t))^T = \underline{T}(x, t),$$

where $(x, t) \in \mathcal{I}$.

Obviously, the second equation is equivalent to

$$(\underline{T}_x(X, t))^T = \underline{T}_x(X, t), \quad (X, t) \in \mathcal{K}(\mathcal{B}) \times \mathcal{A}.$$

¹See Theorems B3.4, 3 and 4.

Unfortunately, because of the divergence operator, the first equation of motion is not so easy to transform. A natural starting point is to write

$$\underline{T}(\underline{x}, t) = \underline{T}_x(\underline{X}_x^{-1}(\underline{x}, t), t).$$

Then we can calculate $\text{div}_x \underline{T}$ via the chain rule. The calculation is most readily carried out in terms of components and coordinates as was done occasionally in §§ A5.14 and A6.9. Accordingly, let $\{\mathcal{O}, \underline{e}_{\langle i \rangle}\}$ be a reference frame for $\mathbb{E}_3$ so that

$$T^{\langle ij \rangle}(\underline{x}, t) = T_x^{\langle ij \rangle}(\underline{X}_x^{-1}(\underline{x}, t), t).$$

Then by the elementary form of the chain rule in terms of ordinary partial derivatives

$$T^{\langle ij \rangle}_{,j}(\underline{x}, t) = T_x^{\langle ij \rangle}_{,k}(\underline{X}_x^{-1}(\underline{x}, t), t) (\underline{X}_x^{-1})^{\langle k \rangle}_{,j}(\underline{x}, t).$$

But by the point variant of Theorem A5.14.1¹,

$$(\underline{X}_x^{-1})^{\langle k \rangle}_{,j}(\underline{x}, t) = (D_1 \underline{X}_x^{-1}(\underline{x}, t))^{\langle k j \rangle},$$

and by Theorem B1.1.1 (vi) and Definition B1.12.4,

$$D_1 \underline{X}_x^{-1}(\underline{x}, t) = (D_1 \underline{X}_x(\underline{x}, t))^{-1} = (F_n(\underline{x}, t))^{-1}.$$

¹cf. also the discussion on pp A6.6.1 and 2.

So by Theorem A6.7.2,

$$\begin{aligned} (\operatorname{div}_x T(x, t))^{<i>} &= \sum_{j,k=1}^3 T_{x_{jk}}^{<ij>}(x, t) (F_x^{-1}(x, t))^{<kj>} \\ &= \sum_{j,k=1}^3 T_{x_{jk}}^{<ij>}(x, t) (F_x^{-T}(x, t))^{<jk>} \quad (\text{Theorems A3.6.4 and A3.9.12}). \end{aligned}$$

The remaining terms in the first equation of motion are easier to deal with. By notational agreement,

$$a_x(x, t) = a_x(X_x^{-1}(x, t), t)$$

and

$$b(x, t) = b_x(X_x^{-1}(x, t), t).$$

By the referential density equation (Theorem B2.1.5),

$$\rho(x, t) = \frac{f_x(X_x^{-1}(x, t))}{J_x(X_x^{-1}(x, t), t)}.$$

Gathering these results together, we have that

$$\operatorname{div}_x T + \rho b = f a_x \quad \text{in } \mathcal{B}$$

is equivalent to

$$\text{'Of course, } F_x^{-1}(x, t) := (F_x(x, t))^{-1}.$$

$$J_x(x, t) \sum_{j,k=1}^3 T_{x,jk}^{(ij)}(x, t) (F_x^{-T}(x, t))^{(jk)} \\ + f_x(x) b_x^{(i)}(x, t) = f_x(x) c_x^{(i)}(x, t)$$

for $(x, t) \in \mathcal{M}(\mathcal{B}) \times \mathcal{Q}$. The first term in this equation does not correspond to anything obvious in direct notation unless we introduce some new tensor operations involving third-order tensors. However, we can get a remarkable simplification by defining

$$\underline{S}_x^{(ik)}(x, t) = J_x(x, t) \sum_{j=1}^3 T_{x,jk}^{(ij)}(x, t) (F_x^{-T}(x, t))^{(jk)},$$

equivalent by¹

$$\underline{S}_x(x, t) = J_x(x, t) T_x(x, t) F_x^{-T}(x, t).$$

Then

$$\begin{aligned} J_x \sum_{j,k=1}^3 T_{x,jk}^{(ij)} (F_x^{-T})^{(jk)} &= \sum_{k=1}^3 \underline{S}_{x,jk}^{(ik)} - \sum_{j,k=1}^3 T_{x,jk}^{(ij)} (J_x F_x^{-T})^{(jk)} \\ &= (\text{div}_j \underline{S}_x)^{(i)} - (T_x \text{div}_j (J_x F_x^{-T}))^{(i)} \quad \left(\begin{array}{l} \text{Theorems} \\ A6.7.2 \\ \text{and A3.2.9} \end{array} \right). \end{aligned}$$

Thus,

$$\text{div}_j T + f b = f c \quad \text{on } \mathcal{M}$$

is equivalent to

$$\text{div}_j \underline{S}_x - T_x \text{div}_j (J_x F_x^{-T}) + f_x b_x = f_x c_x \quad \text{on } \mathcal{M}(\mathcal{B}) \times \mathcal{Q}.$$

¹ Cf. Theorem A3.5.4.

There is additional simplification due to

Theorem B3.6.1, (PIOLA's Identity (1825)).

$$\operatorname{div}_x (J_x F_x^{-1}) = 0 \text{ on } \mathcal{X}(B) \times \mathbb{R}^2.$$

Proof. Recall from Theorem A6.7.1 that the divergence of a differentiable tensor field $\underline{K}$ on an open set $D \subset \mathbb{E}_3$ is the unique vector field $\operatorname{div} \underline{K}$ on D with the property that

$$(\operatorname{div} \underline{K}) \cdot \underline{a} = \operatorname{div} (\underline{K}^T \underline{a}) \text{ on } D$$

for every uniform $\underline{a} \in \mathcal{V}_3$. Thus, in the present circumstances, it follows from Theorems A3.9.12 and A3.6.3 (i) that

$$(\operatorname{div}_t (J_{\tilde{x}} F_n^{-T})) \cdot \underline{a} = \operatorname{div} (\varphi \underline{w}),$$

where, for each $t \in \mathcal{S}$, $\varphi: \mathcal{M}(\mathcal{B}) \rightarrow \mathbb{R}$ and $\underline{w}: \mathcal{M}(\mathcal{B}) \rightarrow \mathcal{V}_3$ are defined by the compositions

$$\varphi = \mathcal{V} \circ \tilde{F}_n(\cdot, t) \quad (1)$$

and

$$\underline{w} = \underline{y} \circ \tilde{F}_n(\cdot, t) \quad (2)$$

with $\mathcal{V}: \operatorname{Inv} \mathcal{V}_3 \rightarrow \mathbb{R}$ and $\underline{y}: \operatorname{Inv} \mathcal{V}_3 \rightarrow \mathcal{V}_3$ defined by

$$\mathcal{V}(\tilde{F}) = |\det \tilde{F}| \quad (3)$$

and

$$\underline{y}(\tilde{F}) = \tilde{F}^{-1} \underline{a}. \quad (4)$$

PIOLA's identity will be established if it is shown that

$$\operatorname{div} (\varphi \underline{w}) = 0 \text{ on } \overline{\mathcal{M}(\mathcal{B})} \quad \forall \text{ uniform } \underline{a} \in \mathcal{V}_3.$$

The extension to the boundary is by the Extension Theorem

(Theorem B1.2.2). For the most part, t has been suppressed above, and that will be the case below.

To begin with, $J_n = |\det \underline{F}_n| \neq 0$, and $\therefore$ by continuity $\det \underline{F}_n$ never changes sign. For definiteness, suppose that $\det \underline{F}_n > 0$ so that $J_n = \det \underline{F}_n$ and

$$\vartheta(\underline{F}_n) = \det \underline{F}_n. \quad (5)$$

If $\det \underline{F}_n < 0$, then $\vartheta(\underline{F}_n) = -\det \underline{F}_n$, and the following argument goes through with corresponding changes.

By Theorem A6.9.7,

$$\operatorname{div}(\varphi \underline{w}) = (\nabla \varphi) \cdot \underline{w} + \varphi \operatorname{div} \underline{w}. \quad (6)$$

The terms on the r.h.s. of (6) will be examined individually.

By the definition of the gradient of a scalar field (see p. A6.5.1) and the chain rule (Theorem A5.7.1), it follows from (1) that for any $\mathbb{X} \in \overline{\mathcal{D}(\mathcal{B})}$ and for any $\underline{u} \in \mathcal{V}_3$,

$$\nabla \varphi(\mathbb{X}) \cdot \underline{u} = D\varphi(\mathbb{X})[\underline{u}] = D\vartheta(\underline{F}_n(\mathbb{X}))[D\underline{F}_n(\mathbb{X})[\underline{u}]]. \quad (7)$$

By (5) and Theorem A5.8.1,

$$D\vartheta(\underline{F}_n)[\underline{U}] = (\det \underline{F}_n) \operatorname{tr}(\underline{F}_n^{-1} \underline{U})$$

for any $\underline{U} \in \operatorname{Lin} \mathcal{V}_3$. $\therefore$, suppressing $\mathbb{X}$, one has that

$$(\nabla\varphi) \cdot \underline{u} = (\det \underline{F}_n) \operatorname{tr} (\underline{F}_n^{-1} D\underline{F}_n [\underline{u}]).$$

The choice

$$\underline{u} = \underline{w} = \underline{y}(\underline{F}_n) = \underline{F}_n^{-1} \underline{a}$$

yields

$$(\nabla\varphi) \cdot \underline{w} = (\det \underline{F}_n) \operatorname{tr} (\underline{F}_n^{-1} D\underline{F}_n [\underline{F}_n^{-1} \underline{a}]). \quad (8)$$

Similarly, (2) implies that for any $\mathbb{X} \in \overline{\mathcal{H}(\mathcal{B})}$ and any $\underline{u} \in \mathcal{V}_3$,

$$D\underline{w}(\mathbb{X})[\underline{u}] = D\underline{y}(\underline{F}_n(\mathbb{X})) [D\underline{F}_n(\mathbb{X})[\underline{u}]].$$

By (4),

$$\underline{y}(\underline{F}_n) = \underline{J}(\underline{F}_n) \underline{a},$$

where $\underline{J}: \operatorname{Inv} \mathcal{V}_3 \rightarrow \operatorname{Inv} \mathcal{V}_3$ is defined by

$$\underline{J}(\underline{F}) = \underline{F}^{-1}.$$

By Theorems A5.6.6 and A5.5.2,

$$D\underline{y}(\underline{F}_n)[\underline{U}] = D\underline{J}(\underline{F}_n)[\underline{U}] \underline{a};$$

and by Theorem A5.8.3,

$$D\underline{J}(\underline{F}_n)[\underline{U}] = -\underline{F}_n^{-1} \underline{U} \underline{F}_n^{-1}.$$

Thus, suppressing $\mathbb{X}$, we have

$$D\underline{W}[\underline{u}] = -\underline{F}_n^{-1}(D\underline{F}_n[\underline{u}])\underline{F}_n^{-1}\underline{a}. \quad (9)$$

Now, $\underline{F}_n$ itself is a derivative so that

$$D\underline{F}_n = D(D\mathbb{X}_n) = D^2\mathbb{X}_n.$$

Of course, $D^2\mathbb{X}_n$ is a linear operator on $\mathcal{V}_3$ to $\text{Lin}\mathcal{V}_3$. Here, it is convenient to interpret $D^2\mathbb{X}_n$ as the bilinear mapping of $\mathcal{V}_3 \times \mathcal{V}_3$ into $\mathcal{V}_3$ defined by

$$D^2\mathbb{X}_n[\underline{u}, \underline{z}] = (D^2\mathbb{X}_n[\underline{u}])\underline{z}, \quad)'$$

Since $\mathbb{X}_n \in C^2$ by the smoothness Axiom (M3), it follows from Theorem A5.10.2 that $D^2\mathbb{X}_n$ is symmetric in the sense that

$$D^2\mathbb{X}_n[\underline{u}, \underline{z}] = D^2\mathbb{X}_n[\underline{z}, \underline{u}].$$

Incorporating this symmetry into (9), one gets

$$D\underline{W}[\underline{u}] = -\underline{F}_n^{-1}(D\underline{F}_n[\underline{F}_n^{-1}\underline{a}])\underline{u};$$

and since this holds for all $\underline{u} \in \mathcal{V}_3$,

¹G. Theorem A5.10.1 and the surrounding discussion.

$$D_{\underline{w}} = -F_n^{-1} D_{\underline{F}_n} [F_n^{-1} \underline{a}]. \quad (10)$$

Then since $\operatorname{div} \underline{w} = t_0 D_{\underline{w}}$ by Definition A6.6.1, it follows from (10) and (5) that

$$\varphi \operatorname{div} \underline{w} = -(\det F_n) t_0 (F_n^{-1} D_{\underline{F}_n} [F_n^{-1} \underline{a}]). \quad (11)$$

By (11), (8), and (6),

$$\operatorname{div} (\varphi \underline{w}) = 0 \text{ on } \overline{\mathcal{M}(\mathcal{B})}. \quad \square$$

In view of PIOLA's identity, the result on the bottom of p. B3.6.4 becomes

$$\operatorname{div}_n \underline{T} + \rho_n \underline{b} = \rho_n \underline{a}_n \text{ on } \mathcal{Y}$$

is equivalent to

$$\operatorname{div}_n \underline{S}_n + \rho_n \underline{b}_n = \rho_n \underline{a}_n \text{ on } \mathcal{M}(\mathcal{B}) \times \mathcal{I},$$

To investigate the second stress equation of motion,

$$\underline{T}_n^T(\mathbb{X}, t) = \underline{T}_n(\mathbb{X}, t),$$

we invert

$$\underline{S}_n = \underline{J}_n \underline{T}_n \underline{F}_n^{-T}$$

by means of Theorems A3.9.12, A3.9.3, and A3.5.1 to get

$$\underline{T}_n = \frac{1}{J_n} \underline{S}_n \underline{F}_n^T.$$

$\therefore$ by parts (iii), (iv) and (i) of Theorem A3.6.3,

$$\underline{T}_n^T = \frac{1}{J_n} \underline{F}_n \underline{S}_n^T.$$

Consequently,

$$\underline{T}^T = \underline{T} \text{ on } \mathcal{V}$$

is equivalent to

$$\underline{F} \underline{S}^T = \underline{S} \underline{F}^T \text{ on } \mathcal{V}(\mathcal{B}) \times \mathcal{Q}.$$

In summary, we have

Theorem A3.6.2. In terms of the asymmetric
nominal stress tensor $\underline{S}_n$, which is defined on
 $\mathcal{V}(\mathcal{B}) \times \mathcal{Q}$ by

$$\underline{S}_n(\mathbb{X}, t) = J_n(\mathbb{X}, t) \underline{T}_n(\mathbb{X}, t) \underline{F}_n^{-T}(\mathbb{X}, t),$$

the stress equations of motion

$$\text{div}_n \underline{T} + \rho \underline{b} = \rho \underline{a}_n$$

and

$$\underline{T}^T = \underline{T}$$

Often, first PIOLA-KIRCHHOFF stress tensor; sometimes,
asymmetric engineering stress tensor.

on $\mathcal{V}$ are equivalent¹ to

$$\operatorname{div}_x \underline{S}_n + \rho_n \underline{b}_n = \rho_n \underline{a}_n$$

and

$$\underline{F}_n \underline{S}_n^T = \underline{S}_n \underline{F}_n^T$$

on $\mathcal{N}(\mathcal{B}) \times \mathcal{Q}$, respectively.

It is common in the literature to denote the divergence above by $\operatorname{Div} \underline{S}_n$ as a reminder to the reader that the divergence is calculated w.r.t. the referential variable $\mathbf{x}$ as opposed to the spatial variable $\mathbf{x}$ used in $\operatorname{div} \underline{T}$. We avoid such complications by being precise about the domains of functions and by using different symbols for different functions.²

It is important to realize that while the introduction of the asymmetric nominal stress tensor cleans up the first stress equation of motion and, in fact, provides an equation in "divergence form" (cf. the corresponding equation in $\underline{T}_n$ on p. B3.6.4), a price is paid in the second equation — as its name indicates, the asymmetric nominal stress tensor is not symmetric. In practice, this generally is not a problem; in fact, the condition

¹Here, of course, the customary smoothness assumptions and conservation of mass are taken for granted.

²Cf. our notational agreements on pp. B1.8.7 and B1.13.6.

$$\underline{F} \underline{S}^T = \underline{S} \underline{F}^T \quad (\text{equivalently } \underline{T}^T = \underline{T})$$

usually is taken care of automatically by building it into the form of the constitutive equation being employed.¹ PIOLA introduced another "stress tensor" appropriate for the referential description which is symmetric, but then, as expected, the corresponding first equation of motion is complicated; however, not so much as the one for $\underline{T}$.

Theorem A3.6.3. In terms of the symmetric
nominal stress tensor² $\underline{P}$, which is defined on
 $\mathcal{K}(\mathcal{B}) \times \mathcal{I}$ by

$$\underline{P}(\mathcal{X}, t) = \underline{F}^{-1}(\mathcal{X}, t) \underline{S}(\mathcal{X}, t),$$

the stress equations of motion

$$\operatorname{div} \underline{T} + \rho \underline{b} = \rho \underline{a}$$

and

$$\underline{T}^T = \underline{T}$$

on $\mathcal{I}$ are equivalent to

¹ Cf. Definition B4.1.2

² Often, second PIOLA-KIRCHHOFF stress tensor; sometimes symmetric engineering stress tensor.

$$\operatorname{div}_x (\underline{F} \underline{P}) + \rho_n \underline{b}_n = \rho_n \underline{a}_n$$

and

$$\underline{P}_n^T = \underline{P}_n$$

on $\mathcal{K}(\mathcal{B}) \times \mathcal{Q}$, respectively.

Proof. Exercise B3.6.1. $\square$

In order to interpret the asymmetric nominal stress $\underline{S}_n$ physically, we consider the total force

$$\underline{f}(\mathcal{P}, t) = \int_{\partial \mathcal{P}_t} \underline{t}_{\underline{n}(\underline{x}, t)}(\underline{x}, t) dA_x + \int_{\mathcal{P}_t} \underline{b}(\underline{x}, t) \rho(\underline{x}, t) dV_x \quad)^1$$

acting on part $\mathcal{P}$ of body $\mathcal{B}$ at time t . Now

$$\int_{\partial \mathcal{P}_t} \underline{t}_{\underline{n}(\underline{x}, t)}(\underline{x}, t) dA_x = \int_{\partial \mathcal{P}_t} \underline{T}(\underline{x}, t) \underline{n}(\underline{x}, t) dA_x \quad (\text{existence of stress tensor})$$

$$= \int_{\mathcal{P}_t} \operatorname{div}_x \underline{T}(\underline{x}, t) dV_x \quad (\text{Divergence Theorem})$$

$$= \int_{\mathcal{K}(\mathcal{P})} \operatorname{div}_x \underline{T}(\underline{x}(\underline{X}, t), t) \underline{J}_x(\underline{X}, t) dV_X \quad (\text{change of Variable Theorem})$$

¹ See Definition B3.1.2.

$$= \int_{\mathcal{V}(\mathcal{P})} \operatorname{div}_x \underline{S}_x(\mathbb{X}, t) dV_x \quad (\text{preceding analysis})^1$$

$$= \int_{\partial \mathcal{V}(\mathcal{P})} \underline{S}_x(\mathbb{X}, t) \underline{m}(\mathbb{X}) dA_x \quad (\text{Divergence Theorem}),$$

where $\underline{m}(\mathbb{X})$ is the outward unit normal to $\partial \mathcal{V}(\mathcal{P})$ at $\mathbb{X}$.

Similarly,

$$\int_{\mathcal{B}_t} b(\alpha, t) \rho(\alpha, t) dV_\alpha = \int_{\mathcal{V}(\mathcal{P})} b(\mathbb{X}_x(\mathbb{X}, t), t) \rho(\mathbb{X}_x(\mathbb{X}, t), t) J_x(\mathbb{X}, t) dV_x \quad (\text{Change of Variable Theorem})$$

$$= \int_{\mathcal{V}(\mathcal{P})} b_x(\mathbb{X}, t) \rho(\mathbb{X}_x(\mathbb{X}, t), t) J_x(\mathbb{X}, t) dV_x \quad (\text{notational agreement})$$

$$= \int_{\mathcal{V}(\mathcal{P})} b_x(\mathbb{X}, t) \rho_x(\mathbb{X}) dV_x \quad (\text{referential density equation}).$$

Hence, we have established

Theorem B3.6.4. The total force acting on part $\mathcal{P}$ at time t is given by

¹Work this out as Exercise B3.6.2. Hint: Calculate $J_x \operatorname{div}_x \underline{T}$ from CAUCHY's first stress equation of motion and compare the result with the equation involving $\operatorname{div}_x \underline{S}_x$ in Theorem B3.6.2.

$$\underline{f}(\mathcal{P}, t) = \int_{\partial \mathcal{K}(\mathcal{P})} \underline{S}_x(\mathcal{K}, t) \underline{m}(\mathcal{K}) dA_x + \int_{\mathcal{K}(\mathcal{P})} \underline{b}_x(\mathcal{K}, t) \rho_x(\mathcal{K}) dV_x,$$

where $\underline{m}(\mathcal{K})$ is the outward unit normal to $\partial \mathcal{K}(\mathcal{P})$ at $\mathcal{K}$.

In view of this result, one can interpret $\underline{S}_x(\mathcal{K}, t) \underline{m}(\mathcal{K})$ as the surface traction on part $\mathcal{P}$ at time t acting at the point $\mathcal{K} = \mathcal{K}_x(\mathcal{K}, t)$ on $\partial \mathcal{P}_t$ but measured per unit area of the reference surface $\partial \mathcal{K}(\mathcal{P})$ at the corresponding point $\mathcal{K}$.

Similarly, $\underline{b}_x(\mathcal{K}, t)$ is the body force on part $\mathcal{P}$ at time t acting at the point $\mathcal{K} = \mathcal{K}_x(\mathcal{K}, t)$ in $\mathcal{P}_t$ but measured per unit mass of the reference configuration $\mathcal{K}$ at the corresponding point $\mathcal{K}$. These interpretations may well have led to more confusion than enlightenment. Remember that the CAUCHY or true stress is the direct mathematical embodiment of the physical concept of distributed contact force. The nominal stresses are best viewed as mathematical transformations of some of the dependent variables (namely, the CAUCHY stress) which are made to ease the solutions of certain classes of problems. After a nominal stress field has been found, one should transform back to the CAUCHY stress and seek a physical interpretation in terms of it.

Sometimes it is useful to have an expression for the total moment on a part in terms of the asymmetric nominal stress.

Theorem B3.6.5. The total moment¹ about point y acting on part $\mathcal{P}$ at time t is given by

$$\begin{aligned} \underline{m}(\mathcal{P}, t, y) = & \int_{\partial \mathcal{K}(\mathcal{P})} [\underline{x}_x(\mathcal{K}, t) - y] \times \underline{S}_x(\mathcal{K}, t) \underline{m}(\mathcal{K}) dA_x \\ & + \int_{\mathcal{K}(\mathcal{P})} [\underline{x}_x(\mathcal{K}, t) - y] \times \underline{b}_x(\mathcal{K}, t) \rho_x(\mathcal{K}) dV_x, \end{aligned}$$

where $\underline{m}(\mathcal{K})$ is the outward unit normal to $\partial \mathcal{K}(\mathcal{P})$ at $\mathcal{K}$.

Proof. Exercise B3.6.3. $\square$

The developments at the beginning of the present section which led to the introduction of the asymmetric nominal stress tensor were natural enough, but they were messy. Modern authors (e.g., TRUESDELL and NOLL in the NFTM and GURTIN in his text) sometimes avoid this by starting with the principles of momentum balance in their original integral forms rather than in their equivalent differential forms (the stress equations of motion). However, in the latter approach, one needs the change of variable theorem for surface integrals (NANSON's formula), and this is not part of the traditional advanced calculus preparation for the study of continuum mechanics. Our calculation of

¹See Definition B3.1.2.

the total force on a part (Theorem B3.6.4) suggests that one could use PIOLA's identity and the divergence theorem to derive variants of NANSON's formula appropriate for continuum physics; indeed, this is the case.¹

Supplementary Reading

CHADWICK, Continuum Mechanics

CIARLET, Mathematical Elasticity, Vol I: Three-Dimensional Elasticity

GURTIN, An Introduction to Continuum Mechanics

MARSDEN and HUGHES, Mathematical Foundations of Elasticity

TRUESDELL, A First Course in Rational Continuum Mechanics, Vol. 1

TRUESDELL and NOLL, NFTM

TRUESDELL and TOUPIN, CFT

¹ See pp. 39-40 of CIARLET's text.

Chapter B4

A Theory of Constitutive Equations

In this chapter, following NOLL (1958), we develop a theory to relate the stress in a body to its motion in such a way as to distinguish the material that makes up the body. The theory will be sufficiently general to include all of the widely studied particular theories that have been used to describe the mechanical behavior of special classes of materials,

B4.1. Simple Materials. The Response Function

We would expect the stress in a body at any given time to depend on what has happened to the body in the past, especially as opposed to what might happen to the body in the future. The following definition will help us to phrase this mathematically.

Definition B4.1.1. Given an m^{th} -order tensor-valued function $\tilde{\mathbf{T}}^{(m)}$ on $(-\infty, t] := \{\tau : -\infty < \tau \leq t\}$, the function $\tilde{\mathbf{T}}^{(m)t}$ defined on $[0, \infty) := \{\tau : 0 \leq \tau < \infty\}$ by

$$\tilde{\mathbf{T}}^{(m)t}(s) = \tilde{\mathbf{T}}^{(m)}(t-s)$$

is the history of $\tilde{\mathbf{T}}^{(m)}$ up to time t .

The variable s represents time units into the past. Small s gives the recent past with $s=0$ corresponding to the present time t , and large s gives the distant past.

In particular, the history of the deformation gradient relative to a reference configuration κ is given by

$$\mathbf{F}_{\kappa}^t(s) = \mathbf{F}_{\kappa}(t-s),$$

Considering only mechanical influences, we would expect that the stress in a body at time t would be determined by

the history of the motion of the body up to time t . Since stress represents contact force, we further expect the stress at a point to be determined by the history of the motion in the immediate neighborhood of the point. Recalling that a deformation is characterized locally by its deformation gradient,¹ we are lead to define the following class of materials or bodies.

Definition B4.1.2. (NOLL, 1958) A body $\mathcal{B}$ is simple
w.r.t. a reference configuration κ if for each particle $X \in \mathcal{B}$
and for each time $t \in \mathbb{R} \exists$ a function

$$\underset{\sim \kappa}{\mathcal{Y}}^{X,t} : C^N([0, \infty), \text{Inv } V_3) \rightarrow \text{Sym } V_3$$

$\Rightarrow$ the stress tensor corresponding to a motion
 $\{\kappa(\cdot, t) : t \in \mathbb{R}\}$ of $\mathcal{B}$ is given by

$$(SM) \quad \underset{\sim \kappa}{T}_m(X, t) = \underset{\sim \kappa}{\mathcal{Y}}^{X,t}(F_{\sim \kappa}^t(\kappa(X), \cdot)) , (X, t) \in \mathcal{B} \times \mathbb{R}.$$

The function $\underset{\sim \kappa}{\mathcal{Y}}^{X,t}$ is called the response function¹
at time t of the particle X of the simple body $\mathcal{B}$, and
(SM) is called the constitutive equation for the
simple body $\mathcal{B}$. $N \geq 2$ is the smoothness index of Axiom (M3).

Thus, a body is simple if the present value of the

¹Sometimes, response functional since the elements of the domain of $\underset{\sim \kappa}{\mathcal{Y}}^{X,t}$ are themselves functions.

stress at a point is determined by the history of the deformation gradient relative to some reference configuration at that point. The adjective simple is used to indicate that the deformation gradient is the only characteristic of the motion involved in the constitutive equation. It is important to realize that for a given particle of a given body we use the same response function to calculate the stress no matter what the motion of the body might be, at least within a certain class of motions such as those with sufficiently small velocities. In other words, the response function is a material property of the body.

Note that we have allowed the response function to depend explicitly on the particle X and the present time t . This permits different particles of the body to have different stress response properties, and it allows these properties to change with time, i.e. to age. We shall see in the next section that this explicit time dependence must drop out.

We have taken the time interval I for the motion to be $\mathbb{R}$ so as to make ^{the} formulation as clean as possible. In practice, this requires more information than generally will be available, and the histories will need to be cut off at some time in the finite past.

The domain of the response function $\underset{\sim}{g}_{\sim n}^{X,t}$ is $C^{NH}([0, \infty), \text{Inv } V_3)$ because of the properties of the

deformation gradient $\tilde{F}_n$. For each $\mathbb{X} \in \mathcal{U}(\mathcal{B})$ and $t \in \mathbb{R}$, the domain of the history $\tilde{F}_n^t(\mathbb{X}, \cdot)$ is $[0, \infty)$ by Definition B4.1.1. Also by this definition and Theorem B1.13.6 (i), $\tilde{F}_n^t(\mathbb{X}, \cdot)$ is a class C^N function. Its values are invertible tensors by Theorems B1.13.7 (ii) and A3.9.9. Moreover, all invertible tensors are included in the range of $\tilde{F}_n^t(\mathbb{X}, \cdot)$ because given any $\tilde{F}_0 \in \text{Inv } \mathcal{V}_3$, the admissible motion¹ defined by

$$\mathbb{X}_n(\mathbb{X}, t) = \mathbb{X}_0 + \tilde{F}_0(\mathbb{X} - \mathbb{X}_0)$$

for constant $\mathbb{X}_0$ and $\mathbb{X}_0$ has the property that

$$\tilde{F}_n^t(\mathbb{X}, s) = \tilde{F}_n(\mathbb{X}, t-s) = \tilde{F}_0.$$

The values of the response function $\tilde{\gamma}_{X,t}$ must be symmetric tensors because of the Second Stress Equation of Motion (Theorem B3.4.4).

Finally, there is a more subtle requirement that must be placed on response functions since their values are stresses. Given a motion, the constitutive equation (SM) produces the set $\{\tilde{T}_m(X, t) : (X, t) \in \mathcal{B} \times \mathbb{R}\}$ as X ranges over all particles of the body and t ranges over all times. The spatial stress field can then be calculated

¹ Admissible in the sense that all of the requirements that we have placed on motions such as Axiom (M3) are met.

on the trajectory by

$$\underline{T}_a(x, t) = \underline{T}_m(\chi^{-1}(x, t), t).$$

But for each t , the tensor field $\underline{T}_a(\cdot, t)$ is smooth on B_t by Theorem B3.4.2. Thus, we have

Theorem B4.1.1. Let the body B be simple w.r.t. the reference configuration $\mathcal{K}$. Then the response function $\underline{g}_{\mathcal{K}}^{X, t}$ is $\exists$ for any motion of B the stress produced by the constitutive equation (SM) has the property that the spatial field $\underline{T}_a(\cdot, t)$ is smooth on $B_t \forall t \in \mathbb{R}$.

Of course, in the practice of continuum mechanics, one has to discover response functions by some fortuitous combination of insight, intelligence, theory, experiment, and anything else that helps. In this spirit, Theorem B4.1.1 provides a requirement that candidate response functions must meet.

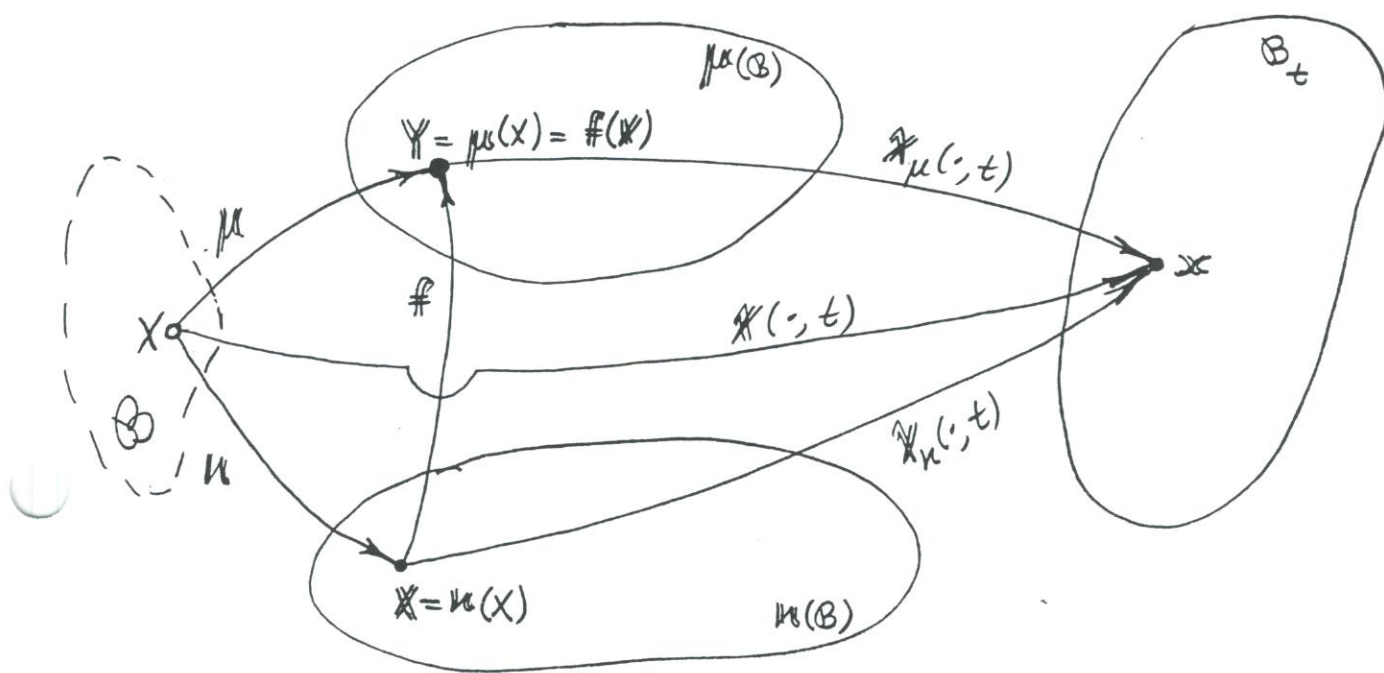
Given a body that is simple w.r.t. one reference configuration, it is natural to ask what happens if we choose to employ a different reference configuration.

Theorem B4.1.2. If a body B is simple w.r.t. one reference configuration $\mathcal{K}$, then it is simple w.r.t. any other reference configuration $\mathcal{K}'$. In fact, if $\mathcal{K}$ and $\mathcal{K}'$ are connected through the diffeomorphism

$\# = \mu \circ \kappa^{-1}$,) then

$$T_m(X, t) = \mathcal{Y}_{\sim \mu}^{X, t} (F_{\mu}^t(\mu(X), \cdot) \underline{G}(\kappa(X))) =: \mathcal{Y}_{\sim \mu}^{X, t} (F_{\mu}^t(\mu(X), \cdot)),$$

where $\underline{G} = D\#$,



Proof. By Theorem B1.13, 15,

$$F_{\kappa}(\kappa, t) = F_{\mu}(\#(\kappa), t) \underline{G}(\kappa);$$

and then by Definition B4.1.1,

$$F_{\kappa}^t(\kappa, s) = F_{\mu}(\#(\kappa), t-s) \underline{G}(\kappa) = F_{\mu}^t(\#(\kappa), s) \underline{G}(\kappa).$$

$\therefore (SM) \Rightarrow$

cf. (B1) of Definition B1.7.1.

$$T_{\sim \mu}(X, t) = \mathcal{Y}_{\sim \mu}^{X, t}(F_{\sim \mu}^t(\kappa(X), \cdot)) = \mathcal{Y}_{\sim \mu}^{X, t}(\underbrace{F_{\sim \mu}^t(\kappa(X), \cdot) \zeta(\kappa(X))}_{\text{response function}}). \quad \square$$

Even though a simple body is simple w.r.t. all reference configurations, the response function depends on the particular reference configuration that is employed. Theorem B4.1.2 provides a transformation rule for finding the new response function when the reference configuration is changed.

Note that if we had allowed for different particles of the body to have different stress response by letting the response function depend explicitly on the position of the particle in the reference configuration, $\# = \kappa(X)$, then the above theorem would have been difficult to formulate. $\oplus_2$

Generally speaking, it is easiest to develop the general theory of constitutive equations in terms of the material description. It then is a trivial matter to obtain the other descriptions whenever they are needed for other purposes.

It is conceivable that a body might have the same stress response properties at all of its particles. To this end, we make

Definition B4.1.3. A simple body $\mathcal{B}$ is homogeneous w.r.t. stress response if $\exists$ a reference configuration $\mathfrak{A}$, called a

homogeneous reference configuration for stress response, ¹ $\exists$ the response function $\underset{\sim}{y}_{\lambda}^{x,t}$ is independent of the particle x ; i.e., for each $t \in \mathbb{R}$,

$$\underset{\sim}{y}_{\lambda}^{x,t} = \underset{\sim}{y}_{\lambda}^{y,t} \quad \forall x, y \in \mathcal{B}. \quad (*)$$

As an interesting application of Theorem B4.1.2, we have

Theorem B4.1.3. Let λ be a homogeneous reference configuration for stress response of a simple body $\mathcal{B}$. Let $\#$ be a homogeneous deformation. Then the configuration $\mu = \# \circ \lambda$ ² also is a homogeneous reference configuration for stress response.

Proof. By Theorem B1.4.2,

$$\underset{\sim}{y}_{\mu}^{x,t}(F_{\mu}^t(\mu(x), \cdot)) = \underset{\sim}{y}_{\lambda}^{x,t}(F_{\mu}^t(\mu(x), \cdot) \underline{G}(\lambda(x))) \quad (*)$$

where $\underline{G} = D\# = \underline{\dot{F}}$ — a uniform tensor field since $\#$ is a homogeneous deformation. Thus,

$$\Rightarrow \underset{\sim}{y}_{\mu}^{x,t}(\underline{I}) = \underset{\sim}{y}_{\lambda}^{x,t}(\underline{I} \underline{G}) \quad \text{where } \underline{I} \in C^{\infty}(\text{core})$$

$$\underset{\sim}{y}_{\mu}^{x,t}(\cdot) = \underset{\sim}{y}_{\lambda}^{x,t}((\cdot) \underline{\dot{F}}).$$

¹ In view of the fact that there might be other important material homogeneities, e.g. the mass density ρ_2 might be uniform, this terminology is not unduly wordy.

² Cf. (B2) of Definition B1.7.1.

$$\therefore \mathcal{Y}_{\sim \mu}^{Y,t}(\cdot) = \mathcal{Y}_{\sim \lambda}^{Y,t}((\cdot) \stackrel{?}{\underset{\sim}{F}}) = \mathcal{Y}_{\sim \lambda}^{X,t}((\cdot) \stackrel{?}{\underset{\sim}{F}}) = \mathcal{Y}_{\sim \mu}^{X,t}(\cdot);$$

i.e., the reference configuration μ is also homogeneous w.r.t. stress response. It is clear from (*) that if the deformation $\#$ were not homogeneous, then particle dependence would be introduced into $\mathcal{Y}_{\sim \mu}^{X,t}$ through the deformation gradient $\underline{G}$. $\square$

Supplementary Reading

Same as for § B2.1 plus

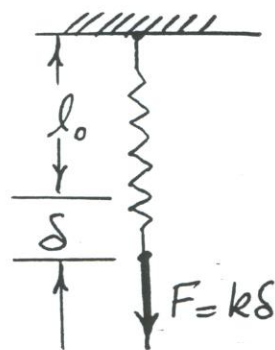
TRUEDEL, Six Lectures on Modern Natural Philosophy (Lecture I, "Rational Mechanics of Materials")

B4.2. The Principle of Material Indifference

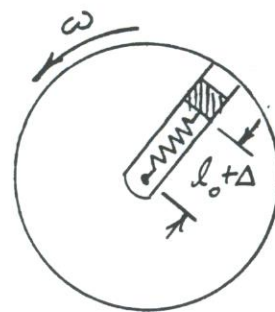
Not every function which maps deformation gradient histories into second-order tensors can serve as a response function. We have already required that the values of $\underline{\underline{g}}_n^{x,t}$ be symmetric tensors because of the Second Stress Equation of Motion, and for any motion the stress field $\underline{\underline{T}}_p(\cdot, t)$ needs to come out to be smooth for each $t \in \mathbb{R}$. The First Stress Equation of Motion does not place any restriction on $\underline{\underline{g}}_n^{x,t}$ because we can always, at least in theory, satisfy this equation by supplying the right body force. The only general principle that we have left, among those already discussed, is conservation of mass; but the various expressions of this principle do not involve the stress. Nevertheless, there is a very subtle condition yet to be placed on the response function, and that is the concern of the present section.

In order to motivate this new principle, we consider the following elementary dynamics problem, which was first suggested by TRUESDALL for this purpose.

Given: A massless, linear spring with spring constant k and unstretched length l_0 is attached at one end to a frictionless pivot at the center of a horizontal turntable and at the other end to a uniform rigid block of mass M . The block slides in a frictionless

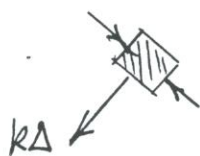


radial groove in the turntable.
The turntable is then spun up
to a steady angular speed ω .



Find: The resulting extension
 Δ of the spring.

Solution: From the radial component of $\underline{f} = m\underline{a}$,



and

$$k\Delta = M(l + \Delta)\omega^2 \quad (l = l_0 + \frac{1}{2} \text{ block width});$$

$$\Delta = \frac{Ml\omega^2}{k - M\omega^2} \quad \text{for } \omega \neq \sqrt{k/M} \quad . \quad \square$$

Note that in solving this problem, we assumed (probably without even realizing it since the assumption is so natural) that the spring law or constitutive equation of the spring is still $F = k\delta$ even when the spring is rotating.

This is an example of the principle that we wish to formulate for general bodies; i.e., the constitutive equation must be invariant under or indifferent to any superposed rigid motion of the whole body. Also bodies do not recognize time zones or the difference between standard time and daylight savings time; i.e., a shift in the origin of the time scale must not affect the stress response. With these notions in mind, we make

Definition B4.2.1. Two motions $\{\mathbb{X}(\cdot, t) : t \in \mathbb{R}\}$ and
 $\{\mathbb{X}^*(\cdot, t) : t \in \mathbb{R}\}$ of a body $\mathcal{B}$ are equivalent w.r.t. material
response if $\exists$ a constant $a \in \mathbb{R}$ $\ni$ for each $t \in \mathbb{R}$
the configurations $\mathbb{X}(\cdot, t)$ and $\mathbb{X}^*(\cdot, \bar{t})$, where $\bar{t} = t - a$,
are connected¹ by a rigid deformation.

Our knowledge of rigid deformations quickly leads us to

Theorem B4.2.1. Two motions $\{\mathbb{X}(\cdot, t) : t \in \mathbb{R}\}$ and
 $\{\mathbb{X}^*(\cdot, t) : t \in \mathbb{R}\}$ of a body $\mathcal{B}$ are equivalent iff $\exists$ a
constant $a \in \mathbb{R}$, a fixed point $\underline{x} \in \mathcal{E}_3$, a function
 $c \in C^N(\mathbb{R}, \mathcal{E}_3)$, and a function $\underline{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } \mathcal{V}_3)$, where
 $N \geq 2$ is the smoothness index of Axiom (M3), $\ni \forall (x, t) \in \mathcal{B} \times \mathbb{R}$

$$\mathbb{X}^*(x, \bar{t}) = c(t) + \underline{Q}(t) [\mathbb{X}(x, t) - \underline{x}], \quad \bar{t} = t - a,$$

Proof. For convenience, write

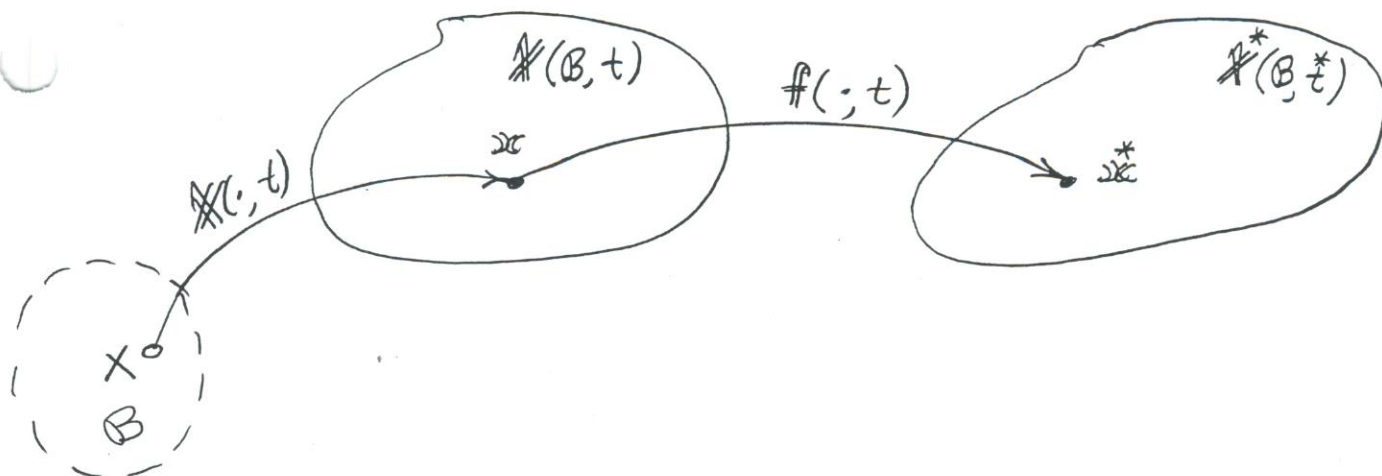
$$x = \mathbb{X}(X, t) \text{ and } \bar{x} = \mathbb{X}^*(X, \bar{t}). \quad (*)$$

Suppose that the motions $\{\mathbb{X}(\cdot, t) : t \in \mathbb{R}\}$ and $\{\mathbb{X}^*(\cdot, t) : t \in \mathbb{R}\}$
 are equivalent. Then by Definitions B4.2.1 and B1.7.1, $\exists$
 $a \in \mathbb{R} \ni$ the deformation $\mathbb{F}$ defined by

$$\bar{x} = \mathbb{X}^* (\mathbb{X}^{-1}(x, t), \bar{t}) =: \mathbb{F}(x, t), \quad (**)$$

¹ Cf. Axiom (B1) of Definition B1.7.1

B4, 2.3a



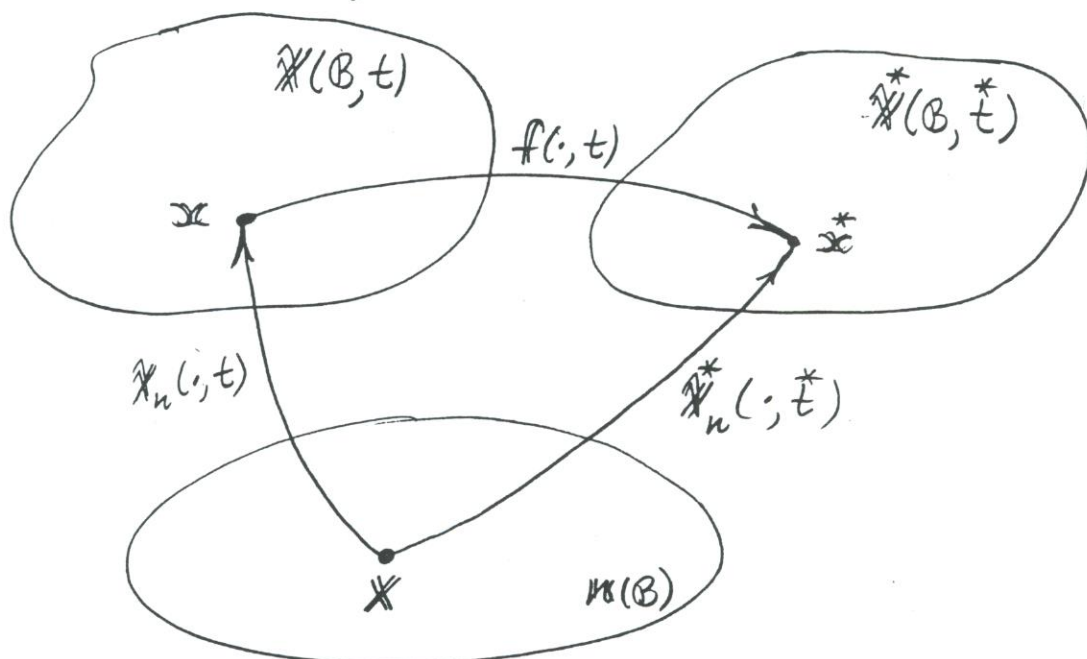
where $\tilde{t} = t - a$, is rigid. Consequently, by Theorem B1.2.1,

$$\#(x, t) = c(t) + Q(t)(x - \tilde{x}), \quad (\#)$$

$$c(t) = \#(\tilde{x}, t).$$

where $\tilde{x}$ is an arbitrary point in E_3 ¹ and $Q(t) \in \text{Orth } V_3$. The claimed relation then follows from (*), (**), and (#).

To get the smoothness of c and Q , we introduce a reference configuration $\mathcal{N}$ so that the smoothness Axiom (M3) can be employed.



Clearly,

$$\#(x, t) = \#_n^*(X_n^{-1}(x, t), t - a), \quad (x, t) \in \mathcal{I}.$$

By Axiom (M3), $\#_n^* \in C^N(\mathcal{N}(B) \times \mathbb{R}, E_3)$; by Theorem B1.13.4, the reference map $X_n^{-1} \in C^N(\mathcal{I}, E_3)$; and by observation, the map $t \mapsto t - a$ is of class $C^\infty(\mathbb{R}, \mathbb{R})$.

¹cf. the remarks following Theorem B1.2.3.

It $\therefore$ follows from the various chain rules of Chapters A5 and A6 that $f \in C^N(\mathcal{I}, \mathcal{E}_3)$.¹ But by (#),

$$C(t) = f(\underline{x}, t)$$

and

$$\underline{Q}(t) = D_1 f(\underline{x}, t) \quad . \quad)^2$$

$\therefore C \in C^N(\mathbb{R}, \mathcal{E}_3)$ and $\underline{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } \mathcal{V}_3)$..

¹Work this out carefully as Exercise B4.2.1.

²Gf. Theorems A6.9.1, A5.5.2, A5.6.6, and A5.5.4 (ii).

Conversely, if the motions $\{\mathbb{X}(\cdot, t) : t \in \mathbb{R}\}$ and $\{\mathbb{X}^*(\cdot, t) : t \in \mathbb{R}\}$ are $\ni$ that $\exists a \in \mathbb{R}$, $\underline{x} \in \mathbb{E}^3$, $\underline{c} \in C^1(\mathbb{R}, \mathbb{E}_3)$, and $\underline{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } \mathbb{V}_3) : \ni$

$$\mathbb{X}^*(\underline{x}, \underline{t}) = \underline{c}(t) + \underline{Q}(t) [\mathbb{X}(\underline{x}, t) - \underline{x}], \quad \underline{t} = t - a,$$

i.e. $\forall \underline{x} \in \mathbb{M}(\mathcal{B}) \quad \underline{x}^* = \underline{c}(t) + \underline{Q}(t) [\underline{x} - \underline{x}^*] \Rightarrow$ smooth, one-one, mapping of a compact region in $\mathbb{E}_3$ to another compact region in $\mathbb{E}_3$ (due to the fact that $\mathbb{M}, \mathbb{M}^*$ are configurations of $\mathcal{B}$)

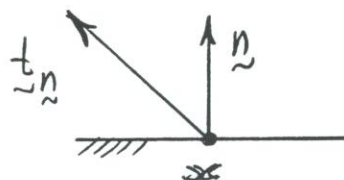
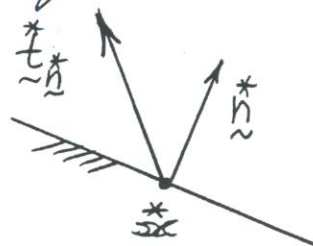
Then the two configurations $\mathbb{X}(\cdot, t)$ and $\mathbb{X}(\cdot, \underline{t})$ are connected by the deformation $\mathbb{F}$ defined by (#). By Theorem B1.2.1, $\mathbb{F}$ is rigid; and then by Definition B4.2.1, the two motions are equivalent w.r.t. stress response. $\square$

If we wish to view the rigid deformation from the configuration $\mathbb{X}(\cdot, t)$ to the configuration $\mathbb{X}(\cdot, \underline{t})$ as a continuous one-parameter family of configurations with the body actually in the two configurations for certain values of the parameter, then the tensor $\underline{Q}(t)$ must be proper orthogonal.¹ This requirement is sometimes imposed.

In order to see how the stress tensors $\underline{T}$ and $\underline{T}^*$ corresponding to equivalent motions $\{\mathbb{X}(\cdot, t) : t \in \mathbb{R}\}$ and $\{\mathbb{X}^*(\cdot, t) : t \in \mathbb{R}\}$ of a body $\mathcal{B}$ should be related, we examine the more physical concept of the associated stress vectors. Accordingly, consider a plane through point $\underline{x}$ with normal $\underline{n}$ and stress vector $\underline{t}_{-\underline{n}}$ in the body in the configuration $\mathbb{X}(\cdot, t)$.

¹ Cf. Theorem B1.14.1.

The corresponding place, normal, and stress vector for the configuration $\mathbb{X}(\cdot, t)$ are denoted by $\underline{x}$, $\underline{n}$, and $\underline{t}_n$.

Configuration $\mathbb{X}(\cdot, t)$ Configuration $\mathbb{X}^*(\cdot, t^*)$

Write $\underline{x} = \mathbb{X}(X, t)$ and $\underline{x}^* = \mathbb{X}^*(X, t^*)$. Then by Theorem B4.2.1,

$$\underline{x}^* = \underline{c} + \underline{Q}(\underline{x} - \underline{x}^*).$$

Certainly, $\underline{n}$ and $\underline{n}^*$ are the same vector except for location and orientation; i.e., $\underline{n}$ gets translated and rotated into $\underline{n}^*$ by the rigid deformation which carries (in imagination) the body from configuration $\mathbb{X}(\cdot, t)$ to configuration $\mathbb{X}^*(\cdot, t^*)$. By Theorem B1.4.4,

$$\underline{n}^* = \underline{Q} \underline{n},$$

a result which was established by considering the mapping of the end points of $\underline{n}$ under the deformation $\underline{x} \mapsto \underline{x}^*$ above.

In general, for any vector $\underline{u}$ that is carried along with the body to $\underline{u}^*$ in the rigid deformation $\underline{x} \mapsto \underline{x}^* = \underline{c} + \underline{Q}(\underline{x} - \underline{x}^*)$ which connects the configurations $\mathbb{X}(\cdot, t)$ and $\mathbb{X}^*(\cdot, t^*)$, we have

$$\underline{u}^* = \underline{Q} \underline{u}, \quad \underline{u} = \underline{Q}^T \underline{u}^*.$$

Now the unit vector $\underline{n}$ is arbitrary, and $\therefore$ the unit vector $\underline{n}^* = \underline{Q} \underline{n}$ also may be viewed as arbitrary; for given $\underline{n}^*$, $\underline{n} = \underline{Q}^T \underline{n}^*$ is the corresponding $\underline{n}$. Hence, the tensors $\underline{T}$ and $\underline{Q} \underline{T} \underline{Q}^T$ agree on the set of all unit vectors, and we conclude from Theorem A3.1.2 that

$$\underline{T}^* = \underline{Q} \underline{T} \underline{Q}^T.$$

Formally, we adopt the following axiom.

The Principle of Material Indifference.¹ The stress tensors $\underline{T}$ and $\underline{T}^*$ corresponding to two motions $\{\chi(\cdot; t) : t \in \mathbb{R}\}$ and $\{\chi^*(\cdot; t) : t \in \mathbb{R}\}$ of a body $\mathcal{B}$ which are equivalent w.r.t. stress response are related through

$$(MI) \quad \underline{T}_m^*(X, \underline{t}) = \underline{Q}(t) \underline{T}_m(X, t) \underline{Q}^T(t) \quad \forall (X, t) \in \mathcal{B} \times \mathbb{R},$$

where

$$\chi^*(X, \underline{t}) = \underline{c}(t) + \underline{Q}(t) [\chi(X, t) - \underline{x}], \quad \underline{t} = t - a$$

¹Often, Material Frame-Indifference (TRUESDALL and NOLL, 1965), Material Objectivity (NOLL, 1958), Observer Independence of Material Response (GURTIN, 1981); rarely, Isotropy of Space (NOLL, 1955).

in accordance with Theorem B4.2.1.

In our approach to Material Indifference (following NOLL, 1955), we imagine two equivalent motions of a body and postulate that the stress is the same in both of them apart from the orientation induced by the rigid motion that connects them. An alternative formulation (NOLL, 1958),¹ considers one motion of the body and postulates that two different observers see the same stress apart from the rotation induced by their different orientations. Of course, the two different observers correspond to the two equivalent motions. Evidently, Material Indifference was first used by CAUCHY^(without explanation), but it was not until after the 1955 thesis of NOLL that the Principle gained wide understanding. For an historical account see pp. 45-47 of TRUESDELL and NOLL's NFM.

Material Indifference is a general principle of continuum mechanics in the sense that we have not used any particular class of materials in its formulation; i.e., we have not made any constitutive assumptions. A characteristic of modern continuum mechanics is the explicit adjustment of constitutive equations so that the Principle of Material Indifference is automatically obeyed by the stresses that they produce.² We

¹ See § B4.3 and also GURTIN's text.

² All of the successful older particular theories of continuum mechanics enjoy this property.

work this out next for simple materials.

By Definition B4.1.2, the constitutive equation for a simple body gives the stress

$$(SM) \quad \underline{T}_m(X, t) = \underline{y}_{\sim n}^{X, t}(\underline{F}_{\sim n}^t(\kappa(X), \cdot)), \quad (X, t) \in \mathcal{B} \times \mathbb{R} \quad (*)$$

for the motion $\{\kappa(\cdot, t) : t \in \mathbb{R}\}$. For the equivalent motion $\{\kappa^*(-, t) : t \in \mathbb{R}\}$ defined by

$$\kappa^*(X, \tilde{t}) = c(t) + Q(t) [\kappa(X, t) - \underline{x}], \quad \tilde{t} = t - a, \quad (**)$$

(SM) gives the stress

$$\underline{T}_{\sim m}^*(X, \tilde{t}) = \underline{y}_{\sim n}^{X, \tilde{t}}(\underline{F}_{\sim n}^{\tilde{t}}(\kappa(X), \cdot)), \quad (X, \tilde{t}) \in \mathcal{B} \times \mathbb{R}. \quad (\#)$$

In order to explore this, we need $\underline{F}_{\sim n}^{\tilde{t}}(\kappa(X), \cdot)$. By Definitions B1.13.4 and B1.13.1,

$$\underline{F}_{\sim n}^*(\mathbb{X}, \tilde{t}) = D_{\mathbb{X}} \kappa_{\sim n}^*(\mathbb{X}, \tilde{t}) \quad \text{and} \quad \kappa_{\sim n}^*(\mathbb{X}, \tilde{t}) = \kappa(\kappa^{-1}(\mathbb{X}), \tilde{t}).$$

Then by (**),

$$\kappa_{\sim n}^*(\mathbb{X}, \tilde{t}) = c(t) + Q(t) [\kappa(\kappa^{-1}(\mathbb{X}), t) - \underline{x}]$$

$$= c(t) + Q(t) [\kappa_{\sim n}(\mathbb{X}, t) - \underline{x}] \quad (\text{Definition B1.13.1});$$

and $\therefore$

$$\begin{aligned}
\tilde{F}_n^*(X, \tilde{t}) &= Q(t) D_1 \tilde{F}_n(X, t) \quad (\text{Exercise B4.2.2}) \\
&= Q(t) \tilde{F}_n(X, t) \quad (\text{Definition B1.13.4}) \\
&= Q(\tilde{t}+a) \tilde{F}_n(X, \tilde{t}+a) \quad (\tilde{t} = t-a).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\tilde{F}_n^{\tilde{t}}(X, s) &= Q(\tilde{t}-s+a) \tilde{F}_n(X, \tilde{t}-s+a) \quad (\text{Definition B4.1.1}) \\
&= Q(t-s) \tilde{F}_n(X, t-s) \quad (\tilde{t} = t-a) \\
&= Q^t(s) \tilde{F}_n^t(X, s) \quad (\text{Definition B4.1.1}).
\end{aligned}$$

Then (#) $\Rightarrow$

$$\tilde{T}_n^*(X, \tilde{t}) = \tilde{Y}_n^{X, \tilde{t}}(Q^t(\cdot) \tilde{F}_n^t(\mathcal{U}(X), \cdot)), \quad \tilde{t} = t-a \quad (\#\#).$$

On placing (\#\#) and (*) into (MI), we get

Theorem B4.2.2. The stress response of a simple body B satisfies the Principle of Material Indifference
iff the response function $\tilde{Y}_n^{X, t}$ meets

$$\tilde{Y}_n^{X, t-a}(Q^t(\cdot) \tilde{F}_n^t(\mathcal{U}(X), \cdot)) = Q(t) \tilde{Y}_n^{X, t}(\tilde{F}_n^t(\mathcal{U}(X), \cdot)) Q^T(t)$$

$$\forall a \in \mathbb{R}, \quad \forall Q \in C^{N+1}(\mathbb{R}, \text{Orth } V_3), \quad \forall X \in \mathcal{B}, \quad \forall t \in \mathbb{R},$$

and $\forall$ deformation gradient histories $\underline{F}_n^t(\mathbb{M}(X), \cdot)$.

This formidable looking functional equation actually can be solved for $\underline{Y}_n^{X,t}$. In order to do so, we arbitrarily fix the particle $X \in \mathcal{B}$ and the time $t \in \mathbb{R}$ and then make special choices of $\underline{a}$ and $\underline{Q}$.

The first special choice is $\underline{a} = t$ (Remember t has been fixed.) and $\underline{Q} \equiv \underline{I}$. This yields

$$\underline{Y}_n^{X,0}(\underline{F}_n^t(\mathbb{M}(X), \cdot)) = \underline{Y}_n^{X,t}(\underline{F}_n^t(\mathbb{M}(X), \cdot)).$$

Hence, the response function $\underline{Y}_n^{X,t}$ cannot depend explicitly on the time t ,¹ and we can write the constitutive equation (SM) as

$$\underline{T}_m(X,t) = \underline{Y}_n^X(\underline{F}_n^t(\mathbb{M}(X), \cdot)), \quad (X,t) \in \mathcal{B} \times \mathbb{R},$$

and state the requirement of Theorem B4.2.2 as

$$\underline{Y}_n^X(\underline{Q}^T(t) \underline{F}_n^t(\mathbb{M}(X), \cdot)) = \underline{Q}(t) \underline{Y}_n^X(\underline{F}_n^t(\mathbb{M}(X), \cdot)) \underline{Q}^T(t), \quad (*)$$

Next we make a polar decomposition of $\underline{F}_n$,

¹ Nevertheless, in practice the response function is occasionally allowed to depend explicitly on time in an attempt to account for the phenomenon of aging in a purely mechanical theory. A fully adequate theory of aging would require nonmechanical considerations such as the wonders of chemistry.

and then we will be in position to make further intelligent choices of $\underline{Q}$. By Definition B1.13.5,

$$\underline{F}_n(\mathbb{X}, t) = \underline{R}_n(\mathbb{X}, t) \underline{U}_n(\mathbb{X}, t), \quad (\mathbb{X}, t) \in \mathbb{W}(\mathbb{B}) \times \mathbb{R},$$

where $\underline{R}_n(\mathbb{X}, t) \in \text{Orth } \mathcal{V}_3$ and $\underline{U}_n(\mathbb{X}, t) \in \text{Psym } \mathcal{V}_3$, and $\therefore$

$$\underline{F}_n^t(\mathbb{X}, s) = \underline{R}_n^t(\mathbb{X}, s) \underline{U}_n^t(\mathbb{X}, s).$$

We capitalize on the orthogonality of $\underline{R}_n$ by choosing

$$\underline{Q}(\tau) = \underline{R}_n^T(\mathbb{W}(X), \tau), \quad \tau \in \mathbb{R}. \quad)^1$$

In view of the differentiability of $\underline{R}_n$ (Theorem B1.13.8) and the transpose operation (Theorem A5.5.4), $\underline{Q}$ has the requisite smoothness (cf. Theorem B4.2.1) by the chain rule (Theorem A5.10.5). Then since $\underline{R}_n^T$ is defined by

$$\underline{R}_n^T(\mathbb{X}, t) = [\underline{R}_n(\mathbb{X}, t)]^T,$$

$$\underline{Q}^t(s) = [\underline{R}_n^t(\mathbb{W}(X), s)]^T;$$

and (*) becomes

$$\underline{J}_n^X(\underline{U}_n^t(\mathbb{W}(X), \cdot)) = \underline{R}_n^T(\mathbb{W}(X), t) \underline{J}_n^X(\underline{F}_n^t(\mathbb{W}(X), \cdot)) \underline{R}_n(\mathbb{W}(X), t).$$

¹ Recall that X has been fixed.

Solving for $\mathcal{Y}_n^X(F_n^T(\kappa(X), \cdot))$, we now can write the constitutive equation (SM) as

$$\underline{T}_m(X, t) = \mathcal{Y}_n^X(F_n^T(\kappa(X), \cdot)) = \underline{R}_n(\kappa(X), t) \mathcal{Y}_n^X(\underline{U}_n^T(\kappa(X), \cdot)) \underline{R}_n^T(\kappa(X), t)$$

Since the Principle of Material Indifference says that the stress response should be indifferent in a certain sense to rigid motions and since the rotation tensor $\underline{R}_n$ represents the rigid part of the deformation gradient, the above result may not be too startling. We do not find that the rotation drops out, but we do find exactly how it enters in.

If we wish to restrict $\underline{Q}$ to be proper orthogonal¹ and if $\underline{R}_n$ should be improper, we can get the above result by choosing $\underline{Q} = -\underline{R}_n^T$. Of course, if we were to insist on proper $\underline{Q}$, we probably would have decided earlier to insist on proper $\underline{R}_n$.

Because of the difficulties associated with calculating the tensors $\underline{R}_n$ and $\underline{U}_n$, we replace them in the above result as follows. First, we note that since $\underline{R}_n = \underline{F}_n \underline{U}_n^{-1}$,

$$\underline{T}_m(X, t) = \underline{F}_n(\kappa(X), t) \underline{U}_n^{-1}(\kappa(X), t) \mathcal{Y}_n^X(\underline{U}_n^T(\kappa(X), \cdot)) \underline{U}_n^{-1}(\kappa(X), t) \underline{F}_n^T(\kappa(X), t).$$

Defining

$$\hat{\mathcal{Y}}_n^X(\underline{U}_n^T(\kappa, \cdot)) = \underline{U}_n^{-1}(\kappa, t) \mathcal{Y}_n^X(\underline{U}_n^T(\kappa, \cdot)) \underline{U}_n^{-1}(\kappa, t),$$

¹cf. the remarks following the proof of Theorem B4.2.1.

we can write

$$\underline{T}_m(X, t) = \underline{F}_n(\kappa(X), t) \hat{\underline{Y}}_n^X(\underline{U}_n^t(\kappa(X), \cdot)) \underline{F}_n^T(\kappa(X), t).$$

Note that the values of $\hat{\underline{Y}}_n^X$ are symmetric tensors because of its definition in terms of $\underline{Y}_n^X$. Also note that the simplified constitutive equation immediately above produces symmetric stress tensors as it must.

Finally, since $\underline{U}_n = \sqrt{\underline{C}_n}$, ¹ we define

$$\bar{\underline{Y}}_n^X(\underline{C}_n^t(\#), \cdot) = \hat{\underline{Y}}_n^X(\sqrt{\underline{C}_n^t(\#), \cdot})$$

to get

$$\underline{T}_m(X, t) = \underline{F}_n(\kappa(X), t) \bar{\underline{Y}}_n^X(\underline{C}_n^t(\kappa(X), \cdot)) \underline{F}_n^T(\kappa(X), \cdot).$$

Again the values of $\bar{\underline{Y}}_n^X$ are symmetric tensors, and the stress tensor comes out to be symmetric.

Since $\underline{C}_n = \underline{F}_n^T \underline{F}_n$ ² is easy to calculate from the motion, we do not seek any further reductions. The above results are summarized in

Theorem B4.2.3. If the constitutive equation

$$(SM) \quad \underline{T}_m(X, t) = \underline{Y}_n^{X, t}(\underline{F}_n^t(\kappa(X), \cdot)), \quad (X, t) \in \mathcal{B} \times \mathbb{R}$$

^{1,2} Cf. Definition B1.13.5 and Theorem B1.5.1.

of a simple body $\mathcal{B}$ satisfies the Principle of Material Indifference, then it can be written in any one of the following reduced forms:

$$(SM-1) \quad \underline{T}_m(X, t) = \underline{R}_n(\kappa(X), t) \underline{Y}_n^X(\underline{U}_n^T(\kappa(X), \cdot)) \underline{R}_n^T(\kappa(X), t), (X, t) \in \mathcal{B} \times \mathbb{R};$$

$$(SM-2) \quad \underline{T}_m(X, t) = \underline{F}_n(\kappa(X), t) \underline{\hat{Y}}_n^X(\underline{U}_n^T(\kappa(X), \cdot)) \underline{F}_n^T(\kappa(X), t), (X, t) \in \mathcal{B} \times \mathbb{R};$$

$$(SM-3) \quad \underline{T}_m(X, t) = \underline{F}_n(\kappa(X), t) \underline{\bar{Y}}_n^X(\underline{U}_n^T(\kappa(X), \cdot)) \underline{F}_n^T(\kappa(X), t), (X, t) \in \mathcal{B} \times \mathbb{R};$$

where each of the three response functions $\underline{Y}_n^X$, $\underline{\hat{Y}}_n^X$, and $\underline{\bar{Y}}_n^X$ is on $C^{N-1}([0, \infty), \text{Psym } \mathcal{V}_3)$ to $\text{Sym } \mathcal{V}_3$. $N \geq 2$ is the smoothness index of (M3).

Conversely, the reduced constitutive equations (SM-1), (SM-2), and (SM-3) each satisfy the Principle of Material Indifference $\forall$ choices of the response functions $\underline{Y}_n^X$, $\underline{\hat{Y}}_n^X$, and $\underline{\bar{Y}}_n^X$.¹

Proof. Only the converse assertion remains to be proved. For any motion $\{\kappa(\cdot, t) : t \in \mathbb{R}\}$ of $\mathcal{B}$, we must show that if the equivalent motion $\{\kappa^*(\cdot, t) : t \in \mathbb{R}\}$ is defined by

¹Of course their domains and codomains must be $C^{N-1}([0, \infty), \text{Psym } \mathcal{V}_3)$ and $\text{Sym } \mathcal{V}_3$, respectively. Also they must be $\exists$ for any motion of $\mathcal{B}$ the spatial stress field $\underline{T}_a(\cdot, t)$ comes out to be smooth in $\mathcal{B}_t$ $\forall t \in \mathbb{R}$; cf. Theorem B4.1.1.

$$\mathbb{X}^*(X, \tilde{t}) = c(t) + Q(t) [\mathbb{X}(X, t) - \tilde{x}], \quad \tilde{t} = t - a,$$

then the corresponding stress tensors $\underline{T}$ and $\underline{\tilde{T}}$ as given by (SM-1), (SM-2), or (SM-3) are related through

$$\underline{\tilde{T}}_{nm}^*(X, \tilde{t}) = Q(t) \underline{T}_{nm}(X, t) Q^T(t).$$

We have already seen that

$$\underline{\tilde{F}}_n^*(\mathbb{X}, \tilde{t}) = Q(t) \underline{F}_n(\mathbb{X}, t) \quad \text{and} \quad \underline{\tilde{F}}_n^{*\tilde{t}}(\mathbb{X}, s) = Q^t(s) \underline{F}_n^t(\mathbb{X}, s).$$

Then since $\underline{F}_n = \underline{R}_n \underline{U}_n$ and $\underline{C}_n = \underline{U}_n^2 = \underline{F}_n^T \underline{F}_n$,

INSERT
material on
p. B4.2.17a

$$\underline{\tilde{C}}_n^{*\tilde{t}}(\mathbb{X}, s) = \underline{C}_n^t(\mathbb{X}, s), \quad)'$$

$$\underline{\tilde{U}}_n^{*\tilde{t}}(\mathbb{X}, s) = \sqrt{\underline{\tilde{C}}_n^{*\tilde{t}}(\mathbb{X}, s)} = \sqrt{\underline{C}_n^t(\mathbb{X}, s)} = \underline{U}_n^t(\mathbb{X}, s),$$

and

$$\underline{\tilde{R}}_n^*(\mathbb{X}, \tilde{t}) = \underline{\tilde{F}}_n^*(\mathbb{X}, \tilde{t}) \underline{\tilde{U}}_n^{*-1}(\mathbb{X}, \tilde{t}) = Q(t) \underline{F}_n(\mathbb{X}, t) \underline{U}_n^{-1}(\mathbb{X}, t) = Q(t) \underline{R}_n(\mathbb{X}, t).$$

With these results, (SM-1) gives

$$\underline{\tilde{T}}_{nm}^*(X, \tilde{t}) = \underline{\tilde{R}}_n^*(m(X), \tilde{t}) \underline{\mathcal{J}}_{nm}^{*\tilde{t}}(\underline{\tilde{U}}_n^{*\tilde{t}}(m(X), \cdot)) \underline{\tilde{R}}_n^{*T}(m(X), \tilde{t})$$

this property
Fields that enjoy this property are sometimes said to be materially
indifferent, objective, or observer independent.

$$\begin{aligned}
 \tilde{C}_n^*(X, \tilde{t}) &= [\tilde{F}_n^*(X, \tilde{t})]^T [\tilde{F}_n^*(X, \tilde{t})] \quad (\text{substitution}) \\
 &= [Q(t) \tilde{F}_n(X, t)]^T [Q(t) \tilde{F}_n(X, t)] \quad (\text{substitution from p. B4.2.11}) \\
 &= [\tilde{F}_n(X, t)]^T [Q(t)]^T Q(t) \tilde{F}_n(X, t) \quad (\text{Theorem A3.6.3 (iv)}) \\
 &= [\tilde{F}_n(X, t)]^T \tilde{F}_n(X, t) \quad (\text{Theorems A3.10.3 and A3.9.3}) \\
 &= \tilde{C}_n(X, t) \quad (\text{Definition B1.13.5}) \\
 &= \tilde{C}_n(X, \tilde{t} + a) \quad (\text{substitution}) ;
 \end{aligned}$$

whence

$$\begin{aligned}
 \tilde{C}_n^{* \tilde{t}}(X, s) &= \tilde{C}_n(X, \tilde{t} - s + a) \quad (\text{Definition B4.1.1}) \\
 &= \tilde{C}_n(X, t - s) \quad (\text{substitution})
 \end{aligned}$$

or

return to p. B4.2.17

$$\begin{aligned}
&= \underline{\underline{Q}}(t) \underline{\underline{R}}_n(\kappa(X), t) \prod_n^X (\underline{\underline{U}}_n^t(\kappa(X), \cdot)) \underline{\underline{R}}_n^T(\kappa(X), t) \underline{\underline{Q}}^T(t) \\
&= \underline{\underline{Q}}(t) \underline{\underline{T}}_m(X, t) \underline{\underline{Q}}^T(t).
\end{aligned}$$

Thus, (SM-1) satisfies the Principle of Material Indifference. In the same way, (SM-2) and (SM-3) may be shown to be indifferent; these demonstrations are left as Exercise B4.2.3. $\square$

The converse assertion of the above theorem tells us that even though we were lead to the reduced constitutive equation (SM-1) by special choices of the α and $\underline{\underline{Q}}$, additional special choices would not lead to further reductions because (SM-1) automatically satisfies the Principle of Material Indifference. The reductions from (SM-1) to (SM-2) and then to (SM-3) were based on kinematical considerations — not material indifference.

INSERT material on p. B4.2.18 a

Theorem B4.2.3 tells the whole story of the Principle of Material Indifference for simple materials. If the body at hand is not simple, then we must go back to the principle as it is stated on p. B4.2.8 and start over. E.g., consider the constitutive equation

$$\underline{\underline{T}}_a(x, t) = \hat{\underline{\underline{T}}}(\rho(x, t), \underline{\underline{v}}_a(x, t), x, t), \quad (x, t) \in \mathcal{I},$$

which we might have tentatively arrived at for a

For applications, the reduced constitutive equation (SM-3) is generally preferred because it is in terms of kinematical variables, F_n and $\underline{\Sigma}_n$, that are easy to calculate for any given motion. It will be of interest to know what happens to this reduced constitutive equation if the reference configuration is changed (cf. Theorem B4.1.2).

Theorem B4.2.4. If the reference configurations κ and μ of a simple body $\mathcal{B}$ are connected by the deformation $\mathbb{F} = \mu \circ \kappa^{-1}$,[†] then

$$F_{\sim \kappa}(\mathbb{X}, t) \bar{J}_{\sim \kappa}^x(C_{\sim \kappa}^t(\mathbb{X}, \cdot)) F_{\sim \kappa}^T(\mathbb{X}, t) = F_{\sim \mu}(\mathbb{Y}, t) \bar{J}_{\sim \mu}^x(C_{\sim \mu}^t(\mathbb{Y}, \cdot)) F_{\sim \mu}^T(\mathbb{Y}, t),$$

where

$$\mathbb{X} = \kappa(x), \quad \mathbb{Y} = \mu(x),$$

$$\bar{J}_{\sim \mu}^x(C_{\sim \mu}^t(\mathbb{Y}, \cdot)) = \underline{G}(\mathbb{X}) \bar{J}_{\sim \kappa}^x(\underline{G}^T(\mathbb{X}) C_{\sim \mu}^t(\mathbb{Y}, \cdot) \underline{G}(\mathbb{X})) \underline{G}^T(\mathbb{X}),$$

and

$$\underline{G} = D\mathbb{F}.$$

Proof. Exercise B4.2.4. $\square$

[†] Cf. (B1) of Definition B1.7.1.

glob of viscous stuff by the following reasoning. The glob appears to flow, and fluids commonly have the density ρ as an independent constitutive variable. Perhaps the viscous effects can be incorporated through dependence on the velocity $\underline{v}$ a la dashpots. The glob appears to be nonuniform; the variable $\underline{x}$ should allow for inhomogeneity. The glob is getting harder as time goes on; hence the explicit dependence on the time t .

As the first step to invoking the Principle of Material Indifference, we convert to the material description and write

$$\underline{T}_m(X, t) = \hat{\underline{T}}(\rho(\underline{x}(X, t), t), \underline{v}_m(X, t), \underline{x}(X, t), t).$$

Further, by the referential density equation (Theorem B2.1.5),

$$\underline{T}_m(X, t) = \hat{\underline{T}}\left(\frac{\rho_n(\underline{x}(X))}{J_n(\underline{x}(X), t)}, \underline{v}_m(X, t), \underline{x}(X, t), t\right) \quad (*)$$

For an equivalent motion $\{\underline{x}^*(\cdot, t) : t \in \mathbb{R}\}$ defined by

$$\underline{x}^*(X, \underline{t}^*) = \underline{c}(t) + \underline{Q}(t)[\underline{x}(X, t) - \underline{x}_0], \quad \underline{t}^* = t - a,$$

this constitutive equation delivers the stress

$$\tilde{J}_m^*(X, \tilde{t}) = \hat{J} \left(\frac{\rho_n(v(X))}{\tilde{J}_n(v(X), \tilde{t})}, \tilde{v}_m^*(X, \tilde{t}), \tilde{X}^*(X, \tilde{t}), \tilde{t} \right). (**)$$

We need the stated quantities on the r.h.s. of (**).

$$\tilde{J}_n^*(X, \tilde{t}) = |\det \tilde{F}_n^*(X, \tilde{t})| \quad (\text{definition})$$

$$= |\det [\tilde{Q}(t) \tilde{F}_n(X, t)]| \quad (p, B4.2.11)$$

$$= |\det \tilde{Q}(t) \det \tilde{F}_n(X, t)| \quad (\text{Theorem A3.8.4})$$

$$= |\pm \det \tilde{F}_n(X, t)| \quad (\text{Theorem A3.10.6})$$

$$= |\det \tilde{F}_n(X, t)| \quad (|\pm a| = |a|)$$

$$= J_n(X, t) \quad (\text{definition}).$$

Time differentiation of

$$\tilde{X}^*(X, t-a) = c(t) + \tilde{Q}(t) [\tilde{X}(X, t) - \tilde{x}]$$

$$\Rightarrow \tilde{X}_{,2}^*(X, t-a) = \dot{c}(t) + \dot{\tilde{Q}}(t) [\tilde{X}(X, t) - \tilde{x}] + \tilde{Q}(t) [\tilde{X}_{,2}(X, t)],$$

The justification of this step via the machinery of Chapters A5 and A6 is left as Exercise B4.2.5. Then

$$\tilde{v}_m^*(X, \tilde{t}) = \dot{c}(t) + \dot{\tilde{Q}}(t) [\tilde{X}(X, t) - \tilde{x}] + \tilde{Q}(t) \tilde{v}_m(X, t).$$

With these results $(**)$ becomes

$$\begin{aligned} \hat{\mathbb{T}}_{\sim m}^*(X, \hat{t}) = \hat{\mathbb{T}}_{\sim} \left(\frac{P_n(\mathbb{N}(X))}{J_n(\mathbb{N}(X), t)}, \dot{\mathbb{C}}(t) + \dot{\mathbb{Q}}(t) [\mathbb{X}(X, t) - \mathbb{X}^0] \right. \\ \left. + \mathbb{Q}(t) \mathbb{V}_m(X, t), \mathbb{C}(t) + \mathbb{Q}(t) [\mathbb{X}(X, t) - \mathbb{X}^0], t-a \right), \end{aligned}$$

so that (MI) requires

$$\begin{aligned} \hat{\mathbb{T}}_{\sim} \left(\frac{P_n(\mathbb{N}(X))}{J_n(\mathbb{N}(X), t)}, \dot{\mathbb{C}}(t) + \dot{\mathbb{Q}}(t) [\mathbb{X}(X, t) - \mathbb{X}^0] + \mathbb{Q}(t) \mathbb{V}_m(X, t), \right. \\ \left. \mathbb{C}(t) + \mathbb{Q}(t) [\mathbb{X}(X, t) - \mathbb{X}^0], t-a \right) \quad (\#) \\ = \mathbb{Q}(t) \hat{\mathbb{T}}_{\sim} \left(\frac{P_n(\mathbb{N}(X))}{J_n(\mathbb{N}(X), t)}, \mathbb{V}_m(X, t), \mathbb{X}(X, t), t \right) \mathbb{Q}^T(t) \end{aligned}$$

$$\forall \mathbb{C} \in C^N(\mathbb{R}, \mathbb{E}_3), \mathbb{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } \mathbb{V}_3), \mathbb{X}^0 \in \mathbb{E}_3, \text{ and } a \in \mathbb{R}.$$

To solve this, we first arbitrarily fix $X \in \mathbb{B}$ and $t \in \mathbb{R}$. Then choose $a = t$, $\mathbb{Q} \equiv \mathbb{I}$ on $\mathbb{R}$, and

$$\mathbb{C}(\tau) = \mathbb{X}^0 + [\mathbb{X}^0 - \mathbb{X}(X, t)] - (\tau - t) \mathbb{V}_m(X, t), \quad \tau \in \mathbb{R}$$

so that

$$\mathbb{C}(t) = \mathbb{X}^0 + [\mathbb{X}^0 - \mathbb{X}(X, t)]$$

and

$$\dot{\mathbb{C}}(t) = -\mathbb{V}_m(X, t).$$

For these choices $(\#) \Rightarrow$

$$\begin{aligned} \hat{\mathbb{T}}_{\sim} \left(\frac{\rho_n(\kappa(X))}{J_n(\kappa(X), t)}, \mathbf{0}, \dot{\mathbf{x}}, 0 \right) \\ = \hat{\mathbb{T}}_{\sim} \left(\frac{\rho_n(\kappa(X))}{J_n(\kappa(X), t)}, v_m(X, t), \mathbb{X}(X, t), t \right), \end{aligned}$$

which on return to the spatial description becomes

$$\hat{\mathbb{T}}_{\sim}(\rho(\mathbf{x}, t), v_s(\mathbf{x}, t), \mathbf{x}, t) = \hat{\mathbb{T}}_{\sim}(\rho(\mathbf{x}, t), \mathbf{0}, \dot{\mathbf{x}}, 0).$$

Thus, the Principle of Material Indifference forces the dependence on the velocity, the place, and the time to drop out, and our original constitutive assumption collapses to

$$\mathbb{T}_s(\mathbf{x}, t) = \bar{\mathbb{T}}(\rho(\mathbf{x}, t)), \quad (\mathbf{x}, t) \in \mathcal{I}. \quad (\#\#).$$

Despite this major reduction, the implications of material indifference have not been exhausted. In view of the simplicity of ($\#\#$), we will not bother to convert to the material description for the rest of this study. From the analysis on p. B4.2.20, the density corresponding to the equivalent motion $\{\mathbb{X}^*(; t) : t \in \mathbb{R}\}$ is

$$\begin{aligned} \check{\rho}(\check{\mathbf{x}}, \check{t}) &= \check{\rho}(\mathbb{X}^*(X, \check{t}), \check{t}) = \frac{\rho_n(\kappa(X))}{J_n^*(\kappa(X), \check{t})} = \frac{\rho_n(\kappa(X))}{J_n(\kappa(X), t)} \\ &= \rho(\mathbb{X}_n(X, t), t) = \rho(\mathbf{x}, t). \end{aligned}$$

Hence,

$$\mathbb{T}_{\sim}^*(\check{\mathbf{x}}, \check{t}) = \bar{\mathbb{T}}(\check{\rho}(\check{\mathbf{x}}, \check{t})) = \bar{\mathbb{T}}(\rho(\mathbf{x}, t)) = \mathbb{T}_s(\mathbf{x}, t).$$

But the Principle of Material Indifference requires that

$$\overset{*}{\underset{\sim}{T}}_s(\overset{*}{x}, \overset{*}{t}) = \underline{Q}(t) \underset{\sim}{T}_s(x, t) \underline{Q}^T(t),$$

and $\therefore$

$$\underline{Q}(t) \tilde{T}(\rho(x, t)) \underline{Q}^T(t) = \tilde{T}(\rho(x, t)) \quad \forall \underline{Q} \in C^{N-1}(\mathbb{R}, \text{Orth } V_3)$$

In particular, on restricting our attention to constant $\underline{Q}$'s and calling

$$\tilde{T}(\rho(x, t)) = \underline{S},$$

we must have

$$\underline{Q} \underline{S} \underline{Q}^T = \underline{S} \quad \forall \underline{Q} \in \text{Orth } V_3.$$

Thus, $\underline{S}$ is an isotropic symmetric tensor, and by Theorem A8.1.2 $\exists \sigma \in \mathbb{R} \exists$

$$\underline{S} = \sigma \underline{I}.$$

Application of this result to the case at hand yields

$$\bar{\mathbf{T}}(\rho(\mathbf{x}, t)) = p(\rho(\mathbf{x}, t)) \bar{\mathbf{I}}$$

for some function $p: \mathbb{R}^+ \rightarrow \mathbb{R}$. Hence, the constitutive equation (##) reduces to

$$\mathbf{T}_s(\mathbf{x}, t) = p(\rho(\mathbf{x}, t)) \bar{\mathbf{I}}, \quad (\mathbf{x}, t) \in \mathcal{J}. \quad (\dagger)$$

It should now be obvious that the constitutive equation (†) automatically meets material indifference so that no further reduction is possible.

Whether or not the constitutive equation (†) adequately describes the stress response of our glob of stuff cannot be answered without considerable additional effort. To begin with we would need to design experiments for the

determination of the response function p .

Expressing (†) in the material description, we get

$$\underline{T}_{im}(X, t) = p \left(\frac{\rho_n(\kappa(X))}{|\det \underline{F}_n(\kappa(X), t)|} \right) \underline{I}, \quad (X, t) \in \mathbb{B} \times \mathbb{R},$$

which makes it clear that this reduced constitutive equation describes a special kind of simple material. In the older literature, it would be called a perfect EULERIAN fluid; in the newer literature it is referred to as an elastic fluid. In the latter context, we shall meet it again in Chapter C2.

Our considerations of the above example are summarized in

Theorem B4.2.6. If the constitutive equation

$$\underline{T}_{\alpha}(\mathbf{x}, t) = \hat{\underline{T}}(\rho(\mathbf{x}, t), \underline{v}_{\alpha}(\mathbf{x}, t), \mathbf{x}, t), \quad (\mathbf{x}, t) \in \mathcal{I}$$

satisfies the Principle of Material Indifference, then it can be written in the reduced form

$$\underline{T}_{\alpha}(\mathbf{x}, t) = p(\rho(\mathbf{x}, t)) \underline{I}, \quad (\mathbf{x}, t) \in \mathcal{I}.$$

Conversely, the reduced constitutive equation satisfies the Principle of Material Indifference $\forall$ choices of the

response function ρ .

From this point forward, we assume without further comment in general that the Principle of Material Indifference holds.

One of the results of our analysis of this example that will be useful in connection with the consideration of various other material models is that as far as the Principle of Material Indifference is concerned, one can always include the mass density ρ as a variable in the constitutive equation for the stress. On the other hand, the velocity $\mathbf{v}$, the current position $\mathbf{x}$, and the current time t are not suitable variables.

Supplementary Reading

Same as for § B4.1

B4.4. Material Indifference in the Relative Description

The reduced forms of the general constitutive equation given in Theorem B4.2.3 are not well suited to the case where the simple material is a fluid. This is because of the significant way that the fixed reference configuration $\mathcal{K}$ enters into the formulation. Hence, it is natural to attempt to formulate the theory in terms of the relative deformation where the configuration at the present time t is used as the reference configuration.

We start by making a polar decomposition of each factor on the r.h.s. of

$$\underline{F}_{\mathcal{K}}(\mathcal{X}, \tau) = \underline{F}_{(t)}(\mathcal{X}, \tau) \underline{F}_{\mathcal{K}}(\mathcal{X}, t) \quad)^2$$

to get

$$\underline{F}_{\mathcal{K}}(\mathcal{X}, \tau) = \underline{R}_{(t)}(\mathcal{X}, \tau) \underline{U}_{(t)}(\mathcal{X}, \tau) \underline{R}_{\mathcal{K}}(\mathcal{X}, t) \underline{U}_{\mathcal{K}}(\mathcal{X}, t) . \quad)^3$$

Then on defining

$$\hat{\underline{U}}_{(t)}(\mathcal{X}, \tau) = \underline{R}_{\mathcal{K}}^T(\mathcal{X}, t) \underline{U}_{(t)}(\mathcal{X}, \tau) \underline{R}_{\mathcal{K}}(\mathcal{X}, t) ,$$

so that

$$\underline{U}_{(t)}(\mathcal{X}, \tau) \underline{R}_{\mathcal{K}}(\mathcal{X}, t) = \underline{R}_{\mathcal{K}}(\mathcal{X}, t) \hat{\underline{U}}_{(t)}(\mathcal{X}, \tau) ;$$

¹ See Definition B1.16.1.

² See Theorem B1.16.1.

³ See Definitions B1.13.5 and B1.16.3.

we have

$$F_n(\mathbb{X}, \tau) = R_{\sim(t)}(\mathbb{X}, \tau) R_n(\mathbb{X}, t) \hat{U}_{(t)}(\mathbb{X}, \tau) U_{\sim n}(\mathbb{X}, t).$$

Setting $\tau = t - s$ and using the history notation of Definition B4.1.1, we get

$$F_n^t(\mathbb{X}, s) = R_{\sim(t)}^t(\mathbb{X}, s) R_n(\mathbb{X}, t) \hat{U}_{\sim(t)}^t(\mathbb{X}, s) U_{\sim n}(\mathbb{X}, t).$$

Next we substitute this expression for the history of F_n into the general constitutive equation

$$T_{\sim m}(X, t) = \mathcal{J}_{\sim x}^X (F_n^t(\mathbb{X}, \cdot)) \quad)^1$$

to get

$$T_{\sim m}(X, t) = \mathcal{J}_{\sim x}^X (R_{\sim(t)}^t(\mathbb{X}, \cdot) R_n(\mathbb{X}, t) \hat{U}_{\sim(t)}^t(\mathbb{X}, \cdot) U_{\sim n}(\mathbb{X}, t)),$$

where, of course, it is understood that

$$\mathbb{X} = \mathbb{X}(X, t) \text{ and } \mathbb{X} = \mathbb{X}(X).$$

The Principle of Material Indifference then requires that²

$$Q(t) T_{\sim m}(X, t) Q^T(t) = \mathcal{J}_{\sim x}^X (Q^t(\cdot) R_{\sim(t)}^t(\mathbb{X}, \cdot) R_n(\mathbb{X}, t) \hat{U}_{\sim(t)}^t(\mathbb{X}, \cdot) U_{\sim n}(\mathbb{X}, t)) (*)$$

$\forall Q \in C^{N-1}(\mathbb{R}, \text{Orth } V_3)$. To "solve" this functional requirement, we arbitrarily fix $(X, t) \in \mathcal{B} \times \mathbb{R}$ and choose Q through

$$Q(\sigma) = R_{\sim n}^T(\mathbb{X}, t) R_{\sim(t)}^T(\mathbb{X}, \sigma),$$

^{1,2} See p. B4.2.12 and the preceding discussion.

so that

$$\underline{Q}(t) = \underline{R}_n^T(\underline{x}, t) \quad)^1$$

and

$$\underline{Q}^t(s) = \underline{R}_n^T(\underline{x}, t) \underline{R}_{(t)}^{tT}(\underline{x}, s)$$

Then (*) becomes

$$\underline{R}_n^T(\underline{x}, t) \underline{T}_m(\underline{x}, t) \underline{R}_n(\underline{x}, t) = \int_{\underline{R}_n}^X (\hat{\underline{U}}_{(t)}^t(\underline{x}, \cdot) \underline{U}_n(\underline{x}, t)) .$$

This, because of Material Indifference, the constitutive equation for a simple material can be written in the form

$$\underline{T}_m(\underline{x}, t) = \underline{R}_n(\underline{x}, t) \int_{\underline{R}_n}^X (\hat{\underline{U}}_{(t)}^t(\underline{x}, \cdot); \underline{U}_n(\underline{x}, t)) \underline{R}_n^T(\underline{x}, t) ,$$

where the semi-colon is used to emphasize that $\underline{U}_n$ enters only through its value at the present time t and not through its history prior to t .

As before², we can get rid of $\underline{R}_n$ by noting that $\underline{R}_n = \underline{F}_n \underline{U}_n^{-1}$. Also, since $\underline{C}_n = \underline{U}_n^2$, we can effectively replace $\underline{U}_n$ by $\underline{C}_n$; and, similarly, $\underline{U}_{(t)}$ can be replaced by $\underline{C}_{(t)}$. Hence, we are led to

Theorem B4.4.1. If the constitutive equation

¹ Here, we have used $\underline{R}_{(t)}(\underline{x}, t) = \underline{I}$ (see Theorem B1.16.5).

² See pp. B4.2.14 and 15.

$$(SM) \quad T_{\sim m}(X, t) = \underset{\sim n}{\mathcal{G}}^{X, t}_{\sim n}(\underset{\sim n}{F}^t(\underset{\sim n}{\mathcal{X}}(X), \cdot)) \quad , \quad (X, t) \in \mathcal{B} \times \mathbb{R}$$

of a simple body $\mathcal{B}$ satisfies the Principle of Material Indifference,
then it can be written in any of the following reduced forms:

$$(SM-4) \quad T_{\sim m}(X, t) = \underset{\sim n}{R}_n(\underset{\sim n}{\mathcal{X}}, t) \underset{\sim n}{\mathcal{H}}^X_{\sim n}(\underset{\sim n}{\hat{U}}^t_{\sim n(t)}(\underset{\sim n}{\mathcal{X}}, \cdot); \underset{\sim n}{U}_n(\underset{\sim n}{\mathcal{X}}, t)) \underset{\sim n}{R}_n^T(\underset{\sim n}{\mathcal{X}}, t);$$

$$(SM-5) \quad T_{\sim m}(X, t) = \underset{\sim n}{F}_n(\underset{\sim n}{\mathcal{X}}, t) \underset{\sim n}{\bar{\mathcal{H}}}^X_{\sim n}(\underset{\sim n}{\hat{U}}^t_{\sim n(t)}(\underset{\sim n}{\mathcal{X}}, \cdot); \underset{\sim n}{U}_n(\underset{\sim n}{\mathcal{X}}, t)) \underset{\sim n}{F}_n^T(\underset{\sim n}{\mathcal{X}}, t);$$

$$(SM-6) \quad T_{\sim m}(X, t) = \underset{\sim n}{R}_n(\underset{\sim n}{\mathcal{X}}, t) \underset{\sim n}{\hat{\mathcal{H}}}^X_{\sim n}(\underset{\sim n}{\hat{C}}^t_{\sim n(t)}(\underset{\sim n}{\mathcal{X}}, \cdot); \underset{\sim n}{C}_n(\underset{\sim n}{\mathcal{X}}, t)) \underset{\sim n}{R}_n^T(\underset{\sim n}{\mathcal{X}}, t);$$

$$(SM-7) \quad T_{\sim m}(X, t) = \underset{\sim n}{F}_n(\underset{\sim n}{\mathcal{X}}, t) \underset{\sim n}{\tilde{\mathcal{H}}}^X_{\sim n}(\underset{\sim n}{\hat{C}}^t_{\sim n(t)}(\underset{\sim n}{\mathcal{X}}, \cdot); \underset{\sim n}{C}_n(\underset{\sim n}{\mathcal{X}}, t)) \underset{\sim n}{F}_n^T(\underset{\sim n}{\mathcal{X}}, t);$$

where

$$\underset{\sim n}{\hat{U}}^t_{\sim n(t)}(\underset{\sim n}{\mathcal{X}}, s) = \underset{\sim n}{R}_n^T(\underset{\sim n}{\mathcal{X}}, t) \underset{\sim n}{U}^t_{\sim n(t)}(\underset{\sim n}{\mathcal{X}}, s) \underset{\sim n}{R}_n(\underset{\sim n}{\mathcal{X}}, t) \quad , \quad s \in [0, \infty),$$

$$\underset{\sim n}{\hat{C}}^t_{\sim n(t)}(\underset{\sim n}{\mathcal{X}}, s) = \underset{\sim n}{R}_n^T(\underset{\sim n}{\mathcal{X}}, t) \underset{\sim n}{C}^t_{\sim n(t)}(\underset{\sim n}{\mathcal{X}}, s) \underset{\sim n}{R}_n(\underset{\sim n}{\mathcal{X}}, t) \quad , \quad s \in [0, \infty),$$

$$\underset{\sim n}{\mathcal{X}} = \underset{\sim n}{\mathcal{X}}(X, t) \quad , \quad \underset{\sim n}{\mathcal{X}} = \underset{\sim n}{\mathcal{X}}(X) \quad ,$$

and where each of the four response functions $\underset{\sim n}{\mathcal{H}}^X_{\sim n}$, $\underset{\sim n}{\bar{\mathcal{H}}}^X_{\sim n}$, $\underset{\sim n}{\hat{\mathcal{H}}}^X_{\sim n}$
and $\underset{\sim n}{\tilde{\mathcal{H}}}^X_{\sim n}$ is on $C^{N-1}([0, \infty), \text{Psym } \mathcal{V}_3) \times \text{Sym } \mathcal{V}_3$ to $\text{Sym } \mathcal{V}_3$,
 $N \geq 2$ being the smoothness index of (M3).

Conversely, the reduced constitutive equations (SM-4)-(SM-7)
each satisfy the Principle of Material Indifference.

B4.4.5

Proof. Only the converse has yet to be established, and that constitutes Exercise B4.4.1. $\square$

Supplementary Reading

Same as for § B4.1.

B4.5. Material Symmetry

In general a transformation of an object which leaves some property of the object invariant is called a symmetry transformation. As an easy example of this concept, we first consider the symmetries of geometrical figures. Specifically, let us consider in-plane (proper) rotations of the square depicted below.



If the square is subjected to a clockwise rotation through 90° about a central axis perpendicular to the page,¹ it ends up appearing the same as before the rotation. Appearance here refers to the shape of the figure and its orientation on the page. Thus, the 90° rotation is a symmetry transformation — it leaves the appearance of the square invariant. Obviously, any rotation about a central axis perpendicular to the page through an integral multiple of 90° and either clockwise or counterclockwise is a symmetry transformation of the square w.r.t. appearance. Next, we note that these symmetry transformations of the square have some special properties.

Most importantly, the composition of any of these symmetry transformations is itself a symmetry transformation. E.g., the 180° rotation followed by the 90° rotation, both

¹ Cf. Definition B1.3.2 and Theorem B1.3.1.

clockwise, is just the clockwise 270° rotation. Thus, the operation of composition of symmetry transformations of the square is closed.

The composition of these symmetry transformations is also associative. E.g.,

$$\underbrace{\underbrace{[(90^\circ \text{cw}) \circ (90^\circ \text{cw})]}_{180^\circ \text{cw}} \circ (270^\circ \text{ccw})}_{90^\circ \text{ccw}} = (90^\circ \text{cw}) \circ \underbrace{\underbrace{[(90^\circ \text{cw}) \circ (270^\circ \text{ccw})]}_{180^\circ \text{ccw}}}_{90^\circ \text{ccw}}$$

The inverses of these symmetry transformations are also symmetry transformations. E.g., the inverse of the 90°cw rotation is just the 90°ccw rotation.

Finally, the identity transformation (i.e., do nothing to the square, or, if you prefer, rotate it through 0°) is a symmetry transformation.¹

On comparison of the above observations with Definition A1.3.1, we see that the set of all symmetry transformations of the square is a group w.r.t. composition.

¹For a complete discussion of the symmetry of the square along these lines, see Chpt. VI of BIRKHOFF and MAC LANE's A Survey of Modern Algebra.

Now let us carry this line of thought over to the consideration of symmetry transformations of material bodies. Bodies are available to us in configurations, and often several configurations of a body will be indistinguishable as far as material properties are concerned — just as certain placements of the square had the same appearance. In the context of continuum mechanics, a body in a configuration has only two material properties: its mass density in the given configuration and its stress response to motions from the given configuration. Thus, the material symmetry of a body is characterized by those density preserving changes of reference configuration which do not alter the response function)¹ — just as the symmetry of the square was described by the subgroup of plane rotations that did not alter its appearance.

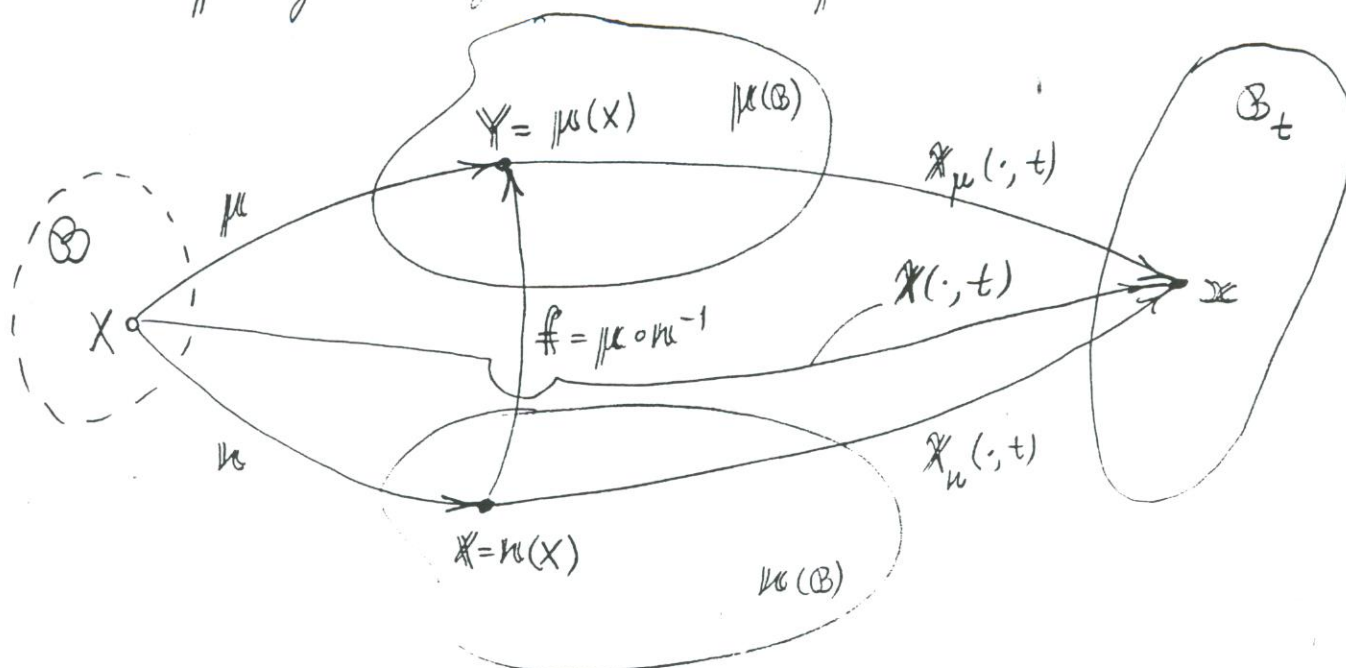
To be specific, consider a body B in a reference configuration κ . The mass density at the place $X = \kappa(X)$ in configuration κ is $\rho_\kappa(X)$, and the stress response at the particle X to time dependent deformations of the body from the configuration κ is given by

$$\underline{T}_m(X, t) = \underline{L}_m^X(\underline{F}_\kappa^t(\kappa(X), \cdot)) , \quad)^2$$

¹We restrict the discussion to simple materials (Definition B4.1.2).

²Nothing is gained at this point in our study of material symmetry by using the more cumbersome reduced forms of the constitutive equation (cf. Theorem B4.2.3); we do, however, drop the explicit time dependence.

Now suppose we change the reference configuration from κ to μ by the deformation $\mathbb{F} = \mu \circ \kappa^{-1}$.¹



By the density equation (Theorem B2.1.1), the mass density at the corresponding place $\Psi = \mu(X) = \mathbb{F}(\Xi)$ (i.e., corresponding to the same particle X) is

$$\rho_\mu(\Psi) = \frac{\rho_\kappa(\Xi)}{|\det \underline{G}(\Xi)|},$$

where $\underline{G}(\Xi) = D\mathbb{F}(\Xi)$. $\therefore$ for the mass density at $\Psi = \mu(X) = \mathbb{F}(\Xi)$ in the configuration μ of B to be the same as the mass density at $\Xi = \kappa(X)$ in the configuration κ of B , it is necessary and sufficient that

$$|\det \underline{G}(\Xi)| = 1.$$

¹G. Axian (B1) of Definition B1.7.1.

This leads us to

Definition B4.5.1. Given an n -dimensional linear space V_n ,
define

$$\text{Unim } V_n = \{ H \in \text{Lin } V_n : |\det H| = 1 \}$$

and

$$\text{Unim}^+ V_n = \{ H \in \text{Lin } V_n : \det H = 1 \}.$$

The elements of $\text{Unim } V_n$ and $\text{Unim}^+ V_n$ are called unimodular and
proper unimodular tensors, respectively.

Theorem B4.5.1. $\text{Unim } V_n$ and $\text{Unim}^+ V_n$ are groups w.r.t.
tensor multiplication, and $\text{Unim}^+ V_n$ is a proper subgroup of
 $\text{Unim } V_n$. $\text{Unim } V_n$ is called the unimodular group (of tensors
on V_n), while $\text{Unim}^+ V_n$ is called the proper unimodular group).¹

Proof. Exercise B4.5.1, $\square$

Theorem B4.5.2. The orthogonal group $\text{Orth } V_n$ ²
is a proper subgroup of the unimodular group $\text{Unim } V_n$,
and the proper orthogonal group $\text{Orth}^+ V_n$ is a proper
subgroup of the proper unimodular group $\text{Unim}^+ V_n$.

Proof. Exercise B4.5.2. $\square$

¹ Sometimes, special unimodular group.

² Cf. Definition A3.10.1 and Theorem A3.10.7.

Turning next to the stress response, we see from Theorem B4.1.2 that the response function at the particle X for the configuration μ is defined by

$$\mathcal{Y}_{\mu}^X(F_{\mu}^+(\mu(X), \cdot)) = \mathcal{Y}_{\mu}^X(F_{\mu}^+(\mu(X), \cdot) G(\mu(X))).$$

For the stress response at the particle X for motions out of the configuration μ to be the same as the stress response at the particle X for motions out of the configuration μ , it is necessary and sufficient that

$$\mathcal{Y}_{\mu}^X(\cdot) = \mathcal{Y}_{\mu}^X(\cdot), \quad (*)$$

which in view of the above requires that

$$\mathcal{Y}_{\mu}^X(F^+(\cdot) G(\mu(X))) = \mathcal{Y}_{\mu}^X(F^+(\cdot))$$

$\forall$ time-dependent deformation gradients F .

Putting the invariance of these two material properties together, we have

Definition B4.5.2. (NOLL, 1958) The material symmetry group¹ of the particle X of the simple body $\mathcal{B}$ relative to the reference configuration μ is the set

¹Often, isotropy group or peer group.

$$\mathcal{D}_n^X = \{ \underline{H} \in \text{Unim } \mathcal{V}_3 : \underline{g}_n^X(\underline{F} \underline{H}) = \underline{g}_n^X(\underline{F}) \quad \forall \underline{F} \in C^{N-1}([0, \infty), \text{Inv } \mathcal{V}_3) \},$$

where N is the smoothness index of Axiom (M3).

It is important to realize that the elements of $\mathcal{D}_n^X$ are the gradients at $\mathbb{X} = \mathbb{K}(X)$ of the deformations of the configuration $\mathbb{K}$ into the new configurations with the same material properties at the particle X . They are not the deformations themselves. It came out this way because both the new density and the new response function are determined by the gradient of the deformation connecting the new configuration to $\mathbb{K}$. Notice also that our considerations are all for the single particle X . Thus, $\underline{H} \in \mathcal{D}_n^X$ is not a field, but rather it is just the gradient at $\mathbb{X} = \mathbb{K}(X)$ of a deformation of $\mathbb{K}$ into a configuration with the same material properties at the particle X .

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As with the plane rotations of the square, the set of symmetry transformations of any object generally turns out to be a group w.r.t. composition of the symmetry transformations.

Theorem B4.5.3. The material symmetry group $\mathcal{D}_n^X$ is, indeed, a group w.r.t. tensor multiplication.¹

¹Of course, tensor multiplication is defined as composition (Definition A3.5.1). And, by the chain rule (Theorem A5.7.1) the gradient of the composition of two deformations is the composition of their gradients.

INSERT for p. B4.5.7

On p. A2.2.14, we noted that two mathematical structures are said to be isomorphic if there is a one-to-one correspondence between them which preserves their structure. In this sense, sometimes two reference configurations κ and μ of a simple body $\mathcal{B}$ are said to be materially isomorphic at a particle X if they have the same material properties at X , i.e., if

$$\rho_{\kappa}(\kappa(X)) = \rho_{\mu}(\mu(X)) \quad \text{and} \quad \mathcal{D}_{\kappa}^X = \mathcal{D}_{\mu}^X.$$

The isomorphism or one-to-one correspondence is simply the connecting deformation $\mathbb{F} = \mu \circ \kappa^{-1}$. In view of the above considerations, we can say that the two reference configurations κ and μ are materially isomorphic at X iff $D(\mu \circ \kappa^{-1}) \in \mathcal{D}_{\kappa}^X$.

Return to p. B4.5.7

Proof. Let $H_1, H_2 \in \mathcal{D}_n^X$. Then necessarily $H_1, H_2 \in \text{Unim } \mathcal{V}_3$, and since $\text{Unim } \mathcal{V}_3$ is a group w.r.t. tensor multiplication (Theorem B4.5.1), $H_1, H_2 \in \text{Unim } \mathcal{V}_3$. Hence, H_1, H_2 is a candidate for membership in $\mathcal{D}_n^X$. For arbitrary $F \in C^{N-1}([0, \infty), \text{Inv } \mathcal{V}_3)$, consider

$$\begin{aligned} \mathcal{Y}_n^X(F \underset{\sim}{(H_1, H_2)}) &= \mathcal{Y}_n^X((F \underset{\sim}{H_1}) \underset{\sim}{H_2}) \quad \left(\begin{array}{l} \text{associativity of} \\ \text{tensor mult.} \end{array} \right) \\ &= \mathcal{Y}_n^X(F \underset{\sim}{H_1}) \quad (H_2 \in \mathcal{D}_n^X)^1 \\ &= \mathcal{Y}_n^X(F) \quad (H_1 \in \mathcal{D}_n^X). \end{aligned}$$

$\therefore H_1, H_2 \in \mathcal{D}_n^X$, i.e. closure is satisfied.

The remaining group axioms (cf. Definition A1.3.1) are easy to check.

((Associativity))
(G1) $H_1, H_2, H_3 \in \mathcal{D}_n^X \Rightarrow H_1, H_2, H_3 \in \text{Unim } \mathcal{V}_3 \subset \text{Lin } \mathcal{V}_3$, and by the associativity of tensor multiplication (Theorem A3.5.1),

$$(H_1, H_2) \underset{\sim}{H_3} = H_1, (H_2, H_3).$$

(G2) (Identity element)²: Trivially, $I \in \mathcal{D}_n^X$.

¹Of course, here we are also using the fact that $F \underset{\sim}{H_1} \in C^{N-1}([0, \infty), \text{Inv } \mathcal{V}_3)$. Prove this as Exercise B4.5.3.

²Since here the combination operation is a multiplication, the neutral element is called the identity element.

$$(G3) (\text{Inverse element})': \underline{H} \in \mathcal{L}_n^X \Rightarrow \underline{H} \in \text{Unim } \mathcal{V}_3 \Rightarrow \\ \exists \underline{H}^{-1} \in \text{Unim } \mathcal{V}_3 \ni \underline{H}^{-1} \underline{H} = \underline{H} \underline{H}^{-1} = \underline{I}$$

since $\text{Unim } \mathcal{V}_3$ is a group. To show that $\underline{H}^{-1} \in \mathcal{L}_n^X$, let $\underline{F} \in C^{N-1}([0, \infty), \text{Inv } \mathcal{V}_3)$ and consider

$$\begin{aligned} \mathcal{Y}_n^X((\underline{F} \underline{H}) \underline{H}^{-1}) &= \mathcal{Y}_n^X(\underline{F} (\underline{H} \underline{H}^{-1})) \quad (\text{associativity of tensor mult.}) \\ &= \mathcal{Y}_n^X(\underline{F}) \quad (\underline{H} \underline{H}^{-1} = \underline{I}) \\ &= \mathcal{Y}_n^X(\underline{F} \underline{H}) \quad (\underline{H} \in \mathcal{L}_n^X), \end{aligned}$$

Set $\underline{F} \underline{H} = \hat{\underline{F}} \in C^{N-1}([0, \infty), \text{Inv } \mathcal{V}_3)$ (cf. Exercise B4.5.3) to get

$$\mathcal{Y}_n^X(\hat{\underline{F}} \underline{H}^{-1}) = \mathcal{Y}_n^X(\hat{\underline{F}}).$$

This holds $\forall \hat{\underline{F}} \in C^{N-1}([0, \infty), \text{Inv } \mathcal{V}_3)$, since given any such $\hat{\underline{F}}$, $\exists \underline{F} \in C^{N-1}([0, \infty), \text{Inv } \mathcal{V}_3)$, namely $\underline{F} = \hat{\underline{F}} \underline{H}^{-1}$, $\ni \underline{F} \underline{H} = \hat{\underline{F}}$, $\therefore \underline{H}^{-1} \in \mathcal{L}_n^X$. $\square$

The formulation of material symmetry is in terms of an underlying reference configuration $\mathcal{R}$, and it is natural to ask of the effect of using a different reference configuration μ in the formulation.

¹Again, since the combination operation is a multiplication, the reverse elements are called inverse elements.

Theorem B4.5.4. Let the reference configurations $\mathcal{U}$ and μ of a simple body $\mathcal{B}$ be connected by the deformation $\mathbb{F} = \mu \circ \mathcal{U}^{-1}$ and write $\underline{G} = D\mathbb{F}$. Then for any $X \in \mathcal{B}$,

$$\underline{H} \in \mathcal{D}_{\mu}^X \iff \underline{G}(X) \underline{H} \underline{G}^{-1}(X) \in \mathcal{D}_{\mu}^X, \quad)^2$$

where $X = \mathcal{U}(x)$,

Proof. Suppose $\underline{H} \in \mathcal{D}_{\mu}^X$. Then $\underline{H} \in \text{Unim } \mathcal{V}_3$, and

$$\det [\underline{G}(X) \underline{H} \underline{G}^{-1}(X)] = (\det \underline{G}(X)) (\det \underline{H}) (\det \underline{G}^{-1}(X)) \quad (\text{Theorem A3.8.4})$$

$$= \det \underline{H} \quad (\text{Theorem A3.9.10})$$

$$= \pm 1 \quad (\underline{H} \in \text{Unim } \mathcal{V}_3),$$

Thus, $\underline{G}(X) \underline{H} \underline{G}^{-1}(X) \in \text{Unim } \mathcal{V}_3$, and it is a candidate for membership in $\mathcal{D}_{\mu}^X$.

By Theorem B4.1.2,

$$\mathcal{D}_{\mu}^X(\underline{F}) = \mathcal{D}_{\mu}^X(\underline{F} \underline{G}(\mathcal{U}(X)))$$

¹ G. Axiom (B1) of Definition B1.7.1.

² Often, this is suggestively written as $\mathcal{D}_{\mu}^X = \underline{G}(X) \mathcal{D}_{\mu}^X \underline{G}^{-1}(X)$ and referred to as NOLL's rule.

for any $F \in C^{N-1}([0, \infty), \text{Inv } \mathcal{V}_3)$. In particular, for $F \in \mathcal{G}(\mathbb{X}) \mathcal{H} \mathcal{G}^{-1}(\mathbb{X})$,

$$\begin{aligned} \mathcal{Y}_\mu^X(F \mathcal{G}(\mathbb{X}) \mathcal{H} \mathcal{G}^{-1}(\mathbb{X})) &= \mathcal{Y}_n^X(F \mathcal{G}(\mathbb{X}) \mathcal{H} \mathcal{G}^{-1}(\mathbb{X}) \mathcal{G}(\mathbb{X})) \\ &= \mathcal{Y}_n^X(F \mathcal{G}(\mathbb{X}) \mathcal{H}) \quad (\mathcal{G} \mathcal{G}^{-1} = \mathcal{I}) \\ &= \mathcal{Y}_n^X(F \mathcal{G}(\mathbb{X})) \quad (\mathcal{H} \in \mathcal{D}_n^X) \\ &= \mathcal{Y}_\mu^X(F) \quad (\text{Theorem B4.1.2}). \end{aligned}$$

Thus, $\mathcal{G}(\mathbb{X}) \mathcal{H} \mathcal{G}^{-1}(\mathbb{X}) \in \mathcal{D}_\mu^X$.

Conversely, if $\mathcal{G}(\mathbb{X}) \mathcal{H} \mathcal{G}^{-1}(\mathbb{X}) \in \mathcal{D}_\mu^X$, then a similar argument shows that $\mathcal{H} \in \mathcal{D}_n^X$. The details are left as Exercise B4.5.4. $\square$

Exercise B4.5.5. Show that changes of reference configuration ($\kappa \rightarrow \mu$) defined by deformations of the form

$$\mathbb{F}(\mathbb{X}) = \mathring{\mathbb{Y}} + \alpha(\mathbb{X} - \mathring{\mathbb{X}}), \quad \alpha \neq 0, \quad \mathring{\mathbb{X}}, \mathring{\mathbb{Y}} \in \mathcal{E}_3$$

do not affect the material symmetry group of a simple body (i.e., $\mathcal{D}_\mu^X = \mathcal{D}_n^X$). Deformations of this type are called dilatations if $\alpha > 0$ and central inversions if $\alpha < 0$.

Orthogonal members of the material symmetry group are of major importance.)¹ For them, we can get special results by incorporating the Principle of Material Indifference,

Theorem B.4.5.5. Let $\underline{Q} \in \text{Orth } V_3$. Then $\underline{Q} \in \mathcal{D}_n^x$ iff

$$\underline{y}_n^x(\underline{Q}\underline{F}\underline{Q}^T) = \underline{Q}\underline{y}_n^x(\underline{F})\underline{Q}^T \quad \forall \underline{F} \in C^{N-1}([0, \infty), \text{Inv } V_3),$$

where N is the smoothness index of Axiom (M3).

Proof. Suppose that $\underline{Q} \in \mathcal{D}_n^x$. Then by Definition B4.5.2,

$$\underline{y}_n^x(\underline{F}) = \underline{y}_n^x(\underline{F}\underline{Q})$$

$\forall \underline{F} \in C^{N-1}([0, \infty), \text{Inv } V_3)$. In particular, for $\underline{Q}\underline{F}\underline{Q}^T$,

$$\begin{aligned} \underline{y}_n^x(\underline{Q}\underline{F}\underline{Q}^T) &= \underline{y}_n^x(\underline{Q}\underline{F}\underline{Q}^T\underline{Q}) \\ &= \underline{y}_n^x(\underline{Q}\underline{F}) \quad (\underline{Q}^T\underline{Q} = \underline{I}) \\ &= \underline{Q}\underline{y}_n^x(\underline{F})\underline{Q}^T \quad (\text{Material Indifference}). \end{aligned}$$

Conversely, suppose that

¹In fact, before the present theory was laid down by NOLL in 1958, only orthogonal elements of the material symmetry group were considered.

² Cf. Exercise B4.5.3. ³ Cf. Theorem B4.2.2. Of course, here the tensor $\underline{Q}$ is time-independent.

$$\underline{\mathcal{Y}}_n^x(\underline{Q} \underline{F} \underline{Q}^T) = \underline{Q} \underline{\mathcal{Y}}_n^x(\underline{F}) \underline{Q}^T \quad \forall \underline{F} \in C^{N-1}([0, \infty), \text{Inv } V_3).$$

Then by the Principle of Material Indifference,

$$\underline{\mathcal{Y}}_n^x(\underline{Q} \underline{F} \underline{Q}^T) = \underline{\mathcal{Y}}_n^x(\underline{Q} \underline{F})$$

$\forall \underline{F} \in C^{N-1}([0, \infty), \text{Inv } V_3)$. In particular, for $\underline{Q}^T \underline{F} \underline{Q}$, this becomes

$$\underline{\mathcal{Y}}_n^x(\underline{F}) = \underline{\mathcal{Y}}_n^x(\underline{F} \underline{Q}),$$

which $\Rightarrow \underline{Q} \in \mathcal{D}_n^x$ since $\underline{Q} \in \text{Unim } V_3$ ¹. $\square$

Sometimes in the formulation of the concept of material symmetry, the deformation $\#$ which connects the configuration $\mathbb{B}$ to the configuration $\# \circ \mathbb{B}$ with the same material properties is viewed as a one-parameter family of configurations with the body actually in $\mathbb{B}$ and $\# \circ \mathbb{B}$ for certain values of the parameter. Then the members of $\mathcal{D}_n^x$ must be proper unimodular tensors.)² The following ^(obvious) corollary of Theorem B4.5.5 shows that in so far as orthogonal elements of $\mathcal{D}_n^x$ are concerned, such worries turn out to be superfluous. This

¹ Cf. Theorem B4.5.2.

² Cf. Theorem B1.14.2.

is because the stress and the deformation gradient are second-order tensors. In theories involving odd-order tensors, e.g. heat conduction, such concerns are nonempty.

Theorem B4.5.6. Let $\underline{Q} \in \text{Orth } V_3$. Then

$$\underline{Q} \in \mathcal{I}_n^X \Rightarrow -\underline{Q} \in \mathcal{I}_n^X.$$

Before NOLL's fundamental work of 1958, material symmetry studies were based from the start on equations of the form

$$\underline{Y}_n^X(\underline{Q} \underline{F} \underline{Q}^T) = \underline{Q} \underline{Y}_n^X(\underline{F}) \underline{Q}^T$$

for $\underline{Q}$ in various subgroups of $\text{Orth } V_3$. The connection of this with the physical notion of material symmetry was somewhat hazy. It was usually said that rotating the strain by an element of the symmetry group rotated the stress in the same way. NOLL's approach leaves little doubt that the mathematical formulation of material symmetry does, indeed, correspond to the physical idea. The familiar formula above then appears as a theorem.

Also, NOLL's use of nonorthogonal transformations of the reference configuration made it possible for him to give precise definitions of the concepts of a material being fluid or solid. We consider this in the next section.

The two previous results are consequences of the Principle of Material Indifference, and as such they serve as a reminder that our formulation of material symmetry has been based on the constitutive equation

$$(SM) \quad \underline{T}_m(X, t) = \underline{Y}_n^X(\underline{F}_n^T(\underline{w}(X), \cdot)),$$

which does not fully account for Material Indifference. The most useful reduced form which automatically satisfies Material Indifference is ¹

$$(SM-3) \quad \underline{T}_m(X, t) = \underline{F}_n(\underline{w}(X), t) \underline{Y}_n^X(\underline{C}_n^T(\underline{w}(X), \cdot)) \underline{F}_n^T(\underline{w}(X), t).$$

By Definition B4.5.2, a unimodular tensor $\underline{H}$ belongs to the material symmetry group $\mathcal{D}_n^X$ iff

$$\underline{Y}_n^X(\underline{F} \underline{H}) = \underline{Y}_n^X(\underline{F}) \quad \forall \underline{F} \in C^{N-1}([0, \infty), \text{Inv } \mathcal{V}_3).$$

Since $\underline{C}_n = \underline{F}_n^T \underline{F}_n$, this requirement in terms of $\underline{Y}_n^X$ is

$$\underline{F}(t) \underline{H} \underline{Y}_n^X((\underline{F} \underline{H})^T (\underline{F} \underline{H})) (\underline{F}(t) \underline{H})^T = \underline{F}(t) \underline{Y}_n^X(\underline{C}) \underline{F}^T(t); \quad ^2$$

or since $\underline{F}$ is invertible,

$$\underline{H} \underline{Y}_n^X(\underline{H}^T \underline{C} \underline{H}) \underline{H}^T = \underline{Y}_n^X(\underline{C}).$$

¹ See Theorem B4.2.3.

² Cf. Theorem B4.2.4.

Since all of the above steps are reversible, we have established

Theorem B4.5.7. In terms of the reduced response function
 $\bar{\mathcal{G}}_n^X$, the material symmetry group of the particle X of the
simple body B relative to the reference configuration is given by

$$\mathcal{Q}_n^X = \{ \underline{H} \in \text{Unim } \mathcal{V}_3 : \underline{H} \bar{\mathcal{G}}_n^X (\underline{H}^T \underline{C} \underline{H}) \underline{H}^T = \bar{\mathcal{G}}_n^X (\underline{C}) \quad \forall \underline{C} \in C^{N-1}([0, \infty), \text{Psym } \mathcal{V}_3) \}$$

In particular, suppose that orthogonal $\underline{Q} \in \mathcal{Q}_n^X$. Then since $\mathcal{Q}_n^X$ is a group w.r.t. tensor multiplication, $\underline{Q}^{-1} \in \mathcal{Q}_n^X$. But by the orthogonality $\underline{Q}^{-1} = \underline{Q}^T$, and then Theorem B4.5.7 and the fact that $(\underline{Q}^T)^T = \underline{Q}$ yield the following companion to Theorem B4.5.5.

Theorem B4.5.8. Let $\underline{Q} \in \text{Orth } \mathcal{V}_3$. Then $\underline{Q} \in \mathcal{Q}_n^X$ iff

$$\bar{\mathcal{G}}_n^X (\underline{Q} \underline{C} \underline{Q}^T) = \underline{Q} \bar{\mathcal{G}}_n^X (\underline{C}) \underline{Q}^T \quad \forall \underline{C} \in C^{N-1}([0, \infty), \text{Psym } \mathcal{V}_3).$$

Since Theorem B4.5.7 fully incorporates the Principle of Material Indifference, the trivial nature of Theorem B4.5.3 should not have been unexpected.

While the detailed development of this section has been restricted to simple bodies, the same approach to material symmetry, i.e., invariance of material properties under a change of reference configuration, can be applied to any class of constitutive equations.

Supplementary Reading
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Same as for § B4.1