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Chapter 6The General Principles of Continuum Thermomechanics

In this chapter, we develop a general theory of continuum thermomechanics. We caution the reader that while our approach to the first law of thermodynamics is universally accepted, the same cannot be said for our approach (or anyone else's) to the second law of thermodynamics.

6.1. Introduction

It is only within the last few years that continuum thermodynamics has reached a stage of development that allows for a rational presentation comparable to our treatment of continuum mechanics. We will be introducing some new concepts such as heat flux, entropy, temperature, etc. These will be undefined primitives of the same rank as stress, mass, time, etc. Whenever you reach a mental block because you do not have a "feeling" for the thermodynamical notions and associated principles, ask yourself if you really understand force and balance of linear momentum. Our comfortable feeling about mechanics is due to familiarity and a knowledge of several centuries of successes with the theory.

Some comments on the modern literature are in order. The first three items on the supplementary reading list for this section are quite readable research papers which got the modern developments started. The 1964 paper by Coleman and Mizel is probably the best, although the 1963 paper by Coleman and Noll is the most significant in that it was first. We have made no explicit reference to the many papers that followed these three. Most of them are rather difficult because of the mathematics involved. Probably the best place to start is with the recent text

Rational Thermodynamics by Truesdell, As with most of Truesdell's expository works, this monograph is entertaining as well as informative.

It should be emphasized that continuum thermomechanics is in a rapid state of development. It can now be presented rationally and it has enjoyed numerous successes, but it may not be in its final form.

Supplementary Reading

COLEMAN and MIZEL, Thermodynamics and departures from Fourier's law of heat conduction, Archive for Rational Mechanics and Analysis 13, 1963.

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TRUESDELL, The Elements of Continuum Mechanics.

TRUESDELL and NOLL, AFTM.

TRUESDELL and TOUPIN, CFT.

Gurtin, Modern Cont. Thermodynamics in Mechanics Today, 1972

edit by S. Nemat-Nasser

(vol. 1)

6.2. Conservation of Energy. The First Law

The first law of thermodynamics is a statement of energy balance which calls for the introduction of three new thermodynamical concepts. We chose to do this in terms of the spatial variables $(\underline{x}, t)$.¹

For the remainder of this section, we consider a body $\mathcal{B}$ and a motion $\{\underline{\chi}(\cdot; t)\}$. As before, $\mathcal{P}$ denotes an arbitrary part² of $\mathcal{B}$; and $\mathcal{P}_t$ is the region of E_3 that $\mathcal{P}$ occupies at time t . The trajectory³ of $\mathcal{B}$ under $\{\underline{\chi}(\cdot; t)\}$ is denoted by $\mathcal{T}$.

We assume that

$\exists$ a scalar-valued function ε on $\mathcal{T} \ni$

$$\int_{\mathcal{P}_t} \varepsilon(\underline{x}, t) \rho(\underline{x}, t) d\underline{x} = E(\mathcal{P}_t)$$

difficultly with
internal body force
or internal radiation

also by used for
 $N(E)$
Euler's mass

represents the internal energy of part $\mathcal{P}$ at time t . The function ε is smooth in $\underline{x}$ and t . The scalar $\varepsilon(\underline{x}, t)$ is called the specific internal energy (per unit mass) at the point $\underline{x}$ at time t .

¹Definition 2.4.5.

²Definition 2.6.7.

³Definition 2.4.3.

In order to handle the flow of heat across surfaces in the body, we assume

For each unit vector $\underline{n}$, $\exists$ a scalar-valued function $q_{\underline{n}}$ on $\mathcal{P} \ni$

$$\int_{\partial \mathcal{P}_t} q_{\underline{n}(\underline{x},t)}(\underline{x},t) d\underline{x},$$

(E₂) where $\underline{n}(\underline{x},t)$ is the unit outward normal to $\partial \mathcal{P}_t$ at $\underline{x}$, represents the rate of energy flow into part $\mathcal{P}$ at time t due to heat conduction across $\partial \mathcal{P}_t$. For each $\underline{n}$ and t , the field $q_{\underline{n}}(\cdot, t)$ is smooth. The scalar $q_{\underline{n}}(\underline{x}, t)$ is called the heat flux at the point $\underline{x}$ on "the" surface through $\underline{x}$ with unit normal $\underline{n}$ at $\underline{x}$ at time t .

It is, of course, appropriate to think of $q_{\underline{n}}$ as the "surface heat supply per unit area per unit time".

It should be clear that the heat flux $q_{\underline{n}}$ is similar in concept to the stress vector $\underline{t}_{\underline{n}}$.¹ As in continuum mechanics, measure theory is the most appropriate vehicle for dealing with these concepts. Following Noll's approach to the stress vector, Gurtin and Williams² have shown that the heat

¹Cf., (F₁) on p. 2.11.2.

²Foundations of continuum thermodynamics, Arch. Rational Mech. Anal. 22, 1967.

flux can depend on no characteristic of the surface other than its normal.

Analogous to the body force, we assume that

$\exists$ a scalar-valued function r on $\mathcal{V} \ni$

$$\int_{\mathcal{P}_t} r(\underline{x}, t) \rho(\underline{x}, t) d\underline{x}$$

(E₃)

represents the rate of heat flow into part $\mathcal{P}$ at time t due to other bodies (than $\mathcal{B}$). For each t , the field $r(\cdot, t)$ is continuous. The scalar $r(\underline{x}, t)$ is called the extrinsic heat supply¹ (per unit mass and per unit time) at the point $\underline{x}$ at time t .

Combining (E₂) and (E₃), we have

Definition 6.2.1. The rate of energy flow into part $\mathcal{P}$ at time t due to thermal effects is

$$Q(\mathcal{P}_t) = \int_{\partial \mathcal{P}_t} \underline{q}_{\underline{n}(\underline{x}, t)}(\underline{x}, t) d\underline{x} + \int_{\mathcal{P}_t} r(\underline{x}, t) \rho(\underline{x}, t) d\underline{x}.$$

We also need two familiar² mechanical terms.

¹Often heat radiation.

²See § 2.11.

Definition 6.2.2. The kinetic energy of part P at time t is

$$K(P_t) = \frac{1}{2} \int_{P_t} |\hat{v}(\underline{x}, t)|^2 \rho(\underline{x}, t) d\underline{x}.$$

The power¹ of the forces on part P at time t is

$$P(P_t) = \int_{\partial P_t} \underline{t}_{\underline{n}(\underline{x}, t)} \cdot \hat{v}(\underline{x}, t) d\underline{x} + \int_{P_t} \underline{b}(\underline{x}, t) \cdot \hat{v}(\underline{x}, t) \rho(\underline{x}, t) d\underline{x}.$$

These quantities are related through the following axiom.

Principle of Conservation of Energy²

$$(T_1) \quad \frac{d}{dt} [K(P_t) + E(P_t)] = P(P_t) + Q(P_t)$$

$$\forall P \subset B \text{ and } \forall \text{ motions } \{K(\cdot, t)\}.$$

We note that an "equivalence" of heat and work is implicit in this axiom. We assume from now on that energy is balanced.

Get K and P out now before local arguments.

In order to establish the local form of (T_1) , we observe the analogy between the heat flux q_n and

¹ Often rate of work.

² Often principle of energy balance or first law of thermodynamics.

the stress vector $\underline{t}_n$. Thus we follow the developments in § 2.11 that lead from a global statement of balance of linear momentum to a local statement of momentum balance.

Applying balance of energy to a pill-box-shaped part leads to. Need polygonal pillbox if don't assume cont. dependence on $\underline{n}$

Theorem 6.2.1. For each unit vector $\underline{n}$ and $\forall (\underline{x}, t) \in \mathcal{I}$,

$$\underline{q}_{-\underline{n}}(\underline{x}, t) = -\underline{q}_{\underline{n}}(\underline{x}, t).$$

Proof. Exercise 6.2.1. $\square$

Next let $\{\underline{e}_{\langle i \rangle}\}$ be an orthonormal basis for $\mathcal{V}_3$ and introduce the scalars

$$q_{\langle i \rangle}(\underline{x}, t) = \underline{q}_{\underline{e}_{\langle i \rangle}}(\underline{x}, t).$$

Using this notation and applying (T_1) to a part shaped as a tetrahedron with sides normal to the base vectors and slant face normal to $\underline{n}$ yields (with the aid of Theorem 6.2.1)

Theorem 6.2.2. $\exists$ a vector-valued function $\underline{q}$ on $\mathcal{I}$
 $\ni$ for each unit vector $\underline{n}$

$$\underline{q}_{\underline{n}}(\underline{x}, t) = \underline{q}(\underline{x}, t) \cdot \underline{n} \quad \forall (\underline{x}, t) \in \mathcal{I},$$

Moreover, for each t , the field $\underline{q}(\cdot, t)$ is smooth on $\mathcal{B}_t$.
 $\underline{q}(\underline{x}, t)$ is called the heat flux vector at the place
 $\underline{x}$ at time t .¹

Proof. Exercise 6.2.2. $\square$

In terms of the heat flux vector, we can express the total heat flow as

$$Q(\mathcal{B}_t) = \int_{\partial \mathcal{B}_t} \underline{q}(\underline{x}, t) \cdot \underline{n}(\underline{x}, t) d\underline{x} + \int_{\mathcal{B}_t} r(\underline{x}, t) \rho(\underline{x}, t) d\underline{x};$$

} might be best to separate out as Thm.

or, using the divergence theorem,²

$$Q(\mathcal{B}_t) = \int_{\mathcal{B}_t} [\nabla \cdot \underline{q}(\underline{x}, t) + r(\underline{x}, t) \rho(\underline{x}, t)] d\underline{x}$$

As an immediate consequence of the reduced transport theorem,³ we have

$$\frac{d}{dt} E(\mathcal{B}_t) = \frac{d}{dt} \int_{\mathcal{B}_t} \varepsilon(\underline{x}, t) \rho(\underline{x}, t) d\underline{x} = \int_{\mathcal{B}_t} \dot{\varepsilon}(\underline{x}, t) \rho(\underline{x}, t) d\underline{x},$$

¹ The notation $\underline{q} = \underline{h}$ or $\underline{q} = -\underline{h}$ is also quite common.

² Theorem 1.23.1.

³ Exercise 2.10.5.

where the superposed dot, $(\dot{})$, denotes the material time derivative:

$$\dot{\varepsilon}(\underline{x}, t) = \frac{\partial \varepsilon}{\partial t}(\underline{x}, t) + \underline{\hat{v}}(\underline{x}, t) \cdot \nabla \varepsilon(\underline{x}, t).^1$$

Finally, in the present notation, the energy transfer theorem² states that

$$\frac{d}{dt} K(\mathcal{Q}_t) = P(\mathcal{Q}_t) - \int_{\mathcal{Q}_t} \text{tr} [\hat{\underline{T}}(\underline{x}, t) \hat{\underline{D}}(\underline{x}, t)] d\underline{x},$$

keep as $\hat{\underline{T}}$ at this point?

where $\underline{T}$ is the stress tensor and $\underline{D}$ is the symmetric part of the spatial velocity gradient,

Putting the above results into (T_1) yields

$$\int_{\mathcal{Q}_t} \left\{ \dot{\varepsilon}(\underline{x}, t) \rho(\underline{x}, t) - \text{tr} [\hat{\underline{T}}(\underline{x}, t) \hat{\underline{D}}(\underline{x}, t)] - \nabla \cdot \underline{q}(\underline{x}, t) - r(\underline{x}, t) \rho(\underline{x}, t) \right\} d\underline{x} = 0.$$

Since this must hold $\forall \mathcal{Q} \subset \mathcal{B}$ and since all of the steps are reversible, we have proven.

Theorem 6.2.3. (Local Energy Balance) Energy is conserved³ iff.

¹ Recall our second notational agreement (p. 2.4.6).

² Theorem 2.11.5.

³ i.e., (T_1) holds.

$$\nabla \cdot \hat{\underline{q}}(\underline{x}, t) + \rho(\underline{x}, t) \hat{r}(\underline{x}, t) + \tau_0 [\hat{T}(\underline{x}, t) \hat{D}(\underline{x}, t)] \\ = \rho(\underline{x}, t) \hat{\dot{E}}(\underline{x}, t)$$

$$\forall (\underline{x}, t) \in \mathcal{V}.$$

In Theorem 6.2.3, the superscript caret, $(\hat{\quad})$, emphasizes the use of the spatial variables.¹

Supplementary Reading

Same as for §6.1

¹ Recall our first notational agreement (p.2.4.5).

6.3. The Dissipation Inequality. The Second Law

First we need the concept of temperature.¹ Temperature measures how "hot" a body is, just as the notion of point tells us "where" a body is. Experience indicates that bodies cannot get too "cold", i.e., the temperature is bounded below. Taking this bound to be zero gives us an absolute temperature scale.

We assume that

(E₄) $\exists$ a strictly positive scalar-valued function Θ on $\mathcal{D} \ni \Theta(x, t) > 0$ is the absolute temperature at the place x at the time t . The function Θ is smooth in $\mathcal{D}$ and t together.

The second law of thermodynamics is usually put forth as representing a certain irreversibility in natural processes. For example, it is observed that while mechanical work can always be totally dissipated as heat energy, heat cannot always be totally converted into mechanical work; also while a body generally becomes hotter if we put energy (work and/or heat) into it, a corresponding cooling does not necessarily release this same

¹Temperature arises naturally as a derived quantity in the axiomatic development of Eshelby and Williams, *Foundations of Continuum Thermodynamics*, Arch. Rational Mech. Anal. 26, 1967.

amount of energy.

In order to proceed, we follow Tensesll's account¹ of C. ECKART'S motivation for the notion of entropy and the second law. We momentarily revert to homogeneous processes where all of the relevant fields are uniform over the body. The observation that there is a limit to the rate at which heat can be put into a body can be stated mathematically as

$$Q \leq B.$$

Since $\Theta > 0$, we can define a new quantity H (the entropy) by

$$H = \int_{t_0}^t \frac{B}{\Theta} dt.$$

Then

$$\dot{H} = \frac{B}{\Theta},$$

and $Q \leq B$ is equivalent to

$$\dot{H} \geq \frac{Q}{\Theta}.$$

This is the Clausius - Planck inequality; it is generally accepted as the appropriate statement of the second law of thermodynamics for homogeneous processes.

¹ Rational Thermodynamics, p. 11.

The "derivation" above is subject to the criticism that there are many upper bounds like B throughout continuum physics, but they seldom make an explicit appearance in the theory — only in the practice of the theory. For example, velocities in Newtonian mechanics are bounded by the speed of light; but the speed of light never appears in classical mechanics. On the other hand, the second law of thermodynamics attempts to model a physical notion vastly different from the usual balance principles with which we are more familiar.

With the Clausius-Planck inequality in mind, we assume that

$\exists$ a scalar-valued function η on $\mathcal{I} \ni$

$$(E_4) \quad \int_{P_t} \eta(\underline{x}, t) \rho(\underline{x}, t) d\underline{x} = H(P_t)$$

represents the entropy of part P at time t . The function η is smooth in $\underline{x}$ and t together. The scalar $\eta(\underline{x}, t)$ is called the specific entropy (per unit mass) at the point $\underline{x}$ at time t .

We generalize the Clausius-Planck inequality to general thermo-mechanical processes with the following axiom.

The Dissipation Inequality'

$$(T_2) \quad \frac{d}{dt} H(\mathcal{D}_t) \geq \int_{\mathcal{D}_t} \frac{q(x,t)}{\theta(x,t)} dx + \int_{\mathcal{D}_t} \frac{r(x,t)}{\theta(x,t)} \rho(x,t) dx$$

$$\forall \mathcal{D} \subset \mathcal{B} \text{ and } \forall \text{ motions } \{\underline{x}(\cdot, t)\}.$$

We note that putting the temperature under the integral signs in passing from the Clausius-Planck inequality to the dissipation inequality is similar to putting the lever arm under the integral sign in passing from the angular momentum theorem of particle mechanics to the angular momentum postulate of continuum mechanics.

Even though the dissipation inequality has led to numerous major successes and advances in continuum thermomechanics, it has not (yet?) met universal acceptance as the appropriate form of the second law of thermodynamics.

Exercise 6.3.1. An alternative (but not equivalent) generalization of the Clausius-Planck inequality is

¹Often the entropy inequality, the second law of thermodynamics, the Clausius-Duhem inequality, or the Clausius-Duhem-Toupin inequality.

$$\rho \dot{\eta} \geq \frac{\nabla \cdot \tilde{q}}{\theta} + \rho \frac{\dot{\theta}}{\theta}.$$

In what sense is this a generalization of $\dot{H} \geq \frac{Q}{\theta}$?

Theorem 6.3.1. (Local Dissipation Inequality) The dissipation inequality (T_2) is satisfied iff

$$\rho(\underline{x}, t) \dot{\hat{\eta}}(\underline{x}, t) \geq \nabla \cdot \left(\frac{\hat{\tilde{q}}(\underline{x}, t)}{\hat{\theta}(\underline{x}, t)} \right) + \rho(\underline{x}, t) \frac{\hat{\dot{\theta}}(\underline{x}, t)}{\hat{\theta}(\underline{x}, t)} \quad \forall (\underline{x}, t) \in J,$$

where $\dot{\hat{\eta}}(\underline{x}, t) = \frac{\partial \hat{\eta}}{\partial t}(\underline{x}, t) + \hat{\underline{v}}(\underline{x}, t) \cdot \nabla \hat{\eta}(\underline{x}, t)$ is the material time derivative.

Proof, Exercise 6.3.2. $\square$

It is sometimes convenient to combine the local dissipation inequality with local energy balance¹ to obtain

Theorem 6.3.2. Granting local energy balance, the local dissipation inequality is equivalent to

$$\rho(\theta \dot{\eta} - \dot{\epsilon}) + \text{tr}(\underline{T} \underline{D}) + \frac{1}{\theta} \tilde{q} \cdot \nabla \hat{\theta} \geq 0 \quad ?^2$$

Proof, Exercise 6.3.3. $\square$

keep as $\hat{\eta}$ at this stage

¹Theorem 6.2.3.

²As usual, the superposed dots denote material time derivatives.

It is often mathematically convenient to replace the specific internal energy ε by the specific free energy ψ as defined below, when using θ as independent thermodynamical variable — Legendre transformation

Definition 6.3.1. The specific free energy is

$$\psi = \varepsilon - \theta \eta.$$

Since $\dot{\psi} = \dot{\varepsilon} - \dot{\eta}\theta - \varepsilon\dot{\theta}$,¹ it follows from Theorems 6.2.3 and 6.3.2 that local energy balance and local dissipation can be expressed in terms of the free energy as given below.

Theorem 6.3.3. The local energy is balanced iff

$$\nabla \cdot \hat{q} + \rho r + \text{tr}(\underline{T} \underline{D}) = \rho (\dot{\psi} + \dot{\theta} \eta + \theta \dot{\eta}).$$

Wanting local energy balance, the local dissipation inequality is equivalent to

$$-\rho (\dot{\psi} + \eta \dot{\theta}) + \text{tr}(\underline{T} \underline{D}) + \frac{1}{\theta} \hat{q} \cdot \nabla \theta \geq 0.$$

The inequalities of Theorems 6.3.2 and 6.3.3 are sometimes called reduced local dissipation inequalities. They are "reduced" by taking into account local energy balance.

The major contribution of Coleman and Noll in continuum thermomechanics was their utilization of the local dissipation

¹ Show this using the spatial description as Exercise 6.3.4.

inequality as a restriction on constitutive equations. In preparation for this in subsequent chapters, we make the following definition.

Definition 6.3.2. Given a body $\mathcal{B}$,¹ the set of eight functions

- 1) $\underline{\chi}$ --- the motion,
- 2) $\underline{T}$ --- the stress tensor,
- 3) $\underline{b}$ --- the body force,
- 4) $\underline{\varepsilon}$ --- the specific internal energy (or $\underline{\psi}$ --- the specific free energy),
- 5) $\underline{q}$ --- the heat flux vector,
- 6) $\underline{r}$ --- the extrinsic heat supply,
- 7) θ --- the absolute temperature,
- 8) η --- the specific entropy.

is said to constitute a thermomechanical process if all of our smoothness hypotheses are satisfied and if the linear momentum is balanced, the angular momentum is balanced, and the energy is balanced.

Exercise 6.3.5. Roughly speaking, the term adiabatic in thermodynamics refers to "no heat flow". In order to be precise, we consider three variations and describe them

¹It is understood that the mass density ρ is part of the prescription of the body $\mathcal{B}$ and that mass is conserved.

in rather nonuniversal terminology:

$$\underline{\dot{q}} = 0, r = 0 \quad \dots \text{completely adiabatic};$$

$$Q(\mathcal{B}_t) = 0 \quad \dots \text{globally adiabatic};$$

$$\nabla \cdot \underline{\hat{q}} + pr = 0 \quad \dots \text{locally adiabatic}.$$

Show that a completely adiabatic thermomechanical process is compatible with the second law of thermodynamics¹ iff $\tilde{\eta}$ [$\tilde{\eta}(X, t) = \hat{\eta}(\underline{x}(X, t), t)$] is a monotone non-decreasing function of t for each particle $X \in \mathcal{B}$, i.e., the entropy does not decrease in a completely adiabatic process,

Supplementary Reading

Same as for §6.1

¹i.e., (T_2) is satisfied.

Chapter 7Thermoelasticity

The full implications of the second law of thermodynamics cannot be realized without consideration of the constitutive equations for the body under study. The thermomechanics of simple materials that have a memory of the past is mathematically very difficult. In this chapter, we see how the second law works by considering the much easier case of elastic bodies.

7.1. Constitutive Assumptions. Equipresence

Now that we are taking thermal effects into account, we are faced with having to write constitutive equations not only for the stress but also for the heat flux, the entropy, etc. To help us do this, we employ Truesdell's

Principle of Equipresence

An independent variable present in one constitutive equation of a body should be present in all unless its presence is ruled out by material symmetry, material indifference, or the basic laws.

Actually, "principle" is too strong a name to attach to the notion of equipresence. It is really intended as a guide for theorists in constructing constitutive equations. It is just a formal way of saying that if you do not have a good reason for leaving a variable out of a constitutive equation, it is safer to leave it in.

In accordance with equipresence, we make the following constitutive assumptions for thermoelasticity.

Definition 7.1.1. A body $\mathcal{B}$ is said to be thermoelastic with respect to the reference configuration κ if for each particle $x \in \mathcal{B} \exists$ response functions $\bar{T}_x^x, \bar{\Psi}_x^x, \bar{\eta}_x^x,$

and $\bar{f}_n^x \ni$

$$\underline{T} = \bar{T}_n^x (F_n(\underline{x}, t), \theta(\underline{x}, t), g(\underline{x}, t)),$$

$$\psi = \bar{\psi}_n^x (F_n(\underline{x}, t), \theta(\underline{x}, t), g(\underline{x}, t)),$$

$$\eta = \bar{\eta}_n^x (F_n(\underline{x}, t), \theta(\underline{x}, t), g(\underline{x}, t)),$$

$$\underline{q} = \bar{q}_n^x (F_n(\underline{x}, t), \theta(\underline{x}, t), g(\underline{x}, t)).$$

Here

$$g(\underline{x}, t) = \nabla \hat{\theta}(\underline{x}, t)$$

is the spatial temperature gradient, and, as usual, $\underline{x}$, $\underline{\hat{x}}$, and $\underline{x}$ are related through

$$\underline{\hat{x}} = \underline{x}(X)$$

$$\underline{x} = \underline{x}(X, t),$$

with $\underline{x}$ being the motion, The response functions are assumed to be smooth on their common domain D , and the values of $\bar{T}_n^x$ are symmetric.

It is natural to think of the stress in elasticity as dependent on the deformation gradient, and this puts F_n into all of the constitutive equations. Similarly, the temperature gradient is customarily thought of as the main driving mechanism for the heat flux. The temperature itself would seem to be an appropriate variable in all of the constitutive equations. ^{assumed}

We will make no ~~essential~~ use of the symmetry of $\underline{T}$, and we will find out

The right hand sides of the constitutive equations are somewhat scrambled up between the spatial and referential descriptions. This was the reason for not indicating any preference on the left hand side. The following notation fits in with schemes introduced previously.[†]

$$\tilde{\Theta}(\underline{x}, t) = \hat{\Theta}(\underline{\chi}(\underline{x}, t), t),$$

$$\tilde{g}(\underline{x}, t) = \hat{g}(\underline{\chi}(\underline{x}, t), t),$$

$$\Theta_n(\underline{x}, t) = \tilde{\Theta}(\underline{\chi}^{-1}(\underline{x}), t),$$

$$g_n(\underline{x}, t) = \tilde{g}(\underline{\chi}^{-1}(\underline{x}), t).$$

Exercise 7.1.1, Show that

$$\nabla \Theta_n(\underline{x}, t) = \underline{F}_n^T(\underline{x}, t) g_n(\underline{x}, t).$$

We are considering here a class of materials that is elastic in the sense that the dependent variables in the constitutive equations depend on the present values of the independent variables. Since we are attempting to take into account thermal effects, the resulting theory is called thermoelasticity.

[†]See Definitions 2.4.6 and 2.6.2.

It should be clear that because of the body force and intrinsic heat supply, the laws of balance of linear momentum, angular momentum, and energy will place no restrictions on the forms of the response functions except that the values of $\bar{I}_2$ must be symmetric tensors. In the next section, we will see that the dissipation inequality does place restrictions on the response functions.

Supplementary Reading

Same as for §6.1 + Wang and Tuncel

7.2. Consequences of the Dissipation Inequality

We recall from Theorem 6.3.3 that the reduced dissipation inequality can be written in the local form

$$-\rho(\dot{\Psi} + \eta\dot{\Theta}) + \text{tr}(\underline{T}\underline{L}^T) + \underline{\theta} \cdot \nabla \hat{\Theta} \geq 0,$$

where $\dot{\Psi}$ and $\dot{\Theta}$ denote material time derivatives, e.g.,

$$\dot{\hat{\Psi}}(\underline{x}, t) = \frac{\partial \hat{\Psi}}{\partial t}(\underline{x}, t) + \hat{\underline{L}}(\underline{x}, t) \cdot \nabla \hat{\Psi}(\underline{x}, t).$$

Here we have purposely ignored the symmetry of the stress tensor¹ and have written the stress power as $\text{tr}(\underline{T}\underline{L}^T)$ as it first appeared in the energy transfer theorem.²

It will be more convenient here to work with the referential variables, and thus in the usual manner³ we introduce

$$\Psi_R(\underline{X}, t) = \hat{\Psi}(\underline{X}_R(\underline{x}, t), t) = \tilde{\Psi}(\underline{x}^{-1}(\underline{x}), t).$$

Exercise 7.2.1. Using the above notation, show that⁴

$$\dot{\Psi} = \frac{\partial \Psi_R}{\partial t}(\underline{X}, t) = \frac{\partial \tilde{\Psi}}{\partial t}(\underline{x}, t).$$

¹Theorem 2.11.4.

²Theorem 2.11.5.

³See Definitions 2.4.6 and 2.6.2.

Our next step is to substitute the constitutive equations of the previous section into the dissipation inequality. To make this substitution as easy as possible, we introduce a fixed orthonormal basis $\{e_{\alpha}^{(i)}\}$ for V_2 and work in terms of components.

Since

$$\begin{aligned}\psi &= \psi_n(\underline{x}, t) = \bar{\Psi}_n^x(F_n(\underline{x}, t), \theta_n(\underline{x}, t), g_n(\underline{x}, t)) \\ &= \bar{\Psi}_n^x(F_n^{(i)}(\underline{x}, t), \theta_n(\underline{x}, t), g_n^{(k)}(\underline{x}, t)),\end{aligned}$$

$$\dot{\psi} = \frac{\partial \bar{\Psi}_n^x}{\partial F_n^{(i)}} \frac{\partial F_n^{(i)}}{\partial t} + \frac{\partial \bar{\Psi}_n^x}{\partial \theta_n} \frac{\partial \theta_n}{\partial t} + \frac{\partial \bar{\Psi}_n^x}{\partial g_n^{(k)}} \frac{\partial g_n^{(k)}}{\partial t}$$

From Theorem 2.6.3,

$$\frac{\partial F_n}{\partial t} = L_n F_n$$

or

$$\frac{\partial F_n^{(i)}}{\partial t} = L_n^{(ik)} F_n^{(kj)},$$

where we are employing the summation convention on all repeated indices. Then

$$\frac{\partial \bar{\Psi}_n^x}{\partial F_n^{(i)}} \frac{\partial F_n^{(i)}}{\partial t} = \frac{\partial \bar{\Psi}_n^x}{\partial F_n^{(i)}} L_n^{(ik)} F_n^{(kj)},$$

^{*}This imprecision should cause no confusion.

We define the tensor $\frac{\partial \bar{\Psi}_n^x}{\partial F_n}$ through

$$\frac{\partial \bar{\Psi}_n^x}{\partial F_n} = \frac{\partial \bar{\Psi}_n^x}{\partial F_n^{<i>}} \underline{e}^{<i>} \otimes \underline{e}^{<j>}$$

or

$$\left(\frac{\partial \bar{\Psi}_n^x}{\partial F_n} \right)^{<ij>} = \frac{\partial \bar{\Psi}_n^x}{\partial F_n^{<j>}} \quad !$$

Then

$$\begin{aligned} \frac{\partial \bar{\Psi}_n^x}{\partial F_n^{<j>}} \frac{\partial F_n^{<j>}}{\partial t} &= F_n^{<kj>} \left(\frac{\partial \bar{\Psi}_n^x}{\partial F_n} \right)^{<ij>} L_n^{<ik>} \\ &= \left[F_n^T \left(\frac{\partial \bar{\Psi}_n^x}{\partial F_n} \right)^T \right]^{<ik>} L_n^{<ik>} \\ \tau_n \left[\left(\frac{\partial \bar{\Psi}_n^x}{\partial F_n} F_n^T \right) L_n^T \right] &= \tau_n \left[F_n^T \left(\frac{\partial \bar{\Psi}_n^x}{\partial F_n} \right)^T L_n^T \right] \end{aligned}$$

Similarly, we define the vector $\frac{\partial \bar{\Psi}_n^x}{\partial g_n}$ through

$$\frac{\partial \bar{\Psi}_n^x}{\partial g_n} = \frac{\partial \bar{\Psi}_n^x}{\partial g_n^{<i>}} \underline{e}^{<i>}$$

or

^{*} This notation agrees with NFTM but not with most of the early papers.

$$\left(\frac{\partial \bar{\Psi}_n^x}{\partial \underline{g}_n} \right)^{<i>} = \frac{\partial \bar{\Psi}_n^x}{\partial \underline{g}_n^{<i>}}$$

Then

$$\frac{\partial \bar{\Psi}_n^x}{\partial \underline{g}_n^{<i>}} \frac{\partial \underline{g}_n^{<i>}}{\partial t} = \frac{\partial \bar{\Psi}_n^x}{\partial \underline{g}_n} \cdot \frac{\partial \underline{g}_n}{\partial t}$$

Thus, we can write

$$\dot{\Psi} = \text{tr} \left[\underline{F}_n^T \left(\frac{\partial \bar{\Psi}_n^x}{\partial \underline{F}_n} \right)^T \underline{L}_n \right] + \frac{\partial \bar{\Psi}_n^x}{\partial \theta_n} \frac{\partial \theta_n}{\partial t} + \frac{\partial \bar{\Psi}_n^x}{\partial \underline{g}_n} \cdot \frac{\partial \underline{g}_n}{\partial t};$$

and hence we have

Theorem 7.2.1. For a thermoelastic body, the reduced local dissipation inequality can be written as

$$\begin{aligned} & \text{tr} \left\{ \left[\bar{\underline{T}}_n^x - \rho \underline{F}_n^T \left(\frac{\partial \bar{\Psi}_n^x}{\partial \underline{F}_n} \right)^T \right] \underline{L}_n \right\} - \rho \left(\bar{\eta}_n^x + \frac{\partial \bar{\Psi}_n^x}{\partial \theta_n} \right) \frac{\partial \theta_n}{\partial t} \\ & - \rho \frac{\partial \bar{\Psi}_n^x}{\partial \underline{g}_n} \cdot \frac{\partial \underline{g}_n}{\partial t} + \frac{1}{\theta_n} \bar{\underline{q}}_n^x \cdot \underline{q}_n \geq 0. \end{aligned}$$

Following Coleman and Noll (1963), we will view this inequality as placing restrictions on the response functions rather than on thermomechanical processes. In other words, we will restrict the response functions in such a way that the dissipation inequality is satisfied for all thermomechanical processes corresponding to the constitutive equations.

Definition 7.2.1. A thermomechanical process¹ is admissible iff it is consistent with the constitutive equations under consideration²; it is local iff the domain of the process is restricted so as to involve a part³ of the body and a finite time interval.

Thus, in the present context, an admissible thermomechanical process must satisfy the constitutive equations of Definition 7.1.1.

The reduced dissipation inequality from Theorem 7.2.1 which accounts for the constitutive then leads us to

Theorem 7.2.2. A necessary and sufficient condition that every admissible, local process corresponding to the thermoelastic constitutive equations

$$\underline{I} = \bar{I}_n^x(\underline{E}_n, \theta, \underline{g}),$$

$$\psi = \bar{\psi}_n^x(\underline{E}_n, \theta, \underline{g}),$$

$$\eta = \bar{\eta}_n^x(\underline{E}_n, \theta, \underline{g}),$$

$$\underline{q} = \bar{q}_n^x(\underline{E}_n, \theta, \underline{g})$$

¹Definition 6.3.2.

²I.e., the functions of the process are related through the constitutive equations.

³Definition 2.6.7.

satisfy the reduced local dissipation inequality is that each of the following four statements hold:

(i) The response functions $\bar{T}_n^x$, $\bar{\Psi}_n^x$, and $\bar{\eta}_n^x$ are independent of the spatial temperature gradient $\underline{g}$, i.e.,

$$\underline{T} = \bar{T}_n^x(\underline{F}_n, \theta), \quad \underline{\Psi} = \bar{\Psi}_n^x(\underline{F}_n, \theta), \quad \underline{\eta} = \bar{\eta}_n^x(\underline{F}_n, \theta).$$

(ii) The response function $\bar{T}_n^x$ is determined from the response function $\bar{\Psi}_n^x$ by the stress relation

$$\bar{T}_n^x(\underline{F}_n, \theta) = \rho \underline{F}_n^T \left[\frac{\partial \bar{\Psi}_n^x(\underline{F}_n, \theta)}{\partial \underline{F}_n} \right]^T.$$

(iii) The response function $\bar{\eta}_n^x$ is determined from the response function $\bar{\Psi}_n^x$ by the entropy relation

$$\bar{\eta}_n^x(\underline{F}_n, \theta) = - \frac{\partial \bar{\Psi}_n^x(\underline{F}_n, \theta)}{\partial \theta}.$$

(iv) The response function $\bar{q}_n^x$ obeys the heat conduction inequality

$$\bar{q}_n^x(\underline{F}_n, \theta, \underline{g}) \cdot \underline{g} \geq 0.$$

Proof. The sufficiency of these four conditions follows immediately from the form of the reduced local dissipation inequality given in Theorem 7.2.1. In order to demonstrate

their necessity, we need the following

Lemma Given $\underline{x}^* \in \underline{x}(\mathcal{B})$, real t^* , $\underline{F}^* \in \mathcal{T}(\underline{V}_2)$ with
 let $\underline{F}^* \neq 0$, $\theta^* > 0$, $\underline{g}^* \in \underline{V}_3$, and also given $\underline{A} \in \mathcal{T}(\underline{V}_2)$,
 α real, $\underline{a} \in \underline{V}_3$; $\exists$ an admissible, local thermomechanical
process with

$$\underline{F}_n(\underline{x}^*, t^*) = \underline{F}^* \quad , \quad \frac{\partial \underline{F}_n}{\partial t}(\underline{x}^*, t^*) = \underline{A} \quad ,$$

$$\theta_n(\underline{x}^*, t^*) = \theta^* \quad , \quad \frac{\partial \theta_n}{\partial t}(\underline{x}^*, t^*) = \alpha \quad ,$$

$$\underline{g}_n(\underline{x}^*, t^*) = \underline{g}^* \quad , \quad \frac{\partial \underline{g}_n}{\partial t}(\underline{x}^*, t^*) = \underline{a} \quad .$$

Proof of the Lemma. Define $\underline{K}_n$ and θ_n by

$$\underline{K}_n(\underline{x}, t) = [\underline{F}^* + (t - t^*)\underline{A}](\underline{x} - \underline{x}^*)$$

and

$$\theta_n(\underline{x}, t) = \theta^* + \alpha(t - t^*) + \left\{ \underline{F}^{*T} [\underline{g}^* + (t - t^*)\underline{a}] \right\} \cdot (\underline{x} - \underline{x}^*) \quad ,$$

Clearly, this definition leads to the desired values at $(\underline{x}^*, t^*)$.
 also, let $\underline{F}_n(\underline{x}, t) \neq 0$ and $\theta_n(\underline{x}, t) > 0$ at least in some
 neighborhood² of $(\underline{x}^*, t^*)$.

We use this choice of $\underline{K}_n$ and θ_n to generate an
 admissible thermomechanical process by getting $\mathcal{I}$, Ψ , η ,

²This is the reason for considering local processes.

¹Exercise 7.2.2.

and $\underline{g}$ from the constitutive equations; and then we choose $\underline{E}$ and $\underline{L}$ so as to balance momentum and energy: $\square$

We now return to the proof of Theorem 7.2.2. Applying the reduced local dissipation inequality of Theorem 7.2.1 to the process constructed in the lemma yields

$$\begin{aligned} & \text{tr} \left\{ \left[\bar{\underline{T}}_x^*(\underline{E}^*, \theta^*, \underline{g}^*) - \rho^* \underline{F}^{*T} \left(\frac{\partial \bar{\Psi}_x^{x^*}}{\partial \underline{E}_x}(\underline{E}_x^*, \theta^*, \underline{g}^*) \right) \right] \underline{L}^{*T} \right\} \\ & - \rho^* \left[\bar{\eta}_x^*(\underline{E}^*, \theta^*, \underline{g}^*) + \frac{\partial \bar{\Psi}_x^{x^*}}{\partial \theta}(\underline{E}^*, \theta^*, \underline{g}^*) \right] \alpha \\ & - \rho^* \frac{\partial \bar{\Psi}_x^{x^*}}{\partial \underline{g}}(\underline{E}^*, \theta^*, \underline{g}^*) \cdot \underline{a} + \frac{1}{\theta^*} \bar{\underline{q}}_x^*(\underline{E}^*, \theta^*, \underline{g}^*) \cdot \underline{g}^* \geq 0, \end{aligned}$$

where $\underline{x}^* = \underline{x}^{-1}(\underline{x}^*)$, $\rho^* = \frac{\rho_x(\underline{x}^*)}{\det \underline{F}^*}$,¹ and $\underline{L}^* = \underline{L}(\underline{E}^*)$.²

The desired results follow because of the arbitrariness of $\underline{x}^*$, $\underline{E}^*$, θ^* , $\underline{g}^*$, $\underline{L}^*$, α , and $\underline{a}$.

First we note that the dissipation inequality can be written as

$$- \rho^* \frac{\partial \bar{\Psi}_x^{x^*}}{\partial \underline{g}}(\underline{E}^*, \theta^*, \underline{g}^*) \cdot \underline{a} + f_1(\underline{x}^*, \underline{E}^*, \theta^*, \underline{g}^*, \underline{L}^*, \alpha) \geq 0.$$

¹Theorem 2.10.7,

²Theorem 2.6.3,

This can hold $\forall \alpha$ iff $\frac{\partial \bar{\Psi}^*}{\partial \underline{g}}(E^*, \theta^*, \underline{g}^*) = 0$. Since x^*, E^*, θ^* , and $\underline{g}^*$ are arbitrary, we can write

$$\frac{\partial \bar{\Psi}^*}{\partial \underline{g}}(E_n, \theta, \underline{g}) = 0 \text{ or } \Psi = \bar{\Psi}_n(E_n, \theta).$$

Exercise 7.2.3. Let $\underline{a}, \underline{b} \in V_n$ and let α be real. Then prove that

$$\underline{b} \cdot \underline{a} + \alpha \geq 0 \quad \forall \underline{a} \Rightarrow \underline{b} = 0.$$

Next write the remaining portion of the dissipation inequality in the form

$$-\rho^* \left[\bar{\eta}_n^*(E^*, \theta^*, \underline{g}^*) + \frac{\partial \bar{\Psi}_n^*}{\partial \theta}(E^*, \theta) \right] \alpha + f_2(x^*, E^*, \theta^*, \underline{g}^*, \underline{L}^*) \geq 0.$$

This can hold $\forall \alpha$ iff $\bar{\eta}_n^*(E^*, \theta^*, \underline{g}^*) + \frac{\partial \bar{\Psi}_n^*}{\partial \theta}(E^*, \theta) = 0$. Since x^*, E^*, θ^* , and $\underline{g}^*$ are arbitrary, we have

$$\eta = \bar{\eta}_n^*(E, \theta) = - \frac{\partial \bar{\Psi}_n^*}{\partial \theta}(E_n, \theta).$$

Exercise 7.2.4. Let α, β, δ be real. Prove that

$$\beta \alpha + \delta \geq 0 \quad \forall \alpha \Rightarrow \beta = 0.$$

Now write the remaining portion of the dissipation inequality as

$$tr \left\{ \left[\bar{\mathcal{I}}_n^x(E^*, \theta^*, g^*) - \rho^* F^* \left(\frac{\partial \bar{\Psi}_n^x}{\partial F_n}(E^*, \theta^*) \right) \right] L^* \right\} + \frac{1}{2} (x^*, E^*, \theta^*, g^*) \geq 0.$$

$$\text{This can hold } \forall L^* \text{ iff } \bar{\mathcal{I}}_n^x(E^*, \theta^*, g^*) - \rho^* F^* \left(\frac{\partial \bar{\Psi}_n^x}{\partial F_n}(E^*, \theta^*) \right) = 0.$$

By the arbitrariness of x^*, E^*, θ^* , and g^* , we have

$$\bar{\mathcal{I}} = \bar{\mathcal{I}}_n^x(F_n, \theta) = \rho F_n \left[\frac{\partial \bar{\Psi}_n^x}{\partial F_n}(F_n, \theta) \right]$$

Exercise 7.2.5. Let $A, B \in \mathcal{T}(V_n)$ and let α be real. Prove that

$$tr(BA) + \alpha \geq 0 \quad \forall A \Rightarrow B = 0.$$

Finally, the dissipation inequality has been reduced to

$$\frac{1}{\theta^*} \bar{\mathcal{I}}_n^x(E^*, \theta^*, g^*) \cdot g^* \geq 0.$$

Since x^*, E^*, θ^* , and g^* are arbitrary, we can write

$$\bar{\mathcal{I}}_n^x(F_n, \theta, g) \cdot g \geq 0. \quad \square$$

We could make considerably more general constitutive assumptions for thermoelasticity as far as dependence on strain type quantities is concerned and still get conditions

(i)-(iii). However, condition (iv) would be more complicated.

According to the heat conduction inequality,

$$\underline{q} \cdot \underline{q} = |\underline{q}| |\underline{q}| \cos(\underline{q}, \underline{q}) \geq 0$$

$$\Rightarrow \cos(\underline{q}, \underline{q}) \geq 0$$

$$\Rightarrow -\pi/2 \leq \angle(\underline{q}, \underline{q}) \leq +\pi/2$$

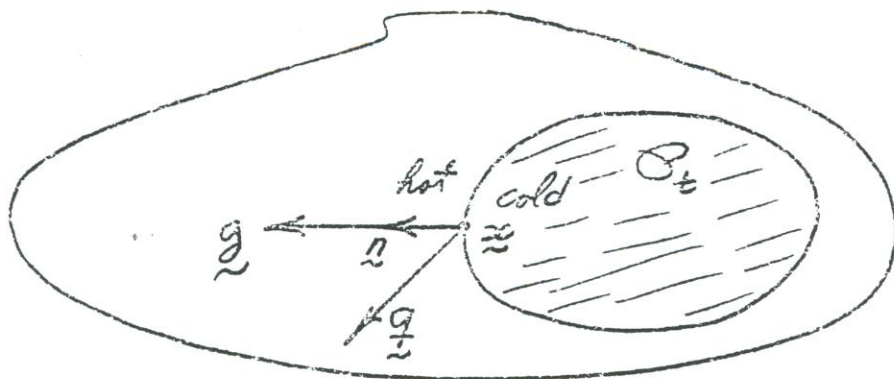
except in degenerate cases where $\underline{q}$ or $\underline{q} = \underline{0}$. Since $\underline{q} = \nabla \hat{\theta}$ points in the direction of increasing temperature and since $\underline{q}$ measures the rate of heat flow into a part, we can interpret the heat conduction inequality as implying that heat flows from hot to cold². This result may be demonstrated by arbitrarily choosing $\underline{x}$ and t and then choosing a part P whose surface at time t passes through $\underline{x}$ and has a unit outer normal $\underline{n}$ that parallels $\underline{q}(\underline{x}, t)$. Then

$$\dot{Q}_P = \underline{q} \cdot \underline{n} = \underline{q} \cdot \frac{\underline{q}}{|\underline{q}|} \geq 0,$$

¹See Gurtin's work on spatial interaction: Arch. Rational Mech. Anal. (1965), 339-352 ; J. Math. Anal. Appl. (1967), 39-44.

²Of course, here we are only considering heat flow due to conduction and only thermoelastic bodies.

and we conclude that positive heat flows into $\mathcal{Q}$.



As another interesting implication of the heat conduction inequality, we have

Theorem 7.2.3. ¹ If the heat conduction inequality holds, then

$$\underline{q} = \underline{0} \Rightarrow \underline{q} = \underline{0} ;$$

i.e., no temperature gradient means no heat flow (due to conduction). (Strain by itself will not cause heat flow in elastic bodies.)

Proof. For fixed E_n and θ define the function φ through

$$\varphi(\underline{q}) = \underline{q} \cdot \underline{\bar{q}} \hat{n}(E_n, \theta, \underline{q}).$$

By the heat conduction inequality, φ has a minimum at $\underline{0}$ since

$$\varphi(\underline{q}) \geq 0 = \varphi(\underline{0}).$$

¹ Pipkin and Rivlin, The formulation of constitutive equations in continuum physics, Div. Appl. Math., Brown Univ. Report, Sept. 1958.

$$\mathcal{Q} = \nabla \phi(\mathcal{Q}) = \bar{q}_n^x(E_n, \theta, \mathcal{Q}). \quad \square$$

Conditions (i)-(iii) also have a number of interesting consequences. For example, they allow us to write

$$\begin{aligned} \dot{\Psi} &= \tau_n \left(\frac{\partial \bar{\Psi}_n^x}{\partial E_n} \dot{E}_n^T \right) + \frac{\partial \bar{\Psi}_n^x}{\partial \theta} \dot{\theta} \\ &= \tau_n \left[\left(\frac{\partial \bar{\Psi}_n^x}{\partial E_n} E_n^T \right) \dot{E}_n^T \right] + \frac{\partial \bar{\Psi}_n^x}{\partial \theta} \dot{\theta} \quad)^2 \\ &= \frac{1}{\rho} \tau_n (I \dot{L}^T) - \eta \dot{\theta}. \end{aligned}$$

But from $\Psi = \varepsilon - \theta \eta$, we get

$$\dot{\varepsilon} = \dot{\Psi} + \eta \dot{\theta} + \dot{\eta} \theta;$$

and $\therefore$

$$\rho \dot{\varepsilon} = \tau_n (I \dot{L}^T) + \theta \rho \dot{\eta}.$$

Substitution of this result into the energy equation³ leads to

¹Exercise 7.2.6. Show that $\nabla \phi(\mathcal{Q}) = \bar{q}_n^x(E_n, \theta, \mathcal{Q})$.

²C.f., the analysis on p. 7.2.3,

³Theorem 6.3.3,

Theorem 7.2.4, In thermoelectricity, the energy equation can be written in the form

$$\nabla \cdot \underline{\hat{q}} + \rho r = \rho \theta \dot{\eta} . ^1$$

As immediate corollaries, we have

Theorem 7.2.5, In thermoelectricity, a process is locally adiabatic² iff it is isentropic ($\dot{\eta} \equiv 0$).

Proof, Exercise 7.2.7, $\square$

Theorem 7.2.6, In thermoelectricity and for an insulated body ($\underline{q}_n \equiv 0$ on ∂B_t and $r \equiv 0$),

$$\frac{d}{dt} \int_{B_t} \rho \eta \, dx = \int_{B_t} \frac{\underline{q} \cdot \underline{q}}{\theta^2} \, dx \geq 0 .$$

Thus, the total entropy of an insulated body does not decrease.

Proof, Exercise 7.2.8, $\square$

We started out assuming that Ψ_n^* was smooth, but

¹ Note that this is the inequality of Exercise 6.3.1 with equality holding, (so thermoelectricity is reversible in the sense of Exercise 6.3.1).

² See Exercise 6.3.5.

(i) - (iii) together with the smoothness assumptions on $\bar{T}_n^x$ and $\bar{\eta}_n^x$ actually imply that $\bar{\Psi}_n^x$ is two times continuously differentiable. Therefore,

$$\frac{\partial}{\partial \theta} \frac{\partial}{\partial \underline{F}_n} \bar{\Psi}_n^x(\underline{F}_n, \theta) = \frac{\partial}{\partial \underline{F}_n} \frac{\partial}{\partial \theta} \bar{\Psi}_n^x(\underline{F}_n, \theta).$$

Substitution of the stress and entropy relations into this equation yields

Theorem 7.2.7. In thermoelectroelasticity, the response functions for stress and entropy satisfy the Maxwell relation

$$\frac{\partial}{\partial \theta} \left[\frac{1}{\rho} \bar{T}_n^x(\underline{F}_n, \theta) (\underline{F}_n^T)^{-1} \right] = - \frac{\partial}{\partial \underline{F}_n} \bar{\eta}_n^x(\underline{F}_n, \theta).$$

Supplementary Reading

Same as for § 7.1

7.3. Entropy as the Independent Thermodynamical Variable

Suppose that we can invert the relation

$$\eta = \bar{\eta}_x^x(E_x, \theta)$$

w.r.t. θ so as to write

$$\theta = \bar{\theta}_x^x(E_x, \eta).$$

Of course, we want the function $\bar{\theta}_x^x$ to be smooth in E_x and η . We know from calculus that this smooth inversion is possible iff

$$\frac{\partial \bar{\eta}_x^x}{\partial \theta}(E_x, \theta) \neq 0.$$

Definition 7.3.1. The scalar

Would be best to
start with Thm 7.3.2

$$c = \bar{c}_x^x(E_x, \theta) = \theta \frac{\partial \bar{\eta}_x^x}{\partial \theta}(E_x, \theta)$$

is the specific heat (at fixed strain).

Thus, our earlier remarks may be formalized as

Theorem 7.3.1. A necessary and sufficient condition for the smooth invertibility of $\eta = \bar{\eta}_x^x(E_x, \theta)$ w.r.t. θ is that the specific heat not vanish.

We can obtain some physical feeling for the specific heat by considering

$$\varepsilon = \psi + \theta \eta \quad . \quad)^1$$

Differentiation w.r.t. θ yields

$$\frac{\partial \varepsilon}{\partial \theta} = \frac{\partial \psi}{\partial \theta} + \eta + \theta \frac{\partial \eta}{\partial \theta} = c$$

in light of the entropy relation² and Definition 7.3.1.
Hence, we have

Theorem 7.3.2. Write

$$\varepsilon = \bar{\Psi}_n^x(F_n, \theta) + \theta \bar{\eta}_n^x(F_n, \theta) = \bar{\varepsilon}_n^x(F_n, \theta).$$

Then

$$c = \frac{\partial \bar{\varepsilon}_n^x}{\partial \theta}(F_n, \theta).$$

Thus, the specific heat is the rate of change of the specific internal energy w.r.t. temperature (at fixed strain).

This last remark makes it quite plausible to assume that the specific heat does not vanish, and we make this assumption for the remainder of this section. Then we can write

$$\theta = \bar{\theta}_n^x(F_n, \eta),$$

¹Definition 6.3.1.

²Theorem 7.2.2,

and from $\psi = \varepsilon - \theta\eta$ we get

$$\begin{aligned}\varepsilon &= \bar{\Psi}_n^x(E_n, \bar{\Theta}_n^x(E_n, \eta)) + \eta \bar{\Theta}_n^x(E_n, \eta) \\ &= \bar{\varepsilon}_n^x(E_n, \eta).\end{aligned}$$

Differentiation w.r.t. η yields

$$\begin{aligned}\frac{\partial \bar{\varepsilon}_n^x(E_n, \eta)}{\partial \eta} &= \frac{\partial \bar{\Psi}_n^x(E_n, \bar{\Theta}_n^x(E_n, \eta))}{\partial \theta} \frac{\partial \bar{\Theta}_n^x(E_n, \eta)}{\partial \eta} \\ &\quad + \bar{\Theta}_n^x(E_n, \eta) + \eta \frac{\partial \bar{\Theta}_n^x(E_n, \eta)}{\partial \eta},\end{aligned}$$

but

$$\frac{\partial \bar{\Psi}_n^x(E_n, \bar{\Theta}_n^x(E_n, \eta))}{\partial \theta} = -\eta$$

by the entropy relation. Therefore, we have

$$\theta = \bar{\Theta}_n^x(E_n, \eta) = \frac{\partial \bar{\varepsilon}_n^x(E_n, \eta)}{\partial \eta}.$$

In the same manner differentiation w.r.t. E_n leads to

$$\frac{\partial \bar{\varepsilon}_n^x(E_n, \eta)}{\partial E_n} = \frac{\partial \bar{\Psi}_n^x(E_n, \bar{\Theta}_n^x(E_n, \eta))}{\partial E_n},$$

and hence

$$\mathcal{I} = \rho \frac{\partial \bar{\varepsilon}_n^x(E_n, \eta)}{\partial E_n} E_n^T = \bar{\mathcal{I}}_n^x(E_n, \eta).$$

Therefore, we have established

Theorem 7.3.3. Let the hypotheses of Theorem 7.2.2
hold and suppose that $c \neq 0$. Then

$$\varepsilon = \bar{\varepsilon}_n^x(E_n, \eta) ,$$

and

$$\theta = \bar{\theta}_n^x(E_n, \eta) = \frac{\partial \bar{\varepsilon}_n^x}{\partial \eta}(E_n, \eta) \dots \text{the } \underline{\text{temperature relation}},$$

and

$$\underline{T} = \bar{T}_n^x(E_n, \eta) = \rho \frac{\partial \bar{\varepsilon}_n^x}{\partial E_n}(E_n, \eta) E_n^T \dots \text{the}$$

alternative stress relation,

Of course, we would have come out with these last two relations for the constitutive equations for temperature and stress if we had started out at the very beginning with η rather than θ as the thermodynamical variable. (independent)

Supplementary Reading

Same as for § 7.1

ε as independent variable

Legendre transformation approach

7.4. Consequences of Material Indifference

The principle of material indifference¹ requires that the response functions of Theorem 7.2.2 meet

$$\bar{I}_n^x(Q\bar{E}, \theta) = Q \bar{I}_n^x(\bar{E}, \theta) Q^T,$$

$$\bar{\Psi}_n^x(Q\bar{E}, \theta) = \bar{\Psi}_n^x(\bar{E}, \theta),$$

$$\bar{\eta}_n^x(Q\bar{E}, \theta) = \bar{\eta}_n^x(\bar{E}, \theta),$$

$$\bar{I}_n^x(Q\bar{E}, \theta, Q\bar{g}) = Q \bar{I}_n^x(\bar{E}, \theta, \bar{g})$$

$\forall$ invertible $\bar{E}$, $\forall$ positive θ , $\forall \bar{g}$, and $\forall$ orthogonal Q . The transformation laws for $\bar{I}$ and $\bar{E}$ were developed in §3.2. The fields θ , Ψ , and η transform into themselves since they are scalars. The transformation laws for $\bar{g}$ and $\bar{g}$ are considered below.

To see how $\bar{g}$ transforms, we consider

$$g_n = \bar{g} \cdot \bar{n}.$$

Since g_n is a scalar, we assume that

$$g_n^* = g_n.$$

¹ See § 3.2,

Then since $\underline{\tilde{q}}^* = \underline{Q} \underline{q}$,¹

$$\underline{q} \cdot \underline{n} = \underline{\tilde{q}}^* \cdot \underline{Q} \underline{n} = (\underline{Q}^T \underline{\tilde{q}}^*) \cdot \underline{n} .^2$$

And because this holds $\forall$ unit vectors $\underline{n}$,

$$\underline{q} = \underline{Q}^T \underline{\tilde{q}}^* \iff \underline{\tilde{q}}^* = \underline{Q} \underline{q} .^3$$

The transformation law for $\underline{g}$ is most clearly derived through the relation

$$\nabla \Theta_x(\underline{x}, t) = \underline{F}_x^T(\underline{x}, t) \underline{g}_x(\underline{x}, t) .^4$$

For the equivalent "stated" motion, we would have

$$\nabla \tilde{\Theta}_x(\underline{x}, t) = \underline{\tilde{F}}_x^T(\underline{x}, t) \underline{\tilde{g}}_x(\underline{x}, t) .$$

But $\tilde{\Theta}_x = \Theta_x$, and we have already seen that $\underline{\tilde{F}}_x = \underline{Q} \underline{F}_x$.⁵
Therefore,

$$\nabla \Theta_x = (\underline{Q} \underline{F}_x)^T \underline{\tilde{g}}_x = \underline{F}_x^T \underline{Q}^T \underline{\tilde{g}}_x .^6$$

¹ See p. 3.2.5.

² Definition 1.10.1.

³ Theorem 1.13.2.

⁴ Exercise 7.1.1.

⁵ See p. 3.2.8.

⁶ Theorem 1.10.2.

But we started with $\nabla \theta_n = \underline{F}_n^T \underline{g}_n$, whence

$$\underline{g} = \underline{Q}^T \underline{g}^* \iff \underline{g}^* = \underline{Q} \underline{g}.$$

Turning now to the response functions, we first consider the free energy. We assume that material indifference holds so that

$$\bar{\Psi}_n^x(\underline{Q}\underline{F}, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta).$$

We make the usual polar decomposition $\underline{F} = \underline{R}\underline{U}$,¹ and choose $\underline{Q} = \underline{R}^T$ to get

$$\bar{\Psi}_n^x(\underline{F}, \theta) = \bar{\Psi}_n^x(\underline{R}^T \underline{R} \underline{U}, \theta) = \bar{\Psi}_n^x(\underline{U}, \theta) = \bar{\bar{\Psi}}_n^x(\underline{C}, \theta).²$$

Conversely, the constitutive equation

$$\psi = \bar{\Psi}_n^x(\underline{C}_n, \theta)$$

obviously satisfies the principle of material indifference because $\theta = \theta^*$ and $\underline{C}_n = \underline{C}_n^*$.³

By the entropy relation,⁴

¹Theorem 2.6.1.

²C.f., the argument on p. 3.2.11.

³See p. 3.2.13.

⁴Theorem 7.2.2.

$$\eta = \bar{\eta}_n^x(\underline{F}_n, \theta) = -\frac{\partial \bar{\Psi}_n^x(\underline{F}_n, \theta)}{\partial \theta} = -\frac{\partial \bar{\Psi}_n^x(\underline{C}_n, \theta)}{\partial \theta};$$

and this reduced form is easily seen to satisfy material indifference.

By the stress relation,¹

$$\underline{T} = \underline{T}_n^x(\underline{F}_n, \theta) = \rho \frac{\partial \bar{\Psi}_n^x(\underline{F}_n, \theta)}{\partial \underline{F}_n} \underline{F}_n^T = \rho \frac{\partial \bar{\Psi}_n^x(\underline{C}_n, \theta)}{\partial \underline{F}_n} \underline{F}_n^T.$$

We fall back on components relative to an orthonormal basis to calculate $\frac{\partial \bar{\Psi}_n^x(\underline{C}_n, \theta)}{\partial \underline{F}_n}$.²

$$\left(\frac{\partial \bar{\Psi}_n^x}{\partial \underline{F}_n} \right)^{\langle ij \rangle} = \frac{\partial \bar{\Psi}_n^x}{\partial F_n^{\langle ij \rangle}} = \frac{\partial \bar{\Psi}_n^x}{\partial C_n^{\langle mn \rangle}} \frac{\partial C_n^{\langle mn \rangle}}{\partial F_n^{\langle ij \rangle}}.$$

Since $C_n^{\langle mn \rangle} = F_n^{\langle lm \rangle} F_n^{\langle ln \rangle}$,³

(and here we sum on all repeated indices)

¹ Theorem 7.2.2,

² See p. 7.2.3,

³ Exercise 2.6.3, Theorem 1.10.5, Theorem 1.9.3.

$$\begin{aligned}
\frac{\partial C_n^{<mn>}}{\partial F_n^{<ij>}} &= \frac{\partial F_n^{<lm>}}{\partial F_n^{<ij>}} F_n^{<ln>} + F_n^{<lm>} \frac{\partial F_n^{<ln>}}{\partial F_n^{<ij>}} \\
&= \delta_{li} \delta_{mj} F_n^{<ln>} + F_n^{<lm>} \delta_{li} \delta_{nj} \\
&= \delta_{mj} F_n^{<in>} + \delta_{nj} F_n^{<im>},
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\frac{\partial \bar{\Psi}_n^x}{\partial F_n} \right)^{<ij>} &= \frac{\partial \bar{\Psi}_n^x}{\partial C_n^{<mn>}} (\delta_{mj} F_n^{<in>} + \delta_{nj} F_n^{<im>}) \\
&= \frac{\partial \bar{\Psi}_n^x}{\partial C_n^{<in>}} F_n^{<in>} + \frac{\partial \bar{\Psi}_n^x}{\partial C_n^{<mj>}} F_n^{<im>},
\end{aligned}$$

or

$$\frac{\partial \bar{\Psi}_n^x}{\partial F_n} = F_n \left(\frac{\partial \bar{\Psi}_n^x}{\partial C_n} \right)^T + F_n \frac{\partial \bar{\Psi}_n^x}{\partial C_n}.$$

Then the stress relation becomes

$$\mathcal{I} = \bar{\mathcal{I}}_n^x(F_n, \theta) = \rho F_n \left[\frac{\partial \bar{\Psi}_n^x}{\partial C_n} + \left(\frac{\partial \bar{\Psi}_n^x}{\partial C_n} \right)^T \right] F_n^T.$$

Since C_n is symmetric,

$$\left(\frac{\partial \bar{\Psi}_n^x}{\partial C_n} \right)^T = \frac{\partial \bar{\Psi}_n^x}{\partial C_n}.$$

(Component-wise, we agree to view $\bar{\Psi}_n^x$ as a function of

the six independent components of $\underline{C}_n$. Hence,

$$\underline{T} = \bar{\underline{T}}^x(\underline{F}, \theta) = 2\rho \underline{F}_n \frac{\partial \bar{\Psi}_n^x}{\partial \underline{C}_n}(\underline{C}_n, \theta) \underline{F}_n^T.$$

In order to demonstrate that this reduced stress relation satisfies material indifference, consider

$$\begin{aligned} \bar{\underline{T}}^x(\underline{Q}\underline{F}, \theta) &= 2\rho(\underline{Q}\underline{F}) \frac{\partial \bar{\Psi}_n^x}{\partial \underline{C}}(\underline{C}, \theta) (\underline{Q}\underline{F})^T \\ &= \underline{Q} \left[2\rho \underline{F} \frac{\partial \bar{\Psi}_n^x}{\partial \underline{C}}(\underline{C}, \theta) \underline{F}^T \right] \underline{Q}^T \\ &= \underline{Q} \bar{\underline{T}}^x(\underline{F}, \theta) \underline{Q}^T. \end{aligned}$$

Finally, we consider the heat flux. Assume that material indifference is satisfied so that

$$\bar{\underline{q}}_n^x(\underline{F}, \theta, \underline{g}) = \underline{Q}^T \bar{\underline{q}}_n^x(\underline{Q}\underline{F}, \theta, \underline{Q}\underline{g}).$$

We choose $\underline{Q} = \underline{R}^T$ ($\underline{F} = \underline{R}\underline{U}$) to get

$$\begin{aligned} \bar{\underline{q}}_n^x(\underline{F}, \theta, \underline{g}) &= \underline{R} \bar{\underline{q}}_n^x(\underline{U}, \theta, \underline{R}^T \underline{g}) \\ &= \underline{F} \underline{U}^{-1} \bar{\underline{q}}_n^x(\underline{U}, \theta, \underline{U}^{-1} \underline{F}^T \underline{g}) \\ &= \underline{F} \bar{\underline{q}}_n^x(\underline{C}, \theta, \underline{F}^T \underline{g}). \end{aligned}$$

¹ See pp. 5.1.3 and 5.1.4 for the indifference of ρ .

Conversely, suppose that

$$\bar{\underline{q}}_n^x(\underline{F}, \theta, \underline{g}) = \underline{F} \bar{\underline{q}}_n^x(\underline{C}, \theta, \underline{F}^T \underline{g}).$$

Then

$$\begin{aligned} \bar{\underline{q}}_n^x(\underline{Q}\underline{F}, \theta, \underline{Q}\underline{g}) &= \underline{Q}\underline{F} \bar{\underline{q}}_n^x(\underline{C}, \theta, \underline{F}^T \underline{Q}^T \underline{Q}\underline{g}) \\ &= \underline{Q}\underline{F} \bar{\underline{q}}_n^x(\underline{C}, \theta, \underline{F}^T \underline{g}) \\ &= \underline{Q} \bar{\underline{q}}_n^x(\underline{F}, \theta, \underline{g}), \end{aligned}$$

and material indifference is satisfied.

From Exercise 7.1.1, $\underline{F}_n^T \underline{g} = \nabla \theta_n$; and $\therefore$ we can write the reduced constitutive equation for the heat flux in the more convenient form

$$\underline{q} = \underline{F}_n \bar{\underline{q}}_n^x(\underline{C}_n, \theta_n, \nabla \theta_n).$$

These results can be summed up as

Theorem 7.4.1. The constitutive equations obtained in Theorem 7.2.2 meet the principle of material indifference iff they assume the reduced forms

$$\begin{aligned} \psi &= \bar{\psi}_n^x(\underline{C}_n, \theta), \\ \eta &= -\frac{\partial \bar{\psi}_n^x}{\partial \theta}(\underline{C}_n, \theta), \end{aligned}$$

$$\underline{I} = 2\rho \underline{F}_n \frac{\partial \bar{\Psi}_n^x}{\partial \underline{C}_n}(\underline{C}_n, \theta) \underline{F}_n^T,$$

$$\underline{q} = \underline{F}_n \bar{\underline{q}}_n^x(\underline{C}_n, \theta, \nabla \theta_n).$$

Exercise 7.4.1 Show that the heat conduction inequality of Theorem 7.2.2 reduces to

$$\bar{\underline{q}}_n^x(\underline{C}_n, \theta, \nabla \theta_n) \cdot \nabla \theta_n \geq 0.$$

It is interesting to note that

$$\underline{I} = 2\rho \underline{F}_n \frac{\partial \bar{\Psi}_n^x}{\partial \underline{C}_n}(\underline{C}_n, \theta) \underline{F}_n^T$$

is automatically symmetric.¹ Thus, we have²

Theorem 7.4.2. In thermoelasticity, balance of angular momentum is a consequence of balance of linear momentum, the two laws of thermomechanics, and the principle of material indifference.

Supplementary Reading

Same as for § 7.1

¹C.f., Theorem 2.11.4.

²C.f., footnote on p. 7.1.2.

see Liu's paper in

Int. J. Engng. Sci., 20, #10, 1982

7.5.1

7.5. Material Symmetry . Isotropy

We begin by writing the constitutive equations of thermoelasticity in the form¹

$$\underline{T} = \underline{\bar{T}}^X_n(\underline{E}_n, \theta),$$

$$\psi = \underline{\bar{\Psi}}^X_n(\underline{E}_n, \theta),$$

$$\eta = \underline{\bar{\eta}}^X_n(\underline{E}_n, \theta),$$

$$\underline{q} = \underline{\bar{q}}^X_n(\underline{E}_n, \theta, \underline{g}).$$

In §3.4, we introduced the notion of symmetry group to study the symmetry of the stress response. Here we must also consider the symmetries of the free energy response, the entropy response, and the heat flux response. Following §3.4 and noting that θ and $\underline{g}$ are independent of the reference configuration, we have

Definition 7.5.1. The symmetry group of the stress response w.r.t. the reference configuration $\underline{x}$, the symmetry group of the free energy response w.r.t. the reference configuration $\underline{x}$, etc. are defined by

$$\underline{\mathcal{L}}^X_n(\underline{T}) = \{ \underline{H} \mid \underline{H} \in \mathcal{U}, \underline{\bar{T}}^X_n(\underline{E}_H, \theta) = \underline{\bar{T}}^X_n(\underline{E}, \theta) \ \forall \underline{E}, \theta \},$$

¹See Theorem 7.2.2.

$$\mathcal{I}_n^X(\Psi) = \{ \underline{H} \mid \underline{H} \in \mathcal{U}, \bar{\Psi}_n^X(\underline{E}\underline{H}, \underline{\theta}) = \bar{\Psi}_n^X(\underline{E}, \underline{\theta}) \forall \underline{E}, \underline{\theta} \},$$

$$\mathcal{I}_n^X(\eta) = \{ \underline{H} \mid \underline{H} \in \mathcal{U}, \bar{\eta}_n^X(\underline{E}\underline{H}, \underline{\theta}) = \bar{\eta}_n^X(\underline{E}, \underline{\theta}) \forall \underline{E}, \underline{\theta} \},$$

$$\mathcal{I}_n^X(\underline{q}) = \{ \underline{H} \mid \underline{H} \in \mathcal{U}, \bar{\underline{q}}_n^X(\underline{E}\underline{H}, \underline{\theta}, \underline{q}) = \bar{\underline{q}}_n^X(\underline{E}, \underline{\theta}, \underline{q}) \forall \underline{E}, \underline{\theta}, \underline{q} \},$$

respectively.¹

As in Theorem 3.4.1, each of the above groups really is a group w.r.t. tensor multiplication.

Definition 7.5.2. The complete symmetry group $\mathcal{I}_n^X$ (with respect to the reference configuration $\underline{x}$) is the intersection of the separate symmetry groups:

$$\mathcal{I}_n^X = \mathcal{I}_n^X(\underline{I}) \cap \mathcal{I}_n^X(\Psi) \cap \mathcal{I}_n^X(\eta) \cap \mathcal{I}_n^X(\underline{q}).$$

We apply Noll's classification² to the complete symmetry group. Thus

Definition 7.5.3. Isotropy, fluidity, and solidity of a thermoelastic material are defined by

$$\begin{aligned} \text{isotropic} &: \mathcal{I}_n^X \supset \mathcal{O}; \quad (\underline{x} \text{ an undistorted state}) \\ \text{fluid} &: \mathcal{I}_n^X = \mathcal{U}; \quad (\forall \underline{x}) \\ \text{solid} &: \mathcal{I}_n^X \subset \mathcal{O}, \quad (\underline{x} \text{ an undistorted state}) \end{aligned}$$

¹Of course, in these definitions $\underline{E}$ is restricted to be invertible and $\underline{\theta}$ must be strictly positive,

²Definitions 3.4.2 - 3.4.4.

Orthogonal members of the symmetry groups satisfy more familiar conditions.

Theorem 7.5.1. Let $\underline{Q} \in \mathcal{O}$. Then $\underline{Q} \in \mathcal{D}_n^x(\underline{I})$ iff

$$\bar{I}_n^x(\underline{Q} \underline{F} \underline{Q}^T, \theta) = \underline{Q} \bar{I}_n^x(\underline{F}, \theta) \underline{Q}^T \quad \forall \underline{F}, \theta.$$

Proof. This theorem is a special case of theorem 3.4.3. $\square$

We note that if $\underline{Q} \in \mathcal{D}_n^x(\underline{I})$, then by theorem 7.5.1 $-\underline{Q} \in \mathcal{D}_n^x(\underline{I})$. Thus, $\mathcal{D}_n^x(\underline{I})$ can contain improper orthogonal tensors; indeed, it always contains $-\underline{I}$.

Theorem 7.5.2. Let $\underline{Q} \in \mathcal{O}$. Then $\underline{Q} \in \mathcal{D}_n^x(\Psi)$ iff

$$\bar{\Psi}_n^x(\underline{Q} \underline{F} \underline{Q}^T, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta) \quad \forall \underline{F}, \theta.$$

An identical result holds for $\mathcal{D}_n^x(\gamma)$.

Proof. Suppose $\underline{Q} \in \mathcal{D}_n^x(\Psi)$. Then

$$\bar{\Psi}_n^x(\underline{F} \underline{Q}, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta) \quad \forall \underline{F}, \theta.$$

Since this must hold $\forall \underline{F}$, it holds when $\underline{F}$ above is replaced by $\underline{Q} \underline{F} \underline{Q}^T$. Therefore,

$$\bar{\Psi}_n^x(\underline{Q} \underline{F} \underline{Q}^T \underline{Q}, \theta) = \bar{\Psi}_n^x(\underline{Q} \underline{F} \underline{Q}^T, \theta) \quad \forall \underline{F}, \theta;$$

or

$$\bar{\Psi}_n^x(\underline{Q} \underline{F}, \theta) = \bar{\Psi}_n^x(\underline{Q} \underline{F} \underline{Q}^T, \theta) \quad \forall \underline{F}, \theta.$$

But by material indifference,¹

$$\bar{\Psi}_n^x(\underline{Q}\underline{F}, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta);$$

and $\therefore$

$$\bar{\Psi}_n^x(\underline{Q}\underline{F}\underline{Q}^T, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta) \quad \forall \underline{F}, \theta.$$

Conversely, suppose that

$$\bar{\Psi}_n^x(\underline{Q}\underline{F}\underline{Q}^T, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta) \quad \forall \underline{F}, \theta.$$

Then by material indifference

$$\bar{\Psi}_n^x(\underline{Q}\underline{F}\underline{Q}^T, \theta) = \bar{\Psi}_n^x(\underline{Q}\underline{F}, \theta) \quad \forall \underline{F}, \theta.$$

Replace $\underline{F}$ by $\underline{Q}^T \underline{F}$ to get — poor choice — see next theorem

$$\bar{\Psi}_n^x(\underline{F}\underline{Q}^T, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta) \quad \forall \underline{F}, \theta.$$

Therefore, $\underline{Q}^T \in \mathcal{D}_n^x(\Psi)$. But by one of the group axioms,²

$$(\underline{Q}^T)^T = \underline{Q} \in \mathcal{D}_n^x(\Psi), \quad \square$$

We note again that $\underline{Q} \in \mathcal{D}_n^x(\Psi) \Rightarrow -\underline{Q} \in \mathcal{D}_n^x(\Psi)$.

Theorem 7.5.3. Let $\underline{Q} \in \mathcal{O}$. Then $\underline{Q} \in \mathcal{D}_n^x(\underline{q})$ iff

¹See p. 7.4.1

²(G₃) of Definition 1.13.3.

$$\bar{q}_n^x(\underline{Q}\underline{F}\underline{Q}^T, \theta, \underline{Q}\underline{g}) = \underline{Q} \bar{q}_n^x(\underline{F}, \theta, \underline{g}) \quad \forall \underline{F}, \theta, \underline{g}.$$

Proof. Suppose $\underline{Q} \in \mathcal{D}_n^x(\underline{q})$. Then

$$\bar{q}_n^x(\underline{F}\underline{Q}, \theta, \underline{g}) = \bar{q}_n^x(\underline{F}, \theta, \underline{g}) \quad \forall \underline{F}, \theta, \underline{g}.$$

Since this must hold $\forall \underline{F}$ and $\underline{g}$, we can replace them by $\underline{Q}\underline{F}\underline{Q}^T$ and $\underline{Q}\underline{g}$, respectively, to get

$$\bar{q}_n^x(\underline{Q}\underline{F}, \theta, \underline{Q}\underline{g}) = \bar{q}_n^x(\underline{Q}\underline{F}\underline{Q}^T, \theta, \underline{Q}\underline{g}) \quad \forall \underline{F}, \theta, \underline{g}.$$

But by material indifference,

$$\bar{q}_n^x(\underline{Q}\underline{F}, \theta, \underline{Q}\underline{g}) = \underline{Q} \bar{q}_n^x(\underline{F}, \theta, \underline{g}),$$

so that

$$\bar{q}_n^x(\underline{Q}\underline{F}\underline{Q}^T, \theta, \underline{Q}\underline{g}) = \underline{Q} \bar{q}_n^x(\underline{F}, \theta, \underline{g}) \quad \forall \underline{F}, \theta, \underline{g}.$$

Conversely, suppose that

$$\bar{q}_n^x(\underline{Q}\underline{F}\underline{Q}^T, \theta, \underline{Q}\underline{g}) = \underline{Q} \bar{q}_n^x(\underline{F}, \theta, \underline{g}) \quad \forall \underline{F}, \theta, \underline{g}.$$

Then by material indifference,

$$\bar{q}_n^x(\underline{Q}\underline{F}\underline{Q}^T, \theta, \underline{Q}\underline{g}) = \bar{q}_n^x(\underline{Q}\underline{F}, \theta, \underline{Q}\underline{g}) \quad \forall \underline{F}, \theta, \underline{g}.$$

Since this holds $\forall \underline{F}$ and $\underline{g}$, we can write

$$\bar{q}_n^x(\underline{Q}(\underline{Q}^T \underline{F} \underline{Q}) \underline{Q}^T, \theta, \underline{Q}(\underline{Q}^T \underline{g})) = \bar{q}_n^x(\underline{Q}(\underline{Q}^T \underline{F} \underline{Q}), \theta, \underline{Q}(\underline{Q}^T \underline{g})),$$

or

$$\bar{f}_n^x(E, \theta, \underline{g}) = \bar{f}_n^x(E, \theta, \underline{g}).$$

Thus, $\underline{Q} \in \mathcal{D}_n^x(\underline{q})$. $\square$

In contrast with our results concerning the symmetry groups for $\underline{I}$, Ψ , and η , we note that we evidently cannot conclude from Theorem 7.5.3 that if $\underline{Q} \in \mathcal{D}_n^x(\underline{q})$ then $-\underline{Q} \in \mathcal{D}_n^x(\underline{q})$. This is, apparently, due to the fact that $\underline{I}$, Ψ and η are even-order tensors while $\underline{q}$ is an odd-order tensor.

One of the consequences of the heat conduction inequality was $\bar{f}_n^x(E, \theta, \underline{Q}) = \underline{Q}$.¹ According to the next theorem, if $-\underline{I} \in \mathcal{D}_n^x(\underline{q})$, then we have this result without recourse to the second law.

Theorem 7.5.4. If $-\underline{I} \in \mathcal{D}_n^x(\underline{q})$, then

$$\bar{f}_n^x(E, \theta, \underline{Q}) = \underline{Q}.$$

Proof. By Theorem 7.5.3, $-\underline{I} \in \mathcal{D}_n^x(\underline{q}) \Rightarrow$

$$\bar{f}_n^x(E, \theta, -\underline{g}) = -\bar{f}_n^x(E, \theta, \underline{g}).$$

Choosing $\underline{g} = \underline{Q}$ yields the desired result. $\square$

¹Theorem 7.2.3.

While we have used material indifference in our study of the orthogonal elements of the symmetry groups, we have not made use of the reduced constitutive equations of Theorem 7.4.1 which take into account the principle of material indifference once and for all.

By Theorem 7.4.1, we can write

$$\begin{aligned}\bar{T}_n^x(E_n, \theta) &= E_n \bar{T}_n^x(C_n, \theta) E_n^T \\ &= E_n \bar{T}_n^x(E_n^T E_n, \theta) E_n^T.\end{aligned}$$

Suppose $H \in \mathcal{D}_n^x(I)$. Then by Definition 7.5.1,

$$\bar{T}_n^x(EH, \theta) = \bar{T}_n^x(E, \theta)$$

or

$$EH \bar{T}_n^x(H^T E^T E H, \theta) H^T E^T = E \bar{T}_n^x(E^T E, \theta) E^T.$$

or

$$H \bar{T}_n^x(H^T C H, \theta) H^T = \bar{T}_n^x(C, \theta).$$

Since all of the above steps are reversible, a unimodular H which satisfies the last equation $\forall C$ and θ is in $\mathcal{D}_n^x(I)$.

We can use Theorem 7.4.1 and Definition 7.5.1 on $\mathcal{D}_n^x(\psi)$, $\mathcal{D}_n^x(\eta)$, and $\mathcal{D}_n^x(\zeta)$ in the same way.

First consider

$$\bar{\Psi}_n^x(E_n, \theta) = \bar{\Psi}_n^x(C_n, \theta) = \bar{\Psi}_n^x(E_n^T E_n, \theta).$$

Suppose $H \in \mathcal{D}_n^x(\psi)$. Then

$$\bar{\Psi}_n^x(EH, \theta) = \bar{\Psi}_n^x(E, \theta)$$

or

$$\bar{\Psi}_n^x(\underbrace{H^T E^T E H}_I, \theta) = \bar{\Psi}_n^x(Q, \theta)$$

or

$$\bar{\Psi}_n^x(H^T Q H, \theta) = \bar{\Psi}_n^x(Q, \theta).$$

Conversely, a unimodular H which satisfies this equation $\forall Q$ and θ is in $\mathcal{D}_n^x(\Psi)$. Clearly, an identical result holds for $\mathcal{D}_n^x(\gamma)$.

Finally, consider

$$\begin{aligned} \bar{\mathcal{Q}}_n^x(E, \theta, \underline{g}) &= E_n \bar{\mathcal{Q}}_n^x(Q_n, \theta, E_n^T \underline{g}) \\ &= E_n \bar{\mathcal{Q}}_n^x(E_n^T E_n, \theta, E_n^T \underline{g}). \end{aligned}$$

Suppose $H \in \mathcal{D}_n^x(\underline{g})$. Then

$$\bar{\mathcal{Q}}_n^x(EH, \theta, \underline{g}) = \bar{\mathcal{Q}}_n^x(E, \theta, \underline{g})$$

or

$$E_n \bar{\mathcal{Q}}_n^x(\underbrace{H^T E^T E H}_I, \theta, \underbrace{H^T E^T \underline{g}}_{E_n^T \underline{g}}) = E_n \bar{\mathcal{Q}}_n^x(Q, \theta, E_n^T \underline{g})$$

or

$$\bar{\mathcal{Q}}_n^x(H^T Q H, \theta, H^T E^T \underline{g}) = \bar{\mathcal{Q}}_n^x(Q, \theta, E^T \underline{g}).$$

Conversely, a unimodular H which satisfies this equation $\forall Q, \theta$, and $\underline{g}$ is in $\mathcal{D}_n^x(\underline{g})$.

We now have results up to

Theorem 7.5.5. Let $\underline{H} \in \mathcal{O}$. Then $\underline{H} \in \mathcal{D}_n^X$ iff

$$\underline{H} \underline{\bar{I}}_n^X(\underline{H}^T \underline{C} \underline{H}, \theta) \underline{H}^T = \underline{\bar{I}}_n^X(\underline{C}, \theta),$$

$$\underline{\bar{\Psi}}_n^X(\underline{H}^T \underline{C} \underline{H}, \theta) = \underline{\bar{\Psi}}_n^X(\underline{C}, \theta),$$

$$\underline{\bar{\gamma}}_n^X(\underline{H}^T \underline{C} \underline{H}, \theta) = \underline{\bar{\gamma}}_n^X(\underline{C}, \theta),$$

$$\underline{H} \underline{\bar{g}}_n^X(\underline{H}^T \underline{C} \underline{H}, \theta, \underline{H}^T \underline{F}^T \underline{g}) = \underline{\bar{g}}_n^X(\underline{C}, \theta, \underline{F}^T \underline{g})$$

$$\forall \underline{C}, \theta, \underline{g}.$$

Again orthogonal elements of $\mathcal{D}_n^X$ satisfy simpler conditions.

Theorem 7.5.6. Let $\underline{Q} \in \mathcal{O}$. Then $\underline{Q} \in \mathcal{D}_n^X$ iff

$$\underline{\bar{I}}_n^X(\underline{Q} \underline{C} \underline{Q}^T, \theta) = \underline{Q} \underline{\bar{I}}_n^X(\underline{C}, \theta) \underline{Q}^T,$$

$$\underline{\bar{\Psi}}_n^X(\underline{Q} \underline{C} \underline{Q}^T, \theta) = \underline{\bar{\Psi}}_n^X(\underline{C}, \theta),$$

$$\underline{\bar{\gamma}}_n^X(\underline{Q} \underline{C} \underline{Q}^T, \theta) = \underline{\bar{\gamma}}_n^X(\underline{C}, \theta),$$

$$\underline{\bar{g}}_n^X(\underline{Q} \underline{C} \underline{Q}^T, \theta, \underline{Q} \underline{F}^T \underline{g}) = \underline{Q} \underline{\bar{g}}_n^X(\underline{C}, \theta, \underline{F}^T \underline{g})$$

$$\forall \underline{C}, \theta, \underline{g}.$$

Proof. It is easy to give a direct proof as for Theorem 7.5.5. However, it is quicker to note that orthogonal $\underline{Q} \in \mathcal{D}_n^X \iff \underline{Q}^{-1} = \underline{Q}^T \in \mathcal{D}_n^X$. Then let $\underline{H} = \underline{Q}^T$ in Theorem 7.5.5. $\square$

Up to this point we have ignored the stress and entropy relations in an study of symmetry. Since we derive $\underline{I}$ and η from Ψ , we would expect $\mathcal{D}_n^x(\underline{I})$ and $\mathcal{D}_n^x(\eta)$ to be related to $\mathcal{D}_n^x(\Psi)$.

By Theorem 7.2.2,

$$\underline{T}_n^x(\underline{E}_n, \theta) = \rho \frac{\partial \bar{\Psi}_n^x}{\partial \underline{E}_n}(\underline{E}_n, \theta) \underline{E}_n^T.$$

Suppose that $\underline{H} \in \mathcal{D}_n^x(\underline{I})$. Then by Definition 7.5.1,

$$\underline{T}_n^x(\underline{E}\underline{H}, \theta) = \underline{T}_n^x(\underline{E}, \theta) \quad \forall \underline{E}, \theta$$

or

$$\rho \frac{\partial \bar{\Psi}_n^x}{\partial \underline{E}}(\underline{E}\underline{H}, \theta) \underline{H}^T \underline{E}^T = \rho \frac{\partial \bar{\Psi}_n^x}{\partial \underline{E}}(\underline{E}, \theta) \underline{E}^T$$

or

$$\frac{\partial \bar{\Psi}_n^x}{\partial \underline{E}}(\underline{E}\underline{H}, \theta) \underline{H}^T = \frac{\partial \bar{\Psi}_n^x}{\partial \underline{E}}(\underline{E}, \theta), \quad \text{notation problems}$$

It is easy to see (Exercise 7.5.1) that

$$\frac{\partial \bar{\Psi}_n^x}{\partial \underline{E}}(\underline{E}\underline{H}, \theta) \underline{H}^T = \frac{\partial}{\partial \underline{E}} [\bar{\Psi}_n^x(\underline{E}\underline{H}, \theta)],$$

and therefore

$$\frac{\partial}{\partial \underline{E}} [\bar{\Psi}_n^x(\underline{E}\underline{H}, \theta) - \bar{\Psi}_n^x(\underline{E}, \theta)] = 0.$$

Thus, the quantity in brackets is the same $\forall \underline{E}$.
Evaluation at $\underline{E} = \underline{I}$ yields

$$\bar{\Psi}_n^x(FH, \theta) - \bar{\Psi}_n^x(E, \theta) = \bar{\Psi}_n^x(H, \theta) - \bar{\Psi}_n^x(\underline{1}, \theta).$$

Since all of the above steps are reversible, any unimodular H which satisfies this last equation is in $\mathcal{D}_n^x(\underline{1})$. Hence, we have proved

Theorem 7.5.7. (Tuesdell') Let $H \in \mathcal{U}$. Then $H \in \mathcal{D}_n^x(\underline{1})$ iff

$$\bar{\Psi}_n^x(FH, \theta) = \bar{\Psi}_n^x(E, \theta) + \bar{\Psi}_n^x(H, \theta) - \bar{\Psi}_n^x(\underline{1}, \theta) \quad \forall E, \theta.$$

Tuesdell's theorem has a number of important corollaries.

Theorem 7.5.8. $\mathcal{D}_n^x(\Psi) \subset \mathcal{D}_n^x(\underline{1})$.

Proof. Suppose $H \in \mathcal{D}_n^x(\Psi)$. Then by Definition 7.5.1,

$$\bar{\Psi}_n^x(FH, \theta) = \bar{\Psi}_n^x(E, \theta) \quad \forall E, \theta,$$

In particular (choosing $E = \underline{1}$),

$$\bar{\Psi}_n^x(H, \theta) = \bar{\Psi}_n^x(\underline{1}, \theta);$$

and $\therefore$

$$\bar{\Psi}_n^x(FH, \theta) = \bar{\Psi}_n^x(E, \theta) + \bar{\Psi}_n^x(H, \theta) - \bar{\Psi}_n^x(\underline{1}, \theta).$$

Consequently, $H \in \mathcal{D}_n^x(\underline{1})$ by Theorem 7.5.7. Thus,

$$H \in \mathcal{D}_n^x(\Psi) \Rightarrow H \in \mathcal{D}_n^x(\underline{I}) \text{ or } \mathcal{D}_n^x(\Psi) \subset \mathcal{D}_n^x(\underline{I}). \quad \square$$

Theorem 7.5.9. Let $\underline{Q} \in \mathcal{O}$. Then

$$\underline{Q} \in \mathcal{D}_n^x(\underline{I}) \Rightarrow \underline{Q} \in \mathcal{D}_n^x(\Psi).$$

Proof. Suppose $\underline{Q} \in \mathcal{O}$ and $\underline{Q} \in \mathcal{D}_n^x(\underline{I})$. Then by Theorem 7.5.7,

$$\bar{\Psi}_n^x(\underline{F}\underline{Q}, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta) + \bar{\Psi}_n^x(\underline{Q}, \theta) - \bar{\Psi}_n^x(\underline{1}, \theta).$$

But by material indifference¹,

$$\bar{\Psi}_n^x(\underline{Q}, \theta) = \bar{\Psi}_n^x(\underline{1}, \theta);$$

and $\therefore$

$$\bar{\Psi}_n^x(\underline{F}\underline{Q}, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta) \Rightarrow \underline{Q} \in \mathcal{D}_n^x(\Psi). \quad \square$$

The implication in Theorem 7.5.9 obviously holds in the other direction by Theorem 7.5.8, and this leads us to

Theorem 7.5.10. The orthogonal subsets of $\mathcal{D}_n^x(\underline{I})$ and $\mathcal{D}_n^x(\Psi)$ are identical, i.e.,

$$\mathcal{D}_n^x(\underline{I}) \cap \mathcal{O} = \mathcal{D}_n^x(\Psi) \cap \mathcal{O}.$$

Proof. Suppose $\underline{Q} \in \mathcal{D}_n^x(\underline{I}) \cap \mathcal{O}$. Then $\underline{Q} \in \mathcal{D}_n^x(\underline{I})$ and $\underline{Q} \in \mathcal{O}$. Then by Theorem 7.5.9, $\underline{Q} \in \mathcal{D}_n^x(\Psi)$. Thus,

$$\mathcal{D}_n^x(\underline{I}) \cap \mathcal{O} \subset \mathcal{D}_n^x(\Psi) \cap \mathcal{O}.$$

¹ See p. 7.4.1.

On the other hand, suppose that $\underline{Q} \in \mathcal{D}_n^x(\Psi) \cap \mathcal{O}$. Then $\underline{Q} \in \mathcal{D}_n^x(\Psi)$ and $\underline{Q} \in \mathcal{O}$, and by theorem 7.5.8, $\underline{Q} \in \mathcal{D}_n^x(\underline{I})$. Thus, we also have

$$\mathcal{D}_n^x(\Psi) \cap \mathcal{O} \subset \mathcal{D}_n^x(\underline{I}) \cap \mathcal{O};$$

and we conclude that

$$\mathcal{D}_n^x(\Psi) \cap \mathcal{O} = \mathcal{D}_n^x(\underline{I}) \cap \mathcal{O}. \quad \square$$

The next three theorems indicate situations in which the symmetry groups for the stress and the free energy agree completely.

Theorem 7.5.11. If $\mathcal{D}_n^x(\underline{I}) = \mathcal{U}$, then $\mathcal{D}_n^x(\underline{I}) = \mathcal{D}_n^x(\Psi)$.

Proof. Let $\underline{H} \in \mathcal{D}_n^x(\underline{I})$. Then from the proof of Theorem 7.5.7, we have that

$$\frac{\partial}{\partial \underline{E}} [\bar{\Psi}_n^x(\underline{E}\underline{H}, \theta) - \bar{\Psi}_n^x(\underline{E}, \theta)] = 0.$$

When $\mathcal{D}_n^x(\underline{I}) = \mathcal{U}$, this must hold for any $\underline{H} \in \mathcal{U}$. In particular, it must hold for $\underline{H} = |\det \underline{E}|^{1/3} \underline{E}^{-1}$. This choice yields

$$\frac{\partial}{\partial \underline{E}} [\bar{\Psi}_n^x(|\det \underline{E}|^{1/3} \underline{1}, \theta) - \bar{\Psi}_n^x(\underline{E}, \theta)] = 0.$$

Thus, the quantity in brackets is the same $\forall \underline{E}$ and evaluation at $\underline{E} = \underline{1}$ yields

$$\bar{\Psi}_n^x(\underline{E}, \theta) = \bar{\Psi}_n^x(|\det \underline{E}|^{1/3} \underline{1}, \theta) \equiv \check{\Psi}_n^x(|\det \underline{E}|, \theta).$$

Clearly, with this structure for $\bar{\Psi}_n^x$,

interesting this
one needs to
go further
back than
Truesdell's
theorem

$$\bar{\Psi}_n^x(\underline{FH}, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta) \quad \forall \underline{H} \in \mathcal{U}.$$

Hence, $\underline{H} \in \mathcal{U} \Rightarrow \underline{H} \in \mathcal{D}_n^x(\Psi)$ or $\mathcal{U} \subset \mathcal{D}_n^x(\Psi)$. By definition $\mathcal{D}_n^x(\Psi) \subset \mathcal{U}$, and $\therefore \mathcal{D}_n^x(\Psi) = \mathcal{U}$. $\square$

In the purely mechanical case, an elastic material is said to be hyperelastic when the stress is derived from a scalar field Ψ through the stress relation. Also in the purely mechanical case, the condition $\mathcal{D}_n^x(\underline{I}) = \mathcal{U}$ is the definition of a fluid. Therefore, in the absence of thermal effects, Theorem 7.5.11 states that the symmetry groups for the stress and the free energy are both equal to the unimodular group for a hyperelastic fluid.

When thermal effects are included, we have chosen to define fluid, solid, etc. in terms of the complete symmetry group.¹ However, the above result for hyperelastic fluids still holds; and, in fact, it is more readily obtainable.

Theorem 7.5.12, If the body is fluid, then

$$\mathcal{D}_n^x(\underline{I}) = \mathcal{D}_n^x(\Psi) = \mathcal{D}_n^x(\underline{\eta}) = \mathcal{D}_n^x(\underline{q}) = \mathcal{U}.$$

Proof. For a fluid body,

$$\mathcal{D}_n^x = \mathcal{D}_n^x(\underline{I}) \cap \mathcal{D}_n^x(\Psi) \cap \mathcal{D}_n^x(\underline{\eta}) \cap \mathcal{D}_n^x(\underline{q}) = \mathcal{U}.$$

Thus, $\mathcal{U} \subset \mathcal{D}_n^x(\underline{I})$; but by definition, $\mathcal{D}_n^x(\underline{I}) \subset \mathcal{U}$. $\therefore$

Definition 7.5.3 (Noll's classification).

$\mathcal{D}_n^x(\underline{I}) = \mathcal{U}$. Similar remarks hold for the other separate symmetry groups. $\square$

In the case of solids, there generally is not equality of the separate symmetry groups. However, we do have the following result.

Theorem 7.5.13. If $\mathcal{D}_n^x(\underline{I}) \subset \mathcal{O}$, then $\mathcal{D}_n^x(\underline{I}) = \mathcal{D}_n^x(\Psi)$.

Proof. Suppose $\mathcal{D}_n^x(\underline{I}) \subset \mathcal{O}$. Then $\underline{H} \in \mathcal{D}_n^x(\underline{I}) \Rightarrow \underline{H} \in \mathcal{O}$, and by Theorem 7.5.9, $\underline{H} \in \mathcal{D}_n^x(\Psi)$. Thus, $\mathcal{D}_n^x(\underline{I}) \subset \mathcal{D}_n^x(\Psi)$. But by Theorem 7.5.8, $\mathcal{D}_n^x(\Psi) \subset \mathcal{D}_n^x(\underline{I})$; and $\therefore \mathcal{D}_n^x(\underline{I}) = \mathcal{D}_n^x(\Psi)$. $\square$

We note that $\mathcal{D}_n^x(\underline{I}) \subset \mathcal{O}$ defines a solid in the purely mechanical case, but in thermoelasticity we need the complete symmetry group $\mathcal{D}_n^x \subset \mathcal{O}$ for a solid.

Theorem 7.5.14. A body is isotropic w.r.t. stress response ($\mathcal{O} \subset \mathcal{D}_n^x(\underline{I})$) iff it is isotropic w.r.t. free energy response ($\mathcal{O} \subset \mathcal{D}_n^x(\Psi)$).

Proof. Suppose the body is isotropic w.r.t. stress response. Then $\mathcal{O} \subset \mathcal{D}_n^x(\underline{I})$, and since the orthogonal subsets of $\mathcal{D}_n^x(\underline{I})$ and $\mathcal{D}_n^x(\Psi)$ agree by Theorem 7.5.10, $\mathcal{O} \subset \mathcal{D}_n^x(\Psi)$. The same argument establishes the converse. $\square$

This last theorem means that we can define isotropy of the stress response through either $\mathcal{D}_n^x(\underline{I})$ or $\mathcal{D}_n^x(\Psi)$.

This is important because $\mathcal{D}_n^x(\Psi)$ is much simpler to work with than $\mathcal{D}_n^x(\mathcal{I})$.

Next we consider the entropy relation. By Theorem 7.2.2,

$$\bar{\eta}_n^x(\underline{E}_n, \theta) = -\frac{\partial \bar{\Psi}_n^x(\underline{E}_n, \theta)}{\partial \theta}.$$

Suppose that $\underline{H} \in \mathcal{D}_n^x(\gamma)$. Then by Definition 7.5.1,

$$\bar{\eta}_n^x(\underline{E}_H, \theta) = \bar{\eta}_n^x(\underline{E}, \theta) \quad \forall \underline{E}, \theta$$

$$\text{or} \quad -\frac{\partial \bar{\Psi}_n^x(\underline{E}_H, \theta)}{\partial \theta} = -\frac{\partial \bar{\Psi}_n^x(\underline{E}, \theta)}{\partial \theta}$$

$$\text{or} \quad \frac{\partial}{\partial \theta} [\bar{\Psi}_n^x(\underline{E}_H, \theta) - \bar{\Psi}_n^x(\underline{E}, \theta)] = 0.$$

Thus, the quantity in brackets is the same $\forall \theta$.
Evaluation at $\theta = \theta_0$ yields

$$\bar{\Psi}_n^x(\underline{E}_H, \theta) - \bar{\Psi}_n^x(\underline{E}, \theta) = \bar{\Psi}_n^x(\underline{E}_H, \theta_0) - \bar{\Psi}_n^x(\underline{E}, \theta_0).$$

Since all of the above steps are reversible, any unimodular $\underline{H}$ which satisfies this last equation is in $\mathcal{D}_n^x(\gamma)$.
Hence, we have

Theorem 7.5.15, (Austin and Williams') Let $\underline{H} \in \mathcal{U}$.

¹Arch. Rational Mech. Anal. 23, 163-172, 1966.

Then $H \in \mathcal{D}_n^x(\eta)$ iff

$$\bar{\Psi}_n^x(\underline{F}H, \theta) = \bar{\Psi}_n^x(\underline{F}, \theta) + \bar{\Psi}_n^x(\underline{F}H, \theta_0) - \bar{\Psi}_n^x(\underline{F}, \theta_0) \quad \forall \underline{F}, \theta$$

for some $\theta_0 > 0$.

Gurtin and Williams' theorem does not have as many corollaries as Truesdell's, but we do have the next two results.

Theorem 7.5, 16. $\mathcal{D}_n^x(\psi) \subset \mathcal{D}_n^x(\eta)$.

Proof. Exercise 7.5, 2. $\square$

Theorem 7.5, 17. If a ^{thermodynamic} body is isotropic w.r.t. free energy response ($\mathcal{O} \subset \mathcal{D}_n^x(\psi)$), then it is isotropic w.r.t. entropy response ($\mathcal{O} \subset \mathcal{D}_n^x(\eta)$).

Proof. Suppose the body is isotropic w.r.t. free energy response. Then by Theorem 7.5, 16,

$$\mathcal{O} \subset \mathcal{D}_n^x(\psi) \subset \mathcal{D}_n^x(\eta).$$

Since $\mathcal{O} \subset \mathcal{D}_n^x(\eta)$, the entropy response is isotropic. $\square$

Next we suppose that the body is isotropic, and we determine the appropriate representation formulas for the response functions. Of course, if the body is isotropic, then

$$\Theta \subset \mathcal{D}_n^x = \mathcal{D}_n^x(\underline{I}) \cap \mathcal{D}_n^x(\Psi) \cap \mathcal{D}_n^x(\eta) \cap \mathcal{D}_n^x(\underline{q})$$

$$\iff \Theta \subset \mathcal{D}_n^x(\underline{I}), \Theta \subset \mathcal{D}_n^x(\Psi), \Theta \subset \mathcal{D}_n^x(\eta), \Theta \subset \mathcal{D}_n^x(\underline{q}).$$

Thus, isotropy of the body requires that each of the separate responses be isotropic. By Theorems 7.5.14 and 7.5.17, isotropy of the full energy response ensures that the stress and entropy response are isotropic.

We begin then by assuming that the full energy response is isotropic, i.e., $\Theta \subset \mathcal{D}_n^x(\Psi)$. By Theorem 7.5.6, we must have

$$\bar{\Psi}_n^x(\underline{Q} \underline{Q}^T, \theta) = \bar{\Psi}_n^x(\underline{C}, \theta) \quad \forall \underline{Q} \in \Theta,$$

Thus, $\bar{\Psi}_n^x$ is an isotropic scalar-valued function of one symmetric tensor (θ is carried along parametrically). We developed a representation for such functions in the proof of Theorem 4.2.1.¹ The function $\bar{\Psi}_n^x$ must have the form

$$\bar{\Psi}_n^x(\underline{C}_n, \theta) = \bar{\Psi}_n^x(\underline{I}_c, \underline{II}_c, \underline{III}_c, \theta),$$

where $\underline{I}_c, \underline{II}_c, \underline{III}_c$ are the principal invariants of $\underline{C}_n$.

For the stress relation, we need $\frac{\partial \bar{\Psi}_n^x}{\partial \underline{C}_n}$. In the present case,

¹ See p. 4.2.7.

$$\frac{\partial \bar{\Psi}_n^x}{\partial \underline{C}_n} = \frac{\partial \bar{\Psi}_n^x}{\partial I_c} \frac{\partial I_c}{\partial \underline{C}_n} + \frac{\partial \bar{\Psi}_n^x}{\partial II_c} \frac{\partial II_c}{\partial \underline{C}_n} + \frac{\partial \bar{\Psi}_n^x}{\partial III_c} \frac{\partial III_c}{\partial \underline{C}_n}$$

by an obvious chain rule. It is easy to show using components (Exercise 7.5.3)¹ that

$$\frac{\partial I_c}{\partial \underline{C}_n} = \underline{1},$$

$$\frac{\partial II_c}{\partial \underline{C}_n} = I_c \underline{1} - \underline{C}_n,$$

$$\frac{\partial III_c}{\partial \underline{C}_n} = III_c \underline{C}_n^{-1}.$$

Then the reduced stress relation² becomes

$$\underline{T} = 2\rho \underline{F}_n \left[\frac{\partial \bar{\Psi}_n^x}{\partial I_c} \underline{1} + \frac{\partial \bar{\Psi}_n^x}{\partial II_c} (I_c \underline{1} - \underline{C}_n) + \frac{\partial \bar{\Psi}_n^x}{\partial III_c} III_c \underline{C}_n^{-1} \right] \underline{F}_n^T.$$

Now, as on p. 4.2.9,

$$\underline{F}_n \underline{F}_n^T = \underline{B}_n,$$

$$\underline{F}_n \underline{C}_n \underline{F}_n^T = \underline{F}_n \underline{F}_n^T \underline{F}_n \underline{F}_n^T = \underline{B}_n^2,$$

$$\underline{F}_n \underline{C}_n^{-1} \underline{F}_n^T = \underline{F}_n (\underline{F}_n^T \underline{F}_n)^{-1} \underline{F}_n^T = \underline{1};$$

¹ Relative to an eigenvector basis it is extra easy.

² Theorem 7.4.1.

and also¹

$$I_B = I_C, \quad \Pi_B = \Pi_C, \quad \mathbb{I}_B = \mathbb{I}_C.$$

Thus, we are lead to.

Theorem 7.5.18. If $\mathcal{O} \subset \mathcal{O}_n^x(\Psi)$, then

$$\Psi = \hat{\Psi}_n^x(I_B, \Pi_B, \mathbb{I}_B, \theta)$$

where I_B, Π_B , and $\mathbb{I}_B$ are the fundamental invariants of $\mathcal{B}_n$; and the stress and entropy relations become

$$\mathcal{I} = 2\rho \left[\mathbb{I}_B \frac{\partial \hat{\Psi}_n^x}{\partial \mathbb{I}_B} \underline{\underline{1}} + \left(\frac{\partial \hat{\Psi}_n^x}{\partial I_B} + I_B \frac{\partial \hat{\Psi}_n^x}{\partial \mathbb{I}_B} \right) \underline{\underline{B}}_n - \frac{\partial \hat{\Psi}_n^x}{\partial \Pi_B} \underline{\underline{B}}_n^2 \right]$$

and

$$\eta = - \frac{\partial \hat{\Psi}_n^x}{\partial \theta},$$

respectively,

Conversely, if the above relations hold then the body is isotropic with respect to free energy response, stress response, and entropy response.

Proof. Only the converse assertion has yet to be proved, and we leave this as Exercise 7.5.4. $\square$

Exercise 7.5.5. Use the Hamilton-Cayley Theorem (Theorem

¹See Exercise 4.2.9.

1.14.12) to put the above stress relations in the form

$$\underline{T} = 2\rho \left[\left(\underline{II}_B \frac{\partial \Psi_x}{\partial \underline{II}_B} + \underline{III}_B \frac{\partial \Psi_x}{\partial \underline{III}_B} \right) \underline{I} + \frac{\partial \Psi_x}{\partial \underline{I}_B} \underline{B}_x - \underline{II}_B \frac{\partial \Psi_x}{\partial \underline{II}_B} \underline{B}_x^{-1} \right].$$

Of course, in these stress relations we can always use the density equation¹

$$\rho = \frac{1}{\sqrt{\underline{II}_B}} \rho_x$$

to replace ρ by the density in the reference configuration x .

It still remains to investigate the heat flux. By Theorem 7.5.6 and Exercise 7.1.1, the heat flux response is isotropic iff

$$\underline{\bar{q}}_x^x(\underline{Q} \underline{C} \underline{Q}^T, \theta, \underline{Q} \nabla \theta_x) = \underline{Q} \underline{\bar{q}}_x^x(\underline{C}_x, \theta, \nabla \theta_x) \quad \forall \underline{Q} \in \mathcal{O}.$$

Thus, $\underline{\bar{q}}_x^x$ is an isotropic vector-valued function of one symmetric tensor and one vector; and hence it admits the representation

$$\underline{\bar{q}}_x^x(\underline{C}_x, \theta, \nabla \theta_x) = (\delta_x^x \underline{I} + \gamma_x^x \underline{C}_x + \gamma_x^{xx} \underline{C}_x^2) \nabla \theta_x$$

where the δ 's are scalar-valued functions of the six invariants

$$\underline{I}_C, \underline{II}_C, \underline{III}_C, \nabla \theta_x \cdot \nabla \theta_x, \nabla \theta_x \cdot \underline{C}_x \nabla \theta_x, \nabla \theta_x \cdot \underline{C}_x^2 \nabla \theta_x$$

¹ Theorem 2.10.4.

and the temperature. We will defer the proof of this representation theorem to the end of the present section.

By Theorem 7.4.1, the reduced constitutive equation for the heat flux becomes

$$\underline{q} = \underline{F}_n ({}_0 \delta_n^x \underline{1} + {}_1 \delta_n^x \underline{C}_n + {}_2 \delta_n^x \underline{C}_n^2) \nabla \theta_n .$$

Using the Hamilton-Cayley Theorem 1.14.12 to replace $\underline{C}_n^2$ by $\underline{C}_n^{-1}$ and noting that

$$\nabla \theta_n = \underline{F}_n^T \underline{g} ,^1$$

we have

$$\underline{q} = \underline{F}_n ({}_0 \Gamma_n^x \underline{1} + {}_1 \Gamma_n^x \underline{C}_n + {}_2 \Gamma_n^x \underline{C}_n^{-1}) \underline{F}_n^T \underline{g} .$$

Distributing the $\underline{F}_n$ and $\underline{F}_n^T$ factors as on p. 7.5.19 leads to

$$\underline{q} = ({}_0 \Gamma_n^x \underline{B}_n + {}_1 \Gamma_n^x \underline{B}_n^2 + {}_2 \Gamma_n^x \underline{1}) \underline{g} .$$

Now, the Γ 's are functions of the temperature and $\underline{I}_C, \underline{II}_C, \underline{III}_C, \nabla \theta_n \cdot \nabla \theta_n, \nabla \theta_n \cdot \underline{C}_n \nabla \theta_n, \nabla \theta_n \cdot \underline{C}_n^2 \nabla \theta_n$.

However, we already know that the invariants of $\underline{C}_n$ and $\underline{B}_n$ are equal, and it is left as Exercise 7.5.6 to show that

¹ Exercise 7.1.1.

$$\nabla \theta_n \cdot \nabla \theta_n = \underline{q} \cdot \underline{B}_n \underline{q} \quad ,$$

$$\nabla \theta_n \cdot \underline{C}_n \nabla \theta_n = \underline{q} \cdot \underline{B}_n^2 \underline{q} \quad ,$$

$$\nabla \theta_n \cdot \underline{C}_n^2 \nabla \theta_n = \underline{I}_B \underline{q} \cdot \underline{B}_n^2 \underline{q} - \underline{II}_B \underline{q} \cdot \underline{B}_n \underline{q} + \underline{III}_B \underline{q} \cdot \underline{q} \quad .$$

Thus, we have established

Theorem 7.5.19. If $\mathcal{OCD}_n^x(\underline{q})$, the reduced heat flux constitutive equation becomes

$$\underline{q} = (\underline{h}_n^x \underline{1} + {}_1\underline{h}_n^x \underline{B}_n + {}_2\underline{h}_n^x \underline{B}_n^2) \underline{q}$$

where the $\underline{h}$'s are scalar-valued functions of θ and the invariants

$$\underline{I}_B, \underline{II}_B, \underline{III}_B, \underline{q} \cdot \underline{q}, \underline{q} \cdot \underline{B}_n \underline{q}, \underline{q} \cdot \underline{B}_n^2 \underline{q} \quad .$$

Conversely, if the heat flux is given by the above constitutive equation, then the body is isotropic w.r.t. heat flux responses.

Of course, it follows immediately from the Hamilton-Cayley theorem that the above constitutive equation can be written as

$$\underline{q} = ({}_0\underline{h}_n^x \underline{B}_n^{-1} + {}_1\underline{h}_n^x \underline{1} + {}_2\underline{h}_n^x \underline{B}_n) \underline{q}$$

where the $\underline{h}$'s are functions of θ and the invariants

$$\underline{I}_B, \underline{II}_B, \underline{III}_B, \underline{q} \cdot \underline{q}, \underline{q} \cdot \underline{B}_n \underline{q}, \underline{q} \cdot \underline{B}_n^{-1} \underline{q} \quad .$$

Exercise 7.5.7. Establish the converse assertion of Theorem 7.5.19 by direct computation.

Now let us suppose that the elastic body under consideration is fluid so that¹

$$\mathcal{I}_n^x(\underline{I}) = \mathcal{I}_n^x(\Psi) = \mathcal{I}_n^x(\underline{\eta}) = \mathcal{I}_n^x(\underline{q}) = \mathcal{U}.$$

According to Theorems 7.5.8 and 7.5.16, it is sufficient to require that

$$\mathcal{I}_n^x(\Psi) = \mathcal{I}_n^x(\underline{q}) = \mathcal{U}.$$

Since $\mathcal{O} \subset \mathcal{U}$, we can start with the representations that we have already established for isotropic bodies.

Exercise 7.5.8. Prove that

$$\mathcal{I}_n^x(\Psi) = \mathcal{U} \text{ iff } \Psi = \tilde{\Psi}_n^x(|\det \underline{E}_n|, \theta)$$

and

$$\mathcal{I}_n^x(\underline{q}) = \mathcal{U} \text{ iff } \underline{q} = \tilde{q}_n^x(|\det \underline{E}_n|, \underline{q} \cdot \underline{q}, \theta) \underline{q}.$$

(Hint: See the proofs of Theorems 4.2.3 and 7.5.11.)

It will be convenient to replace $|\det \underline{E}_n|$ by $\sqrt{\text{III}}_B$, and this leads us to

¹See Theorem 7.5.12.

$$\psi = \check{\psi}^x_n(\mathbb{I}_B, \theta),$$

$$\underline{I} = 2\rho \mathbb{I}_B \frac{\partial \check{\psi}^x_n}{\partial \mathbb{I}_B} \underline{1},$$

$$\eta = - \frac{\partial \check{\psi}^x_n}{\partial \theta},$$

$$\underline{g} = \check{g}^x_n(\mathbb{I}_B, \underline{g} \cdot \underline{g}, \theta) \underline{g}.$$

Since $\mathbb{I}_B$ determines ρ through the referential density equation¹

$$\rho = \frac{\rho_n}{\sqrt{\mathbb{I}_B}},$$

we can write

$$\psi = \check{\psi}^x_n(\rho, \theta)$$

and

$$\underline{g} = \check{g}^x_n(\rho, \underline{g} \cdot \underline{g}, \theta) \underline{g}.$$

¹ Theorem 2.10.4.

Then

$$\underline{I} = 2\rho \underline{\Pi}_B \frac{\partial \check{\Psi}_n^x}{\partial \underline{\Pi}_B} \underline{1}$$

$$= 2\rho \left(\frac{\rho_n}{\rho}\right)^2 \frac{\partial \check{\Psi}_n^x}{\partial \rho} \frac{\partial \rho}{\partial \underline{\Pi}_B} \underline{1}$$

$$= 2\rho^{-1} \rho_n^2 \left(-\frac{1}{2} \rho_n \underline{\Pi}_B^{-3/2}\right) \frac{\partial \check{\Psi}_n^x}{\partial \rho} \underline{1}$$

$$= -\rho^{-1} \left(\frac{\rho_n}{\sqrt{\underline{\Pi}_B}}\right)^3 \frac{\partial \check{\Psi}_n^x}{\partial \rho} \underline{1}$$

$$= -\rho^2 \frac{\partial \check{\Psi}_n^x}{\partial \rho} \underline{1}.$$

Since ρ , θ , and g are unaltered by a change of the

reference configuration from $\underline{n}$ to $\underline{\mu}$, we have

$$\check{\Psi}_n^x = \check{\Psi}_\mu^x \equiv \check{\Psi}^x, \quad \check{q}_n^x = \check{q}_\mu^x \equiv \check{q}^x.$$

We sum these results up as

Theorem 7.5.20. A thermoelastic body is a
fluid iff

$$\psi = \bar{\psi}^x(\rho, \theta),$$

$$\underline{T} = -\rho^2 \frac{\partial \bar{\psi}^x}{\partial \rho} \underline{1},$$

$$\eta = -\frac{\partial \bar{\psi}^x}{\partial \theta},$$

$$\underline{g} = \bar{g}^x(\rho, \theta, \underline{g} \cdot \underline{g}) \underline{g}.$$

In the purely mechanical context, we found¹
 an elastic fluid to be Euler's compressible perfect fluid.

However, in a thermomechanical setting a thermoelastic
 fluid is evidently more general than a perfect gas.

It will be instructive to extract the constitutive equations

¹Theorem 4.2.3.

for a perfect gas from Theorem 7.5.20.

For a perfect gas, we have the "equation of state" $pv = RT$. Here, p is the pressure, v is the specific volume $v = 1/\rho$; T is the absolute temperature $T = \theta$; and R is a positive constant known as the gas constant. Thus, in our notation, the equation of state comes out as

$$p = R\rho\theta.$$

But, according to Theorem 7.5.20, $p = \rho^2 \frac{\partial \Psi^x}{\partial \rho}(\rho, \theta)$.

Thus, we have the differential equation

$$\frac{\partial \Psi^x}{\partial \rho}(\rho, \theta) = R\rho^{-1}\theta;$$

which may be integrated to yield

$$\Psi^x(\rho, \theta) = R\theta \ln \rho + f(\theta).$$

Turning to the entropy relation, we have

$$\eta = -\frac{\partial \mathcal{F}^x}{\partial \theta}(p, \theta) = -R \ln p + f'(\theta).$$

We can get a handle on $f'(\theta)$ through the specific heat¹

$$c_v = \theta \frac{\partial \eta}{\partial \theta}.$$

Here we have added the subscript v to c as to agree with common usage. On substituting for η , we get

$$-\theta f''(\theta) = c_v.$$

For a perfect gas, c_v is assumed to be a constant,

whence,

$$f'(\theta) = -c_v \ln \theta + \text{const};$$

and

$$\eta = -R \ln p + c_v \ln \theta + \text{const.}$$

Finally, in gas dynamics the constitutive

¹ See Definition 7.3.1 and Theorem 7.3.2.

equation for the heat flux $\underline{q} = \underline{q}^0(p, \theta, \underline{q} \cdot \underline{q}) \underline{q}$

is specialized to

$$\underline{q} = k(\theta) \underline{q}$$

with the additional special cases

$$k(\theta) = K = \text{const.}$$

and

$$k(\theta) = 0$$

being rather common. In the last special case

above, we would refer to the gas as being a

nonconductor.

At this point, we have a sufficiently complete

constitutive theory for perfect gases to quit. However,

with just a little more effort we can obtain a

number of interesting and familiar relations whose

derivations are sometimes obscured in the traditional treatments.

From $\varepsilon = \psi + \theta \eta$ and our constitutive equations for ψ and η given on pages 7.5.28 and 7.5.29, we get

$$\varepsilon = f(\theta) - \theta f'(\theta).$$

Thus, for a perfect gas the specific internal energy depends only on the absolute temperature.

The ~~specific~~ specific enthalpy is defined by

$$h = \varepsilon + p v,$$

When the enthalpy is given by a constitutive

equation of the form $h = \bar{h}(p, \theta)$, the ~~specific~~ specific

heat at constant pressure is defined by

$$c_p = c_p(p, \theta) = \frac{\partial h}{\partial \theta}(p, \theta).$$

For a perfect gas, $pv = R\theta$ and $\varepsilon = \bar{\varepsilon}(\theta)$;

and therefore

$$h = \bar{\varepsilon}(\theta) + R\theta$$

and

$$c_p = \frac{d\bar{\varepsilon}}{d\theta}(\theta) + R = c_v + R,$$

where we have used Theorem 7.3.2. Consequently, for

a perfect gas, the enthalpy depends only on the

absolute temperature and

$$R = c_p - c_v.$$

That the gas constant is the difference in the specific heats allows us to rewrite the constitutive equation

$$\eta = -R \ln p + c_v \ln \theta + \text{const.}$$

First we make the replacement $p = 1/v$ to get

$$\eta = -R \ln v^{-1} + c_v \ln \theta + \text{const.}$$

$$= R \ln v + c_v \ln \theta + \text{const.}$$

$$= \ln v^R + \ln \theta^{c_v} + \text{const.}$$

$$= \ln(v^R \theta^{c_v}) + \text{const.}$$

Since $\theta = pv/R$, we can write

$$\eta = \ln \left[v^R \left(\frac{pv}{R} \right)^{c_v} \right] + \text{const.}$$

$$= \ln \left[(v^{R+c_v}) (p^{c_v}) (R^{-c_v}) \right] + \text{const.}$$

$$= \ln \left[(v^{R+c_v}) (p^{c_v}) \right] + \ln R^{-c_v} + \text{const.}$$

$$= \ln \left[(v^{R+c_v}) (p^{c_v}) \right] + \text{const.}$$

Here, of course, the value of the additive const. has changed.

We can get additional simplifications by recognizing that

$R+c_v = c_p$ and introducing the customary notation

$$\gamma = c_p / c_v.$$

Then

$$\eta = \ln [v^{c_p} p^{c_v}] + \text{const.}$$

$$= \ln [v^{c_p/c_v} p]^{c_v} + \text{const.},$$

$$= c_v \ln (p v^\gamma) + \text{const.}$$

This is the usual text book formula for the entropy of a perfect gas.

We now return to the representation theorem needed on p. 7.5.21 for an isotropic vector-valued function of a symmetric tensor and a vector.

Exercise 7.5.9. Define $\mathcal{S}(\mathcal{V}_n) = \{ \underline{S} \mid \underline{S} \in \mathcal{T}(\mathcal{V}_n), \underline{S} = \underline{S}^T \}$.

Show that $\mathcal{S}(\mathcal{V}_n)$ is a subspace of $\mathcal{T}(\mathcal{V}_n)$.

Theorem 7.5.21. Let $\underline{v} : \mathcal{S}(\mathcal{V}_3) \times \mathcal{V}_3 \rightarrow \mathcal{V}_3$

be isotropic, i.e.,

$$\underline{v}(\underline{Q} \underline{A} \underline{Q}^T, \underline{Q} \underline{u}) = \underline{Q} \underline{v}(\underline{A}, \underline{u}) \quad \forall \underline{Q} \in \mathcal{O},$$

then $\underline{v}$ admits the representation

$$\underline{v}(\underline{A}, \underline{u}) = (\gamma_0 \underline{1} + \gamma_1 \underline{A} + \gamma_2 \underline{A}^2) \underline{u}$$

where the γ 's are functions of the joint invariants ^{(or simultaneous,}

$$\underline{I}_A, \underline{II}_A, \underline{III}_A, \underline{u} \cdot \underline{u}, \underline{u} \cdot \underline{A} \underline{u}, \underline{u} \cdot \underline{A}^2 \underline{u}.$$

Conversely, a function $\underline{v}$ of this form is isotropic

Proof. The proof of the converse assertion is

straightforward and constitutes Exercise 7.5.10.

To establish the main assertion, we suppose that $\underline{v}$ is an isotropic function and write

$$\underline{b} = \underline{v}(\underline{A}, \underline{u}),$$

Next let $\{\underline{\tilde{e}}_{\langle i \rangle}^*\}$ be an eigenvector basis for $\underline{A}$

and denote the components of $\underline{b}$, $\underline{u}$, and $\underline{A}$ relative to this

basis by $\delta^{* \langle i \rangle}$, $u^{* \langle i \rangle}$, and $\text{diag}(a_1, a_2, a_3)$.

Consider the equations

$$\delta^{* \langle 1 \rangle} = (\gamma_0 + \gamma_1 a_1 + \gamma_2 a_1^2) u^{* \langle 1 \rangle},$$

$$\delta^{* \langle 2 \rangle} = (\gamma_0 + \gamma_1 a_2 + \gamma_2 a_2^2) u^{* \langle 2 \rangle}, \quad (*)$$

$$\delta^{* \langle 3 \rangle} = (\gamma_0 + \gamma_1 a_3 + \gamma_2 a_3^2) u^{* \langle 3 \rangle}.$$

If each $u^{* \langle i \rangle} \neq 0$, then we can divide by the u 's

and view (*) as a set of linear algebraic equations

for the γ 's. The determinant of the coefficient matrix

$$\text{is } \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} = (a_1 - a_2)(a_2 - a_3)(a_3 - a_1).$$

Thus, if we also assume that the eigenvalues of A

are distinct, then the γ 's are well-defined by (*).

In order to interpret (*), we use the spectral

resolutions

$$\underline{1} = \sum_{i=1}^3 \underline{\underline{e}}_{\langle i \rangle}^* \otimes \underline{\underline{e}}_{\langle i \rangle}^*$$

$$\underline{A} = \sum_{i=1}^3 a_i \underline{\underline{e}}_{\langle i \rangle}^* \otimes \underline{\underline{e}}_{\langle i \rangle}^*$$

$$\underline{A}^2 = \sum_{i=1}^3 a_i^2 \underline{\underline{e}}_{\langle i \rangle}^* \otimes \underline{\underline{e}}_{\langle i \rangle}^*$$

to write

$$\gamma_0 \underline{1} + \gamma_1 \underline{A} + \gamma_2 \underline{A}^2 = \sum_{i=1}^3 (\gamma_0 + \gamma_1 a_i + \gamma_2 a_i^2) \underline{\underline{e}}_{\langle i \rangle}^* \otimes \underline{\underline{e}}_{\langle i \rangle}^* .$$

Then since

$$\underline{u} = \underline{\underline{u}}^{\langle j \rangle} \otimes \underline{\underline{e}}_{\langle j \rangle}^*$$

and

$$(\underline{\underline{e}}_{\langle i \rangle}^* \otimes \underline{\underline{e}}_{\langle i \rangle}^*) \underline{\underline{e}}_{\langle j \rangle}^* = \delta_{ij} \underline{\underline{e}}_{\langle i \rangle}^* \quad (\text{no sum}),$$

$$(\gamma_0 \underline{1} + \gamma_1 \underline{A} + \gamma_2 \underline{A}^2) \underline{u} = \sum_{i=1}^3 (\gamma_0 + \gamma_1 a_i + \gamma_2 a_i^2) \underline{\underline{u}}^{\langle i \rangle} \otimes \underline{\underline{e}}_{\langle i \rangle}^* .$$

Therefore,

$$[(\gamma_0 \underline{1} + \gamma_1 \underline{A} + \gamma_2 \underline{A}^2) \underline{u}] \cdot \underline{\underline{e}}_{\langle i \rangle}^* = (\gamma_0 + \gamma_1 a_i + \gamma_2 a_i^2) \underline{\underline{u}}^{\langle i \rangle} \quad (\text{no sum}).$$

Thus, (*) implies

$$\underline{b} = (\gamma_0 \underline{1} + \gamma_1 \underline{A} + \gamma_2 \underline{A}^2) \underline{u} ;$$

and since $\underline{b} = \underline{v}(\underline{A}, \underline{u})$ by definition, we are close to the desired representation.

Clearly the δ 's as defined by (*) will depend on $\underline{A}$ and $\underline{u}$, and we write $\delta_o = \delta_o(\underline{A}, \underline{u})$ etc. In order to investigate this dependence, we put

$$\underline{A}' = \underline{Q} \underline{A} \underline{Q}^T, \quad \underline{u}' = \underline{Q} \underline{u}, \quad \underline{b}' = \underline{v}(\underline{A}', \underline{u}')$$

where $\underline{Q}$ is an arbitrary orthogonal tensor. By the isotropy,

$$\underline{b}' = \underline{Q} \underline{b},$$

We can now proceed just as before through an eigenvector basis for $\underline{A}'$ to the representation

$$\underline{b}' = (\delta'_0 \underline{1} + \delta'_1 \underline{A}' + \delta'_2 \underline{A}'^2) \underline{u}',$$

where the δ' 's are determined by

$$b'^{\langle 1 \rangle} = (\gamma_0' + \gamma_1' a_1' + \gamma_2' a_1'^2) u'^{\langle 1 \rangle},$$

$$b'^{\langle 2 \rangle} = (\gamma_0' + \gamma_1' a_2' + \gamma_2' a_2'^2) u'^{\langle 2 \rangle}, \quad (*)'$$

$$b'^{\langle 3 \rangle} = (\gamma_0' + \gamma_1' a_3' + \gamma_2' a_3'^2) u'^{\langle 3 \rangle}.$$

Now suppose that $(a, \underline{e})$ is an eigenpair for $\underline{A}$ and consider

$$\underline{A}'(\underline{Q}\underline{e})$$

$$= (\underline{Q}\underline{A}\underline{Q}^T)(\underline{Q}\underline{e}) = \underline{Q}\underline{A}\underline{e} = a(\underline{Q}\underline{e}).$$

Thus, $(a, \underline{Q}\underline{e})$ is an eigenpair of $\underline{A}'$. In particular

$$a_i' = a_i \quad (i=1,2,3),^1$$

and $\{\underline{Q}\underline{e}_{\langle i \rangle}^*\}$ is an eigenvector basis for $\underline{A}'$.⁷

Taking $\{\underline{Q}\underline{e}_{\langle i \rangle}^*\}$ to be the eigenvector basis for

$\underline{A}'$ used in $(*)'$, we have

¹ Here we have an alternative approach to Exercise 4.2.

$$\begin{aligned} b'^{<i>} &= \underline{b}' \cdot \underline{Q} \underline{\tilde{e}}_{<i>}^* = \underline{\tilde{e}}_{<i>}^* \cdot \underline{Q}^T \underline{b}' \\ &= \underline{\tilde{e}}_{<i>}^* \cdot \underline{Q}^T \underline{Q} \underline{b} = \underline{\tilde{e}}_{<i>}^* \cdot \underline{b} = \underline{b}^{*<i>}. \end{aligned}$$

Similarly,

$$u'^{<i>} = \underline{u}' \cdot \underline{Q} \underline{\tilde{e}}_{<i>}^* = \underline{u}^{*<i>},$$

Thus, the γ' 's are determined by the same equations as the γ 's, i.e.,

$$\gamma_0(\underline{Q} \underline{A} \underline{Q}^T, \underline{Q} \underline{u}) = \gamma_0(\underline{A}, \underline{u}) \quad \forall \underline{Q} \in \mathcal{Q}.$$

That is γ_0 is an isotropic scalar-valued function of a symmetric tensor and a vector. Consequently, by a representation theorem¹ whose proof we defer, γ_0

depends on $\underline{A}$ and $\underline{u}$ only through the joint invariants

$$I_A, II_A, III_A, \underline{u} \cdot \underline{u}, \underline{u} \cdot \underline{A} \underline{u}, \underline{u} \cdot \underline{A}^2 \underline{u}.$$

¹Theorem 7.5.22,

This completes the proof of Theorem 7.5.21 in the case where none of the components of $\underline{u}$ relative to $\{\underline{\hat{e}}_{\langle i \rangle}\}$ vanish and where the eigenvalues of $\underline{A}$ are distinct. We now turn to the removal of these restrictions.

Consider the isotropy condition

$$\chi(\underline{Q}\underline{A}\underline{Q}^T, \underline{Q}\underline{u}) = \underline{Q}\chi(\underline{A}, \underline{u}) \quad \forall \underline{Q} \in \mathcal{O}$$

and choose $\underline{Q}$ to be a reflection in the $\underline{\hat{e}}_{\langle 1 \rangle}, \underline{\hat{e}}_{\langle 3 \rangle}$ -plane.

$$\text{Then } \underline{Q}\underline{\hat{e}}_{\langle 1 \rangle}^* = -\underline{\hat{e}}_{\langle 1 \rangle}^*, \quad \underline{Q}\underline{\hat{e}}_{\langle 2 \rangle}^* = \underline{\hat{e}}_{\langle 2 \rangle}^*, \quad \underline{Q}\underline{\hat{e}}_{\langle 3 \rangle}^* = \underline{\hat{e}}_{\langle 3 \rangle}^*,$$

$$\text{and } \underline{Q}\underline{u} = \underline{Q}(\underline{u}^{\langle 1 \rangle} \underline{\hat{e}}_{\langle 1 \rangle}^*) = -\underline{u}^{\langle 1 \rangle} \underline{\hat{e}}_{\langle 1 \rangle}^* + \underline{u}^{\langle 2 \rangle} \underline{\hat{e}}_{\langle 2 \rangle}^* + \underline{u}^{\langle 3 \rangle} \underline{\hat{e}}_{\langle 3 \rangle}^* ;$$

with a similar result holding for $\underline{Q}\underline{b}$; but by Exercise 4.2.2, $\underline{Q}\underline{A}\underline{Q}^T = \underline{A}$. Consequently, for this choice of $\underline{Q}$

the isotropy condition implies

¹ See p. 4.2.3.

$$\nabla(A, -\tilde{a}^{* \langle 1 \rangle} \tilde{e}_{\langle 1 \rangle}^{*} + \tilde{a}^{* \langle 2 \rangle} \tilde{e}_{\langle 2 \rangle}^{*} + \tilde{a}^{* \langle 3 \rangle} \tilde{e}_{\langle 3 \rangle}^{*}) = -\tilde{b}^{* \langle 1 \rangle} \tilde{e}_{\langle 1 \rangle}^{*} + \tilde{b}^{* \langle 2 \rangle} \tilde{e}_{\langle 2 \rangle}^{*} + \tilde{b}^{* \langle 3 \rangle} \tilde{e}_{\langle 3 \rangle}^{*}$$

But also

$$\nabla(A, \tilde{a}^{* \langle 1 \rangle} \tilde{e}_{\langle 1 \rangle}^{*} + \tilde{a}^{* \langle 2 \rangle} \tilde{e}_{\langle 2 \rangle}^{*} + \tilde{a}^{* \langle 3 \rangle} \tilde{e}_{\langle 3 \rangle}^{*}) = \tilde{b}^{* \langle 1 \rangle} \tilde{e}_{\langle 1 \rangle}^{*} + \tilde{b}^{* \langle 2 \rangle} \tilde{e}_{\langle 2 \rangle}^{*} + \tilde{b}^{* \langle 3 \rangle} \tilde{e}_{\langle 3 \rangle}^{*}$$

From these two equations, we see that

$$\tilde{a}^{* \langle 1 \rangle} = 0 \Rightarrow \tilde{b}^{* \langle 1 \rangle} = -\tilde{b}^{* \langle 1 \rangle} \Rightarrow \tilde{b}^{* \langle 1 \rangle} = 0,$$

By other obvious choices for $\tilde{Q}$, we get similar results

for the second and third components. Thus,

$$\text{for each } i, \tilde{a}^{* \langle i \rangle} = 0 \Rightarrow \tilde{b}^{* \langle i \rangle} = 0.$$

Therefore, if $\tilde{a}^{* \langle i \rangle} = 0$ for some i , the corresponding equation of (*),

$$\tilde{b}^{* \langle i \rangle} = (\gamma_0 + \gamma_1 a_i + \gamma_2 a_i^2) \tilde{a}^{* \langle i \rangle},$$

is trivially satisfied independent of $\gamma_0 + \gamma_1 a_i + \gamma_2 a_i^2$.

In such a case, we set

$$\gamma_0 + \gamma_1 a_i + \gamma_2 a_i^2 = 0$$

in order to complete our set of equations for the $\tilde{x}$'s, this scheme works if more than one of the $\tilde{u}^*$'s vanishes. Of course, if they all vanish, then $\underline{u} = \underline{0}$ and the representation holds trivially since $\chi(\underline{A}, \underline{0}) = \underline{0}$.

Exercise 7.5.10. We can read off that $\chi(\underline{A}, \underline{0}) = \underline{0}$ from

the result on the previous page that $\tilde{u}^{(i)} = 0 \Rightarrow \tilde{b}^{(i)} = 0$.

Show that $\chi(\underline{A}, \underline{0}) = \underline{0}$ more directly by making the choice $\underline{Q} = -\underline{1}$ in the isotropy condition.

Granting the next theorem, we complete the present proof by leaving the case of repeated eigenvalues of $\underline{A}$ as Exercise 7.5.12. $\square$

$$\text{if } a_2 = a_3, \text{ pick } \tilde{e}_{(2)}^* = \frac{\underline{u} - (\underline{u} \cdot \tilde{e}_{(1)}^*) \tilde{e}_{(1)}^*}{|\underline{u} - (\underline{u} \cdot \tilde{e}_{(1)}^*) \tilde{e}_{(1)}^*|}$$

to set $\tilde{u}_{(2)}^* = 0 \Rightarrow \tilde{b}_{(2)} = 0$ taking care of 3rd $\tilde{e}_j$.

Theorem 7.5, 22. Let $\gamma: S(V_3) \times V_3 \rightarrow \text{reals}$ be
isotropic, i.e.,

$$\gamma(\underline{Q} \underline{A} \underline{Q}^T, \underline{Q} \underline{u}) = \gamma(\underline{A}, \underline{u}) \quad \forall \underline{Q} \in \mathcal{O},$$

then γ admits the representation

$$\gamma(\underline{A}, \underline{u}) = \varphi(I_A, II_A, III_A, \underline{u} \cdot \underline{u}, \underline{u} \cdot \underline{A} \underline{u}, \underline{u} \cdot \underline{A}^2 \underline{u}).$$

Conversely, a function γ of this form is isotropic.

Proof. The proof of the converse assertion presents
 no difficulties.¹

Let $\{e_{\langle i \rangle}\}$ be an arbitrary but fixed orthonormal
 basis for V_3 and write

$$\gamma(\underline{A}, \underline{u}) = \gamma(A^{\langle ij \rangle}, u^{\langle i \rangle}),$$

where $A^{\langle ij \rangle}$ and $u^{\langle i \rangle}$ are the components of $\underline{A}$ and $\underline{u}$

¹cf. Exercise 7.5.10.

relative to $\{\underline{e}_{\langle i \rangle}\}$. Then the isotropy condition becomes

$$\chi(\underline{(QAQ^T)}^{\langle ij \rangle}, \underline{(Qu)}^{\langle i \rangle}) = \chi(\underline{A}^{\langle ij \rangle}, \underline{u}^{\langle i \rangle}). \quad (*)$$

Choose $\underline{Q}$ to be the orthogonal tensor which maps the eigenvector basis $\{\underline{e}_{\langle i \rangle}^*\}$ for $\underline{A}$ into $\{\underline{e}_{\langle i \rangle}\}$, i.e.,

$$\underline{Q} \underline{e}_{\langle i \rangle}^* = \underline{e}_{\langle i \rangle}. \quad \text{By Exercise 4.2.6,}$$

$$[\underline{(QAQ^T)}^{\langle ij \rangle}] = \text{diag}(a_1, a_2, a_3).$$

Also

$$\underline{(Qu)}^{\langle i \rangle} = \underline{e}_{\langle i \rangle} \cdot \underline{Qu} = \underline{u} \cdot \underline{Q}^T \underline{e}_{\langle i \rangle} = \underline{u} \cdot \underline{e}_{\langle i \rangle}^* = \underline{u}^{\langle i \rangle}.$$

Hence,

$$\chi(\underline{A}^{\langle ij \rangle}, \underline{u}^{\langle i \rangle}) = \chi(\delta_{ij} a_j, \underline{u}^{\langle i \rangle}) \quad (**)$$

Now return to (*) and choose $\underline{Q}$ to be a reflection in the $\underline{e}_{\langle 2 \rangle}, \underline{e}_{\langle 3 \rangle}$ -plane. Then following the argument

¹ See Theorem 1.13.4.

on pp. 7.5.41-42, we see that

$$\gamma(A^{<ij>}, -u^{<1>}, u^{<2>}, u^{<3>}) = \gamma(A^{<ij>}, u^{<1>}, u^{<2>}, u^{<3>}).$$

In general, γ is even in each of the components of $\underline{u}$ (relative to an orthonormal basis), and consequently

$$\gamma(A^{<ij>}, u^{<i>}) = \gamma(A^{<ij>}, (u^{<i>})^2).$$

Exercise 7.5.13. Prove the following theorem; Let f be a real-valued function on the reals $\ni f(-x) = f(x)$, i.e. let f be even. Then $\exists$ another function g (on the reals into the reals) $\ni f(x) = g(x^2)$.

In particular, we can write (***) as

$$\gamma(\underline{A}, \underline{u}) = \sum (a_i, (\underline{u}^{<i>})^2). \quad (***)$$

Next consider

$$\underline{u} \cdot \underline{A} \underline{u} = \underline{u} \cdot [a_1 \underline{e}_{<1>}^* \otimes \underline{e}_{<1>}^* + \dots] \underline{u}$$

$$= \underline{u} \cdot [a_1 \underline{e}_{<1>}^* (\underline{e}_{<1>}^* \cdot \underline{u}) + \dots]$$

$$= a_1 (\underline{e}_{<1>}^* \cdot \underline{u})^2 + \dots$$

$$= a_1 (\underline{u}^{*<1>})^2 + a_2 (\underline{u}^{*<2>})^2 + a_3 (\underline{u}^{*<3>})^2$$

With similar considerations for $\underline{u} \cdot \underline{A}^2 \underline{u}$ and $\underline{u} \cdot \underline{u}$,
we have the identities

$$(\underline{u}^{*<1>})^2 + (\underline{u}^{*<2>})^2 + (\underline{u}^{*<3>})^2 = \underline{u} \cdot \underline{u},$$

$$a_1 (\underline{u}^{*<1>})^2 + a_2 (\underline{u}^{*<2>})^2 + a_3 (\underline{u}^{*<3>})^2 = \underline{u} \cdot \underline{A} \underline{u}, \quad (\#)$$

$$a_1^2 (\underline{u}^{*<1>})^2 + a_2^2 (\underline{u}^{*<2>})^2 + a_3^2 (\underline{u}^{*<3>})^2 = \underline{u} \cdot \underline{A}^2 \underline{u}.$$

Viewing (#) as a set of linear equations for the $\underline{u}^{*2}$,
the determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{vmatrix} = (a_1 - a_2)(a_2 - a_3)(a_3 - a_1).$$

Assuming that the eigenvalues of $\underline{A}$ are distinct, we see

from (#) that the $(u^{<i>})^2$ are determined by the a_i and $u \cdot u$, $u \cdot Au$, $u \cdot A^2 u$. But, of course, the a_i are determined by the fundamental invariants I_A, II_A, III_A .

Combining the above result with (***) we have the desired representation

$$\chi(A, u) = \varphi(I_A, II_A, III_A, u \cdot u, u \cdot Au, u \cdot A^2 u) -$$

The removal of the restriction of distinct eigenvalues is left as Exercise 7.5.14. $\square$

Shorter but less transparent proofs of Theorems 7.5.21 and 7.5.22 have been provided by Noll[†] and

[†]W. Noll: Representations of certain isotropic tensor functions. Archiv der Mathematik 21, 87-90, (1970).

Wang.¹ The proofs given here are patterned after Serin's proof of Theorem 4.2.1.

Supplementary Reading

Same as for § 7.1

¹C.-C. Wang: On representations for isotropic functions
Parts I and II. Arch. Rational Mech. Anal. 33, 249-287, (196

See also

G.F. Smith: On a fundamental error in two papers of
C.-C. Wang, Arch. Rational Mech. Anal. 36, 161-165, (1970).

C.-C. Wang: A new representation theorem for isotropic functions
Part I and II. Arch. Rational Mech. Anal. 36, 166-223, (1970)

G.F. Smith: On isotropic functions of symmetric tensors, and
symmetric tensors and vectors, Int. J. Engng. Sci. 9, 899-916, (19

7.6. Internal Constraints. ~~Impossibility~~

Since thermomechanical constraints have been investigated only relatively recently and since the several investigations have taken different starting points and have reached different end points, some comments on the references given at the end of the section are in order. We are following the 1970 paper by Green, Naghdi, and Trapp¹ with some minor differences in technique and assumptions. The 1973 paper by Andreussi and Guidugli takes a starting point close to Noll's theory of mechanical constraints (our § 3.6), but this

¹The 1971 paper by Trapp uses this theory to model materials reinforced with inextensible cords.

writer finds the analysis unsatisfactory and is unable to correct it without changing the starting point. The 1973 paper by Gurtin and Guidugli is difficult to relate to because of its abstractness; it probably gives the Green, Naghdi, Trapp theory a more general setting (especially their second alternative).

Definition 7.6.1. A body B' is said to be
thermomechanically constrained at the particle X
with respect to the reference configuration $\approx$ iff $\exists$
functions $A_x^x: \mathcal{T}(\mathcal{V}_3) \times \mathbb{R}^+ \rightarrow \mathcal{S}(\mathcal{V}_3)$, $b_x^x: \mathcal{T}(\mathcal{V}_3) \times \mathbb{R}^+$
 $\rightarrow \mathbb{R}$, $c_x^x: \mathcal{T}(\mathcal{V}_3) \times \mathbb{R}^+ \rightarrow \mathcal{V}_3 \ni$ the rate of

¹ Not necessarily thermoelastic.

material time derivative² of the

deformation tensor¹ $\underline{D}$, the temperature $\dot{\Theta}$, and the spatial temperature gradient³ $\underline{g}$ are required to satisfy

$$(TMC) \quad \underline{A}_n^x(\underline{F}_n(\underline{x}, t), \Theta_n(\underline{x}, t)) \cdot \underline{D}_n(\underline{x}, t) + \underline{b}_n^x(\underline{F}_n(\underline{x}, t), \Theta_n(\underline{x}, t)) \dot{\Theta}_n(\underline{x}, t) + \underline{c}_n^x(\underline{F}_n(\underline{x}, t), \Theta_n(\underline{x}, t)) \cdot \underline{g}_n(\underline{x}, t) = 0 \quad \forall t.$$

In the above definition we are using the notations:

$$\underline{A}_n^x \cdot \underline{D}_n = \text{tr}(\underline{A}_n^x \underline{D}_n^T);^4$$

$$\mathcal{T}^+(\mathcal{V}_n) = \{ \underline{I} \mid \underline{I} \in \mathcal{T}(\mathcal{V}_n) \text{ and } \det \underline{I} > 0 \};$$

$$\mathbb{R} = \{ n \mid n \text{ real} \};$$

$$\mathbb{R}^+ = \{ n \mid n \in \mathbb{R} \text{ and } n > 0 \};$$

¹ Definition 2.4.7.

² See, e.g., Theorem 6.3.1

³ Definition 7.1.1.

⁴ Theorem 1.11.2

$$\underline{\underline{X}} = \underline{\underline{x}}(X).$$

Generally, we shall write (TMC) in the abbreviated form

$$\underline{\underline{A}} \cdot \underline{\underline{D}} + b \dot{\theta} + \underline{\underline{c}} \cdot \underline{\underline{g}} = 0,$$

The constraint of incompressibility is stated through (TMC) by taking $\underline{\underline{A}} = \underline{\underline{1}}$, $b = 0$, $\underline{\underline{c}} = \underline{\underline{0}}$ to get

$$\underline{\underline{1}} \cdot \underline{\underline{D}} = \text{tr } \underline{\underline{D}} = 0$$

and then noting Exercise 2.4.2 and Theorem 2.6.6,

Theorem 7.6.1. If a body $\mathcal{B}$ is thermomechanically constrained w.r.t. one reference configuration, then it is thermomechanically constrained w.r.t. any other reference configuration.

Proof. Suppose that $\mathcal{B}$ is thermomechanically constrained w.r.t. $\underline{\underline{x}}$ so that

$$\tilde{A}_n^x(F_n(\tilde{x}, t), \Theta_n(\tilde{x}, t)) \cdot \tilde{D}_n(\tilde{x}, t) + b\dot{\Theta} + \underline{\rho} \cdot \underline{g} = 0.$$

(We shall just work on the first term; the others are handled in an identical manner.) Let μ be an other reference configuration. It is connected to $\tilde{x}$ through the deformation $\tilde{f} = \mu \circ \tilde{x}^{-1}$.¹ By Theorem 2.6.2,

$$\tilde{F}_n(\tilde{x}, t) = \tilde{F}_\mu(\tilde{y}, t) \nabla \tilde{f}(\tilde{x})$$

where $\tilde{y} = \mu(\tilde{x})$. Also

$$\Theta_n(\tilde{x}, t) = \tilde{\Theta}(\tilde{x}, t) = \tilde{\Theta}(\mu^{-1}(\tilde{y}), t) = \Theta_\mu(\tilde{y}, t), \quad ^2$$

and in the same way $\tilde{D}_n(\tilde{x}, t) = \tilde{D}_\mu(\tilde{y}, t)$. Thus,

$$\begin{aligned} & \tilde{A}_n^x(F_n(\tilde{x}, t), \Theta_n(\tilde{x}, t)) \cdot \tilde{D}_n(\tilde{x}, t) \\ &= \tilde{A}_n^x(\tilde{F}_\mu(\tilde{y}, t) \nabla \tilde{f}(\tilde{x}), \Theta_\mu(\tilde{y}, t)) \cdot \tilde{D}_\mu(\tilde{y}, t) \end{aligned}$$

¹ Definition 2.1.2, (B_1) .

$$= A_{\mu}^x(F_{\mu}(\underline{Y}, t), \theta_{\mu}(\underline{Y}, t)) \cdot D_{\mu}(\underline{Y}, t)$$

where

$$A_{\mu}^x(\cdot, \cdot) = A_x^x((\cdot) \nabla f(\underline{x}), \cdot)$$

Consequently,

$$A_{\mu}^x(F_{\mu}(\underline{Y}, t), \theta_{\mu}(\underline{Y}, t)) \cdot D_{\mu}(\underline{Y}, t) + b_{\mu}^x(F_{\mu}(\underline{Y}, t), \theta_{\mu}(\underline{Y}, t)) \dot{\theta}_{\mu}(\underline{Y}, t) \\ + c_{\mu}^x(F_{\mu}(\underline{Y}, t), \theta_{\mu}(\underline{Y}, t)) \cdot g_{\mu}(\underline{Y}, t) = 0,$$

i.e., $\mathcal{B}$ is thermomechanically constrained at X w.r.t. the reference configuration μ . $\square$

Since the coefficients in (TMC) are given by constitutive equations, we expect them to be objective; thus, we require that

The coefficients in (TMC) must satisfy the principle of material indifference, i.e.,

$$\underline{A}_{\underline{x}}^x(\underline{Q}\underline{F}, \theta) = \underline{Q} \underline{A}_{\underline{x}}^x(\underline{F}, \theta) \underline{Q}^T,$$

$$\underline{b}_{\underline{n}}^x(\underline{Q}\underline{F}, \theta) = \underline{b}_{\underline{n}}^x(\underline{F}, \theta),$$

$$\underline{\rho}_{\underline{n}}^x(\underline{Q}\underline{F}, \theta) = \underline{Q} \underline{\rho}_{\underline{n}}^x(\underline{F}, \theta)$$

$$\forall \underline{Q} \in \mathcal{O}(\mathcal{V}_3), \underline{F} \in \mathcal{T}^+(\mathcal{V}_3), \theta \in \mathbb{R}^+$$

Fortunately such invariance requirements have already been investigated in Theorems 3.2.2, 3.6.2, and 7.4.1.

We have then

Theorem 7.6.2. If the coefficients in (TMC) satisfy the principle of material indifference, they can be written in the reduced forms:

$$\underline{A} = \underline{A}_{\underline{x}}^x(\underline{F}_{\underline{n}}, \theta_{\underline{x}}) = \underline{F}_{\underline{n}} \underline{\bar{A}}_{\underline{n}}^x(\underline{C}_{\underline{n}}, \theta_{\underline{n}}) \underline{F}_{\underline{n}}^T,$$

$$\underline{b} = \underline{b}_{\underline{n}}^x(\underline{F}_{\underline{n}}, \theta_{\underline{n}}) = \underline{\bar{b}}_{\underline{n}}^x(\underline{C}_{\underline{n}}, \theta_{\underline{n}}),$$

$$\underline{\rho} = \underline{\rho}_{\underline{n}}^x(\underline{F}_{\underline{n}}, \theta_{\underline{n}}) = \underline{F}_{\underline{n}} \underline{\bar{\rho}}_{\underline{n}}^x(\underline{C}_{\underline{n}}, \theta_{\underline{n}}).$$

Conversely, the above reduced constitutive equations satisfy the principle of material indifference.

Since $\underline{D}$, θ , and $\underline{g}$ are objective¹, i.e.,

$$\underline{\tilde{D}}^* = \underline{Q} \underline{D} \underline{Q}^T, \quad \tilde{\theta}^* = \theta, \quad \underline{\tilde{g}}^* = \underline{Q} \underline{g}, \quad ^2$$

it is not surprising that (TMC) itself meets the principle of material indifference (when the coefficients are in reduced form) in the sense that

$$\begin{aligned} & [\underline{\tilde{F}}_{\underline{n}}^* \underline{\tilde{A}}_{\underline{n}}^* (\underline{\tilde{C}}_{\underline{n}}^*, \tilde{\theta}^*) \underline{\tilde{F}}_{\underline{n}}^{*T}] \cdot \underline{\tilde{D}} + \underline{\tilde{b}}_{\underline{n}}^* (\underline{\tilde{C}}_{\underline{n}}^*, \tilde{\theta}^*) \dot{\tilde{\theta}} + [\underline{\tilde{F}}_{\underline{n}}^* \underline{\tilde{c}}_{\underline{n}}^* (\underline{\tilde{C}}_{\underline{n}}^*, \tilde{\theta}^*)] \cdot \underline{\tilde{g}} \\ &= [\underline{F}_{\underline{n}} \underline{A}_{\underline{n}}^* (\underline{C}_{\underline{n}}, \theta) \underline{F}_{\underline{n}}^T] \cdot \underline{D} + \underline{b}_{\underline{n}}^* (\underline{C}_{\underline{n}}, \theta) \dot{\theta} + [\underline{F}_{\underline{n}} \underline{c}_{\underline{n}}^* (\underline{C}_{\underline{n}}, \theta)] \cdot \underline{g} = \end{aligned}$$

We leave the details as Exercise 7.6.1.

¹See Exercise 5.1.4, p. 7.4.1, and p. 7.4.3, respectively.

²Here, as usual, a superscripted * indicates an equivalent motion. See Definition 3.2.1.

In order to proceed, we make the following assumption

Principle of Determinism for Thermomechanically Constrained Materials. The histories $F_n^{(t)}$, $\Theta_n^{(t)}$, $g_n^{(t)}$ determine the stress $\underline{T}$, the free energy Ψ , the entropy η , and the heat flux $\underline{q}$ only to within additive reactions $\underline{T}_R$, Ψ_R , η_R , and $\underline{q}_R$ that do not contribute to entropy production^{*} in any process satisfying the constraint^{*}, i.e.,

$$\left. \begin{aligned} \underline{T} &= \underline{T}_R + \underline{T}_E, \\ \Psi &= \Psi_R + \Psi_E, \\ \eta &= \eta_R + \eta_E, \\ \underline{q} &= \underline{q}_R + \underline{q}_E, \end{aligned} \right\} \quad (\text{TMCM})$$

^{*} See Theorem 6.3.3.

^{*} ... all other ... and the ...

where

$$\underline{T}_R \cdot \underline{D} - \rho \eta_R \dot{\theta} + \frac{1}{\theta} \underline{q}_R \cdot \underline{q} + \overset{-\rho}{\dot{\psi}_R} = 0$$

$\forall \underline{D}, \dot{\theta}, \underline{q}$ that satisfy the thermomechanical constraint $\underline{A} \cdot \underline{D} + b \dot{\theta} + \underline{c} \cdot \underline{q} = 0$, The extra stress, extra free energy, extra entropy, and extra heat flux ($\underline{T}_E, \psi_E, \eta_E$, and $\underline{q}_E$, respectively) are given by constitutive equations in the usual way.

It is important to realize that the constitutive equations for $\underline{T}_E$ etc. may be very general. That is, at this point, our theory of thermomechanical constraints is not restricted to thermoelasticity. On the other hand, the constraint (TMC) might be much more general, the condition of null entropy production and the

constraint itself are most easily investigated if we introduce the set $S(V_3) \times \mathbb{R} \times V_3$. The typical element of this set is an ordered triple whose elements (in order) are a ^{symm} second-order tensor, a real, and a vector.

Exercise 7.6.2. Show that with the obvious definitions for addition, etc. $S(V_3) \times \mathbb{R} \times V_3$ is a vector space.

Exercise 7.6.3, Show that

$$(\underline{T}_1, \alpha_1, \underline{v}_1) \cdot (\underline{T}_2, \alpha_2, \underline{v}_2) \equiv \underline{T}_1 \cdot \underline{T}_2 + \alpha_1 \alpha_2 + \underline{v}_1 \cdot \underline{v}_2$$

defines an inner product on $S(V_3) \times \mathbb{R} \times V_3$.

With the above scheme in mind we write

$$\underline{\Gamma} = (\underline{T}_R, -p \eta_R, \underline{\theta} \underline{q}_R),$$

$$\underline{\Lambda} = (\underline{A}, b, \underline{E}),$$

$$\underline{\Phi} = (\underline{D}, \dot{\theta}, \underline{g}).$$

then our principle of determinism states that

$$\underline{\Gamma} \cdot \underline{\Phi} + \rho \dot{\psi}_R = 0 \quad \forall \underline{\Phi} \in \underline{\Lambda} \cdot \underline{\Phi} = 0.$$

Choosing $\underline{\Phi} = (\underline{D}, 0, \underline{D})$, we get

$$\dot{\psi}_R = 0 \Rightarrow \psi_R \text{ constant in } t.$$

Thus, we are left with

$$\underline{\Gamma} \cdot \underline{\Phi} = 0 \quad \forall \underline{\Phi} \in \underline{\Lambda} \cdot \underline{\Phi} = 0.$$

Then by the supplement to § 1.14,

$$\underline{\Gamma} = \alpha \underline{\Lambda}$$

for some $\alpha \in \mathbb{R}$. We sum these results up as

Theorem 7.6.3. The reactions in a body subject
to the thermomechanical constraint

$$\underline{\Lambda} \cdot \underline{D} + b \dot{\theta} + \underline{\rho} \cdot \underline{g} = 0$$

are given by

$$\tilde{T}_R(x, t) = \alpha(x, t) A_n^x(F_n(x, t), \theta_n(x, t)) ,$$

$$\tilde{\eta}_R(x, t) = -\frac{\alpha(x, t)}{\tilde{\rho}(x, t)} b_n^x(F_n(x, t), \theta_n(x, t)) ,$$

$$\tilde{q}_R(x, t) = \alpha(x, t) \theta_n(x, t) c_n^x(F_n(x, t), \theta_n(x, t)) ,$$

$$\tilde{\psi}_R(x, t) = \beta(x) ,$$

where the functions α and β are arbitrary.

Of course, the response functions for A , b , and c in Theorem 7.6.3 are subject to the principle of material indifference.¹ The functions α and β are analogous to η in Theorem 3.6.3.

Since ψ_R is constant (in time), it drops out of the energy equation and the Clausius-Duhem inequality,

¹ See Theorem 7.6.2.

It is reasonable then to agree to take

$$\psi_R \equiv 0.$$

the theories by Green, Naghdi, and Trapp and by Gurtin and Guichugli assume this at the start (of course, in anticipation of that if the assumption is not made, ψ_R still comes out as a constant).

Next we consider the principle of material indifference for thermomechanically constrained bodies. By (TMCM), Theorem 7.6.2, and Theorem 7.6.3, the constitutive equation for the stress, e.g., is

$$\underline{T} = \alpha \underline{F} \underline{\bar{A}}^* (\underline{C}_n, \underline{\Theta}_n) \underline{F}^T + \underline{T}_E.$$

For the equivalent motion $\{\underline{\bar{X}}^*(\cdot; t)\}$, we have

$$\underline{\bar{T}}^* = \alpha \underline{\bar{F}} \underline{\bar{A}}^* (\underline{\bar{C}}_n, \underline{\bar{\Theta}}_n) \underline{\bar{F}}^T + \underline{\bar{T}}_E.$$

Since $\tilde{F}^* = Q F$, $\tilde{C}_n^* = C_n$, and $\tilde{\Theta}_n^* = \Theta_n$,

$$\tilde{T}^* = \alpha Q F \bar{A}_n^x(C_n, \Theta_n) F^T Q^T + \tilde{T}_E^*$$

But material indifference requires that $\tilde{T}^* = Q T Q^T$,

and therefore

$$\alpha Q F \bar{A}_n^x(C_n, \Theta_n) F^T Q^T + \tilde{T}_E^*$$

$$= \alpha Q F \bar{A}_n^x(C_n, \Theta_n) F^T Q^T + Q T_E Q^T$$

or

$$\tilde{T}_E^* = Q T_E Q^T.$$

A similar result holds for the other fields. Formally, we have

Theorem 7.6.4. Let A , b , and c be expressed in the reduced forms of Theorem 7.6.2. Then the constitutive equations (TMCM) satisfy the principle of material indifference iff the constitutive equations for T_E , ψ_E ,

η_E and g_E satisfy material indifference.

The next result is an immediate consequence of the requirement that the reactions not contribute to the entropy production.

Theorem 7.6.5.¹ A necessary and sufficient condition that every admissible local thermomechanical process corresponding to the constitutive equations (TMC) satisfy the reduced local dissipation inequality is that every admissible local thermomechanical process (consistent with the constraint (TMC)) and corresponding to the constitutive equations for T_E , ψ_E , η_E , and g_E satisfy the reduced local dissipation inequality.

¹Cf. Theorem 7.2.2.

Put more simply: The reactions drop out of the second law, and the Clausius-Duhem inequality need be applied only to the constitutive equations for the extra fields. Of course, for thermomechanically constrained materials, we must be careful not to violate the constraints in constructing special processes as in the lemma associated with Theorem 7.2.2.

Since the deformation gradient enters into the expressions for the reactions, the reactions must be considered in material symmetry considerations. Of course, in some special cases such as incompressibility, the dependence on the deformation gradient will drop out.

To be more specific, we consider the constitutive

equation for the stress

$$\underline{T} = \alpha \bar{A}(\underline{F}, \theta) + \underline{T}_E,$$

where we have used Theorem 7.6.3 to express the traction stress $\underline{T}_R$. Suppose $\underline{H}$ is an element of the symmetry group of the material; then $\underline{H}$ must belong to the symmetry group for the stress response, which means that replacing $\underline{F}$ by $\underline{F}\underline{H}$ in the constitutive equation for the stress does not change the stress.[†] Then

$$\alpha \bar{A}(\underline{F}\underline{H}, \theta) + \underline{T}_E \Big|_{\underline{F}=\underline{F}\underline{H}} = \alpha \bar{A}(\underline{F}, \theta) + \underline{T}_E. \quad (*)$$

Now as far as the theory of constitutive equations is concerned, α is arbitrary. Hence, (*) must hold $\forall \alpha$; an

[†] See § 7.5.

in particular, it must hold for $\alpha=0$. This yields

$$\left. \underline{\underline{T}}_E \right|_{\underline{\underline{F}}=\underline{\underline{FH}}} = \underline{\underline{T}}_E, \quad (**)$$

i.e., the extra stress by itself must meet the symmetry condition just as if the reaction stress were not present. $(**)$ and $(*) \Rightarrow$

$$\alpha \underline{\underline{A}}(\underline{\underline{FH}}, \theta) = \alpha \underline{\underline{A}}(\underline{\underline{E}}, \theta),$$

and so the reaction stress by itself must also satisfy the symmetry requirement.

Since the other constitutive equations can be handled in the same way, we have

Theorem 7.6.6. The reaction responses and the extra responses in (TMCM) must individually satisfy any material symmetry requirements imposed on the body

In view of the general nature of Theorem 7.6.6, we point out again that our deliberations are not yet restricted to thermoelastic materials. Of course, the restricted character of (TMC) is especially suited to thermoelasticity.

Now suppose that the material is isotropic (with $\mathcal{R}$ an undistorted reference configuration) so that $\mathcal{O} \subset \mathcal{O}_n^x$.

Let $\underline{Q} \in \mathcal{O}$, then $\underline{Q}^T \in \mathcal{O}$. By Theorems 7.6.6, 7.6.3, and 7.6.2

$$\alpha(\underline{F}\underline{Q}) \bar{A}_n^x(\underline{Q}\underline{Q}^T, \theta) \underline{Q} \underline{F}^T = \alpha \underline{F} \bar{A}_n^x(\underline{Q}, \theta) \underline{F}^T,$$

$$-\frac{\alpha}{\rho} \bar{b}_n^x(\underline{Q}\underline{Q}^T, \theta) = -\frac{\alpha}{\rho} \bar{b}_n^x(\underline{Q}, \theta),$$

$$\alpha \theta(\underline{F}\underline{Q}) \bar{E}_n^x(\underline{Q}\underline{Q}^T, \theta) = \alpha \theta \underline{F} \bar{E}_n^x(\underline{Q}, \theta).$$

With some rearranging, we get

$$\bar{A}_n^x(\underline{Q}\underline{Q}^T, \theta) = \underline{Q} \bar{A}_n^x(\underline{Q}, \theta) \underline{Q}^T,$$

$$\bar{b}_n^x(\underline{Q}\underline{C}\underline{Q}^T, \theta) = \bar{b}_n^x(\underline{C}, \theta),$$

$$\bar{\underline{c}}_n^x(\underline{Q}\underline{C}\underline{Q}^T, \theta) = \underline{Q} \bar{\underline{c}}_n^x(\underline{C}, \theta).$$

Since the equations must hold $\forall \underline{Q} \in \mathcal{O}$ and $\forall$

$\underline{C} \in \mathcal{S}^+$, $\theta \in \mathbb{R}^+$; $\bar{A}_n^x$, b_n^x , and $\bar{\underline{c}}_n^x$ are isotropic

functions. By the proof of Theorem 4.2.1, $\bar{A}_n^x$ and b_n^x

have the representations

$$\bar{A}_n^x(\underline{C}_n, \theta) = {}_0\bar{a}_n^x(I_C, II_C, III_C, \theta) \underline{1}_n + {}_1\bar{a}_n^x(-) \underline{C}_n + {}_2\bar{a}_n^x(-) \underline{C}_n^2$$

and

$$\bar{b}_n^x(\underline{C}_n, \theta) = {}_0b_n^x(I_C, II_C, III_C, \theta).$$

We have not considered the representation of an

isotropic, vector-valued function of one symmetric tensor

before, but this is surprisingly easy. Choosing $\underline{Q} = -\underline{1}$

in $\bar{\underline{c}}_n^x(\underline{Q}\underline{C}\underline{Q}^T, \theta) = \underline{Q} \bar{\underline{c}}_n^x(\underline{C}, \theta)$ yields

$$\bar{\underline{c}}_n^x(\underline{C}, \theta) = -\bar{\underline{c}}_n^x(\underline{C}, \theta) \Rightarrow \bar{\underline{c}}_n^x(\underline{C}, \theta) \equiv \underline{0}.$$

Placing the expression for $\bar{A}_n^x$ back into Theorem 7.6.2 and following the proof of Theorem 4.2.2 leads to

$$\bar{A}_n^x = {}_0 a_n^x(I_B, II_B, III_B, \theta) \underline{1} + {}_1 a_n^x(-) \underline{B}_n + {}_2 a_n^x(-) \underline{B}_n^2$$

As usual, the corresponding reactions are given by Theorem 7.6.3. We sum these results up in

Theorem 7.6.7, Let a body¹ subjected to the thermomechanical constraint

$$\bar{A}_n^x(F_n, \theta) \cdot \underline{D} + \bar{b}_n^x(F_n, \theta) \dot{\theta} + \bar{c}_n^x(F_n, \theta) \cdot \underline{g} = 0$$

be isotropic at X (with $\underline{n}$ an undistorted reference configuration). Then

$$\bar{A}_n^x(F_n, \theta) = {}_0 a_n^x(I_B, II_B, III_B, \theta) \underline{1} + {}_1 a_n^x(-) \underline{B}_n + {}_2 a_n^x(-) \underline{B}_n^2$$

$$\bar{b}_n^x(F_n, \theta) = {}_0 b_n^x(I_B, II_B, III_B, \theta),$$

¹Not necessarily thermoelastic.

$$\underline{C}_n^x(F_n, \theta) = \underline{Q};$$

and the reactions are given by

$$\underline{T}_R = \alpha \left[{}_0 a_n^x(I_B, II_B, III_B, \theta) \underline{1} + {}_1 a_n^x(-) \underline{B}_n + {}_2 a_n^x(-) \underline{B}_n^2 \right]$$

$$\underline{\eta}_R = -\frac{\alpha}{\rho} {}_0 b_n^x(I_B, II_B, III_B, \theta),$$

$$\underline{Q}_R = \underline{Q},$$

$$\underline{\psi}_R = 0.^1$$

The result that $\underline{Q} = \underline{Q}$ for isotropic materials can be interpreted as requiring the temperature gradient to be unconstrained.² The Gurtin and Shindugli

theory leads to the conclusion that isotropy requires that either the temperature gradient is unconstrained

¹Recalling the agreement on p. 7.6, 14.

²In our special class of thermomechanical constraints

on the material is a perfect conductor. The requirement of being a perfect conductor is a constraint on the temperature gradient, but we will find that the discrepancy noted here is not as serious as it may appear when we look at the constraint of perfect conduction.

Next suppose that the body is fluid. Since fluids are isotropic, we start with the representations of Theorem 7.6.7 and choose $\underline{H} = |\det \underline{F}|^{1/3} \underline{F}^{-1}$.¹ Then as on p. 4.2.10, replacing $\underline{F}$ by $\underline{F}\underline{H}$ leads to replacing $\underline{B}$ by $|\det \underline{F}|^{2/3} \underline{1}$; whence,

$$\underline{A} = \hat{a}_x^x(|\det \underline{F}|, \theta) \underline{1}$$

and

$$b = \hat{b}_x^x(|\det \underline{F}|, \theta).$$

¹Cf. Theorems 4.2.3 and 7.5.11 and Exercise 7.5.8.

Formally, we have

Theorem 7.6.8. Let a body¹ subjected to the

thermomechanical constraint

$$\underline{A}_n^x(\underline{F}_n, \theta) \cdot \underline{D} + \underline{b}_n^x(\underline{F}_n, \theta) \dot{\theta} + \underline{\rho}_n^x(\underline{F}_n, \theta) \cdot \underline{g} = 0$$

be fluid at X . Then

$$\underline{A}_n^x(\underline{F}_n, \theta) = \hat{\underline{a}}_n^x(|\det \underline{F}_n|, \theta) \underline{1},$$

$$\underline{b}_n^x(\underline{F}_n, \theta) = \hat{\underline{b}}_n^x(|\det \underline{F}_n|, \theta),$$

$$\underline{\rho}_n^x(\underline{F}_n, \theta) = \underline{0};$$

and the reactions are given by

$$\underline{T}_R = \alpha \hat{\underline{a}}_n^x(|\det \underline{F}_n|, \theta) \underline{1},$$

$$\eta_R = -\frac{\alpha}{\rho} \hat{\underline{b}}_n^x(|\det \underline{F}_n|, \theta),$$

$$\underline{q}_R = \underline{0},$$

¹Not necessarily thermochastic.

$$\psi_R = 0.^1$$

Of course, we could easily use the density equation to replace $|\det F_n|$ by ρ in the above representations.²

Gurtin and Guidugli find that a fluid can at most be subjected to the constraint of temperature-dependent compressibility.³ What remains of (TMC) for a fluid in our case appears to be slightly more general than temperature-dependent compressibility.

¹ Recalling the agreement on p. 7.6.14,

² Cf., p. 3.5.5 and p. 4.2.11, e.g.

³ See Definition 7.6.2, Theorem 7.6.10, and Exercise 7.6.9.

Next suppose that there are several constraints so that (TMC) is replaced by

$$\tilde{A}^i \cdot \underline{D} + \tilde{b}^i \dot{\Theta} + \tilde{c}^i \cdot \underline{g} = 0 \quad (i=1, 2, \dots, m).$$

Of course, each of these constraint equations is subject to the principle of material indifference as on pp. 7.6.6-7.6.8. Each of the constraints will generate a set of reaction responses, and we replace (TMCM) with

$$\tilde{T} = \sum_{i=1}^m \tilde{T}_R^i + \tilde{T}_E,$$

$$\psi = \sum_{i=1}^m \psi_R^i + \psi_E,$$

$$\underline{\eta} = \sum_{i=1}^m \underline{\eta}_R^i + \underline{\eta}_E,$$

$$\underline{q} = \sum_{i=1}^m \underline{q}_R^i + \underline{q}_E.$$

Each set of reactions should not contribute to the entropy production for any process consistent with the corresponding constraint, and therefore we use Theorem 7.6.3 and the

agreement on p. 7.6.14 to write

$$\underline{T} = \sum_{i=1}^m \alpha_i \underline{\dot{A}}^i + \underline{T}_E,$$

$$\underline{\eta} = - \sum_{i=1}^m \frac{\alpha_i}{\rho} \underline{\dot{b}}^i + \underline{\eta}_E,$$

$$\underline{q} = \sum_{i=1}^m \alpha_i \theta \underline{\dot{c}}^i + \underline{q}_E,$$

$$\underline{\psi} = \underline{\psi}_E$$

as the constitutive equations for a material subjected to no thermomechanical constraints.

Finally, we consider some particular forms for (TMC).

Incompressibility

Here all motions are required to be isochoric, and as noted on p. 7.6.4 this corresponds to taking

$\underline{A} = \underline{1}$, $b = 0$, and $\underline{\rho} = \underline{\rho}_0$ in (TMC). These choices

are consistent with the reduced forms of Theorem 7.6.2

if we take $\bar{b}_n^x(\underline{c}_n, \theta_n) = 0$, $\bar{p}_n^x(\underline{c}_n, \theta_n) = 0$, and

$$\bar{A}_n^x(\underline{c}_n, \theta_n) = \underline{c}_n^{-1} = (\underline{F}_n^T \underline{F}_n)^{-1} = \underline{F}_n^{-1} (\underline{F}_n^T)^{-1}.$$

By Theorem 7.6.3 and the agreement on p. 7.6.14, the reactions for incompressibility are read off as stated below.

Theorem 7.6.9. For an incompressible material,
the reactions are

$$\tilde{T}_R(x, t) = -\tilde{p}(x, t) \underline{1},$$

$$\psi_R = \eta_R = 0,$$

$$\underline{I}_R = 0.$$

¹We note that this result is consistent with Theorem 3.6.5.

Since the deformation gradient has dropped out of these reactions, they need not be considered in material symmetry considerations.¹

Temperature-Dependent Compressibility

As a natural generalization of the constraint of incompressibility, we have

Definition 7.6.2. A body $\mathcal{B}$ is said to be subjected to a constraint of temperature-dependent compressibility at the particle x with respect to the reference configuration κ

iff $\exists$ a function $f_x^x: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \ni$

$$J_x(\tilde{x}, t) = f_x^x(\theta_x(\tilde{x}, t)) \quad \forall t. \quad ^2 \quad (\text{TDC})$$

¹Gf. p. 4.2.11.

²Gf. Definition 2.6.6.

To cast (TDC)₁ into the form (TMC), we differentiate it w.r.t. t :

$$\frac{\partial J_n(\underline{x}, t)}{\partial t} = J_n(\underline{x}, t) \text{ to } \underline{D}_n(\underline{x}, t);$$

$$\frac{\partial}{\partial t} [f_n^x(\theta_n(\underline{x}, t))] = (f_n^x)'(\theta_n(\underline{x}, t)) \frac{\partial \theta_n(\underline{x}, t)}{\partial t},$$

Hence, we have

Theorem 7.6.10. (TDC)₁ $\Rightarrow$

$$\text{to } \underline{D}_n(\underline{x}, t) + \frac{(f_n^x)'(\theta_n(\underline{x}, t))}{f_n^x(\theta_n(\underline{x}, t))} \dot{\theta}_n(\underline{x}, t) = 0 \quad \forall t \quad (\text{TDC})_2$$

← Exercise 7.6.4. Consider the converse of Theorem 7.6.10

We can put (TDC)₂ into the reduced form of

Theorem 7.6.2 by taking $\bar{A}_n^x(\underline{c}_n, \theta_n) = \underline{c}_n^{-1}$, $\bar{B}_n^x(\underline{c}_n, \theta_n) = -\frac{(f_n^x)'(\theta_n(\underline{x}, t))}{f_n^x(\theta_n(\underline{x}, t))}$, $\bar{c}_n^x = 0$. Then by Theorem 7.6.3

¹See Theorem 2.6.5 and Exercise 2.4.2.

and the agreement on p. 7.6.14, we have

Theorem 7.6.11. The reactions corresponding to the
constraint $(TDC)_2$ are

$$(\underline{T})_{R_x}(\underline{x}, t) = -p_x(\underline{x}, t) \underline{1},$$

$$(\underline{\eta})_{R_x}(\underline{x}, t) = \frac{-p_x(\underline{x}, t)}{\rho_x(\underline{x}, t)} (f_x^x)'(\theta_x(\underline{x}, t)),$$

$$\underline{I}_R = \underline{0}, \quad \Psi_R = 0.$$

We observe again that the deformation gradient has dropped out of the expressions for the reactions; and, thus for temperature-dependent incompressibility, the reactions drop out of material symmetry considerations.

Perfect Conductor

We can motivate the notion of a perfect

¹Here we have used the density equation (Theorem 2.10.4,

conductor by considering the classical Fourier's law,

$$\underline{q} = k \underline{g}.$$

For given $\underline{q}$, the higher the conductivity (i.e., the larger

k) the smaller $|\underline{g}|$ must be. In the limit as $k \rightarrow \infty$,

$$\underline{g} \rightarrow \underline{0}.$$

Definition 7.6.3. A body B is said to be a
perfect conductor at the particle x if the spatial
temperature gradient must vanish at x , i.e.,

$$\underline{\tilde{g}}(x, t) = \underline{0}, \quad \forall t \quad (PC)_x$$

Of course, usually a body would be required to be a perfect conductor at all its particles, and thus it could sustain only uniform temperature fields.

We can cast the constraint $(PC)_x$ into three

constraints of the form (TMC) as follows. Let

$\{\underline{e}_i\}$ be a basis for V_3 , then $\{\underline{e}'_i\}$ where

$\underline{e}'_i = \underline{F}_x^T \underline{e}_i$ is also a basis,¹ and $(PC)_1$ is equivalent to

$$\underline{g} \cdot \underline{e}_i = 0 \quad (i=1,2,3). \quad (PC)_2$$

In terms of Theorem 7.6.2, we have

$$\underline{c}_x^i = \underline{F}_x^T \underline{c}_x^i(\underline{c}_n, \theta_n) = \underline{e}_i = (\underline{F}_x^T)^{-1} \underline{e}'_i.$$

Thus, $\underline{c}_x^i(\underline{c}_n, \theta_n) = \underline{c}_x^{-1} \underline{e}'_i$. Also $\underline{A} = \underline{Q}$ and $\underline{b} = \underline{C}$

Then Theorem 7.6.3 and the agreement on p. 7.6.14 yield

$$\underline{T}_R = \underline{0}, \quad \eta_R = 0, \quad \psi_R = 0,$$

$$\underline{g}_R = \theta \sum_{i=1}^3 \alpha_i \underline{e}_i \quad \text{--- arbitrary because of } \underline{e}'_i.$$

We sum these results up in

Theorem 7.6.12. For a perfect conductor, the

¹See the proof of Theorem 1.11.6.

reactions are

$$\underline{T}_R = \underline{0}, \quad \eta_R = 0, \quad \psi_R = 0,$$

$\underline{q}_R$ arbitrary.

The perfect conductor is the thermal counterpart of the rigid body where $\underline{T}_R$ comes out to be arbitrary.

Since the deformation gradient has dropped out of the expressions for the reactions, perfect conductivity is automatically consistent with any kind of material symmetry that we might wish to impose. In particular, a perfect conductor might be isotropic. We did not pick this up in Theorem 7.6.7 because the constraint of perfect conductivity is not expressible by a single constraint equation of the

¹See Exercise 3.6.2.

form (TMC). Of course, in their general theory of constraints Gurtin and Guidugli also found the notions of isotropy and perfect conductivity to be compatible.

Isothermal

A process for which $\dot{\Theta} \equiv 0$ is said to be isothermal. There does not seem to be any special name for bodies that can sustain only isothermal processes (analogous to incompressible bodies being those that can sustain only isochoric motions). Processes in which $\dot{q} \equiv 0$ are also occasionally called isothermal. With these cautions, we proceed.

Definition 7.6.4. A body B is said to be
subjected to the isothermal constraint at the particle

X if the material time derivative of the temperature must vanish at X, i.e.,

$$\tilde{\Theta}(X, t) = 0 \quad \forall t. \quad (IC)$$

The constraint (IC) is of the form (TMC) with $A = Q$, $b = 1$, $c = Q$, and these forms satisfy the principle of material indifference.¹ We can then read off the following result from Theorem 7.6.3 and the agreement on p. 7.6.14.

Theorem 7.6.13. For a body subjected to the
isothermal constraint, the reactions are

$$\tilde{T}_R = Q, \quad \Psi_R = 0, \quad \tilde{I}_R = Q,$$

η_R arbitrary.

¹ Cf., Theorem 7.6.2.

We note again that the reactions are such that the isothermal constraint is compatible with any kind of material symmetry. In particular, a fluid might be subjected to the isothermal constraint. This is consistent with our Theorem 7.6.8; however, since (IC) is not a special case $(TDC)_2$, it is not consistent with Gurtin and Guidugli's observation that a fluid can at most be subjected to the constraint of temperature-dependent compressibility.

Inextensibility¹

We saw in §2.3 that the stretch² of a line

¹ This purely mechanical constraint could have been discussed in §3.6.

² Roughly speaking, stretch $\equiv (\text{length after}) / (\text{length before})$.

in the direction $\underline{e}$ ($|\underline{e}|=1$) in the reference configuration $\underline{x}$ is $\sqrt{\underline{e} \cdot \underline{C}_n \underline{e}}$. In particular, if there is no extension in the direction $\underline{e}$, then $\underline{e} \cdot \underline{C}_n \underline{e} = 1$.

Formally, we have

Definition 7.6.5. A body B is said to be
subjected to the constraint of inextensibility in the
direction $\underline{e}$ ($|\underline{e}|=1$) in the reference configuration at
the particle X iff

$$\underline{e} \cdot \underline{C}_n(\underline{x}, t) \underline{e} = 1 \quad \forall t, \quad (I),$$

To cast (I), into the form (TMC), we differentiate w.r.t. t :

$$\underline{e} \cdot \frac{\partial \underline{C}_n(\underline{x}, t)}{\partial t} \underline{e} = 0;$$

but by Exercise 3.6.1,

$$\frac{\partial \underline{Q}_n}{\partial t} = 2 \underline{F}^T \underline{D} \underline{F}.$$

thus,

$$\underline{e} \cdot \underline{F}^T \underline{D} \underline{F} \underline{e} = 0.$$

We leave it as Exercise 7.6.5 to show that

$$\underline{e} \cdot \underline{F}^T \underline{D} \underline{F} \underline{e} = [(\underline{F} \underline{e}) \otimes (\underline{F} \underline{e})] \cdot \underline{D}.$$

Hence, we have

Theorem 7.6.14. (I), $\Rightarrow$

$$[(\underline{F}_n \underline{e}) \otimes (\underline{F}_n \underline{e})] \cdot \underline{D} = 0 \quad \forall t, \quad (I)_2$$

Exercise 7.6.6. Consider the converse of Theorem 7.6.14.

Of course, (I)₂ is in the form (TMC) with

$\underline{A} = (\underline{F}_n \underline{e}) \otimes (\underline{F}_n \underline{e})$, $b = 0$, $\underline{c} = 0$. We leave it as

Exercise 7.6.7 to show that

$$(\underline{F}_n \underline{e}) \otimes (\underline{F}_n \underline{e}) = \underline{F}_n (\underline{e} \otimes \underline{e}) \underline{F}_n^T.$$

Hence, our expression for $\underline{\underline{A}}$ is consistent with the reduced form of Theorem 7.6.2 if we take $\bar{\underline{\underline{A}}}_n^x(\underline{\underline{e}}_n, \theta) = \underline{\underline{e}} \otimes \underline{\underline{e}}$.

By Theorem 7.6.3 and the agreement on p. 7.6.14, we have

Theorem 7.6.15, The reactions corresponding to the constraint of inextensibility in the direction $\underline{\underline{e}}$ (in the reference configuration) are

$$\underline{\underline{T}}_R = \alpha(\underline{\underline{F}}\underline{\underline{e}}) \otimes (\underline{\underline{F}}\underline{\underline{e}}), \quad (\text{uniaxial tension})$$

$$\eta_R = 0, \quad \underline{\underline{I}}_R = \underline{\underline{0}}, \quad \psi_R = 0.$$

Since the deformation gradient plays a prominent role in $\underline{\underline{T}}_R$, not all material symmetries are compatible with the constraint of inextensibility in the

direction $\underline{e}$. This should be no surprise since the constraint is obviously highly dependant on the particular direction $\underline{e}$.

Since $\underline{e}$ may vary from particle to particle, the direction of inextensibility need not correspond to a straight line. In particular, $\underline{e}$ might be the tangent to a smooth curve.

The constraint of inextensibility provides a relatively simple model of materials reinforced by inextensible cords. Of course, the concept of inextensibility is an idealization. But it may be appropriate if the cord material is much stiffer than the matrix (e.g., rubber reinforced with steel wire).

Perfect Conductivity in the Direction $\underline{e}$

Suppose $\underline{e}$ ($|\underline{e}|=1$) is the tangent vector at $\underline{x}$ to a curve in the reference configuration $\underline{x}$. The vector $\underline{F}_{\underline{x}}(\underline{x}, t) \underline{e}$ is tangent at $\underline{x}(\underline{x}, t)$ to the corresponding curve at time t . Combining this observation with the notion of perfect conductivity leads to

Definition 7.6.6. A body B is said to be a perfect conductor in the direction $\underline{e}$ in the reference configuration $\underline{x}$ at the particle $\underline{x}$ iff

$$[\underline{F}_{\underline{x}}(\underline{x}, t) \underline{e}] \cdot \underline{g}_{\underline{x}}(\underline{x}, t) = 0 \quad \forall t. \quad (PC_{\underline{e}}),$$

Exercise 7.6.8. Show that $(PC_{\underline{e}})_t$ is equivalent to

$$\underline{e} \cdot \nabla \theta(\underline{x}, t) = 0 \quad \forall t. \quad (PC_e)$$

Clearly $(PC_{\underline{e}})_1$ is in the form of (TMC) with $A = \underline{0}$, $b = 0$, $\underline{g} = \underline{F}_n \underline{e}$. We note also that this is consistent with the reduced forms of Theorem 7.6.2,

Theorem 7.6.3 and the agreement on p. 7.6.14 yield

Theorem 7.6.16. For a perfect conductor in the
direction $\underline{e}$ (in the reference configuration), the reactions
are

$$\underline{T}_R = \underline{0}, \quad \eta_R = \psi_R = 0,$$

$$\underline{q}_R = \alpha \underline{F}_n \underline{e}.$$

At the end of the previous subsection, we mentioned rubber reinforced with steel wire as an example of an inextensible material. Since steel is a much better conductor of heat than rubber, it is reasonable to model

rubber reinforced with steel wire in the direction $\underline{e}_2$ as a perfect conductor in the direction $\underline{e}_2$ as well as being inextensible in the direction $\underline{e}_2$. Then the combined reactions are

$$\underline{T}_R = \alpha (F_n \underline{e}_2) \otimes (F_n \underline{e}_2),$$

$$\underline{q}_R = \beta F_n \underline{e}_2,$$

$$\eta_R = \psi_R = 0.$$

Supplementary Reading

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