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Second-Order Tensors

A3.1. Definition of a Second-Order Tensor. The Tensor Product

A second-order tensor is simply a linear transformation[1] on a linear space into the same space. There are many instances of this in continuum mechanics; e.g., the "stress tensor" maps (linearly) "surface normals" into "stress vectors", and the "inertia tensor" of a rigid body maps (linearly) the "angular velocity vector" into the "angular momentum vector". We have already had a brief introduction to linear transformations on a linear space into a generally different linear space. In the present section, we shall mainly just specialize the definitions and results of §A2.4 to the case at hand. An important exception will be the introduction of the "tensor product" at the end of the section.

Definition A3.1.1. A second-order tensor[2] $\underset{\sim}{T}$ on a linear space $\mathcal{V}$ is a linear transformation from $\mathcal{V}$ to $\mathcal{V}$; i.e., $\underset{\sim}{T}: \mathcal{V} \to \mathcal{V}$ has the following properties:

(L1) Additivity. $\forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$

$$\underset{\sim}{T}(\underset{\sim}{u} + \underset{\sim}{v}) = \underset{\sim}{T}\underset{\sim}{u} + \underset{\sim}{T}\underset{\sim}{v} \quad ;$$

[1] See Definition A2.4.1.

(cont.)

(L2) Homogeneity. $\forall \alpha \in \mathbb{R}$ and $\forall \underset{\sim}{u} \in \mathcal{V}$

$$\underset{\sim}{T}(\alpha \underset{\sim}{u}) = \alpha(\underset{\sim}{T}\underset{\sim}{u})$$

Here, $\underset{\sim}{T}\underset{\sim}{u}$ is a more common and (convenient) notation than $\underset{\sim}{T}(\underset{\sim}{u})$ for the function value. We will almost always use boldface upper case letters to denote second-order tensors. Since higher-order tensors do not occur often in general continuum mechanics, we will usually refer to second-order tensors as simply tensors.

Our approach to second-order tensors is especially appropriate for application to continuum mechanics, but it is not the only avenue. Most engineers and physicists evidently prefer the component-transformation rule approach[1], while mathematicians usually use the multilinear function approach of § A4.

Exercise A3.1.1. Let $[a_{ij}]$ be an $n \times n$ matrix of real numbers. Consider n-dimensional number space $\mathbb{R}^n$.)[2] With the list $\underset{\sim}{u} = (u_1, u_2, \cdots, u_n) \in \mathbb{R}^n$, associate the list $\underset{\sim}{v} = \underset{\sim}{A}\underset{\sim}{u} =$

(cont. from p. A3.1.1)

[2] Often, endomorphism, linear operator.

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[1] Cf. the remarks on p. A2.6.6 and in § A3.3.

[2] See Theorem A2.1.1.

$(v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$ defined by

$$v_i = \sum_{j=1}^{n} a_{ij} u_j \quad , \quad i = 1, 2, \cdots, n.$$

Show that $\underset{\sim}{A}$ is a second-order tensor on $\mathbb{R}^n$.

Exercise A3.1.2. Consider the infinite-dimensional space $C^0([a,b])$.[1] With $f \in C^0([a,b])$ associate $g \in C^0([a,b])$, where g is defined by

$$g(x) = \int_a^x f(\xi) d\xi ;$$

symbolically, $g = \int f$. Show that $\int$ is a second-order tensor on $C^0([a,b])$. Is the derivative a second-order tensor on $C^1([a,b])$?)[2]

From Theorems A2.4.1 and 2, we have

Theorem A3.1.1. Let $\underset{\sim}{T}$ be a second-order tensor on a linear space $\mathcal{V}$. Then

(i) $\underset{\sim}{T}\,\underset{\sim}{0} = \underset{\sim}{0}$

and

(ii) $\underset{\sim}{T}(\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}) = \alpha \underset{\sim}{T}\,\underset{\sim}{u} + \beta \underset{\sim}{T}\,\underset{\sim}{v} \quad \forall \alpha, \beta \in \mathbb{R}$ and $\forall \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$.

As in Definition A2.4.4, two tensors are viewed as equal if they are the same function. Thus,

Definition A3.1.2. Two second-order tensors $\underset{\sim}{S}$ and $\underset{\sim}{T}$ on a linear

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[1] See Exercise A2.1.3. | [2] See p. 2.4.2a.

space $\mathcal{V}$ are equal, and we write $\mathbf{S} = \mathbf{T}$, if

$$\mathbf{S}\mathbf{u} = \mathbf{T}\mathbf{u} \quad \forall\, \mathbf{u} \in \mathcal{V}.$$

We shall have several occasions to use the following variant of this.

Theorem A3.1.2. Let $\mathbf{S}$ and $\mathbf{T}$ be second-order tensors on a normed[1] linear space $\mathcal{V}$. Then $\mathbf{S} = \mathbf{T}$ iff

$$\mathbf{S}\mathbf{u} = \mathbf{T}\mathbf{u} \quad \forall\, \mathbf{u} \in \{\mathbf{v} \in \mathcal{V} : \|\mathbf{v}\| = 1\}.$$

Proof. Suppose $\mathbf{S} = \mathbf{T}$. Then by Definition A3.1.2, they necessarily agree on all unit vectors.

Conversely, suppose $\mathbf{S}$ and $\mathbf{T}$ agree on all unit elements. Let $\mathbf{u} \in \mathcal{V}$. Then either $\mathbf{u} \neq \mathbf{0}$ or $\mathbf{u} = \mathbf{0}$. If $\mathbf{u} \neq \mathbf{0}$, then

$$\mathbf{S}\frac{\mathbf{u}}{\|\mathbf{u}\|} = \mathbf{T}\frac{\mathbf{u}}{\|\mathbf{u}\|} \qquad \left(\left\| \frac{\mathbf{u}}{\|\mathbf{u}\|} \right\| = 1 \right)$$

$$\Longrightarrow$$

$$\frac{1}{\|\mathbf{u}\|}\mathbf{S}\mathbf{u} = \frac{1}{\|\mathbf{u}\|}\mathbf{T}\mathbf{u} \qquad (\text{homogeneity})$$

$$\Longrightarrow \qquad \mathbf{S}\mathbf{u} = \mathbf{T}\mathbf{u} \qquad (\text{multiply by } \|\mathbf{u}\|)$$[2].

If $\mathbf{u} = \mathbf{0}$, then $\mathbf{S}\mathbf{u} = \mathbf{T}\mathbf{u} = \mathbf{0}$ by Theorem A3.1.1. Thus, we have

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[1] See Definition A2.3.3.

[2] See Theorem A2.1.9.

$\underset{\sim}{S}\underset{\sim}{u} = \underset{\sim}{T}\underset{\sim}{u} \quad \forall\, \underset{\sim}{u} \in \mathcal{V}$; and by Definition A3.1.2, $\underset{\sim}{S} = \underset{\sim}{T}$. $\square$

From Theorems A2.4.3 and 4, we get

Theorem A3.1.3. Let $\underset{\sim}{S}$ and $\underset{\sim}{T}$ be second-order tensors on a linear space $\mathcal{V}$. Then their sum, $\underset{\sim}{S} + \underset{\sim}{T}$, defined on $\mathcal{V}$ by

$$(\underset{\sim}{S} + \underset{\sim}{T})\underset{\sim}{u} = \underset{\sim}{S}\underset{\sim}{u} + \underset{\sim}{T}\underset{\sim}{u}\,;$$

the scalar multiple of $\underset{\sim}{T}$ by $\alpha \in \mathbb{R}$, $\alpha\underset{\sim}{T}$, defined on $\mathcal{V}$ by

$$(\alpha\underset{\sim}{T})\underset{\sim}{u} = \alpha(\underset{\sim}{T}\underset{\sim}{u})\,;$$

and the zero second-order tensor, $\underset{\sim}{O}$, defined on $\mathcal{V}$ by

$$\underset{\sim}{O}\underset{\sim}{u} = \underset{\sim}{0}$$

are all second-order tensors on $\mathcal{V}$.

Finally, as a special case of Theorem A2.4.5, we have

Theorem A3.1.4. With the above definitions for addition, scalar multiplication, equality, and zero, the set of all second-order tensors on $\mathcal{V}$,

$$\operatorname{Lin}(\mathcal{V},\mathcal{V}) =: \operatorname{Lin}\mathcal{V},$$

<u>is a linear space.</u>

Consequently, all of the general properties of linear spaces, such as those given in Theorems A2.1.2–10, apply to $\operatorname{Lin}\mathcal{V}$. E.g., in the present context, Theorem A2.1.3 says

if $\underset{\sim}{R}, \underset{\sim}{S}, \underset{\sim}{T} \in \operatorname{Lin}\mathcal{V}$, then

$$\underset{\sim}{R} + \underset{\sim}{S} = \underset{\sim}{R} + \underset{\sim}{T} \Rightarrow \underset{\sim}{S} = \underset{\sim}{T}.$$

We shall regularly use such results without additional comment.

Since the identity map on a linear space $\mathcal{V}$ necessarily has values in $\mathcal{V}$, it is a candidate for being a second-order tensor.

Theorem A3.1.5. <u>Let</u> $\mathcal{V}$ <u>be a linear space</u>. <u>Define the function</u> $\underset{\sim}{I} : \mathcal{V} \to \mathcal{V}$ <u>by</u>

$$\underset{\sim}{I}(\underset{\sim}{v}) = \underset{\sim}{v}.$$

<u>Then</u> $\underset{\sim}{I} \in \operatorname{Lin}\mathcal{V}$, <u>and we write</u> $\underset{\sim}{I}(\underset{\sim}{v}) = \underset{\sim}{I}\underset{\sim}{v}$. $\underset{\sim}{I}$ <u>is called the</u> identity tensor.[1]

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[1] Sometimes, the unit tensor, and then, especially, it is denoted by $\underset{\sim}{1}$.

Proof. Exercise A3.1.3. □

We already have two ways of multiplying two elements of a linear space together: the inner product, which yields a scalar; the vector product, which yields an element of the same (three-dimensional) linear space. The next theorem introduces a multiplication which results in a tensor.

Theorem A3.1.6. *Let* $\mathcal{V}$ *be an inner product space. Then the* **tensor product**[1], $\underset{\sim}{u} \otimes \underset{\sim}{v}$, *of two elements* $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$, *defined as a function on* $\mathcal{V}$ *to* $\mathcal{V}$ *by*

$$(\underset{\sim}{u} \otimes \underset{\sim}{v})\underset{\sim}{w} = (\underset{\sim}{v} \cdot \underset{\sim}{w})\underset{\sim}{u} \quad , \quad \underset{\sim}{w} \in \mathcal{V},$$

is a second-order tensor on $\mathcal{V}$.

Proof. Exercise A3.1.4. □

The more important features of the tensor product are given by

Theorem A3.1.7. *The tensor product of two elements* $\underset{\sim}{u}$ *and* $\underset{\sim}{v}$ *of an inner product space* $\mathcal{V}$ *has the following properties:*

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[1] Sometimes, **dyadic product**. In "dyadic notation" the tensor product of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is denoted by $\underset{\sim}{u}\underset{\sim}{v}$.

(i) Homogeneity. $\forall\, \alpha, \beta \in \mathbb{R}$ and $\forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$

$$(\alpha \underset{\sim}{u}) \otimes (\beta \underset{\sim}{v}) = (\alpha\beta)(\underset{\sim}{u} \otimes \underset{\sim}{v});$$

(ii) Distributivity w.r.t. addition. $\forall\, \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathcal{V}$

$$\underset{\sim}{u} \otimes (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \otimes \underset{\sim}{v} + \underset{\sim}{u} \otimes \underset{\sim}{w},$$

$$(\underset{\sim}{u} + \underset{\sim}{v}) \otimes \underset{\sim}{w} = \underset{\sim}{u} \otimes \underset{\sim}{w} + \underset{\sim}{v} \otimes \underset{\sim}{w};$$

(iii) $\forall\, \underset{\sim}{u} \in \mathcal{V}$

$$\underset{\sim}{u} \otimes \underset{\sim}{0} = \underset{\sim}{0} \otimes \underset{\sim}{u} = \underset{\sim}{O};$$

(iv) For $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$

$\underset{\sim}{u} \otimes \underset{\sim}{v} = \underset{\sim}{v} \otimes \underset{\sim}{u}$ iff $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are parallel.[1]

Proof. Exercises A3.1.5-8. □

[1] See Definition A2.3.6 and the surrounding discussion.

Supplementary Reading

Same as for § A2.4 plus

CHADWICK, Continuum Mechanics

COLEMAN, MARKOVITZ, and NOLL, Viscometric Flows of Non-Newtonian Fluids, Appendix on Mathematical Concepts

GURTIN, An Introduction to Continuum Mechanics

NICKERSON, SPENCER, and STEENROD, Advanced Calculus

A3.2. Second-Order Tensors on Finite-Dimensional Linear Spaces

When the underlying linear space is finite-dimensional, considerable simplification occurs in the theory of tensors. As a good example of this we have

Theorem A3.2.1. Two second-order tensors $\underset{\sim}{S}$ and $\underset{\sim}{T}$ on an n-dimensional linear space $\mathcal{V}_n$ are equal iff they agree on any basis $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ for $\mathcal{V}_n$. I.e.,

$$\underset{\sim}{S} = \underset{\sim}{T} \iff \underset{\sim}{S}\underset{\sim}{e}_i = \underset{\sim}{T}\underset{\sim}{e}_i, \quad i = 1, 2, \cdots, n.$$

Proof. Suppose $\underset{\sim}{S} = \underset{\sim}{T}$. Then by Definition A3.1.2,

$$\underset{\sim}{S}\underset{\sim}{u} = \underset{\sim}{T}\underset{\sim}{u} \quad \forall \underset{\sim}{u} \in \mathcal{V}_n.$$

In particular,

$$\underset{\sim}{S}\underset{\sim}{e}_i = \underset{\sim}{T}\underset{\sim}{e}_i, \quad i \in \mathbb{I}_n. \qquad (*)$$

Conversely, suppose that $(*)$ holds. Since $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ is a basis for $\mathcal{V}_n$, any $\underset{\sim}{u} \in \mathcal{V}_n$ admits the representation

$$\underset{\sim}{u} = u^i \underset{\sim}{e}_i.$$

Then

$$\underset{\sim}{S}\underset{\sim}{u} = \underset{\sim}{S}(u^i \underset{\sim}{e}_i) \qquad \text{(substitution)}$$

$$= u^i \underset{\sim}{S}\underset{\sim}{e}_i \qquad \text{(linearity)}$$

$$= u^i \underset{\sim}{T}\underset{\sim}{e}_i \qquad ((*))$$

$$= \underset{\sim}{T}(u^i \underset{\sim}{e}_i) \qquad \text{(linearity)}$$

$$= \underset{\sim}{T}\underset{\sim}{u} \qquad \text{(substitution)};$$

and since this holds $\forall\, \underset{\sim}{u} \in \mathcal{V}_n$, $\underset{\sim}{S} = \underset{\sim}{T}$. $\square$

When the linear space $\mathcal{V}$ is finite-dimensional, then the linear space $\mathrm{Lin}\,\mathcal{V}$ is also finite-dimensional. The tensor product plays a central role in establishing this and in exhibiting the structure of $\mathrm{Lin}\,\mathcal{V}$ in this case. We approach this matter as follows.

Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ and $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^n\}$ be a basis and its dual for an n-dimensional inner product space[1] $\mathcal{V}_n$. Let $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n$ and consider its action, $\underset{\sim}{T}\underset{\sim}{u}$, on an arbitrary element $\underset{\sim}{u} \in \mathcal{V}_n$. By Theorem A2.5.8, $\underset{\sim}{u}$ admits the representations

$$\underset{\sim}{u} = u^i \underset{\sim}{e}_i = u_i \underset{\sim}{e}^i, \quad \text{where } u^i = \underset{\sim}{e}^i \cdot \underset{\sim}{u} \text{ and } u_i = \underset{\sim}{e}_i \cdot \underset{\sim}{u}.$$

As an element of $\mathcal{V}_n$, $\underset{\sim}{T}\underset{\sim}{u}$ may be represented similarly:

$$\underset{\sim}{T}\underset{\sim}{u} = (\underset{\sim}{T}\underset{\sim}{u})^i \underset{\sim}{e}_i = [\underset{\sim}{e}^i \cdot (\underset{\sim}{T}\underset{\sim}{u})]\underset{\sim}{e}_i$$

$$= \{\underset{\sim}{e}^i \cdot [\underset{\sim}{T}(u_j \underset{\sim}{e}^j)]\}\underset{\sim}{e}_i \qquad \text{(substitution)}$$

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[1] Cf. Theorem A2.3.2.

$$= \{\underset{\sim}{e}^i \cdot [u_j(\underset{\sim}{T}\underset{\sim}{e}^j)]\}\underset{\sim}{e}_i \quad \text{(linearity of } \underset{\sim}{T}\text{)}$$

$$= \{u_j[\underset{\sim}{e}^i \cdot (\underset{\sim}{T}\underset{\sim}{e}^j)]\}\underset{\sim}{e}_i \quad \text{(distributivity and homogeneity of inner product)}$$

$$= \{(\underset{\sim}{e}_j \cdot \underset{\sim}{u})[\underset{\sim}{e}^i \cdot (\underset{\sim}{T}\underset{\sim}{e}^j)]\}\underset{\sim}{e}_i \quad \text{(Theorem A2.5.8)}$$

$$= \{[\underset{\sim}{e}^i \cdot (\underset{\sim}{T}\underset{\sim}{e}^j)](\underset{\sim}{e}_j \cdot \underset{\sim}{u})\}\underset{\sim}{e}_i \quad \text{(commutativity of} \times \text{in } \mathbb{R})$$

$$= [\underset{\sim}{e}^i \cdot (\underset{\sim}{T}\underset{\sim}{e}^j)][(\underset{\sim}{e}_j \cdot \underset{\sim}{u})\underset{\sim}{e}_i] \quad \text{(distributivity and associativity of scalar multiplication)}$$

$$= [\underset{\sim}{e}^i \cdot (\underset{\sim}{T}\underset{\sim}{e}^j)][(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)\underset{\sim}{u}] \quad \text{(definition of } \otimes)$$

$$= \{[\underset{\sim}{e}^i \cdot (\underset{\sim}{T}\underset{\sim}{e}^j)](\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)\}\underset{\sim}{u} \quad \text{(definition of addition and scalar multiplication in } \mathrm{Lin}\,\mathcal{V}_n)$$

Since $\underset{\sim}{u}$ is an arbitrary element of $\mathcal{V}_n$, it follows from the definition of equality of tensors[1] that

$$\underset{\sim}{T} = (\underset{\sim}{e}^i \cdot \underset{\sim}{T}\underset{\sim}{e}^j)\,\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \,.$$

In terms of the operations that we have defined, the only interpretation that can be assigned to $\underset{\sim}{e}^i \cdot \underset{\sim}{T}\underset{\sim}{e}^j$ is $\underset{\sim}{e}^i \cdot (\underset{\sim}{T}\underset{\sim}{e}^j)$; so we dropped the parentheses in the last step above.

Similar results are obtained by starting with any of the other representations of $\underset{\sim}{u}$ and $\underset{\sim}{T}\underset{\sim}{u}$. Hence, we have proven

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[1] Definition A3.1.2.

Theorem A3.2.2. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ and $\{\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^n\}$ be a basis and its dual for an n-dimensional inner product space $\mathcal{V}_n$. Then any second-order tensor $\mathbf{T} \in \operatorname{Lin} \mathcal{V}_n$ admits the representations

$$\mathbf{T} = T^{ij}\mathbf{e}_i \otimes \mathbf{e}_j = T_{ij}\,\mathbf{e}^i \otimes \mathbf{e}^j$$

$$= T_i^{\bullet j}\mathbf{e}^i \otimes \mathbf{e}_j = T^i_{\bullet j}\,\mathbf{e}_i \otimes \mathbf{e}^j ,$$

where

$$T^{ij} = \mathbf{e}^i \cdot \mathbf{T}\mathbf{e}^j \quad , \quad T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j ,$$

$$T_i^{\bullet j} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}^j \quad , \quad T^i_{\bullet j} = \mathbf{e}^i \cdot \mathbf{T}\mathbf{e}_j .$$

The dots are inserted in the symbols $T_i^{\bullet j}$ and $T^i_{\bullet j}$ to eliminate confusion as to which index is first.

In the language of Definition A2.2.2, Theorem A3.2.2 says that the set $\{\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_1 \otimes \mathbf{e}_2, \cdots, \mathbf{e}_n \otimes \mathbf{e}_n\}$ spans the linear space $\operatorname{Lin} \mathcal{V}_n$. This observation leads us to

Theorem A3.2.3. Let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ and $\{\mathbf{e}^1, \mathbf{e}^2, \cdots, \mathbf{e}^n\}$ be a basis and its dual for an n-dimensional inner product space $\mathcal{V}_n$. Then each of the sets of n^2 second-order tensors

$$\{\mathbf{e}_i \otimes \mathbf{e}_j\} , \{\mathbf{e}^i \otimes \mathbf{e}^j\} , \{\mathbf{e}^i \otimes \mathbf{e}_j\} , \{\mathbf{e}_i \otimes \mathbf{e}^j\} \quad)^1$$

---

1 Here, of course i and j range over $\mathbb{I}_n$.

<u>is a Basis for</u> $\mathrm{Lin}\,\mathcal{V}_n$, <u>and, consequently,</u> $\mathrm{Lin}\,\mathcal{V}_n$ <u>is</u> n^2-<u>dimensional</u>

Proof. In view of the remark preceding the theorem and Definitions A2.2.2 and A2.2.3, all that remains to be shown is that the set $\{\underset{\sim}{e}^i \otimes \underset{\sim}{e}^j\}$, for example, is linearly independent. Accordingly, consider the linear combination

$$\alpha_{ij}\, \underset{\sim}{e}^i \otimes \underset{\sim}{e}^j = \underset{\sim}{O} .$$

Let $\underset{\sim}{u} \in \mathcal{V}_n$. Then

$$(\alpha_{ij}\, \underset{\sim}{e}^i \otimes \underset{\sim}{e}^j)\underset{\sim}{u} = \underset{\sim}{O}\,\underset{\sim}{u} \quad \text{(substitution)}$$

$$= \underset{\sim}{0} \quad \text{(Theorem A3.1.3)} .$$

But also

$$(\alpha_{ij}\, \underset{\sim}{e}^i \otimes \underset{\sim}{e}^j)\underset{\sim}{u} = \alpha_{ij}\left[(\underset{\sim}{e}^i \otimes \underset{\sim}{e}^j)\underset{\sim}{u}\right] \quad \text{(linearity)}$$

$$= \alpha_{ij}\left[(\underset{\sim}{e}^j \cdot \underset{\sim}{u})\underset{\sim}{e}^i\right] \quad \text{(Theorem A3.1.6)}$$

$$= \alpha_{ij}(u^j \underset{\sim}{e}^i) \quad \text{(Theorem A2.5.8)}$$

$$= (\alpha_{ij}\, u^j)\underset{\sim}{e}^i \quad \text{(linear space axioms)} .$$

$\therefore$ $(\alpha_{ij} u^j)\underset{\sim}{e}^i = \underset{\sim}{0}$, and by Theorem A2.2.12

$$\alpha_{ij}\, u^j = 0, \; i \in \mathcal{I}_n .$$

Since $\underset{\sim}{u}$ is an arbitrary element of $\mathcal{V}_n$, this must hold $\forall\, u^j$.

The choice $u^1 = 1$, $u^2 = u^3 = \cdots = u^n = 0$ leads to $\alpha_{i1} = 0$. With similar other special choices we conclude that

$$\alpha_{ij} = 0 , \quad i,j \in \mathbb{I}_n .$$

Hence, the set $\{\underset{\sim}{e}^i \otimes \underset{\sim}{e}^j\}$ is linearly independent by Theorem A2.2.2. □

The reader should not miss the fact that in order to investigate the tensor $\alpha_{ij}\, \underset{\sim}{e}^i \otimes \underset{\sim}{e}^j$ in the above proofs we checked to see how it mapped a typical element of $\mathcal{V}_n$. Tensors are above all (linear) functions; in order to examine a particular tensor, we just look at how it maps the members of its domain.

In parallel with Definitions A2.2.4 and A2.5.2, it is now appropriate to make

Definition A3.2.1. <u>Refer to the terminology of Theorem A3.2.2. The scalars</u>

T_{ij} <u>are the</u> *covariant components* <u>of $\underset{\sim}{T}$</u>,

T^{ij} <u>are the</u> *contravariant components* <u>of $\underset{\sim}{T}$</u>,

$T_{i}^{\cdot j}$ <u>and</u> $T^{i}_{\cdot j}$ <u>are the</u> *mixed components* <u>of $\underset{\sim}{T}$</u>

<u>w.r.t. the basis system</u> $\{\underset{\sim}{e}_i\}$, $\{\underset{\sim}{e}^i\}$ <u>for</u> $\mathcal{V}_n$.

As an immediate consequence of Definition A3.2.1 and Theorems A3.2.2 and A2.5.7, we have the following analog of Theorem A2.5.9.

Theorem A3.2.4. Relative to an orthonormal basis for an n-dimensional inner product space $\mathcal{V}_n$, the covariant, contravariant, and mixed components a second-order tensor $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$ coalesce; and we write

$$T_{ij} = T^{ij} = T_{i}^{\bullet j} = T^{i}_{\bullet j} = T_{\langle ij \rangle} = T^{\langle ij \rangle}.$$

Since $\operatorname{Lin} \mathcal{V}_n$ is a finite-dimensional linear space, the general results of § A2.2 apply. In particular, from Theorem A2.2.12 we can read off

Theorem A3.2.5. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots, \underset{\sim}{e}_n\}$ and $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \dots, \underset{\sim}{e}^n\}$ be a basis and its dual for an n-dimensional inner product space $\mathcal{V}_n$. Let $\underset{\sim}{S}, \underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$ so that, e.g.,

$$\underset{\sim}{S} = S^{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \quad , \quad \underset{\sim}{T} = T^{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j .$$

Then

(i) For $\alpha, \beta \in \mathbb{R}$,

$$\alpha \underset{\sim}{S} + \beta \underset{\sim}{T} = (\alpha S^{ij} + \beta T^{ij}) \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \;;$$ [1]

(ii) $\underset{\sim}{0} = 0 \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_1 + 0 \underset{\sim}{e}_1 \otimes \underset{\sim}{e}_2 + \dots + 0 \underset{\sim}{e}_n \otimes \underset{\sim}{e}_n$;

(iii) $\underset{\sim}{S} = \underset{\sim}{T} \iff S^{ij} = T^{ij}, \quad i, j \in \mathbb{I}_n$;

[1] I.e., in transparent notation, $(\alpha \underset{\sim}{S} + \beta \underset{\sim}{T})^{ij} = \alpha S^{ij} + \beta T^{ij}$.

with similar statements for each of the other types of components.

We note that (iii) above includes the result that the components (of any type) of a given tensor $\underset{\sim}{T} \in \mathrm{Lin}\, V_n$ w.r.t a given basis system are unique.

The components of the tensor product $\underset{\sim}{u} \otimes \underset{\sim}{v}$ are easily calculated. E.g.,

$$
\begin{aligned}
(\underset{\sim}{u} \otimes \underset{\sim}{v})_{ij} &= \underset{\sim}{e}_i \cdot \left[(\underset{\sim}{u} \otimes \underset{\sim}{v}) \underset{\sim}{e}_j \right] \quad \text{(Theorem A3.2.2)} \\
&= \underset{\sim}{e}_i \cdot \left[(\underset{\sim}{v} \cdot \underset{\sim}{e}_j) \underset{\sim}{u} \right] \quad (\text{definition of } \underset{\sim}{u} \otimes \underset{\sim}{v}) \\
&= \underset{\sim}{e}_i \cdot (v_j \underset{\sim}{u}) \quad \text{(Theorem A2.5.8)} \\
&= v_j (\underset{\sim}{e}_i \cdot \underset{\sim}{u}) \quad \text{(bilinearity of inner product)} \\
&= v_j u_i \quad \text{(Theorem A2.5.8)} \\
&= u_i v_j \quad (\text{commutativity of multiplication in } \mathbb{R})
\end{aligned}
$$

Thus, we have

Theorem A3.2.6. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ and $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^n\}$ be a basis and its dual for an n-dimensional inner product space V_n. Let $\underset{\sim}{u}, \underset{\sim}{v} \in V_n$. Then the components of the tensor product

$\underset{\sim}{u} \otimes \underset{\sim}{v}$ are given by

$$(\underset{\sim}{u} \otimes \underset{\sim}{v})^{ij} = u^i v^j \,, \quad (\underset{\sim}{u} \otimes \underset{\sim}{v})_{ij} = u_i v_j \,,$$

$$(\underset{\sim}{u} \otimes \underset{\sim}{v})_{i}^{\bullet\, j} = u_i v^j \,, \quad (\underset{\sim}{u} \otimes \underset{\sim}{v})^{i}_{\bullet\, j} = u^i v_j \,.$$

As the proof of Theorem A3.2.2 suggests, tensor equations on a finite-dimensional linear space are equivalent to systems of linear algebraic equations. Specifically, we have

Theorem A3.2.9. *Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ and $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^n\}$ be a basis and its dual for an n-dimensional inner product space $\mathcal{V}_n$. Let $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_n$ and $\underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}_n$. Then the equation*

$$\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v}$$

is equivalent to each of the following systems of equations in the components of $\underset{\sim}{u}$, $\underset{\sim}{v}$, and $\underset{\sim}{T}$: for $i \in \mathcal{I}_n$

$$T_{ij} u^j = v_i \quad ; \quad T^{ij} u_j = v^i \,;$$

$$T_i^{\bullet\, j} u_j = v_i \quad ; \quad T^i_{\bullet\, j} u^j = v^i \,.$$

Proof. Exercise A3.2.1. □

INSERT for p. A3.2.13

The components of the identity tensor $\underset{\sim}{I}$ are also easy to calculate :

$$(\underset{\sim}{I})^{ij} = \underset{\sim}{e}^i \cdot \underset{\sim}{I}\underset{\sim}{e}^j \quad (\text{Theorem A3.2.2})$$

$$= \underset{\sim}{e}^i \cdot \underset{\sim}{e}^j \quad (\text{definition of } \underset{\sim}{I})$$

$$= g^{ij} \quad (\text{Definition A2.5.3}).$$

Hence, we have

Theorem A3.2.7. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ and $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^3\}$ be a basis and its dual for an n-dimensional inner product space $\mathcal{V}_n$. Then the identity tensor $\underset{\sim}{I}$ on $\mathcal{V}_n$ has the following component representations:

$$\underset{\sim}{I} = g^{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = g_{ij}\underset{\sim}{e}^i \otimes \underset{\sim}{e}^j$$

$$= g^{i}_{\cdot j}\underset{\sim}{e}_i \otimes \underset{\sim}{e}^j = \delta^i_j \underset{\sim}{e}_i \otimes \underset{\sim}{e}^j = \underset{\sim}{e}_1 \otimes \underset{\sim}{e}^1 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}^2 + \cdots + \underset{\sim}{e}_n \otimes \underset{\sim}{e}^n$$

$$= g_i^{\cdot j}\underset{\sim}{e}^i \otimes \underset{\sim}{e}_j = \delta_i^j \underset{\sim}{e}^i \otimes \underset{\sim}{e}_j = \underset{\sim}{e}^1 \otimes \underset{\sim}{e}_1 + \underset{\sim}{e}^2 \otimes \underset{\sim}{e}_2 + \cdots + \underset{\sim}{e}^n \otimes \underset{\sim}{e}_n .$$

In particular, if $\{\underset{\sim}{e}_{\langle 1\rangle}, \underset{\sim}{e}_{\langle 2\rangle}, \cdots, \underset{\sim}{e}_{\langle n\rangle}\}$ is an orthonormal basis for $\mathcal{V}_n$, then

$$\underset{\sim}{I} = \delta^{ij}\underset{\sim}{e}_{\langle i\rangle} \otimes \underset{\sim}{e}_{\langle j\rangle} = \underset{\sim}{e}_{\langle 1\rangle} \otimes \underset{\sim}{e}_{\langle 1\rangle} + \underset{\sim}{e}_{\langle 2\rangle} \otimes \underset{\sim}{e}_{\langle 2\rangle} + \cdots + \underset{\sim}{e}_{\langle n\rangle} \otimes \underset{\sim}{e}_{\langle n\rangle}.$$

The components of the identity tensor, i.e., the g-symbols, can be used to raise and lower indices on tensor components in the same way as for elements of $\mathcal{V}_n$.)[1] Formally, we have

Theorem A3.2.8. *Let $\underset{\sim}{T} \in \text{Lin}\,\mathcal{V}_n$, where $\mathcal{V}_n$ is an n-dimensional inner product space. Then for $i,j \in \mathbb{I}_n$*

$$T^{ij} = g^{im} g^{jn} T_{mn} = g^{im} T_m{}^{\bullet j} = g^{jm} T^{i}{}_{\bullet m} \, ,$$

$$T_{ij} = g_{im} g_{jn} T^{mn} = g_{im} T^{m}{}_{\bullet j} = g_{jm} T_i{}^{\bullet m} \, ,$$

$$T_i{}^{\bullet j} = g_{im} g^{jn} T^{m}{}_{\bullet n} = g_{im} T^{mj} = g^{jm} T_{im} \, ,$$

$$T^{i}{}_{\bullet j} = g^{im} g_{jn} T_m{}^{\bullet n} = g^{im} T_{mj} = g_{jm} T^{im} .$$

Proof. We consider just the first claim. The rest are proven in the same manner.

$$\begin{aligned}
T^{ij} &= \underset{\sim}{e}^i \cdot \underset{\sim}{T}\, \underset{\sim}{e}^j && \text{(Theorem A3.2.2)} \\
&= \underset{\sim}{e}^i \cdot \left[(T_{mn} \underset{\sim}{e}^m \otimes \underset{\sim}{e}^n) \underset{\sim}{e}^j \right] && \text{(Theorem A3.2.2)} \\
&= \underset{\sim}{e}^i \cdot \left\{ T_{mn} \left[(\underset{\sim}{e}^m \otimes \underset{\sim}{e}^n) \underset{\sim}{e}^j \right] \right\} && \text{(Theorem A3.1.3)} \\
&= \underset{\sim}{e}^i \cdot \left\{ T_{mn} \left[(\underset{\sim}{e}^n \cdot \underset{\sim}{e}^j) \underset{\sim}{e}^m \right] \right\} && \text{(definition of } \underset{\sim}{u} \otimes \underset{\sim}{v}) \\
&= \underset{\sim}{e}^i \cdot \left[T_{mn} (g^{nj}) \underset{\sim}{e}^m \right] && \text{(definition of } g^{nj}) \\
&= T_{mn} g^{nj} (\underset{\sim}{e}^i \cdot \underset{\sim}{e}^m) && \text{(linearity of inner product)} \\
&= T_{mn} g^{nj} g^{im} && \text{(definition of } g^{im}).
\end{aligned}$$

---

[1] Cf. Theorem A2.5.11.

$$= T_{mn} g^{jn} g^{im} \quad (\text{symmetry of } g^{nj})$$

$$= g^{im} g^{jn} T_{mn} \quad (\text{properties of } \mathbb{R}). \quad \square$$

$\boxed{\text{INSERT material on p. A3.2.9}}$

Supplementary Reading

Same as for § A3.1

A.3.3. Transformations of Bases

Of course, the components of a second-order tensor on a finite-dimensional linear space depend on the particular basis chosen for the linear space. The machinery set up in § A2.6 will make it easy to see how the components of a tensor change when the basis is changed.

Accordingly, let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$, $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^n\}$ and $\{\bar{\underset{\sim}{e}}_1, \bar{\underset{\sim}{e}}_2, \cdots, \bar{\underset{\sim}{e}}_n\}$, $\{\bar{\underset{\sim}{e}}^1, \bar{\underset{\sim}{e}}^2, \cdots, \bar{\underset{\sim}{e}}^n\}$ be two bases and their duals for an n-dimensional inner product space $\mathcal{V}_n$. Then from the previous section, $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$ admits the representations

$$\underset{\sim}{T} = T^{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = \bar{T}^{ij}\, \bar{\underset{\sim}{e}}_i \otimes \bar{\underset{\sim}{e}}_j$$

in terms of contravariant components. By Theorem A2.6.1,

$$\underset{\sim}{e}_i = \alpha^j_i\, \bar{\underset{\sim}{e}}_j\ ;$$

and $\therefore$

$$\bar{T}^{ij}\, \bar{\underset{\sim}{e}}_i \otimes \bar{\underset{\sim}{e}}_j = T^{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$$

$$= T^{ij}(\alpha^k_i\, \bar{\underset{\sim}{e}}_k) \otimes (\alpha^l_j\, \bar{\underset{\sim}{e}}_l) \quad \text{(substitution)}$$

$$= \alpha^k_i\, \alpha^l_j\, T^{ij}\, \bar{\underset{\sim}{e}}_k \otimes \bar{\underset{\sim}{e}}_l \quad \text{(Theorem A3.1.7)}$$

$$= \alpha^i_k\, \alpha^j_l\, T^{kl}\, \bar{\underset{\sim}{e}}_i \otimes \bar{\underset{\sim}{e}}_j \quad \text{(labeling)}.$$

Then by the uniqueness of components w.r.t. a given basis,[1]

$$\bar{T}^{ij} = \alpha^i_k \alpha^j_l T^{kl} .$$

All of the other types of components, as well as the transformations in the opposite direction, can be handled in exactly the same way; and thus we have

<u>Theorem A3.3.1.</u> <u>Let</u> $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}, \{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^n\}$ <u>and</u> $\{\bar{\underset{\sim}{e}}_1, \bar{\underset{\sim}{e}}_2, \cdots, \bar{\underset{\sim}{e}}_n\}, \{\bar{\underset{\sim}{e}}^1, \bar{\underset{\sim}{e}}^2, \cdots, \bar{\underset{\sim}{e}}^n\}$ <u>be two bases and their duals for an</u> n<u>-dimensional inner product space</u> $\mathcal{V}_n$. <u>Define</u>

$$\alpha^j_i = \bar{\underset{\sim}{e}}^j \cdot \underset{\sim}{e}_i \quad , \quad \beta^i_j = \bar{\underset{\sim}{e}}_j \cdot \underset{\sim}{e}^i \qquad (i,j \in \mathcal{I}_n).$$

<u>Let</u> $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n$ <u>so that</u>

$$\underset{\sim}{T} = T^{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = T_{ij} \underset{\sim}{e}^i \otimes \underset{\sim}{e}^j = T^i_{\cdot j} \underset{\sim}{e}_i \otimes \underset{\sim}{e}^j = T_i^{\cdot j} \underset{\sim}{e}^i \otimes \underset{\sim}{e}_j .$$

<u>Then</u>, <u>for</u> $i,j \in \mathcal{I}_n$,

$$\bar{T}^{ij} = \alpha^i_k \alpha^j_l T^{kl}$$ ... transformation rule for contravariant components,

$$\bar{T}_{ij} = \beta^k_i \beta^l_j T_{kl}$$ ... transformation rule for covariant components,

$$\left.\begin{aligned} \bar{T}^i_{\cdot j} &= \alpha^i_k \beta^l_j T^k_{\cdot l} \\ \bar{T}_i^{\cdot j} &= \beta^k_i \alpha^j_l T_k^{\cdot l} \end{aligned}\right\}$$ ... transformation rules for mixed components,

---

[1] Theorem A3.2.5 (iii).

and

$$\left.\begin{aligned}
T^{ij} &= \beta^i_k \beta^j_l \bar{T}^{kl} \\
T_{ij} &= \alpha^k_i \alpha^l_j \bar{T}_{kl} \\
T^i_{\cdot j} &= \beta^i_k \alpha^l_j \bar{T}^k_{\cdot l} \\
T_i^{\cdot j} &= \alpha^k_i \beta^j_l \bar{T}_k^{\cdot l}
\end{aligned}\right\} \text{--- inverse transformation rules.}$$

Exercise A3.3.1. Obtain the inverse transformation rules above by starting with the rules in the "forward direction" and then using the relationships

$$\alpha^l_i \beta^k_l = \delta^k_i \ , \ \alpha^i_l \beta^l_k = \delta^i_k$$

of Theorem A2.6.1.

Exercise A3.3.2. Prove Theorem A3.3.1 by noting from Theorem A3.2.9 that the statement $\underset{\sim}{T}\,\underset{\sim}{u} = \underset{\sim}{v}$ is equivalent to each of

$$T^{ij} u_j = v^i \text{ and } \bar{T}^{ij} \bar{u}_j = \bar{v}^i ,$$

for example; then employ the transformation rules of Theorem A2.6.2 for the components of elements of $\mathcal{V}_n$; and finally utilize the arbitrariness of $\underset{\sim}{u}$.

Obviously, if we have a list of n^2 real numbers, say $(T^{11}, T^{12}, \ldots, T^{nn})$, we can generate a second-order tensor on $\mathcal{V}_n$, say $\underset{\sim}{T}$, by letting the T^{ij} be the components of $\underset{\sim}{T}$ w.r.t. some basis for Lin $\mathcal{V}_n$, say $\{\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\}$:

$$\underset{\sim}{T} = T^{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j .$$

As noted on p. A2.6.4, a common occurrence in the physical sciences is that by some natural scheme we have real lists associated with each basis for Lin $\mathcal{V}_n$, say

$$T^{ij} \text{ with } \{\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\}, \quad \bar{T}_{ij} \text{ with } \{\bar{\underset{\sim}{e}}_i \otimes \bar{\underset{\sim}{e}}_j\}, \text{ etc.}$$

This allows us to generate the tensors

$$\underset{\sim}{T} = T^{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j, \quad \bar{\underset{\sim}{T}} = \bar{T}^{ij}\, \bar{\underset{\sim}{e}}_i \otimes \bar{\underset{\sim}{e}}_j, \text{ etc.}$$

By Theorem A3.3.1, if these tensors are all the same (i.e., if $\underset{\sim}{T} = \bar{\underset{\sim}{T}}$, etc.), then the T^{ij}, $\bar{T}^{ij}$, etc. are related through the transformation rules

$$\bar{T}^{ij} = \alpha^i_k\, \alpha^j_l\, T^{kl}, \quad T^{ij} = \beta^i_k\, \beta^j_l\, \bar{T}^{kl},$$

etc.

This leads us to the following second-order tensor counterpart of Theorem A2.6.3.

Theorem A3.3.2. Let the hypotheses of Theorem A3.3.1 hold. Let there be lists of n^2 real numbers associated with each basis for $\mathrm{Lin}\,\mathcal{V}_n$:

$$T^{ij} \text{ with } \{\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\},\quad \bar{T}^{ij} \text{ with } \{\underset{\sim}{\bar{e}}_i \otimes \underset{\sim}{\bar{e}}_j\},\ \text{etc.}$$

The T^{ij} are contravariant)[1] components of a second-order tensor

$$\underset{\sim}{T} = T^{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = \bar{T}^{ij}\underset{\sim}{\bar{e}}_i \otimes \underset{\sim}{\bar{e}}_j = \cdots$$

on $\mathcal{V}_n$ iff they satisfy the transformation rules

$$\bar{T}^{ij} = \alpha^i_k \alpha^j_l T^{kl},\quad T^{ij} = \beta^i_k \beta^j_l \bar{T}^{kl},$$

etc.

Proof. Only the "if" portion remains to be established, and this constitutes *Exercise* A3.3.3. □

As an example of this, associate the g-symbol $g^{ij} = \underset{\sim}{e}^i \cdot \underset{\sim}{e}^j$ with $\{\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\}$, etc. I.e., define the tensors

$$\underset{\sim}{G} = g^{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j,\quad \underset{\sim}{\bar{G}} = \bar{g}^{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j,\ \text{etc.}$$

Now by Theorem A2.6.4, the g-symbols satisfy the

---

[1] Of course, there are strictly analogous results for the covariant and mixed components.

transformation rules, and $\therefore$ $\underset{\sim}{G} = \underset{\sim}{\bar{G}} = \cdots$. In fact, we know from Theorem A3.2.7 that the common value of $\underset{\sim}{G}$, $\underset{\sim}{\bar{G}}$, etc. is the identity tensor $\underset{\sim}{I}$.

Exercise A3.3.4. Refer to the above discussion. Suppose that Theorem A3.2.7 had not been established. Show that $\underset{\sim}{G} = g^{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = \underset{\sim}{I}$ by examining the action of $\underset{\sim}{G}$ on an arbitrary element $\underset{\sim}{u} \in \mathcal{V}_n$.

Because of the "iff" nature of Theorem A3.3.2, it provides us with an alternative approach to second-order tensors that is equivalent to the linear transformation approach of Definition A3.1.1. The linear transformation approach is called the *direct* or *basis-free approach*; for reasons that will become evident, the direct approach is generally preferred in modern continuum mechanics. Theorem A3.3.2 supplies the *component-transformation rule approach* to second-order tensors[1], which is used by the majority of physical scientists. The wise student of mechanics becomes facile with both points of view. Mathematicians generally prefer yet another scheme called the *multilinear function approach*, which we shall introduce in Chapter A4.

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[1] Cf. the corresponding remarks for vectors (elements of $\mathcal{V}_n$) on p. A2.6.6.

Supplementary Reading

Same as for § A3.1

A3.4. Boundedness. Continuity. Operator Norm

Despite being placed in a chapter devoted to second-order tensors, most of the present section pertains to the more general case of linear transformations on a linear space $\mathcal{U}$ to a linear space $\mathcal{V}$; i.e., to the elements of $Lin(\mathcal{U},\mathcal{V})$. In analogy with the notation introduced in §3.1 for $Lin(\mathcal{V},\mathcal{V}) = Lin\,\mathcal{V}$, we shall use upper case, bold face letters to denote elements of $Lin(\mathcal{U},\mathcal{V})$, and we shall drop the parentheses in denoting function values; i.e., we write

$$\underset{\sim}{u} \mapsto \underset{\sim}{L}\underset{\sim}{u}$$

in place of

$$\underset{\sim}{u} \mapsto \underset{\sim}{L}(\underset{\sim}{u}).$$

Definition A3.4.1. Let $\underset{\sim}{L} \in Lin(\mathcal{U},\mathcal{V})$, where $\mathcal{U}$ and $\mathcal{V}$ are normed linear spaces[1]. The linear transformation $\underset{\sim}{L}$ is said to be bounded on $\mathcal{U}$ if $\exists$ a constant $M \geq 0 \ni$

$$\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} \leq M\|\underset{\sim}{u}\|_{\mathcal{U}} \quad \forall \underset{\sim}{u} \in \mathcal{U};\ )^{2}$$

such a number M is called a bound for $\underset{\sim}{L}$.

For linear transformations, boundedness is equivalent to continuity.

---

[1] See Definition A2.3.3.

[2] Here, of course, $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_{\mathcal{V}}$ are the norms on $\mathcal{U}$ and $\mathcal{V}$.

Theorem A3.4.1. Let $\underset{\sim}{L} \in Lin(\mathcal{U}, \mathcal{V})$, where $\mathcal{U}$ and $\mathcal{V}$ are normed linear spaces. Then the linear transformation $\underset{\sim}{L}$ is bounded iff it is continuous.

Proof. First, suppose that $\underset{\sim}{L}$ is bounded. Then by the linearity, for $\underset{\sim}{u}, \underset{\sim}{u}_0 \in \mathcal{U}$,

$$\|\underset{\sim}{L}\underset{\sim}{u} - \underset{\sim}{L}\underset{\sim}{u}_0\|_{\mathcal{V}} = \|\underset{\sim}{L}(\underset{\sim}{u} - \underset{\sim}{u}_0)\|_{\mathcal{V}} \leq M \|\underset{\sim}{u} - \underset{\sim}{u}_0\|_{\mathcal{U}} .$$

Let $\varepsilon > 0$ be given. Set $\delta = \frac{\varepsilon}{M}$.[1] Then $\|\underset{\sim}{u} - \underset{\sim}{u}_0\|_{\mathcal{U}} < \delta \Rightarrow$

$$\|\underset{\sim}{L}\underset{\sim}{u} - \underset{\sim}{L}\underset{\sim}{u}_0\|_{\mathcal{V}} \leq M \|\underset{\sim}{u} - \underset{\sim}{u}_0\|_{\mathcal{U}} < \varepsilon ;$$

i.e., $\underset{\sim}{L}$ is continuous at any $\underset{\sim}{u}_0 \in \mathcal{U}$. In fact, since δ is independent of $\underset{\sim}{u}_0$, $\underset{\sim}{L}$ is uniformly continuous on $\mathcal{U}$.

Conversely, suppose that $\underset{\sim}{L}$ is continuous on $\mathcal{U}$. Then, in particular, it is continuous at the zero element $\underset{\sim}{0}$. So for any $\varepsilon > 0 \ \exists \ \delta > 0 \ni$

$$\|\underset{\sim}{u}' - \underset{\sim}{0}\|_{\mathcal{U}} < \delta \Rightarrow \|\underset{\sim}{L}\underset{\sim}{u}' - \underset{\sim}{L}\underset{\sim}{0}\|_{\mathcal{V}} < \varepsilon .$$

By the linearity, $\underset{\sim}{L}\underset{\sim}{0} = \underset{\sim}{0}$;[2] and, of course, $\underset{\sim}{L}\underset{\sim}{u}' - \underset{\sim}{0} = \underset{\sim}{L}\underset{\sim}{u}'$ and $\underset{\sim}{u}' - \underset{\sim}{0} = \underset{\sim}{u}'$. So with the choice $\varepsilon = 1$, the above becomes

$$\|\underset{\sim}{u}'\|_{\mathcal{U}} < \delta \Rightarrow \|\underset{\sim}{L}\underset{\sim}{u}'\|_{\mathcal{V}} < 1, \qquad (*)$$

---

[1] Consideration of the $M = 0$ case is left as *Exercise* A3.4.0.

[2] See Theorem A2.4.1.

Now, consider any $\underset{\sim}{u} \in \mathcal{U}$. If $\underset{\sim}{u} \neq \underset{\sim}{0}$, define $\underset{\sim}{u}' = \alpha \underset{\sim}{u}$, where

$$\alpha = \frac{\delta'}{\|\underset{\sim}{u}\|_{\mathcal{U}}}, \quad \delta' \in (0, \delta),$$

which makes sense by the positive definiteness of the norm. Then by the positive homogeneity of the norm,

$$\|\underset{\sim}{u}'\|_{\mathcal{U}} = |\alpha| \|\underset{\sim}{u}\|_{\mathcal{U}} = \delta' < \delta;$$

so that by (*)

$$\|\underset{\sim}{L}\underset{\sim}{u}'\|_{\mathcal{V}} = \|\underset{\sim}{L}(\alpha \underset{\sim}{u})\|_{\mathcal{V}} = \|\alpha \underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} = |\alpha| \|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} < 1$$

$$\Rightarrow \quad \|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} < \frac{1}{|\alpha|} = \frac{1}{\delta'} \|\underset{\sim}{u}\|_{\mathcal{U}}.$$

When $\underset{\sim}{u} = \underset{\sim}{0}$,

$$\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} = \frac{1}{\delta'} \|\underset{\sim}{u}\|_{\mathcal{U}}.$$

Thus,

$$\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} \leq \frac{1}{\delta'} \|\underset{\sim}{u}\|_{\mathcal{U}} \quad \forall \underset{\sim}{u} \in \mathcal{U};$$

i.e., $\underset{\sim}{L}$ is bounded on $\mathcal{U}$. $\square$

The next result shows that linear transformations on finite-dimensional spaces (but not necessarily into finite-dimensional spaces) are always bounded, and, consequently by the last theorem, continuous. In particular, second-order tensors on finite-dimensional spaces are always bounded and continuous.

Theorem A3.4.2. Let $\underset{\sim}{L} \in \mathrm{Lin}(\mathcal{U}_m, \mathcal{V})$, where $\mathcal{U}_m$ is an m-dimensional inner product space[1,2] and $\mathcal{V}$ is a normed linear space. Then the linear transformation $\underset{\sim}{L}$ is bounded.

Proof. Let $\{\underset{\sim}{e}_{<1>}, \underset{\sim}{e}_{<2>}, \cdots, \underset{\sim}{e}_{<m>}\}$ be an orthonormal basis[3] for $\mathcal{U}_m$. By the linearity of $\underset{\sim}{L}$,

$$\underset{\sim}{L}\underset{\sim}{u} = \underset{\sim}{L}(u^{<i>}\underset{\sim}{e}_{<i>}) = u^{<i>}\,\underset{\sim}{L}\underset{\sim}{e}_{<i>} \,.\quad)^4$$

Then by the triangle inequality and the positive homogeneity of the norm,

$$\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} \le |u^{<i>}|\,\|\underset{\sim}{L}\underset{\sim}{e}_{<i>}\|_{\mathcal{V}} \le M_0 \sum_{i=1}^{m} |u^{<i>}|\,, \qquad (*)$$

where

$$M_0 := \max_{i \in \mathbb{I}_m} \{\|\underset{\sim}{L}\underset{\sim}{e}_{<i>}\|_{\mathcal{V}}\}.$$

By Theorem A2.3.14,

$$u^{<i>} = \underset{\sim}{e}_{<i>} \cdot \underset{\sim}{u}\;;$$

so by the inner product-magnitude inequality (Theorem A2.3.6),

---

[1] See Theorem A2.3.2.

[2] Here, of course, we use the inner product norm (cf. Theorem A2.3.8)

[3] Cf. Theorem A2.3.13.

[4] See Theorem A2.4.2.

$$|u^{\langle i\rangle}| = |\underset{\sim}{e}_{\langle i\rangle} \cdot \underset{\sim}{u}| \le |\underset{\sim}{e}_{\langle i\rangle}||\underset{\sim}{u}| = |\underset{\sim}{u}|,$$

$\therefore$ (*) becomes

$$\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \le m M_0 |\underset{\sim}{u}| \quad \forall\, \underset{\sim}{u} \in \mathcal{U}_m;$$

i.e., $\underset{\sim}{L}$ is bounded on $\mathcal{U}_m$. $\square$

$\boxed{\text{INSERT material on pp. A3.4.5a–c}}$

The following notation will be useful.

Definition A3.4.2. Let $\mathcal{U}$ and $\mathcal{V}$ be normed linear spaces. Then $Bd(\mathcal{U},\mathcal{V})$[1] is the set of all bounded linear transformations on $\mathcal{U}$ to $\mathcal{V}$; i.e.,

$$Bd(\mathcal{U},\mathcal{V}) = \{\underset{\sim}{L} \in Lin(\mathcal{U},\mathcal{V}) : \underset{\sim}{L} \text{ bounded on } \mathcal{U}\}.$$

It is an easy matter to prove

Theorem A3.4.3. $Bd(\mathcal{U},\mathcal{V})$ is a linear subspace of $Lin(\mathcal{U},\mathcal{V})$.

Proof. *Exercise* A3.4.3. $\square$

In fact, $Bd(\mathcal{U},\mathcal{V})$ is always a normed linear space. To get at this, we note that for any $\underset{\sim}{L} \in Bd(\mathcal{U},\mathcal{V})$ $\exists\, M > 0$

---

[1] Often, $B(\mathcal{U},\mathcal{V})$.

To see that not all linear transformations are bounded, we consider the following (necessarily) infinite-dimensional example. Let

$$\mathcal{U} = C^1([a,b]) \quad \text{and} \quad \mathcal{V} = C^0([a,b]) .$$

We saw in Exercise A2.1.3 that $C^0([a,b])$ is a linear space, and it is easy to see (Exercise A3.4.1) that $C^1([a,b])$ is a linear subspace of $C^0([a,b])$. By a standard theorem in calculus, a continuous real-valued function on a closed interval attains its maximum (and minimum) on the interval;[1] and we leave it as Exercise A3.4.2 to show that

$$\|f\|_0 := \max_{x \in [a,b]} |f(x)|$$

is a norm for $C^0([a,b])$. We also use it for the linear subspace $C^1([a,b])$. Consider the differentiation operator

$$D : C^1([a,b]) \to C^0([a,b])$$

defined by

$$Df = f',$$

Trivially, $D \in \operatorname{Lin}\left(C^1([a,b]), C^0([a,b])\right)$. We will show that D is not continuous at the zero element of $C^1([a,b])$, and consequently in view of Theorem A3.4.1, D is an unbounded linear transformation.

---

[1] See, e.g., BARTLE's <u>The Elements of Real Analysis</u>.

To this end, note that the common zero element of $C^0([a,b])$ and its linear subspace $C^1([a,b])$ is the function $\theta : [a,b] \to \mathbb{R}$ defined by

$$\theta(x) = 0 \quad \forall\, x \in [a,b].$$

Obviously,

$$(D\theta)(x) = 0 \;\; \forall\, x \in [a,b] \Rightarrow D\theta = \theta.$$

Suppose that D is continuous at θ. Then given any $\varepsilon > 0 \;\exists\; \delta > 0 \ni$

$$\|f - \theta\|_0 = \|f\|_0 < \delta \Rightarrow \|Df - \theta\|_0 = \|Df\|_0 < \varepsilon.$$

Take $[a,b] = [-\pi, \pi]$ and consider the function $f_n \in C^1([-\pi,\pi])$ defined by

$$f_n(x) = \frac{1}{n}\sin(nx),$$

where $n \in \{1, 2, 3, \ldots\}$. Then

$$(Df_n)(x) = \cos(nx),$$

and

$$\|f_n\|_0 = \frac{1}{n} \quad , \quad \|Df_n\|_0 = 1.$$

For n sufficiently large, $\|f_n\|_0 < \delta$; but $\|Df_n\|_0 = 1$ contradicts $\|Df_n\|_0 < \varepsilon$ (provided ε was chosen < 1).
$\therefore$ f_n is not continuous at θ.

To demonstrate how strongly such considerations depend on the choice of norm, let us now take the norm for $C^1([a,b])$ to be

$$\|f\|_1 := \max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |f'(x)|$$

(It is left as Exercise A3.4.20 for the reader to show that this is a norm for $C^1([a,b])$.); but we keep $\|\cdot\|_0$ as the norm for $C^0([a,b])$. Then the above definition of $\|\cdot\|_1$ may be written as

$$\|f\|_1 = \|f\|_0 + \|Df\|_0$$

$$\Longrightarrow \quad \|Df\|_0 = \|f\|_1 - \|f\|_0 \le \|f\|_1 .$$

Hence, given $\varepsilon > 0$,

$$\|f\|_1 < \delta := \varepsilon \Longrightarrow \|Df\|_0 < \varepsilon ;$$

i.e., D is continuous at θ. Replacing f in the above implication with $f-g$ and using the linearity of D, we get

$$\|f-g\|_1 < \delta = \varepsilon \Longrightarrow \|Df - Dg\|_0 < \varepsilon .$$

Thus, D is continuous (and hence bounded) at any $g \in C^1([a,b])$ in this case.

Return to p. A3.4.5

$\ni$

$$\|\underset{\sim\sim}{Lu}\|_{\mathcal{V}} \le M \|\underset{\sim}{u}\|_{\mathcal{U}} \ \forall \underset{\sim}{u} \in \mathcal{U}.$$

Clearly, if $\exists$ one such M then $\exists$ many; but the set of all of them is bounded below by 0, and this allows us to select a special bound. Some readers will need a little more background for this. Accordingly, we turn to

Definition A3.4.3. Let $A \subset \mathbb{R}$. If $\exists \ x \in \mathbb{R} \ni$

$$a \in A \Rightarrow x \le a,$$

then x is a *lower bound* for A, and we say that A is *bounded below*.

The notion of an *upper bound* is defined similarly.

Definition A3.4.4. Let $A \subset \mathbb{R}$ be bounded below. If $\exists$ $x \in \mathbb{R} \ni$

(i) x is a lower bound for A

and

(ii) x' is any lower bound for $A \Rightarrow x \ge x'$,

then x is the *greatest lower bound*[1] of A, and we write

$$x = \text{glb}\, A.$$

---

[1] More often, *infimum* of A; and then $x = \inf A$.

For a set B bounded above, the *least upper bound* or *supremum* is similarly defined, and it is denoted by lub B or sup B.

The following theorem justifies our referring to the greatest lower bound.

Theorem A3.4.4. A subset of $\mathbb{R}$ has at most one greatest lower bound (least upper bound).

Proof. *Exercise* A3.4.4. □

It is a property of $\mathbb{R}$ that every nonempty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.[1]

Returning to the task at hand, we are now in position to make

Definition A3.4.5. Let $\underset{\sim}{L} \in Bd(\mathcal{U}, \mathcal{V})$. Then the *operator norm* of $\underset{\sim}{L}$ is

$$\|\underset{\sim}{L}\| = \text{glb}\,\{M \geq 0 : \|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \leq M \|\underset{\sim}{u}\|_{\mathcal{U}} \ \forall\, \underset{\sim}{u} \in \mathcal{U}\}.$$

---

[1] Actually, this property is usually formulated first in terms of the least upper bound for a set bounded above. It is either an axiom for $\mathbb{R}$ or a theorem, depending on the procedure used to construct $\mathbb{R}$ (See, e.g., BARTLE's The Elements of Real Analysis.).

The operator norm would not be very useful if it were not itself a bound. Fortunately, we have

Theorem A3.4.5. Let $\underset{\sim}{L} \in Bd(\mathcal{U}, \mathcal{V})$. Then

$$\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} \le \|\underset{\sim}{L}\|\|\underset{\sim}{u}\|_{\mathcal{U}} \quad \forall \underset{\sim}{u} \in \mathcal{U}.$$

This result is usually taken as obvious, but it is not transparent to the author. Certainly, the open interval $(0,1)$ is a subset of $\mathbb{R}$ which does not contain its greatest lower bound. The following nomenclature will be useful in putting together a proof:

$$\mathcal{S} = \{\underset{\sim}{u} \in \mathcal{U} : \|\underset{\sim}{u}\|_{\mathcal{U}} = 1\},$$ the unit sphere in $\mathcal{U}$;

$$B = \{M \ge 0 : \|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} \le M\|\underset{\sim}{u}\|_{\mathcal{U}} \ \forall \underset{\sim}{u} \in \mathcal{U}\},$$ the set of bounds

for $\underset{\sim}{L}$;

$$B_1 = \{m \ge 0 : \|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} \le m\|\underset{\sim}{u}\|_{\mathcal{U}} = m \ \forall \underset{\sim}{u} \in \mathcal{S}\},$$ the set of bounds

for the restriction[1] of $\underset{\sim}{L}$ to $\mathcal{S}$.

Because of the linearity of $\underset{\sim}{L}$, we have

Lemma. $B_1 = B$.

Proof. Let $M \in B$. Then

---

[1] Cf. Definition A1.2.4.

$$\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} \le M\|\underset{\sim}{u}\|_{\mathcal{U}} \quad \forall\, \underset{\sim}{u} \in \mathcal{U}.$$

In particular, since $\mathcal{S} \subset \mathcal{U}$,

$$\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} \le M\|\underset{\sim}{u}\|_{\mathcal{U}} = M \quad \forall\, \underset{\sim}{u} \in \mathcal{S}.$$

Thus, $M \in \mathcal{B}_1$; i.e., $\mathcal{B} \subset \mathcal{B}_1$.

Let $m \in \mathcal{B}_1$. Then

$$\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} \le m \quad \forall\, \underset{\sim}{u} \in \mathcal{S}.$$

Now for any $\underset{\sim}{v} \in \mathcal{U} \ni \underset{\sim}{v} \ne \underset{\sim}{0}$,

$$\underset{\sim}{L}\underset{\sim}{v} = \underset{\sim}{L}\left(\frac{\|\underset{\sim}{v}\|_{\mathcal{U}}\underset{\sim}{v}}{\|\underset{\sim}{v}\|_{\mathcal{U}}}\right) \qquad \text{(substitution)}$$

$$= \|\underset{\sim}{v}\|_{\mathcal{U}}\underset{\sim}{L}\left(\frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|_{\mathcal{U}}}\right) \quad \text{(homogeneity of } \underset{\sim}{L}\text{)},$$

and

$$\|\underset{\sim}{L}\underset{\sim}{v}\|_{\mathcal{V}} = \|\underset{\sim}{v}\|_{\mathcal{U}}\left\|\underset{\sim}{L}\left(\frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|_{\mathcal{U}}}\right)\right\|_{\mathcal{V}} \quad \text{(positive homogeneity of } \|\cdot\|_{\mathcal{V}}\text{)}$$

$$\le m\|\underset{\sim}{v}\|_{\mathcal{U}} \qquad \left(\frac{\underset{\sim}{v}}{\|\underset{\sim}{v}\|_{\mathcal{U}}} \in \mathcal{S}\right).$$

If $\underset{\sim}{v} = \underset{\sim}{0}$, then trivially $\|\underset{\sim}{L}\underset{\sim}{v}\|_{\mathcal{V}} \le m\|\underset{\sim}{v}\|_{\mathcal{U}}$ is satisfied as an equality. Hence,

$$\|\underset{\sim}{L}\underset{\sim}{v}\|_{\mathcal{V}} \le m\|\underset{\sim}{v}\|_{\mathcal{U}} \quad \forall\, \underset{\sim}{v} \in \mathcal{U},$$

which $\Rightarrow m \in \mathcal{B}$; i.e., $\mathcal{B}_1 \subset \mathcal{B}$. $\therefore\ \mathcal{B}_1 = \mathcal{B}$. $\square$

Proof of Theorem A3.4.5. In the above language, and in view of the Lemma,

$$\|\underset{\sim}{L}\| = \text{glb}\ B = \text{glb}\ B_1\ ;$$

and the Theorem claims

$$\|\underset{\sim}{L}\| \in B.$$

To establish this, suppose $\|\underset{\sim}{L}\| \notin B$. Then by the Lemma, $\|\underset{\sim}{L}\| \notin B_1$, which means $\exists\ \underset{\sim}{u}_0 \in S \ni$

$$\|\underset{\sim}{L}\,\underset{\sim}{u}_0\|_{\mathcal{V}} > \|\underset{\sim}{L}\|. \qquad (*)$$

Let $m \in B_1$. Then

$$\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \le m \quad \forall\,\underset{\sim}{u} \in S.$$

In particular,

$$\|\underset{\sim}{L}\,\underset{\sim}{u}_0\|_{\mathcal{V}} \le m.$$

Since m was an arbitrary element of B_1, this says that $\|\underset{\sim}{L}\,\underset{\sim}{u}_0\|_{\mathcal{V}}$ is a lower bound for B_1. Then $(*)$ contradicts that $\|\underset{\sim}{L}\| = \text{glb}\ B_1$, and we conclude that $\|\underset{\sim}{L}\| \in B$. □

The next result provides alternative definitions of the operator norm. It is important to realize that we get the same $\|\underset{\sim}{L}\|$ — not alternative norms.

<u>Theorem</u> A3.4.6. <u>Let</u> $\underset{\sim}{L} \in Bd(\mathcal{U}, \mathcal{V})$. <u>Then</u>

(i) $\|\underset{\sim}{L}\| = \operatorname{lub}_{\substack{\underset{\sim}{u} \in \mathcal{U} \\ \underset{\sim}{u} \neq \underset{\sim}{0}}} \dfrac{\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}}}{\|\underset{\sim}{u}\|_{\mathcal{U}}} := \operatorname{lub}\left\{\dfrac{\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}}}{\|\underset{\sim}{u}\|_{\mathcal{U}}} : \underset{\sim}{u} \in (\mathcal{U} - \{\underset{\sim}{0}\})\right\}$;

(ii) $\|\underset{\sim}{L}\| = \operatorname{lub}_{\substack{\underset{\sim}{u} \in \mathcal{U} \\ \|\underset{\sim}{u}\|_{\mathcal{U}} \leq 1}} \|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}}$;

(iii) $\|\underset{\sim}{L}\| = \operatorname{lub}_{\substack{\underset{\sim}{u} \in \mathcal{U} \\ \|\underset{\sim}{u}\|_{\mathcal{U}} = 1}} \|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}}$.

<u>Proof</u>. (i): By Theorem A3.4.5,

$$\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} \leq \|\underset{\sim}{L}\|\|\underset{\sim}{u}\|_{\mathcal{U}} \quad \forall\, \underset{\sim}{u} \in \mathcal{U}.$$

$\therefore$ for $\underset{\sim}{u} \neq \underset{\sim}{0}$,

$$\frac{\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}}}{\|\underset{\sim}{u}\|_{\mathcal{U}}} \leq \|\underset{\sim}{L}\|.$$

Thus, $\|\underset{\sim}{L}\|$ is an upper bound for the set

$$A := \left\{ \xi : \xi = \frac{\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}}}{\|\underset{\sim}{u}\|_{\mathcal{U}}},\ \underset{\sim}{u} \in (\mathcal{U} - \{\underset{\sim}{0}\}) \right\}.$$

In fact, $\|\underset{\sim}{L}\|$ is the least upper bound. For, suppose not. Then $\exists\ \overset{*}{\xi} < \|\underset{\sim}{L}\| \ni$

$$\frac{\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}}}{\|\underset{\sim}{u}\|_{\mathcal{U}}} \leq \overset{*}{\xi} \quad \forall\ \underset{\sim}{u} \in \mathcal{U} - \{\underset{\sim}{0}\}$$

Thus,

$$\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \leq \overset{*}{\xi}\,\|\underset{\sim}{u}\|_{\mathcal{U}} \quad \forall\ \underset{\sim}{u} \in \mathcal{U} - \{\underset{\sim}{0}\}\ ;$$

but this inequality is satisfied trivially when $\underset{\sim}{u} = \underset{\sim}{0}$, so we have

$$\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \leq \overset{*}{\xi}\,\|\underset{\sim}{u}\|_{\mathcal{U}} \quad \forall\ \underset{\sim}{u} \in \mathcal{U}\,.$$

Whence,

$$\overset{*}{\xi} \in \{M \geq 0 : \|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \leq M\,\|\underset{\sim}{u}\|_{\mathcal{U}} \quad \forall\ \underset{\sim}{u} \in \mathcal{U}\}\ ;$$

but since $\|\underset{\sim}{L}\|$ is the greatest lower bound of this set,

$$\|\underset{\sim}{L}\| \leq \overset{*}{\xi}\,,$$

which contradicts the starting hypothesis. $\therefore\ \|\underset{\sim}{L}\| = \operatorname{lub} A$.

(iii): Since $\underset{\sim}{L}$ is linear and $\|\cdot\|_{\mathcal{V}}$ is positive homogeneous, we can use (i) to write

$$\|\underset{\sim}{L}\| = \operatorname*{lub}_{\substack{\underset{\sim}{u} \in \mathcal{U} \\ \underset{\sim}{u} \neq \underset{\sim}{0}}} \frac{\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}}}{\|\underset{\sim}{u}\|_{\mathcal{U}}} = \operatorname*{lub}_{\substack{\underset{\sim}{u} \in \mathcal{U} \\ \underset{\sim}{u} \neq \underset{\sim}{0}}} \left\|\underset{\sim}{L}\,\frac{\underset{\sim}{u}}{\|\underset{\sim}{u}\|_{\mathcal{U}}}\right\|_{\mathcal{V}} = \operatorname*{lub}_{\substack{\underset{\sim}{x} \in \mathcal{U} \\ \|\underset{\sim}{x}\| = 1}} \|\underset{\sim}{L}\,\underset{\sim}{x}\|_{\mathcal{V}}\,.$$

(ii): $\forall\ \underset{\sim}{u} \in \mathcal{U}$,

$$\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \le \|\underset{\sim}{L}\|\,\|\underset{\sim}{u}\|_{\mathcal{U}} \;;$$

and $\therefore$ for $\|\underset{\sim}{u}\|_{\mathcal{U}} \le 1$,

$$\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \le \|\underset{\sim}{L}\|.$$

Thus, $\|\underset{\sim}{L}\|$ is an upper bound for the set

$$B := \{\eta : \eta = \|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \,,\ \|\underset{\sim}{u}\|_{\mathcal{U}} \le 1\}.$$

Suppose $\|\underset{\sim}{L}\| \ne \text{lub}\, B$, then

$$\text{lub}\, B < \|\underset{\sim}{L}\|.$$

Since

$$\{\underset{\sim}{x} \in \mathcal{U} : \|\underset{\sim}{x}\|_{\mathcal{U}} = 1\} \subset \{\underset{\sim}{x} \in \mathcal{U} : \|\underset{\sim}{x}\|_{\mathcal{U}} \le 1\},$$

$$\underset{\|\underset{\sim}{x}\|_{\mathcal{U}}=1}{\text{lub}} \|\underset{\sim}{L}\,\underset{\sim}{x}\|_{\mathcal{V}} \le \underset{\|\underset{\sim}{x}\|_{\mathcal{U}}\le 1}{\text{lub}} \|\underset{\sim}{L}\,\underset{\sim}{x}\|_{\mathcal{V}} = \text{lub}\, B < \|\underset{\sim}{L}\|.$$

But by (iii), $$\underset{\|\underset{\sim}{x}\|_{\mathcal{U}}=1}{\text{lub}} \|\underset{\sim}{L}\,\underset{\sim}{x}\| = \|\underset{\sim}{L}\| \;;$$

and we are left with the contradiction that $\|\underset{\sim}{L}\| < \|\underset{\sim}{L}\|$. $\therefore$ $\|\underset{\sim}{L}\| = \text{lub}\, B$. $\square$

Because of the least upper bounds in Theorem A3.4.6, the operator norm is often referred to as the <u>supremum</u>

norm or the sup norm.

Of course, the operator norm is not a norm just because we called it one and used the notation $\|\cdot\|$; indeed, we have

Theorem A3.4.7. The linear space $Bd(\mathcal{U},\mathcal{V})$ equipped with the function

$$\|\cdot\| : Bd(\mathcal{U},\mathcal{V}) \to \mathbb{R}^+$$

defined in Definition A3.4.5 is a normed linear space.

Proof. We need to show that $\|\cdot\|$ meets the axioms of Definition A2.3.3.

By (iii) of Theorem A3.4.6, for any $\alpha \in \mathbb{R}$

$$\|\alpha \underset{\sim}{L}\| = \underset{\|\underset{\sim}{u}\|_{\mathcal{U}}=1}{\text{lub}} \|(\alpha \underset{\sim}{L})\underset{\sim}{u}\|_{\mathcal{V}} = \underset{\|\underset{\sim}{u}\|_{\mathcal{U}}=1}{\text{lub}} \|\alpha(\underset{\sim}{L}\underset{\sim}{u})\|_{\mathcal{V}} = \underset{\|\underset{\sim}{u}\|_{\mathcal{U}}=1}{\text{lub}} \left(|\alpha|\|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}}\right)$$

$$= |\alpha| \underset{\|\underset{\sim}{u}\|_{\mathcal{U}}=1}{\text{lub}} \|\underset{\sim}{L}\underset{\sim}{u}\|_{\mathcal{V}} = |\alpha|\|\underset{\sim}{L}\| .$$

$\therefore$ $\|\cdot\|$ is positive homogeneous. Note that we made use of the positive homogeneity of $\|\cdot\|_{\mathcal{V}}$ to get this result.

Since

$$\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \le \|\underset{\sim}{L}\|\,\|\underset{\sim}{u}\|_{\mathcal{U}} \quad \forall\, \underset{\sim}{u} \in \mathcal{U}, \qquad (*)$$

$\|\underset{\sim}{L}\| \ge 0$. Moreover, if $\|\underset{\sim}{L}\| = 0$, then $(*) \Rightarrow$

$$\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \le 0 \quad \forall\, \underset{\sim}{u} \in \mathcal{U}.$$

But $\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} \ge 0$, so

$$\|\underset{\sim}{L}\,\underset{\sim}{u}\|_{\mathcal{V}} = 0 \quad \forall\, \underset{\sim}{u} \in \mathcal{U}.$$

Then by the positive definiteness of $\|\cdot\|_{\mathcal{V}}$,

$$\underset{\sim}{L}\,\underset{\sim}{u} = \underset{\sim}{0} \quad \forall\, \underset{\sim}{u} \in \mathcal{U};$$

i.e., $\underset{\sim}{L} = \underset{\sim}{0}$. Setting $\alpha = 0$ in the positive homogeneity property and using the linear space property $0\underset{\sim}{L} = \underset{\sim}{0}$, we get $\|\underset{\sim}{0}\| = 0$. Thus, $\|\cdot\|$ is positive definite.

As might be expected by now, the triangle inequality for $\|\cdot\|$ follows from the triangle inequality for $\|\cdot\|_{\mathcal{V}}$. The proof of this is left as Exercise A3.4.7. (Hint: Start with (ii) of Theorem A3.4.6.) $\square$

In view of Theorem A3.4.2, if $\mathcal{V}_n$ is an n-dimensional inner product space, then any $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$ is bounded. Consequently, the operator norm makes sense on $\operatorname{Lin} \mathcal{V}_n$, and renders $\operatorname{Lin} \mathcal{V}_n$ a normed linear space.

In Theorem A3.1.5, we noted that the identity transformation $\underset{\sim}{I}$ defined on a linear space V by

$$\underset{\sim}{I}\,\underset{\sim}{u} = \underset{\sim}{u} \quad \forall\ \underset{\sim}{u} \in V$$

is a linear transformation; i.e., $\underset{\sim}{I} \in \mathrm{Lin}\, V$. It turns out that when V is a normed linear space, then $\underset{\sim}{I}$ is bounded and has unit operator norm.

<u>Theorem A3.4.8.</u> <u>Let $\underset{\sim}{I}$ be the identity second-order tensor on a normed linear space V. Then</u>

$$\underset{\sim}{I} \in Bd(V,V) \quad \underline{\text{and}} \quad \|\underset{\sim}{I}\| = 1.$$

<u>Proof.</u> For any $\underset{\sim}{u} \in V$

$$\|\underset{\sim}{I}\,\underset{\sim}{u}\|_V = \|\underset{\sim}{u}\|_V \quad (\text{substitution})$$

$$\leq \|\underset{\sim}{u}\|_V \quad (\text{definition of } \leq).$$

$\therefore\ \underset{\sim}{I} \in Bd(V,V)$. Then by Theorem A3.4.6(iii),

$$\|\underset{\sim}{I}\| = \underset{\|\underset{\sim}{u}\|_V=1}{\mathrm{lub}} \|\underset{\sim}{I}\,\underset{\sim}{u}\|_V = \underset{\|\underset{\sim}{u}\|_V=1}{\mathrm{lub}} \|\underset{\sim}{u}\|_V = \mathrm{lub}\,\{1\} = 1. \quad \square$$

Supplementary Reading

HALMOS, Finite-Dimensional Vector Spaces

LOOMIS and STERNBERG, Advanced Calculus

LUENBERGER, Optimization by Vector Space Methods

MICHEL and HERGET, Mathematical Foundations in Engineering and Science

NAYLOR and SELL, Linear Operator Theory in Engineering and Science

NOLL, Finite-Dimensional Spaces

ODEN, Applied Functional Analysis

SIMMONS, Introduction to Topology and Modern Analysis

A3.5. Multiplication of Second-Order Tensors

Recall that second-order tensors are just functions, albeit special functions in that they are linear and their domains and codomains are the same linear space. To get the product of two tensors on the same space, we simply compose the two functions. Formally, we have

Definition A3.5.1. Let $\mathcal{V}$ be a linear space. Then the *product* $\mathbf{TS}$ of two second-order tensors $\mathbf{T}, \mathbf{S} \in \operatorname{Lin}\mathcal{V}$ is the composition[1] of the functions $\mathbf{T}$ and $\mathbf{S}$; i.e., for $\mathbf{u} \in \mathcal{V}$

$$(\mathbf{TS})\mathbf{u} = \mathbf{T}(\mathbf{S}\mathbf{u}).$$

Then by definition the product $\mathbf{TS}$ is a function on $\mathcal{V}$ to $\mathcal{V}$. The following result shows that this function itself is linear; thus, the product operation is closed.

Theorem A3.5.1. Let $\mathbf{T}, \mathbf{S} \in \operatorname{Lin}\mathcal{V}$. Then

$$\mathbf{TS} \in \operatorname{Lin}\mathcal{V},$$

and the following properties hold:

(i) *Associativity.* $\forall\ \mathbf{T}, \mathbf{S}, \mathbf{R} \in \operatorname{Lin}\mathcal{V}$

---

[1] See Theorem A1.2.2.

$$(\underset{\sim}{T}\underset{\sim}{S})\underset{\sim}{R} = \underset{\sim}{T}(\underset{\sim}{S}\underset{\sim}{R});$$

(ii) *Distributivity*. $\forall\ \underset{\sim}{T}, \underset{\sim}{S}, \underset{\sim}{R} \in \mathrm{Lin}\,\mathcal{V}$

$$\underset{\sim}{T}(\underset{\sim}{S}+\underset{\sim}{R}) = \underset{\sim}{T}\underset{\sim}{S} + \underset{\sim}{T}\underset{\sim}{R},$$

$$(\underset{\sim}{S}+\underset{\sim}{R})\underset{\sim}{T} = \underset{\sim}{S}\underset{\sim}{T} + \underset{\sim}{R}\underset{\sim}{T};$$

(iii) *Homogeneity*. $\forall\ \alpha, \beta \in \mathbb{R}$ and $\forall\ \underset{\sim}{T}, \underset{\sim}{S} \in \mathrm{Lin}\,\mathcal{V}$

$$(\alpha\underset{\sim}{T})(\beta\underset{\sim}{S}) = (\alpha\beta)(\underset{\sim}{T}\underset{\sim}{S});$$

(iv) *Identity Element*. $\forall\ \underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}$

$$\underset{\sim}{I}\underset{\sim}{T} = \underset{\sim}{T}\underset{\sim}{I} = \underset{\sim}{T}.$$

Proof. The proof of $\underset{\sim}{T}\underset{\sim}{S} \in \mathrm{Lin}\,\mathcal{V}$ is left as *Exercise* A3.5.1.

Property (i) follows from repeated application of the definition. To see this, let $\underset{\sim}{u} \in \mathcal{V}$. Then

$$((\underset{\sim}{T}\underset{\sim}{S})\underset{\sim}{R})\underset{\sim}{u} = (\underset{\sim}{T}\underset{\sim}{S})(\underset{\sim}{R}\underset{\sim}{u}) = \underset{\sim}{T}(\underset{\sim}{S}(\underset{\sim}{R}\underset{\sim}{u})) = \underset{\sim}{T}((\underset{\sim}{S}\underset{\sim}{R})\underset{\sim}{u}) = (\underset{\sim}{T}(\underset{\sim}{S}\underset{\sim}{R}))\underset{\sim}{u},$$

and since this holds $\forall\ \underset{\sim}{u} \in \mathcal{V}$,

$$(\underset{\sim}{T}\underset{\sim}{S})\underset{\sim}{R} = \underset{\sim}{T}(\underset{\sim}{S}\underset{\sim}{R}).\quad)^{1}$$

[1] Note that the linearity of the transformations has not been used; thus, the composition of general functions is associative.

The proofs of (ii) – (iv) are left as *Exercises* A3.5.2 – 4, respectively. □

In view of the associativity, we can leave out the parentheses and write $\underset{\sim}{T}\underset{\sim}{S}\underset{\sim}{R}$ without confusion of any consequence. Of course, $\underset{\sim}{T}\underset{\sim}{S}\underset{\sim}{R}$ appears ambiguous at first. Since the product operation is binary (i.e., on $\mathcal{V} \times \mathcal{V}$), $\underset{\sim}{T}\underset{\sim}{S}\underset{\sim}{R}$ must be interpreted as either $(\underset{\sim}{T}\underset{\sim}{S})\underset{\sim}{R}$ or $\underset{\sim}{T}(\underset{\sim}{S}\underset{\sim}{R})$. The point is that these two interpretations result in the same tensor.

Similarly, because of the homogeneity, we can safely write $\alpha \underset{\sim}{T}\underset{\sim}{S}$ without parentheses.

Along these same lines, $\underset{\sim}{T}\underset{\sim}{S}\underset{\sim}{u}$ could mean $(\underset{\sim}{T}\underset{\sim}{S})\underset{\sim}{u}$ or $\underset{\sim}{T}(\underset{\sim}{S}\underset{\sim}{u})$; but by the very definition of tensor multiplication, these are the same.

The identity element property (iv) shows that the identity tensor $\underset{\sim}{I}$ deserves its name in the multiplicative sense as well as in the functional sense.[1] Of course, from the multiplicative point of view the terminology <u>unit tensor</u> is preferred.

The mathematical structure displayed in Theorem A3.5.1 is common enough to have a standard name.

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[1] Cf. Theorem A3.1.4 and the remark preceding it.

Definition A3.5.2. An algebra is a linear space, say V, equipped with a function on $V \times V$ to V, denoted here by

$$(\underset{\sim}{u}, \underset{\sim}{v}) \mapsto \underset{\sim}{u} \odot \underset{\sim}{v},$$

which satisfies the following axioms:

(A1) Associativity. $\forall \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$

$$(\underset{\sim}{u} \odot \underset{\sim}{v}) \odot \underset{\sim}{w} = \underset{\sim}{u} \odot (\underset{\sim}{v} \odot \underset{\sim}{w});$$

(A2) Distributivity. $\forall \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in V$

$$\underset{\sim}{u} \odot (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \odot \underset{\sim}{v} + \underset{\sim}{u} \odot \underset{\sim}{w},$$

$$(\underset{\sim}{u} + \underset{\sim}{v}) \odot \underset{\sim}{w} = \underset{\sim}{u} \odot \underset{\sim}{w} + \underset{\sim}{v} \odot \underset{\sim}{w};$$

(A3) Homogeneity. $\forall \alpha, \beta \in \mathbb{R}$ and $\forall \underset{\sim}{u}, \underset{\sim}{v} \in V$

$$(\alpha \underset{\sim}{u}) \odot (\beta \underset{\sim}{v}) = (\alpha\beta)(\underset{\sim}{u} \odot \underset{\sim}{v}).$$

If, in addition, $\exists$ an identity element $\underset{\sim}{i} \in V \ni$

$$\underset{\sim}{i} \odot \underset{\sim}{u} = \underset{\sim}{u} \odot \underset{\sim}{i} = \underset{\sim}{u} \quad \forall \underset{\sim}{u} \in V,$$

then V is said to be an algebra with a unit.

In view of Definition A3.5, 2 and Theorem A3.5.1, we have

Theorem A3.5.2. With respect to multiplication, Lin $\mathcal{V}$ is an algebra with a unit.

The product of tensor products has a special form.

Theorem A3.5.3. Let $\mathcal{V}$ be an inner product space. Then $\forall\ \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}, \underset{\sim}{x} \in \mathcal{V}$,

$$(\underset{\sim}{u} \otimes \underset{\sim}{v})(\underset{\sim}{w} \otimes \underset{\sim}{x}) = (\underset{\sim}{v} \cdot \underset{\sim}{w})\, \underset{\sim}{u} \otimes \underset{\sim}{x} .$$

Proof. Exercise A3.5.5. □

When the underlying linear space is finite-dimensional, we have convenient formulas for the various types of components of products of tensors.

Theorem A3.5.4. Let $\underset{\sim}{T}, \underset{\sim}{S} \in \mathrm{Lin}\, \mathcal{V}_n$, where $\mathcal{V}_n$ is an n-dimensional inner product space.[1] Then

$$(\underset{\sim}{T}\underset{\sim}{S})^{ij} = T^{ik} S_k{}^{\bullet j} = T^{i}{}_{\bullet k} S^{kj} ,$$

$$(\underset{\sim}{T}\underset{\sim}{S})_{ij} = T_{ik} S^{k}{}_{\bullet j} = T_i{}^{\bullet k} S_{kj} ,$$

[1] Cf. Theorem A2.3.2.

$$(\underset{\sim}{T}\underset{\sim}{S})^{i}_{\bullet j} = T^{ik} S_{kj} = T^{i}_{\bullet k} S^{k}_{\bullet j},$$

$$(\underset{\sim}{T}\underset{\sim}{S})_{i}^{\bullet j} = T_{ik} S^{kj} = T_{i}^{\bullet k} S_{k}^{\bullet j}.$$

Proof. By Theorem A3.2.2 and Definition A3.5.1,

$$(\underset{\sim}{T}\underset{\sim}{S})^{ij} = \underset{\sim}{e}^{i} \cdot (\underset{\sim}{T}\underset{\sim}{S})\underset{\sim}{e}^{j} = \underset{\sim}{e}^{i} \cdot \underset{\sim}{T}(\underset{\sim}{S}\,\underset{\sim}{e}^{j}).$$

But by Theorem A2.5.3,

$$\underset{\sim}{S}\,\underset{\sim}{e}^{j} = (\underset{\sim}{S}\,\underset{\sim}{e}^{j})_{k}\underset{\sim}{e}^{k} = (\underset{\sim}{e}_{k} \cdot \underset{\sim}{S}\,\underset{\sim}{e}^{j})\underset{\sim}{e}^{k} = S_{k}^{\bullet j}\underset{\sim}{e}^{k};$$

and ∴

$$\begin{aligned}
(\underset{\sim}{T}\underset{\sim}{S})^{ij} &= \underset{\sim}{e}^{i} \cdot \underset{\sim}{T}(S_{k}^{\bullet j}\underset{\sim}{e}^{k}) && \text{(substitution)} \\
&= \underset{\sim}{e}^{i} \cdot (S_{k}^{\bullet j}\underset{\sim}{T}\,\underset{\sim}{e}^{k}) && \text{(linearity of } \underset{\sim}{T}) \\
&= S_{k}^{\bullet j}(\underset{\sim}{e}^{i} \cdot \underset{\sim}{T}\,\underset{\sim}{e}^{k}) && \text{(linearity of inner product)} \\
&= S_{k}^{\bullet j} T^{ik} && \text{(Theorem A2.5.3)} \\
&= T^{ik} S_{k}^{\bullet j} && \text{(commutativity of multiplication in } R).
\end{aligned}$$

The remaining relations are proved in the same way. □

Note that, in terms of components, tensor multiplication reduces to matrix multiplication. The second (or column) index on the first factor is summed off against the first (or row) index of the second factor; i.e., the "interior" indices are summed. Keeping this in mind, we can easily remember the above formulas by matching up free indices across the equal sign and using the summation convention correctly.

Since matrix multiplication is noncommutative, the above observation leads us to

Theorem A3.5.5. Multiplication of second-order tensors is generally noncommutative; i.e., in general if $\underset{\sim}{S}, \underset{\sim}{T} \in \operatorname{Lin} V$, then

$$\underset{\sim}{T}\underset{\sim}{S} \neq \underset{\sim}{S}\underset{\sim}{T}.$$

Proof. Exercise A3.5.6. (Hint: Find an example in $\operatorname{Lin} V_2$.) $\square$

The following result will be useful when we have occasion to consider the product of bounded tensors.

Theorem A3.5.6. Let $\underset{\sim}{S}, \underset{\sim}{T} \in Bd(V, V)$. Then

$$\underset{\sim}{S}\,\underset{\sim}{T} \in Bd(V, V)$$

and

$$\|\underset{\sim}{S}\,\underset{\sim}{T}\| \leq \|\underset{\sim}{S}\|\|\underset{\sim}{T}\|.$$

Proof. By Theorem A3.5.1, $\underset{\sim}{S}\underset{\sim}{T} \in \operatorname{Lin} V = \operatorname{Lin}(V, V)$. Then for any $\underset{\sim}{u} \in V$,

$$\|(\underset{\sim}{S}\underset{\sim}{T})\underset{\sim}{u}\|_{\mathcal{V}} = \|\underset{\sim}{S}(\underset{\sim}{T}\underset{\sim}{u})\|_{\mathcal{V}} \quad \text{(definition of tensor multiplication)}$$

$$\leq \|\underset{\sim}{S}\|\|\underset{\sim}{T}\underset{\sim}{u}\|_{\mathcal{V}} \quad \text{(definition of operator norm)}$$

$$\leq \|\underset{\sim}{S}\|\|\underset{\sim}{T}\|\|\underset{\sim}{u}\|_{\mathcal{V}} \quad \text{(definition of operator norm).}$$

$\therefore \|\underset{\sim}{S}\|\|\underset{\sim}{T}\|$ is a bound for $\underset{\sim}{S}\underset{\sim}{T}$, so $\underset{\sim}{S}\underset{\sim}{T} \in Bd(\mathcal{V},\mathcal{V})$.

Furthermore,

$$\|\underset{\sim}{S}\,\underset{\sim}{T}\| = \underset{\|\underset{\sim}{u}\|_{\mathcal{V}}=1}{\text{lub}} \|(\underset{\sim}{S}\underset{\sim}{T})\underset{\sim}{u}\| \quad \text{(Theorem A3.4.6 (iii))}$$

$$\leq \underset{\|\underset{\sim}{u}\|_{\mathcal{V}}=1}{\text{lub}} \{\|\underset{\sim}{S}\|\|\underset{\sim}{T}\|\|\underset{\sim}{u}\|_{\mathcal{V}}\} \quad \text{(last inequality above)}$$

$$= \|\underset{\sim}{S}\|\|\underset{\sim}{T}\| \,. \quad \square$$

In words, Theorem A3.5.6 asserts that the product of bounded tensors is a bounded tensor and that the operator norm of the product (of bounded tensors) is less than or equal to the product of the operator norms.

Supplementary Reading

the union of the Supplementary Readings for §§ A3.1 and A3.4

A3.6. Transposition. Symmetric Tensors

In this section we shall develop the notion of the transpose of a second-order tensor on an inner product space. Transposition can be defined more generally for elements of $Lin(\mathcal{U}, \mathcal{V})$, where neither $\mathcal{U}$ nor $\mathcal{V}$ are inner product spaces. For this, the reader is referred to the books by HALMOS and NOLL listed in the Supplementary Reading at the end of the section.

Theorem A3.6.1. Let $\underset{\sim}{T} \in Lin\,\mathcal{V}$, where $\mathcal{V}$ is an inner product space. If $\exists$ a function $\underset{\sim}{t} : \mathcal{V} \to \mathcal{V} \ni$

$$\underset{\sim}{u} \cdot \underset{\sim}{T}\underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{t}(\underset{\sim}{u}) \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V} \quad ,$$

then $\underset{\sim}{t} \in Lin\,\mathcal{V}$; i.e., $\underset{\sim}{t}$ is a second-order tensor on $\mathcal{V}$, which we call the transpose[1] of $\underset{\sim}{T}$ and denote by $\underset{\sim}{T}^T$.)[2]

Proof. Suppose such a $\underset{\sim}{t}$ exists. Then for $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\underset{\sim}{u}_1, \underset{\sim}{u}_2, \underset{\sim}{v} \in \mathcal{V}$

$$\underset{\sim}{v} \cdot \underset{\sim}{t}(\alpha_1 \underset{\sim}{u}_1 + \alpha_2 \underset{\sim}{u}_2) = (\alpha_1 \underset{\sim}{u}_1 + \alpha_2 \underset{\sim}{u}_2) \cdot \underset{\sim}{T}\underset{\sim}{v}$$

---

[1] Often, adjoint.

[2] Other common notations for the transpose are $\underset{\sim}{T}'$, $\underset{\sim}{T}^*$, $\tilde{\underset{\sim}{T}}$, and $\underset{\sim}{T}^{\dagger}$. In terms of $\underset{\sim}{T}^T$, the defining condition is $\underset{\sim}{u} \cdot \underset{\sim}{T}\underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{T}^T\underset{\sim}{u} \;\; \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$.

$$= \alpha_1(\underset{\sim}{u}_1 \cdot \underset{\sim}{T}\underset{\sim}{v}) + \alpha_2(\underset{\sim}{u}_2 \cdot \underset{\sim}{T}\underset{\sim}{v}) \quad \text{(linearity of inner product)}$$

$$= \alpha_1(\underset{\sim}{v} \cdot \underset{\sim}{t}(\underset{\sim}{u}_1)) + \alpha_2(\underset{\sim}{v} \cdot \underset{\sim}{t}(\underset{\sim}{u}_2)) \quad \text{(defining condition)}$$

$$= \underset{\sim}{v} \cdot (\alpha_1 \underset{\sim}{t}(\underset{\sim}{u}_1) + \alpha_2 \underset{\sim}{t}(\underset{\sim}{u}_2)) \quad \text{(linearity of inner product}$$

Since this holds $\forall\, \underset{\sim}{u} \in \mathcal{V}$, Theorem A2.3.4 $\Rightarrow$

$$\underset{\sim}{t}(\alpha_1 \underset{\sim}{u}_1 + \alpha_2 \underset{\sim}{u}_2) = \alpha_1 \underset{\sim}{t}(\underset{\sim}{u}_1) + \alpha_2 \underset{\sim}{t}(\underset{\sim}{u}_2) \quad \forall \alpha_1, \alpha_2 \in \mathbb{R} \text{ and } \forall\, \underset{\sim}{u}_1, \underset{\sim}{u}_2 \in \mathcal{V};$$

and we conclude from Theorem A2.4.2 that $\underset{\sim}{t}$ is linear. □

It is interesting to note that the above proof rests on the linearity of the inner product and makes no use of the linearity of $\underset{\sim}{T}$, so we could have stated it more generally. Nevertheless, the concept of transpose is applied only to <u>linear</u> operators. Most formulations of this topic require a priori that the transpose be linear. The point of Theorem A3.6.1 is that this is not necessary — if the transpose exists, it is automatically linear.

The following result justifies our speaking of <u>the</u> transpose of $\underset{\sim}{T}$.

<u>Theorem</u> A3.6.2. <u>When the transpose exists, it is unique.</u>

<u>Proof</u>. Given $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}$, where $\mathcal{V}$ is an inner product space,

suppose that both $\underset{\sim}{T}^T$ and $\underset{\sim}{S}$ are transposes of $\underset{\sim}{T}$. Then

$$(\underset{\sim}{T}^T\underset{\sim}{u} - \underset{\sim}{S}\underset{\sim}{u})\cdot\underset{\sim}{v} = 0 \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in V.$$

Since this holds $\forall\, \underset{\sim}{v} \in V$, it follows from Theorem A2.3.7 that

$$\underset{\sim}{T}^T\underset{\sim}{u} = \underset{\sim}{S}\underset{\sim}{u};$$

and since this holds $\forall\, \underset{\sim}{u} \in V$, $\underset{\sim}{T}^T = \underset{\sim}{S}$ by the definition of equality of tensors (Definition A3.1.2). □

The following theorem gives the more important rules for manipulating transposes.

Theorem A3.6.3. Let $\underset{\sim}{S}, \underset{\sim}{T} \in \mathrm{Lin}\, V$, where V is an inner product space. If $\underset{\sim}{S}^T$ and $\underset{\sim}{T}^T$ both exist, then

(i) $(\underset{\sim}{T}^T)^T$ exists and

$$(\underset{\sim}{T}^T)^T = \underset{\sim}{T};$$

(ii) $(\underset{\sim}{S} + \underset{\sim}{T})^T$ exists and

$$(\underset{\sim}{S} + \underset{\sim}{T})^T = \underset{\sim}{S}^T + \underset{\sim}{T}^T;$$

(iii) $(\alpha\underset{\sim}{T})^T$ exists and

$$(\alpha\underset{\sim}{T})^T = \alpha\underset{\sim}{T}^T;$$

(iv) $(\underset{\sim}{S}\underset{\sim}{T})^T$ exists and

$$(\underset{\sim}{S}\,\underset{\sim}{T})^T = \underset{\sim}{T}^T\underset{\sim}{S}^T ;$$

(v) $\underset{\sim}{I}^T$ and $\underset{\sim}{O}^T$ exist and

$$\underset{\sim}{I}^T = \underset{\sim}{I} , \quad \underset{\sim}{O}^T = \underset{\sim}{O} ;$$

(vi) for any $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$, $(\underset{\sim}{u} \otimes \underset{\sim}{v})^T$ exists and

$$(\underset{\sim}{u} \otimes \underset{\sim}{v})^T = \underset{\sim}{v} \otimes \underset{\sim}{u}.$$

Proof. (ii): For any $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$,

$$\begin{aligned}
\underset{\sim}{u} \cdot (\underset{\sim}{S}+\underset{\sim}{T})\underset{\sim}{v} &= \underset{\sim}{u} \cdot [\underset{\sim}{S}\underset{\sim}{v} + \underset{\sim}{T}\underset{\sim}{v}] \quad (\text{definition of } \underset{\sim}{S}+\underset{\sim}{T}) \\
&= \underset{\sim}{u} \cdot \underset{\sim}{S}\underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{T}\underset{\sim}{v} \quad (\text{linearity of inner product}) \\
&= \underset{\sim}{v} \cdot \underset{\sim}{S}^T\underset{\sim}{u} + \underset{\sim}{v} \cdot \underset{\sim}{T}^T\underset{\sim}{u} \quad (\text{definition of transpose}) \\
&= \underset{\sim}{v} \cdot (\underset{\sim}{S}^T + \underset{\sim}{T}^T)\underset{\sim}{u} \quad (\text{linearity of inner product}).
\end{aligned}$$

$\therefore$ $\underset{\sim}{S}^T + \underset{\sim}{T}^T$ is the transpose of $\underset{\sim}{S}+\underset{\sim}{T}$.

The proofs of (i), (iii), (iv), (v), and (vi) are left as *Exercises* A3.6.1 – 5, respectively. □

The results above are easy to remember as word statements.

For example, loosely speaking, (i) asserts that the transpose of the transpose is the original tensor, (ii) asserts that the transpose of a sum is the sum of the transposes, etc.

Exercise A3.6.6. Let $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$ and $\underset{\sim}{S}, \underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}$, where $\mathcal{V}$ is an inner product space. Let $\underset{\sim}{T}^T$ exist. Then show that

$$\underset{\sim}{S}(\underset{\sim}{u} \otimes \underset{\sim}{v})\underset{\sim}{T} = (\underset{\sim}{S}\underset{\sim}{u}) \otimes (\underset{\sim}{T}^T \underset{\sim}{v}).$$

Useful special cases are obtained by setting either $\underset{\sim}{S} = \underset{\sim}{I}$ or $\underset{\sim}{T} = \underset{\sim}{I}$:

$$(\underset{\sim}{u} \otimes \underset{\sim}{v})\underset{\sim}{T} = \underset{\sim}{u} \otimes \underset{\sim}{T}^T \underset{\sim}{v};$$

$$\underset{\sim}{S}(\underset{\sim}{u} \otimes \underset{\sim}{v}) = (\underset{\sim}{S}\underset{\sim}{u}) \otimes \underset{\sim}{v}.$$

Exercise A3.6.7. Note that the last result above does not involve the transpose. State and prove it directly.

Tensors on finite-dimensional spaces always have transposes.

Theorem A3.6.4. <u>Let $\mathcal{V}_n$ be an n-dimensional inner product space.</u>[1] <u>Then $\underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}_n$ has a transpose, and the components of $\underset{\sim}{T}^T$ are given by</u>

$$(\underset{\sim}{T}^T)^{ij} = T^{ji}, \quad (\underset{\sim}{T}^T)_{ij} = T_{ji},$$

---

[1] Cf. Theorem A2.3.2.

$$(\underset{\sim}{T}^T)_{\bullet j}^{\;i} = T_j^{\;\bullet i}\;,\quad (\underset{\sim}{T}^T)_i^{\;\bullet j} = \underset{\sim}{T}^{j}_{\;\bullet i}.$$

Proof. By Theorem A3.2.2, $\underset{\sim}{T}$ has the representation

$$\underset{\sim}{T} = T^{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\;.$$

Thus, $\underset{\sim}{T}$ is a linear combination of tensors each of which has a transpose by Theorem A3.6.3 (vi). Consequently, by repeated application (ii) and (iii) of Theorem A.3.6.3, $\underset{\sim}{T}^T$ exists[1], and

$$\underset{\sim}{T}^T = T^{ij}(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)^T$$

$$= T^{ij}\, \underset{\sim}{e}_j \otimes \underset{\sim}{e}_i \quad \text{(Theorem A3.6.3 (vi))}$$

$$= T^{ji}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \quad \text{(labeling)}.$$

$\therefore$ by the uniqueness[2] of components,

$$(\underset{\sim}{T}^T)^{ij} = T^{ji}.$$

The other relations are obtained in the same way. □

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[1] At this point then, we know that $\underset{\sim}{T}^T$ exists, and the proof could proceed by calculating the components of $\underset{\sim}{T}^T$ by Theorem A3.2.2. Carry this out as *Exercise* A3.6.8.

[2] See the remark following the statement of Theorem A3.2.5.

We note from Theorem A3.6.5 that in the finite-dimensional context tensor transposition corresponds to matrix transposition of the matrices of the various types of components. Notice also that in the cases of mixed components, transposition corresponds to switching the order of the indices as opposed to a literal switch of the indices themselves. For example,

$$(\mathbf{T}^T)^{i}_{\cdot j} = T_j^{\cdot i} \neq T^j_{\cdot i}, \text{ in general.}$$

Next, we turn to the notion of symmetric tensors.

Definition A3.6.1. Let $\mathcal{V}$ be an inner product space. Then a second-order tensor $\mathbf{T}$ on $\mathcal{V}$ is *symmetric*[1] if $\mathbf{T}^T$ exists and $\mathbf{T}^T = \mathbf{T}$. It is *skew*[2] if $\mathbf{T}^T$ exists and $\mathbf{T}^T = -\mathbf{T}$.

Theorem A3.6.5. Let $\mathcal{V}$ be an inner product space. Let $\mathbf{T} \in \mathrm{Lin}\,\mathcal{V}$ have a transpose. Then $\mathbf{T}$ admits the unique decomposition

$$\mathbf{T} = \mathbf{S} + \mathbf{W}$$

where $\mathbf{S}$ is symmetric and $\mathbf{W}$ is skew. In fact,

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[1] *Self-adjoint* if $\mathbf{T}^T$ is called the adjoint of $\mathbf{T}$.

[2] Often, *skew-symmetric* or *anti-symmetric*.

$$\underset{\sim}{S} = \tfrac{1}{2}\left(\underset{\sim}{T} + \underset{\sim}{T}^{T}\right), \quad \underset{\sim}{W} = \tfrac{1}{2}\left(\underset{\sim}{T} - \underset{\sim}{T}^{T}\right).$$

$\underset{\sim}{S}$ and $\underset{\sim}{W}$ are called the symmetric part and the skew part of $\underset{\sim}{T}$, respectively; and we often write

$$\underset{\sim}{S} = \operatorname{sym} \underset{\sim}{T}, \qquad \underset{\sim}{W} = \operatorname{skw} \underset{\sim}{T}.$$

Proof. Suppose that $\underset{\sim}{T}$ admits such a decomposition; i.e.,

$$\underset{\sim}{T} = \underset{\sim}{S} + \underset{\sim}{W}, \qquad (1)$$

where $\underset{\sim}{S}$ is symmetric and $\underset{\sim}{W}$ is skew. Then by (ii) of Theorem A3.6.3,

$$\underset{\sim}{T}^T = \underset{\sim}{S}^T + \underset{\sim}{W}^T,$$

which in view of the properties of $\underset{\sim}{S}$ and $\underset{\sim}{W}$ becomes

$$\underset{\sim}{T}^T = \underset{\sim}{S} - \underset{\sim}{W}. \qquad (2)$$

$$(1), (2) \Rightarrow \underset{\sim}{S} = \tfrac{1}{2}(\underset{\sim}{T} + \underset{\sim}{T}^T), \quad \underset{\sim}{W} = \tfrac{1}{2}(\underset{\sim}{T} - \underset{\sim}{T}^T).$$

Thus, if $\underset{\sim}{T}$ admits the decomposition, $\underset{\sim}{S}$ and $\underset{\sim}{W}$ must have the above forms. In particular, this establishes the uniqueness.

It remains to show that this decomposition works. Obviously,

$$\underset{\sim}{T} = \tfrac{1}{2}(\underset{\sim}{T} + \underset{\sim}{T}^T) + \tfrac{1}{2}(\underset{\sim}{T} - \underset{\sim}{T}^T) = \underset{\sim}{S} + \underset{\sim}{W}.$$

Moreover,

$$\underset{\sim}{S} = \tfrac{1}{2}(\underset{\sim}{T} + \underset{\sim}{T}^T) \Rightarrow \underset{\sim}{S}^T = \underset{\sim}{S},$$

and

$$\underset{\sim}{W} = \tfrac{1}{2}(\underset{\sim}{T} - \underset{\sim}{T}^T) \Rightarrow \underset{\sim}{W}^T = -\underset{\sim}{W}. \quad \square$$

When the underlying linear space is finite-dimensional, Theorems A3.6.5 and A3.6.4 yield

Theorem A3.6.6. Let $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$, where $\mathcal{V}_n$ is an n-dimensional inner product space. Then

$$(\operatorname{sym} \underset{\sim}{T})_{ij} = \tfrac{1}{2}(T_{ij} + T_{ji}), \quad (\operatorname{skw} \underset{\sim}{T})_{ij} = \tfrac{1}{2}(T_{ij} - T_{ji}),$$

with similar formulas for the other types of components.

In line with $\operatorname{Lin} \mathcal{V}$ standing for the set of all second-order tensors on the linear space $\mathcal{V}$, we introduce

Definition A3.6.2. Let $\mathcal{V}$ be an inner product space. Then

$$\operatorname{Sym} \mathcal{V} = \{ \underset{\sim}{T} \in \operatorname{Lin} \mathcal{V} : \underset{\sim}{T}^T = \underset{\sim}{T} \}$$

and

$$\operatorname{Skw} \mathcal{V} = \{ \underset{\sim}{T} \in \operatorname{Lin} \mathcal{V} : \underset{\sim}{T}^T = -\underset{\sim}{T} \}.$$

Thus, $\operatorname{Sym} \mathcal{V}$ ($\operatorname{skw} \mathcal{V}$) is the set of all symmetric (skew) tensors on $\mathcal{V}$.

Theorem A3.6.7. Let $\mathcal{V}$ be an inner product space. Then $\operatorname{Sym} \mathcal{V}$ and $\operatorname{Skw} \mathcal{V}$ are linear subspaces of $\operatorname{Lin} \mathcal{V}$.

Proof. Exercise A3.6.9. □

Of course, when $\mathcal{V}$ is finite-dimensional, $\operatorname{Lin} \mathcal{V}$ is is finite-dimensional as must be Sym $\mathcal{V}$ and Skew $\mathcal{V}$.)[1] In particular, we have

Theorem A3.6.8. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ be a basis for a 3-dimensional inner product space $\mathcal{V}_3$. Then

$$\{\underset{\sim}{e}_1\otimes\underset{\sim}{e}_1,\ \underset{\sim}{e}_2\otimes\underset{\sim}{e}_2,\ \underset{\sim}{e}_3\otimes\underset{\sim}{e}_3,\ \underset{\sim}{e}_1\otimes\underset{\sim}{e}_2+\underset{\sim}{e}_2\otimes\underset{\sim}{e}_1,\ \underset{\sim}{e}_2\otimes\underset{\sim}{e}_3+\underset{\sim}{e}_3\otimes\underset{\sim}{e}_2,\ \underset{\sim}{e}_3\otimes\underset{\sim}{e}_1+\underset{\sim}{e}_1\otimes\underset{\sim}{e}_3\}$$

and

$$\{\underset{\sim}{e}_1\otimes\underset{\sim}{e}_2-\underset{\sim}{e}_2\otimes\underset{\sim}{e}_1,\ \underset{\sim}{e}_2\otimes\underset{\sim}{e}_3-\underset{\sim}{e}_3\otimes\underset{\sim}{e}_2,\ \underset{\sim}{e}_3\otimes\underset{\sim}{e}_1-\underset{\sim}{e}_1\otimes\underset{\sim}{e}_3\}$$

are bases for Sym $\mathcal{V}_3$ and Skew $\mathcal{V}_3$, respectively. Thus, the linear subspaces Sym $\mathcal{V}_3$ and Skew $\mathcal{V}_3$ are 6-dimensional and 3-dimensional, respectively.

Proof. Exercise A3.6.10. $\square$

Since Skew $\mathcal{V}_3$ is 3-dimensional, it elements should be representable in terms of elements of $\mathcal{V}_3$. We also note that, independent of dimension, a skew tensor $\underset{\sim}{W}$ satisfies

$$\underset{\sim}{u}\cdot\underset{\sim}{W}\underset{\sim}{u} = \underset{\sim}{u}\cdot\underset{\sim}{W}^T\underset{\sim}{u} = -\underset{\sim}{u}\cdot\underset{\sim}{W}\underset{\sim}{u} = 0.$$

Thus, $\underset{\sim}{W}\underset{\sim}{u}$ is always orthogonal to $\underset{\sim}{u}$. Those observations lead us to

Theorem A3.6.9. Let the 3-dimensional inner product space $\mathcal{V}_3$ be oriented by Δ. Let $\underset{\sim}{W}\in$ Skew $\mathcal{V}_3$; then $\exists$ a unique $\underset{\sim}{\omega}\in\mathcal{V}_3$ $\ni$

$$\underset{\sim}{W}\underset{\sim}{u} = \underset{\sim}{\omega}\times\underset{\sim}{u} \quad \forall\ \underset{\sim}{u}\in\mathcal{V}_3.$$

[1] Cf. Theorem A2.2.10.

<u>Conversely</u>, <u>let</u> $\underset{\sim}{w} \in \mathcal{V}_3$; <u>then</u> $\exists$ <u>a unique</u> $\underset{\sim}{W} \in \mathrm{Skw}\,\mathcal{V}_3 \ni$

$$\underset{\sim}{W}\,\underset{\sim}{u} = \underset{\sim}{w} \times \underset{\sim}{u} \quad \forall \; \underset{\sim}{u} \in \mathcal{V}_3 .$$

In fact, if $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$ is a basis for $\mathcal{V}_3$, then

$$\omega^i = \frac{-1}{2\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)} \varepsilon^{ijk} W_{jk} \qquad)^1$$

and

$$W_{ij} = -\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)\, \varepsilon_{ijk}\, \omega^k.$$

$\underset{\sim}{\omega}$ is called the axial vector[2] of the skew tensor $\underset{\sim}{W}$; conversely, the skew tensor $\underset{\sim}{W}$ is the axial tensor[3] of the vector $\underset{\sim}{\omega}$. We shall write

$$\underset{\sim}{\omega} = \text{ax}\,\underset{\sim}{W} \quad \text{and} \quad \underset{\sim}{W} = \text{ax}\,\underset{\sim}{\omega}. \qquad)^3$$

Proof. Let $\underset{\sim}{W} \in \text{skw}\,\mathcal{V}_3$ be given and suppose $\exists\, \underset{\sim}{\omega} \in \mathcal{V}_3 \ni$

$$\underset{\sim}{\omega} \times \underset{\sim}{u} = \underset{\sim}{W}\,\underset{\sim}{u} \quad \forall\, \underset{\sim}{u} \in \mathcal{V}_3.$$

In components, we have from Theorems A3.2.9 and A2.8.2 that this is equivalent to

$$W_{ik}\, u^k = \Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)\, \varepsilon_{ijk}\, \omega^j u^k \qquad \forall\, u^k \in \mathbb{R},$$

By the arbitrariness of the u^k,

---

[1] Here, ε^{ijk} is simply the 3-dimensional alternating symbol (Definition A2.7.2) written with superscripts.

[2] Sometimes, dual vector.

[3] Sometimes, dual tensor.

[4] Often, $\underset{\sim}{\omega} = \underset{\sim}{W}_\times$ and $\underset{\sim}{W} = \underset{\sim}{\omega}_\times$.

$$W_{ik} = \Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)\, \varepsilon_{ijk}\, \omega^j$$

$$= -\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)\, \varepsilon_{ikj}\, \omega^j \quad (\text{definition of } \varepsilon_{ijk}).$$

$\therefore$

$$\varepsilon^{lik} W_{ik} = -\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)\, \varepsilon^{lik} \varepsilon_{ikj}\, \omega^j .$$

But again by the definition of the alternating symbol,

$$\varepsilon^{lik} = -\varepsilon^{ilk} = +\varepsilon^{ikl} \quad ;$$

and

$$\varepsilon^{lik} \varepsilon_{ikj} = \varepsilon^{ikl} \varepsilon_{ikj} = 2\,\delta^l_j \quad (\text{Theorem A2.7.8 (iii)}).$$

Consequently,

$$\varepsilon^{lik} W_{ik} = -\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)\, 2\,\delta^l_j\, \omega^j = -2\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)\, \omega^l .$$

Hence, if $\underset{\sim}{\omega}$ exists, its contravariant components are given by the stated formula. Since we have an explicit unambiguous formula for $\underset{\sim}{\omega}$ which is implied by the assumption that $\underset{\sim}{\omega}$ exists, the uniqueness of $\underset{\sim}{\omega}$ is established. We leave it as Exercise A3.6.11 to confirm by direct substitution that this $\underset{\sim}{\omega}$ does, indeed, have the property that $\underset{\sim}{\omega} \times \underset{\sim}{u} = \underset{\sim}{W}\underset{\sim}{u} \;\; \forall\, \underset{\sim}{u} \in V_3$.

¶ The same machinery establishes the converse assertion; the details are left as Exercise A3.6.12. □

Exercise A3.6.13. The constructive proof above for the existence of an axial vector of a skew tensor automatically established its uniqueness. Provide an independent proof of the uniqueness.

So far we have three different ways of multiplying two elements $\underset{\sim}{u}$ and $\underset{\sim}{v}$ of the same linear space together: the inner product $\underset{\sim}{u}\cdot\underset{\sim}{v}$; the cross product $\underset{\sim}{u}\times\underset{\sim}{v}$ (restricted to 3-dimensional spaces); and the tensor product $\underset{\sim}{u}\otimes\underset{\sim}{v}$ (like $\underset{\sim}{u}\cdot\underset{\sim}{v}$, restricted to inner product spaces). An additional product is provided by

Definition A3.6.3. Let $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$, where $\mathcal{V}$ is an inner product space. Then the skew product[1] of $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is the skew tensor

$$\underset{\sim}{u}\wedge\underset{\sim}{v} = 2\,\mathrm{skw}(\underset{\sim}{u}\otimes\underset{\sim}{v}) = \underset{\sim}{u}\otimes\underset{\sim}{v} - \underset{\sim}{v}\otimes\underset{\sim}{u}.$$

Of course, the last equation above is an immediate consequence of Theorems A3.6.5 and A3.6.3 (vi).

By its very definition the skew product is intimately related to the tensor product; in three-dimensions, there is a close connection to the vector product as well.

Theorem A3.6.10. Let $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_3$, where $\mathcal{V}_3$ is a 3-dimensional inner product space. Then

[1] Often, exterior or wedge product.

$$\underset{\sim}{u} \times \underset{\sim}{v} = -\operatorname{ax}(\underset{\sim}{u} \wedge \underset{\sim}{v}).$$

Proof. Exercise A3.6.14. □

While the vector product traditionally has played a strong role in mechanics, it does have the undesirable features of being nonassociative and of requiring that the space be oriented. The above theorem suggests that the vector product could probably be eliminated from mechanics through judicious use of the skew product.

Finally, we consider the transpose of a bounded tensor.

Theorem A3.6.11. Let $\mathcal{V}$ be an inner product space and assign it the inner product norm. Let $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}$ be bounded and have transpose $\underset{\sim}{T}^T$. Then $\underset{\sim}{T}^T$ is bounded and

$$\|\underset{\sim}{T}^T\| = \|\underset{\sim}{T}\|.$$

Proof. For any $\underset{\sim}{u} \in \mathcal{V}$,

$$\|\underset{\sim}{T}^T\underset{\sim}{u}\|_{\mathcal{V}}^2 = |\underset{\sim}{T}^T\underset{\sim}{u}|^2 = \underset{\sim}{T}^T\underset{\sim}{u} \cdot \underset{\sim}{T}^T\underset{\sim}{u} \quad \text{(inner product norm)}$$

$$= \underset{\sim}{u} \cdot (\underset{\sim}{T}^T)^T(\underset{\sim}{T}^T\underset{\sim}{u}) \quad \text{(definition of transpose)}$$

$$= \underset{\sim}{u} \cdot \underset{\sim}{T}(\underset{\sim}{T}^T\underset{\sim}{u}) \quad ((\underset{\sim}{T}^T)^T = \underset{\sim}{T})$$

$$\leq |\underset{\sim}{u}|\,|\underset{\sim}{T}(\underset{\sim}{T}^T\underset{\sim}{u})| \quad \text{(inner product-magnitude inequality)}$$

$$\leq |\underset{\sim}{u}| \, \|\underset{\sim}{T}\| \, |\underset{\sim}{T}^T \underset{\sim}{u}| \quad \text{(definition of operator norm)}.$$

$\therefore$ if $|\underset{\sim}{T}^T \underset{\sim}{u}| \neq 0$,

$$|\underset{\sim}{T}^T \underset{\sim}{u}| \leq \|\underset{\sim}{T}\| \, |\underset{\sim}{u}| \, ;$$

but this inequality obviously holds if $|\underset{\sim}{T}^T \underset{\sim}{u}| = 0$. Hence, $\|\underset{\sim}{T}\|$ is a bound for $\underset{\sim}{T}^T$. So $\underset{\sim}{T}^T$ is bounded and

$$\|\underset{\sim}{T}^T\| \leq \|\underset{\sim}{T}\|$$

since

$$\|\underset{\sim}{T}^T\| = \text{glb} \{ M : |\underset{\sim}{T}^T \underset{\sim}{u}| \leq M |\underset{\sim}{u}| \ \forall \ \underset{\sim}{u} \in V \} \, ,$$

Applying the same argument to $\underset{\sim}{T}^T$, we get $\|(\underset{\sim}{T}^T)^T\| \leq \|\underset{\sim}{T}^T\|$, or since $(\underset{\sim}{T}^T)^T = \underset{\sim}{T}$,

$$\|\underset{\sim}{T}\| \leq \|\underset{\sim}{T}^T\| .$$

$\therefore$

$$\|\underset{\sim}{T}^T\| = \|\underset{\sim}{T}\| . \quad \square$$

Continuing on with the hypotheses of the above theorem, we get from Theorem A3.5.6 that the tensor $\underset{\sim}{T}^T \underset{\sim}{T}$ is bounded and

$$\|\underset{\sim}{T}^T \underset{\sim}{T}\| \leq \|\underset{\sim}{T}^T\| \, \|\underset{\sim}{T}\| = \|\underset{\sim}{T}\|^2 .$$

But also for any $\underset{\sim}{u} \in V$

$$|\underset{\sim}{T}\underset{\sim}{u}|^2 = \underset{\sim}{T}\underset{\sim}{u} \cdot \underset{\sim}{T}\underset{\sim}{u} \quad \text{(inner product norm)}$$

$$= u \cdot \underset{\sim}{T}^T(\underset{\sim}{T}\underset{\sim}{u}) \quad (\text{definition of } \underset{\sim}{T}^T)$$

$$= \underset{\sim}{u} \cdot (\underset{\sim}{T}^T\underset{\sim}{T})\underset{\sim}{u} \quad \text{(definition of tensor multiplication)}$$

$$\leq |\underset{\sim}{u}|\,|(\underset{\sim}{T}^T\underset{\sim}{T})\underset{\sim}{u}| \quad \text{(inner product-magnitude inequality)}$$

$$\leq |\underset{\sim}{u}|\,\|\underset{\sim}{T}^T\underset{\sim}{T}\|\,|\underset{\sim}{u}| \quad \text{(definition of operator norm)}.$$

$\therefore$

$$|\underset{\sim}{T}\underset{\sim}{u}| \leq \sqrt{\|\underset{\sim}{T}^T\underset{\sim}{T}\|}\;|\underset{\sim}{u}|,$$

which says that $\sqrt{\|\underset{\sim}{T}^T\underset{\sim}{T}\|}$ is a bound for $\underset{\sim}{T}$. By definition, $\|\underset{\sim}{T}\|$ is the greatest lower bound of the set of all bounds for $\underset{\sim}{T}$; whence

$$\|\underset{\sim}{T}\| \leq \sqrt{\|\underset{\sim}{T}^T\underset{\sim}{T}\|} \quad \text{or} \quad \|\underset{\sim}{T}\|^2 \leq \|\underset{\sim}{T}^T\underset{\sim}{T}\|.$$

On comparison with our earlier result that $\|\underset{\sim}{T}\|^2 \geq \|\underset{\sim}{T}^T\underset{\sim}{T}\|$, we have

$$\|\underset{\sim}{T}^T\underset{\sim}{T}\| = \|\underset{\sim}{T}\|^2,$$

In exactly the same way, $\|\underset{\sim}{T}\underset{\sim}{T}^T\| = \|\underset{\sim}{T}\|^2$. Thus, we have proven

Theorem A3.6.12. <u>Let the hypotheses of Theorem A3.6.11 hold. Then $\underset{\sim}{T}^T\underset{\sim}{T}$ and $\underset{\sim}{T}\underset{\sim}{T}^T$ are bounded, and</u>

$$\|\underset{\sim}{T}^T \underset{\sim}{T}\| = \|\underset{\sim}{T}\,\underset{\sim}{T}^T\| = \|\underset{\sim}{T}\|^2 = \|\underset{\sim}{T}^T\|^2 .$$

Supplementary Reading

same as for § A3.5

A3.7. The Trace of a Second-Order Tensor. The Trace Inner Product

The trace of a tensor is a special scalar-valued function of a tensor which occurs frequently in continuum mechanics.

Definition A3.7.1. Let $\mathcal{V}$ be an inner product space. Then a linear functional on $Lin\,\mathcal{V}$, denoted by

$$\underset{\sim}{T} \mapsto tr\,\underset{\sim}{T},$$

is called a trace operation on $Lin\,\mathcal{V}$ if it satisfies

$$tr\,(\underset{\sim}{u} \otimes \underset{\sim}{v}) = \underset{\sim}{u} \cdot \underset{\sim}{v} \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}. \quad)^{1}$$

In the finite-dimensional case, a unique trace operation always exists. For the remainder of this section, $\mathcal{V}_n$ will always denote an n-dimensional inner product space.)[2]

Theorem A3.7.1. $\exists$ one, and only one, trace operation on $Lin\,\mathcal{V}_n$. In fact, for $\underset{\sim}{T} \in Lin\,\mathcal{V}_n$,

$$tr\,\underset{\sim}{T} = g_{ij} T^{ij} = g^{ij} T_{ij} = T^{i}_{\cdot i} = T_{i}^{\cdot i}.$$

---

[1] See MARTIN and MIZEL's Introduction to Linear Algebra for a geometrical interpretation of trace operation in the finite-dimensional case.

[2] Cf. Theorem A2.3.2.

Proof. Suppose tr is a trace operation on Lin $\mathcal{V}_n$. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ be a basis for $\mathcal{V}_n$. Then by Theorem A3.2.2, $\underset{\sim}{T} \in \text{Lin}\,\mathcal{V}_n$ has the representation

$$\underset{\sim}{T} = T^{ij}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j .$$

Then

$$\begin{aligned} \operatorname{tr} \underset{\sim}{T} &= \operatorname{tr}\left(T^{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\right) \quad \text{(substitution)} \\ &= T^{ij} \operatorname{tr}\left(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\right) \quad \text{(linearity of tr)} \\ &= T^{ij} g_{ij} \quad \left(\operatorname{tr}\left(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j\right) = \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j\right). \end{aligned}$$

Hence, if a trace operation on Lin $\mathcal{V}_n$ exists, it must have the form

$$\operatorname{tr} \underset{\sim}{T} = g_{ij} T^{ij} .$$

This establishes the uniqueness. The alternative expressions follow from shifting indices via Theorem A3.2.8.

We leave it as Exercise A3.7.1 to show that the functional on Lin $\mathcal{V}_n$ defined by $\operatorname{tr} \underset{\sim}{T} = g_{ij} T^{ij}$ meets all the requirements of the definition. □

Exercise A3.7.2. In the component-transformation rule approach, the trace would be defined by in terms of components as

$$\operatorname{tr} \underset{\sim}{T} := g_{ij} T^{ij} .$$

Of course, in this approach one needs to be concerned about the definition being tied fundamentally to the basis used in the definition. Use the transformation rules to show that

$$\bar{g}_{ij}\bar{T}^{ij} = g_{ij}T^{ij} .$$

Of course, in our basis-free approach, such questions never arise; and

$$\operatorname{tr} \underset{\sim}{T} = \bar{g}_{ij}\bar{T}^{ij}$$

follows directly from Theorem A3.7.1.

<u>We note that in general the trace of a second-order tensor on $\mathcal{V}_n$ is not the trace of a corresponding matrix of components.</u> But these two notions do agree if we consider a matrix of mixed components or if the basis chosen for $\mathcal{V}_n$ is orthonormal.

Two important additional features of the trace beyond its defining properties are given by

<u>Theorem</u> A3.7.2. <u>Let</u> $\underset{\sim}{S}, \underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$. <u>Then</u>

$$\text{(i)} \quad \operatorname{tr} \underset{\sim}{T}^T = \operatorname{tr} \underset{\sim}{T}$$

<u>and</u>

$$\text{(ii)} \quad \operatorname{tr}(\underset{\sim}{S}\underset{\sim}{T}) = \operatorname{tr}(\underset{\sim}{T}\underset{\sim}{S}).$$

<u>Proof</u>. <u>Exercises</u> A3.7.3 and A3.7.4. □

The trace operation can be used to generate an inner product on Lin $\mathcal{V}_n$.

Theorem A3.7.3. The n^2-dimensional linear space Lin $\mathcal{V}_n$ equipped with the function on Lin $\mathcal{V}_n \times$ Lin $\mathcal{V}_n$ to $\mathbb{R}$ defined by

$$(\underset{\sim}{S}, \underset{\sim}{T}) \mapsto \underset{\sim}{S} \cdot \underset{\sim}{T} = \mathrm{tr}(\underset{\sim}{S}\, \underset{\sim}{T}^T),$$

in components

$$\underset{\sim}{S} \cdot \underset{\sim}{T} = S^{ij} T_{ij} = S_{ij} T^{ij} = g_{im} g_{jn} S^{mn} T^{ij}, \text{ etc.},$$

is an inner product space.

Proof. First, we note that the component formulas are just calculations based on Theorems A3.5.4, A3.6.4, and A3.7.1. Of course, the indices can be shifted via Theorem A3.2.8.

To prove the theorem, we must show that $\underset{\sim}{S} \cdot \underset{\sim}{T}$ satisfies the inner product axioms laid out in Definition A2.3.1. This is an easy task, and it is left as Exercise A3.7.5. (Hint: To establish the positive definiteness, use components relative to an orthonormal basis for $\mathcal{V}_n$.) $\square$

There are, of course, other inner products for Lin $\mathcal{V}_n$; but we shall reserve the notation $\underset{\sim}{S} \cdot \underset{\sim}{T}$ for the one introduced above.[1]

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[1] In dyadic notation, our $\underset{\sim}{S} \cdot \underset{\sim}{T}$ would appear as $\underset{\sim}{S} : \underset{\sim}{T}$; while $\underset{\sim}{S} \cdot \underset{\sim}{T}$ would denote our product $\underset{\sim}{S}\,\underset{\sim}{T}$.

which is called the *trace inner product*.

Of course, this particular example of an inner product space enjoys all of the general properties developed in § A2.3. We shall not list these here, but we do explicitly note that we carry over the notation

$$|\underset{\sim}{T}| = \sqrt{\underset{\sim}{T} \cdot \underset{\sim}{T}}$$

for the inner product norm, and we call $|\underset{\sim}{T}|$ the *magnitude* of $\underset{\sim}{T}$.

Some useful identities associated with this inner product are given by

Theorem A3.7.4. Let $\mathcal{V}_n$ be an n-dimensional inner product space. Then

$$(i)\quad \underset{\sim}{X} \cdot \underset{\sim}{Y} = 0 \quad \forall\ \underset{\sim}{X} \in \mathrm{Sym}\,\mathcal{V}_n,\ \underset{\sim}{Y} \in \mathrm{Skw}\,\mathcal{V}_n,$$

$$(ii)\quad \underset{\sim}{S} \cdot \underset{\sim}{T} = (\mathrm{sym}\,\underset{\sim}{S}) \cdot (\mathrm{sym}\,\underset{\sim}{T}) + (\mathrm{skw}\,\underset{\sim}{S}) \cdot (\mathrm{skw}\,\underset{\sim}{T}) \quad \forall\ \underset{\sim}{S}, \underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n,$$

$$(iii)\quad \underset{\sim}{u} \cdot \underset{\sim}{T}\underset{\sim}{v} = (\underset{\sim}{u} \otimes \underset{\sim}{v}) \cdot \underset{\sim}{T} \quad \forall\ \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_n \text{ and } \forall\ \underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n,$$

$$(iv)\quad (\underset{\sim}{u} \otimes \underset{\sim}{v}) \cdot (\underset{\sim}{w} \otimes \underset{\sim}{x}) = (\underset{\sim}{u} \cdot \underset{\sim}{w})(\underset{\sim}{v} \cdot \underset{\sim}{x}) \quad \forall\ \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}, \underset{\sim}{x} \in \mathcal{V}_n,$$

$$(v)\quad |\underset{\sim}{u} \otimes \underset{\sim}{v}| = |\underset{\sim}{u}||\underset{\sim}{v}| \quad \forall\ \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_n.$$

Proof. *Exercises* A3.7.6–10. □

The inner product $\underset{\sim}{S}\cdot\underset{\sim}{T} = \operatorname{tr}(\underset{\sim}{S}\underset{\sim}{T}^T)$ has the desirable property that relative to it an orthonormal basis for $\mathcal{V}_n$ generates an orthonormal basis for $\operatorname{Lin}\mathcal{V}_n$. Formally, we have

Theorem A3.7.5. Let $\{\underset{\sim}{e}_{\langle 1\rangle}, \underset{\sim}{e}_{\langle 2\rangle}, \cdots, \underset{\sim}{e}_{\langle n\rangle}\}$ be an orthonormal basis for an n-dimensional inner product space $\mathcal{V}_n$. Then relative to the inner product $\underset{\sim}{S}\cdot\underset{\sim}{T} = \operatorname{tr}(\underset{\sim}{S}\underset{\sim}{T}^T)$, $\{\underset{\sim}{e}_{\langle i\rangle}\otimes\underset{\sim}{e}_{\langle j\rangle}\}$ is an orthonormal basis for $\operatorname{Lin}\mathcal{V}_n$; i.e., for $i,j,k,l \in \mathcal{I}_n$

$$(\underset{\sim}{e}_{\langle i\rangle}\otimes\underset{\sim}{e}_{\langle j\rangle})\cdot(\underset{\sim}{e}_{\langle k\rangle}\otimes\underset{\sim}{e}_{\langle l\rangle}) = \delta_{ik}\,\delta_{jl}\,.$$

Proof. In view of Theorem A3.2.3, only the orthonormality needs to be checked. This is an immediate consequence of identity (iv) of Theorem A3.7.4. □

The identity tensor, of course, has special properties, with respect to the trace operation, some of them are given by

Theorem A3.7.6. Let $\underset{\sim}{I}$ denote the identity tensor on an n-dimensional inner product space $\mathcal{V}_n$. Then

(i) $\operatorname{tr}\underset{\sim}{I} = n$,

(ii) $\operatorname{tr}\underset{\sim}{T} = \underset{\sim}{I}\cdot\underset{\sim}{T} \quad \forall\ \underset{\sim}{T}\in\operatorname{Lin}\mathcal{V}_n$,

(iii) $|\underset{\sim}{I}| = \sqrt{n}$.

Proof. Exercises A3.7.11-13. □

At first glance, Theorem A 3.7.6 appears to be of interest mainly with regard to calculations. Actually, it contains a wealth of deeper information. To begin with, since the trace is a linear functional on Lin V_n, the representation theorem (Theorem A2.5.1) tells us that $\exists$ a unique tensor $\hat{\underset{\sim}{T}} \in \text{Lin}\, V_n$ $\ni$

$$\text{tr}\, \underset{\sim}{T} = \hat{\underset{\sim}{T}} \cdot \underset{\sim}{T} \quad \forall\ \underset{\sim}{T} \in \text{Lin}\, V_n .$$

By (ii) of Theorem A3.7.6, this special tensor $\hat{\underset{\sim}{T}}$ is the identity tensor $\underset{\sim}{I}$.

Next, we not that

$$|\text{tr}\, \underset{\sim}{T}| = |\underset{\sim}{I} \cdot \underset{\sim}{T}| \quad (\text{Theorem A3.7.6 (ii)})$$

$$\leq |\underset{\sim}{I}||\underset{\sim}{T}| \quad (\text{inner product-magnitude inequality})$$

$$= \sqrt{n}\, |\underset{\sim}{T}| \quad (\text{Theorem A 3.7.6 (iii)}) .$$

Thus, we have proven

Theorem A 3.7.7. Let $\underset{\sim}{T} \in \text{Lin}\, V_n$, where V_n is an n-dimensional inner product space. Then

$$|\text{tr}\, \underset{\sim}{T}| \leq \sqrt{n}\, |\underset{\sim}{T}| .$$

In Theorem A3.4.8, we saw that $\|\underset{\sim}{I}\| = 1$. Hence, (iii) of theorem A3.7.6 makes it clear that the operator norm and the inner product norm of the same tensor generally have different values. Let us explore this for a general tensor $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$. For any $\underset{\sim}{u} \in \mathcal{V}_n$

$$
\begin{aligned}
|\underset{\sim}{T}\underset{\sim}{u}|^2 &= (\underset{\sim}{T}\underset{\sim}{u}) \cdot (\underset{\sim}{T}\underset{\sim}{u}) \quad \text{(inner product norm)} \\
&= ((\underset{\sim}{T}\underset{\sim}{u}) \otimes \underset{\sim}{u}) \cdot \underset{\sim}{T} \quad \text{(Theorem A3.7.4 (iii))} \\
&\leq |(\underset{\sim}{T}\underset{\sim}{u}) \otimes \underset{\sim}{u}|\,|\underset{\sim}{T}| \quad \text{(inner product-magnitude ineq.)} \\
&= |\underset{\sim}{T}\underset{\sim}{u}|\,|\underset{\sim}{u}|\,|\underset{\sim}{T}| \quad \text{(Theorem A3.7.4 (v))}.
\end{aligned}
$$

$\therefore$ if $|\underset{\sim}{T}\underset{\sim}{u}| \neq 0$,

$$|\underset{\sim}{T}\underset{\sim}{u}| \leq |\underset{\sim}{T}|\,|\underset{\sim}{u}|.$$

But this inequality is trivially true if $|\underset{\sim}{T}\underset{\sim}{u}| = 0$, and $\therefore$ we are led to

Theorem A3.7.8. Let $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$, where $\mathcal{V}_n$ is an n-dimensional inner product space. Then

(i) $|\underset{\sim}{T}\underset{\sim}{u}| \leq |\underset{\sim}{T}|\,|\underset{\sim}{u}| \quad \forall\, \underset{\sim}{u} \in \mathcal{V}_n$;

and

(ii) $\|\underset{\sim}{T}\| \leq |\underset{\sim}{T}|$.

Proof. (i): Established by the preceding discussion.

(ii): By (i),

$$|\underset{\sim}{T}| \in \underset{\sim}{B} := \{M \geq 0 : |\underset{\sim}{T}\underset{\sim}{u}| \leq M|\underset{\sim}{u}| \ \forall \underset{\sim}{u} \in \mathcal{V}_n\}.$$

But by definition $\|\underset{\sim}{T}\| = \text{glb}\, \underset{\sim}{B}$, and $\therefore \|\underset{\sim}{T}\| \leq |\underset{\sim}{T}|$. $\square$

We note that our earlier observation for the identity tensor on $\mathcal{V}_n$ that $\|\underset{\sim}{I}\| = 1$ and $|\underset{\sim}{I}| = \sqrt{n}$ is in agreement with (ii) of Theorem A3.7.8.

By Theorem A3.5.6, the operator norm has the property that

$$\|\underset{\sim}{S}\underset{\sim}{T}\| \leq \|\underset{\sim}{S}\|\|\underset{\sim}{T}\|.$$

This is also a feature of the inner product norm

$$|\underset{\sim}{S}| = \sqrt{\underset{\sim}{S}\cdot\underset{\sim}{S}} = \sqrt{tr(\underset{\sim}{S}\underset{\sim}{S}^T)}.$$

__Theorem__ A3.7.9. <u>Let $\underset{\sim}{S}, \underset{\sim}{T} \in Lin\, \mathcal{V}_n$, where $\mathcal{V}_n$ is an n-dimensional inner product space with inner product norm. Then</u>

$$|\underset{\sim}{S}\underset{\sim}{T}| \leq \|\underset{\sim}{S}\||\underset{\sim}{T}| \leq |\underset{\sim}{S}||\underset{\sim}{T}|$$

Proof. First, let $\{\underset{\sim}{e}_{\langle 1\rangle}, \underset{\sim}{e}_{\langle 2\rangle}, \cdots, \underset{\sim}{e}_{\langle n\rangle}\}$ be any orthonormal basis for $\mathcal{V}_n$. Then for any $\underset{\sim}{L} \in \mathcal{V}_n$

$$|\underset{\sim}{L}|^2 = \underset{\sim}{L} \cdot \underset{\sim}{L} = tr(\underset{\sim}{L}\underset{\sim}{L}^T) \quad \text{(inner product norm)}$$

$$= tr(\underset{\sim}{L}\underset{\sim}{I}\underset{\sim}{L}^T) \quad \text{(definition of } \underset{\sim}{I} \text{ and associativity of tensor multiplication)}$$

$$= tr\left(\underset{\sim}{L}\left(\sum_{i=1}^{n} \underset{\sim}{e}_{\langle i \rangle} \otimes \underset{\sim}{e}_{\langle i \rangle}\right)\underset{\sim}{L}^T\right) \quad \text{(Theorems A3.2.7 and A2.5.7)}$$

$$= tr \sum_{i=1}^{n} \left(\underset{\sim}{L}(\underset{\sim}{e}_{\langle i \rangle} \otimes \underset{\sim}{e}_{\langle i \rangle})\underset{\sim}{L}^T\right) \quad \text{(distributivity of tensor mult.)}$$

$$= tr \sum_{i=1}^{n} (\underset{\sim}{L}\underset{\sim}{e}_{\langle i \rangle}) \otimes (\underset{\sim}{L}\underset{\sim}{e}_{\langle i \rangle}) \quad \text{(Exercise A3.6.6 and } (\underset{\sim}{L}^T)^T = \underset{\sim}{L})$$

$$= \sum_{i=1}^{n} tr\left((\underset{\sim}{L}\underset{\sim}{e}_{\langle i \rangle}) \otimes (\underset{\sim}{L}\underset{\sim}{e}_{\langle i \rangle})\right) \quad \text{(linearity of trace)}$$

$$= \sum_{i=1}^{n} (\underset{\sim}{L}\underset{\sim}{e}_{\langle i \rangle}) \cdot (\underset{\sim}{L}\underset{\sim}{e}_{\langle i \rangle}) \quad \text{(definition of trace)}.$$

Application of this result to $\underset{\sim}{S}\underset{\sim}{T}$ yields

$$|\underset{\sim}{S}\underset{\sim}{T}|^2 = \sum_{i=1}^{n} ((\underset{\sim}{S}\underset{\sim}{T})\underset{\sim}{e}_{\langle i \rangle}) \cdot ((\underset{\sim}{S}\underset{\sim}{T})(\underset{\sim}{e}_{\langle i \rangle}))$$

$$= \sum_{i=1}^{n} (\underset{\sim}{S}(\underset{\sim}{T}\underset{\sim}{e}_{\langle i \rangle})) \cdot (\underset{\sim}{S}(\underset{\sim}{T}\underset{\sim}{e}_{\langle i \rangle})) \quad \text{(tensor multiplication)}$$

$$= \sum_{i=1}^{n} |\underset{\sim}{S}(\underset{\sim}{T}\underset{\sim}{e}_{\langle i \rangle})|^2 \quad \text{(definition of magnitude)}$$

$$\leq \sum_{i=1}^{n} \|\underset{\sim}{S}\|^2 |\underset{\sim}{T}\underset{\sim}{e}_{\langle i \rangle}|^2 \quad \text{(definition of } \|\underset{\sim}{S}\|)$$

$$= \|\underset{\sim}{S}\|^2 \sum_{i=1}^{n} |\underset{\sim}{T}\underset{\sim}{e}_{\langle i \rangle}|^2 \quad (\|\underset{\sim}{S}\|^2 \text{ independent of } i)$$

$$= \|\underset{\sim}{S}\|^2 \sum_{i=1}^{n} (\underset{\sim}{T}\underset{\sim}{e}_{<i>}) \cdot (\underset{\sim}{T}\underset{\sim}{e}_{<i>}) \quad \text{(definition of magnitude)}$$

$$= \|\underset{\sim}{S}\|^2 |\underset{\sim}{T}|^2 \text{ (above result)},$$

$\therefore$ by the monotonicity of the square root

$$|\underset{\sim}{S}\underset{\sim}{T}| \le \|\underset{\sim}{S}\| |\underset{\sim}{T}|.$$

The second inequality follows from Theorem A3.7.9 (ii); i.e.,

$$\|\underset{\sim}{S}\| \le |\underset{\sim}{S}| \Rightarrow \|\underset{\sim}{S}\| |\underset{\sim}{T}| \le |\underset{\sim}{S}| |\underset{\sim}{T}|. \quad \square$$

Taking $\underset{\sim}{T} = \underset{=}{I}$ in Theorem A3.7.9 yields

$$|\underset{\sim}{S}| = |\underset{\sim}{S}\underset{\sim}{I}| \le \|\underset{\sim}{S}\| |\underset{\sim}{I}| = \sqrt{n}\, \|\underset{\sim}{S}\|,$$

where have used Theorem A3.7.6 (iii). On combining this with Theorem A3.7.8 (iii), we get

Theorem A3.7.10. Let $\underset{\sim}{T} \in \text{Lin}\, \mathcal{V}_n$, where $\mathcal{V}_n$ is an n-dimensional inner product space. Then

$$\|\underset{\sim}{T}\| \le |\underset{\sim}{T}| \le \sqrt{n}\, \|\underset{\sim}{T}\|.$$

Supplementary Reading

Same as for § A3.5 + BODEWIG, Matrix Calculus

A3.8. The Determinant of a Second-Order Tensor

Throughout this section $\mathcal{V}_n$ denotes an n-dimensional inner product space[1] which is oriented[2] by the volume orientation function Δ.

By Definition A2.7.5, the volume spanned by a basis $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ for $\mathcal{V}_n$ is $|\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n)|$. If $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n$, then the image of $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ under $\underset{\sim}{T}$ is $\{\underset{\sim}{T}\underset{\sim}{e}_1, \underset{\sim}{T}\underset{\sim}{e}_2, \cdots, \underset{\sim}{T}\underset{\sim}{e}_n\}$; and the volume spanned by this set of image elements is $|\Delta(\underset{\sim}{T}\underset{\sim}{e}_1, \underset{\sim}{T}\underset{\sim}{e}_2, \cdots, \underset{\sim}{T}\underset{\sim}{e}_n)|$. Apart from sign, the determinant of the tensor $\underset{\sim}{T}$ is simply the ratio of these volumes. In other words, the determinant of $\underset{\sim}{T}$ is a direct measure of the volume change caused by the linear mapping $\underset{\sim}{T}$. Formally, we have

Theorem A3.8.1. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ and Δ be a basis and a volume orientation function for $\mathcal{V}_n$. Then for any $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n$, the scalar

$$\det \underset{\sim}{T} = \frac{\Delta(\underset{\sim}{T}\underset{\sim}{e}_1, \underset{\sim}{T}\underset{\sim}{e}_2, \cdots, \underset{\sim}{T}\underset{\sim}{e}_n)}{\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n)} \qquad)^3$$

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[1] Cf. Theorem A2.3.2.

[2] See Definition A2.7.4.

[3] By Theorem A2.7.4, there is no chance of the denominator vanishing.

is independent of the choice of $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ and of the choice of Δ. It is called the determinant of the second-order tensor $\underset{\sim}{T}$.

Proof. By Theorem A2.7.2, the two choices of Δ differ by a factor of -1, which is divided out in the ratio defining $\det \underset{\sim}{T}$.

To show that $\det \underset{\sim}{T}$ also is independent of the choice of $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$, we introduce the function

$$\Delta_T : \underbrace{\mathcal{V}_n \times \mathcal{V}_n \times \cdots \times \mathcal{V}_n}_{n \text{ times}} \longrightarrow R$$

defined by

$$\Delta_T(\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_n) = \Delta(\underset{\sim}{T}\underset{\sim}{u}_1, \underset{\sim}{T}\underset{\sim}{u}_2, \cdots, \underset{\sim}{T}\underset{\sim}{u}_n).$$

By the properties of Δ and the linearity of $\underset{\sim}{T}$, Δ_T satisfies Axioms (O_1) and (O_2) of Definition A2.7.3 for volume orientation functions. Consequently, by Theorem A2.7.1,

$$\Delta_T(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n) = \varepsilon_{\sigma_1 \sigma_2 \cdots \sigma_n} e_1^{\sigma_1} e_2^{\sigma_2} \cdots e_n^{\sigma_n} \Delta_T(\bar{\underset{\sim}{e}}_1, \bar{\underset{\sim}{e}}_2, \cdots, \bar{\underset{\sim}{e}}_n),$$

where $\{\bar{\underset{\sim}{e}}_1, \bar{\underset{\sim}{e}}_2, \cdots, \bar{\underset{\sim}{e}}_n\}$ is any basis for $\mathcal{V}_n$ and $e_j^{\sigma_i} = \bar{\underset{\sim}{e}}^{\sigma_i} \cdot \underset{\sim}{e}_j$.

Thus,

$$\Delta(\underset{\sim}{T}\underset{\sim}{e}_1, \underset{\sim}{T}\underset{\sim}{e}_2, \cdots, \underset{\sim}{T}\underset{\sim}{e}_n) = \varepsilon_{\sigma_1 \sigma_2 \cdots \sigma_n} e_1^{\sigma_1} e_2^{\sigma_2} \cdots e_n^{\sigma_n} \Delta(\underset{\sim}{T}\bar{\underset{\sim}{e}}_1, \underset{\sim}{T}\bar{\underset{\sim}{e}}_2, \cdots, \underset{\sim}{T}\bar{\underset{\sim}{e}}_n).$$

But application of Theorem A2.7.1 to Δ itself yields

$$\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n) = \varepsilon_{\tau_1 \tau_2 \cdots \tau_n} e_1^{\tau_1} e_2^{\tau_2} \cdots e_n^{\tau_n} \Delta(\bar{\underset{\sim}{e}}_1, \bar{\underset{\sim}{e}}_2, \cdots, \bar{\underset{\sim}{e}}_n);$$

and $\therefore$

$$\frac{\Delta(\underset{\sim}{T}\underset{\sim}{e}_1, \underset{\sim}{T}\underset{\sim}{e}_2, \cdots, \underset{\sim}{T}\underset{\sim}{e}_n)}{\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n)} = \frac{\Delta(\underset{\sim}{T}\bar{\underset{\sim}{e}}_1, \underset{\sim}{T}\bar{\underset{\sim}{e}}_2, \cdots, \underset{\sim}{T}\bar{\underset{\sim}{e}}_n)}{\Delta(\bar{\underset{\sim}{e}}_1, \bar{\underset{\sim}{e}}_2, \cdots, \bar{\underset{\sim}{e}}_n)} . \; \square$$

Theorem A3.8.1 shows that det is a well-defined function on $\mathrm{Lin}\, \mathcal{V}_n$ to $\mathbb{R}$; i.e., a scalar-valued function of a tensor.

Theorem A3.8.2. The determinant function on $\mathrm{Lin}\, \mathcal{V}_n$ has the following properties:

(i) $\det \underset{\sim}{O} = 0$;

(ii) $\det \underset{\sim}{I} = 1$;

(iii) $\det(\underset{\sim}{u} \otimes \underset{\sim}{v}) = 0 \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_n$;

(iv) $\det(\alpha \underset{\sim}{T}) = \alpha^n \det \underset{\sim}{T} \quad \forall\, \alpha \in \mathbb{R}$ and $\forall\, \underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}_n$.

Proof. Exercises A3.8.1–4. $\square$

$\boxed{\text{INSERT material on p. A3.8.3a}}$

The next result is a variant of the definition of $\det \underset{\sim}{T}$ appropriate to the case where the $\underset{\sim}{e}$'s are not a basis.

Theorem A3.8.3. Let $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_n\} \subset \mathcal{V}_n$. Then for any $\underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}_n$,

$$\Delta(\underset{\sim}{T}\underset{\sim}{u}_1, \underset{\sim}{T}\underset{\sim}{u}_2, \cdots, \underset{\sim}{T}\underset{\sim}{u}_n) = \det \underset{\sim}{T} \; \Delta(\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_n).$$

INSERT for p. A3.8.3 (immediately after proof of Theorem A3.8.2)

Since there are many tensors with nonzero determinant (e.g., the identity tensor $\underset{\sim}{I}$), (iii) above provides conclusive evidence of the fact that <u>not every second-order tensor can be expressed as the tensor product of two vectors.</u>

return to p. A3.8.3

Proof. If the set $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_n\}$ is linearly independent, then by Theorem A2.2.8 it is a basis for $\mathcal{V}_n$; and the theorem follows from the definition of $\det \underset{\sim}{T}$ as given in Theorem A3.8.1.

If the set $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_n\}$ is linearly dependent, the set $\{\underset{\sim}{T}\underset{\sim}{u}_1, \underset{\sim}{T}\underset{\sim}{u}_2, \cdots, \underset{\sim}{T}\underset{\sim}{u}_n\}$ is also linearly dependent since

$$\underset{\sim}{T}(\alpha^i \underset{\sim}{u}_i) = \alpha^i (\underset{\sim}{T}\underset{\sim}{u}_i) \quad \forall \alpha^i \in \mathbb{R}.$$

In this case, by Theorem A2.7.4,

$$\Delta(\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_n) = \Delta(\underset{\sim}{T}\underset{\sim}{u}_1, \underset{\sim}{T}\underset{\sim}{u}_2, \cdots, \underset{\sim}{T}\underset{\sim}{u}_n) = 0;$$

and trivially,

$$\Delta(\underset{\sim}{T}\underset{\sim}{u}_1, \underset{\sim}{T}\underset{\sim}{u}_2, \cdots, \underset{\sim}{T}\underset{\sim}{u}_n) = \det \underset{\sim}{T}\, \Delta(\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_n). \quad \square$$

As an important corollary of Theorem A3.8.3, we have

Theorem A3.8.4. Let $\underset{\sim}{S}, \underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}_n$. Then

$$\det(\underset{\sim}{S}\underset{\sim}{T}) = (\det \underset{\sim}{S})(\det \underset{\sim}{T}).$$

Proof. Exercise A3.8.5. $\square$

In words, Theorem A3.8.4 asserts that the determinant of a product is the product of the determinant. This has a a familiar ring because the same result holds for determinants of matrices. However, we must be careful here. The next theorem shows exactly how the determinant of a tensor is related to the determinants of its various matrices of components.

Theorem A3.8.5. Let $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$. In terms of components, $\det \underset{\sim}{T}$ has the representations

$$\det \underset{\sim}{T} = \det [T_{i}^{\cdot j}] = \det [g_{im} T^{mj}] = \det [g_{ij}] \det [T^{kl}]$$

$$= \det [T^{i}_{\cdot j}] = \det [g^{im} T_{mj}] = \det [g^{ij}] \det [T_{kl}] .$$

Proof. By definition (Theorem A3.8.1),

$$\det \underset{\sim}{T} = \frac{\Delta(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_n)}{\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n)} , \quad \underset{\sim}{v}_i = \underset{\sim}{T}\underset{\sim}{e}_i , \quad i \in \mathbb{I}_n .$$

By Theorem A2.7.1,

$$\Delta(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_n) = \varepsilon_{\sigma_1 \sigma_2 \cdots \sigma_n} (\underset{\sim}{v}_1)^{\sigma_1} (\underset{\sim}{v}_2)^{\sigma_2} \cdots (\underset{\sim}{v}_n)^{\sigma_n} \Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n) ,$$

where

$$(\underset{\sim}{v}_i)^j = \underset{\sim}{e}^j \cdot \underset{\sim}{v}_i = \underset{\sim}{e}^j \cdot \underset{\sim}{T}\underset{\sim}{e}_i = T^{j}_{\cdot i}$$

by Theorems A2.5.8 and A3.2.2. Thus,

$$\Delta(\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_n) = \underbrace{\varepsilon_{\sigma_1 \sigma_2 \cdots \sigma_n} T^{\sigma_1}_{\cdot 1} T^{\sigma_2}_{\cdot 2} \cdots T^{\sigma_n}_{\cdot n}}_{\det [T^{i}_{\cdot j}]} \Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n) .$$

Consequently,
$$\det \underset{\sim}{T} = \det [T^{i}_{\cdot j}] .$$

By Theorem A3.2.8,
$$T^{i}_{\cdot j} = g^{im} T_{mj} ;$$
whence,
$$\det [T^{i}_{\cdot j}] = \det [g^{im} T_{mj}] = \det [g^{ij}] \det [T_{kl}] .$$

Here we have used the well-known fact that the determinant of a <u>matrix</u> product is the product of the determinants of the factors.

The remaining results are proven in the same fashion. □

As with the trace operation, we note that <u>in general the determinant of a second-order tensor is not the determinant of a corresponding matrix of components</u>. However, the values do agree if we consider a matrix of mixed components. Also we get agreement for all of the component matrices if the basis for $\mathcal{V}_n$ is chosen $\ni \det [g_{ij}] = 1$. In particular, this is the case if the basis is orthonormal.

As with matrices, the determinant of the transpose of a tensor equals the determinant of the original tensor.

<u>Theorem</u> A3.8.6. <u>Let $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n$. Then</u>

$$\det \underset{\sim}{T}^{T} = \det \underset{\sim}{T} .$$

Proof. By Theorem A3.8.5

$$\det \underset{\sim}{T}^{T} = \det [(\underset{\sim}{T}^{T})_{i}^{\cdot j}]$$

$$= \det [T^{j}_{\cdot i}] \quad (\text{Theorem A3.6.4})$$

$$= \det \underset{\sim}{T} \quad (\text{Theorem A3.8.5}). \ \square$$

The following definition and subsequent theorem will play an important role in the next section on the inverse of a tensor.

Definition A3.8.1. A second-order tensor $\underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}_n$ is singular if $\det \underset{\sim}{T} = 0$.

Theorem A3.8.7. Let $\underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}_n$. Then the homogeneous equation

$$\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{0}$$

has a nontrivial solution (i.e., a solution $\underset{\sim}{u} \neq \underset{\sim}{0}$) iff $\underset{\sim}{T}$ is singular.

Proof. Suppose $\underset{\sim}{T}$ is singular. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ be a basis for $\mathcal{V}_n$. Then by Theorems A3.8.3 and A2.7.4, the set $\{\underset{\sim}{T}\underset{\sim}{e}_1, \underset{\sim}{T}\underset{\sim}{e}_2, \cdots, \underset{\sim}{T}\underset{\sim}{e}_n\}$ is linearly dependent. Thus, $\exists\, \alpha^i \in \mathbb{R}$ not all zero $\ni$

$$\alpha^i (\underset{\sim}{T}\underset{\sim}{e}_i) = \underset{\sim}{0},$$

which by the linearity of $\underset{\sim}{T}$ $\Rightarrow$

$$\underset{\sim}{T}(\alpha^i \underset{\sim}{e}_i) = \underset{\sim}{0} .$$

Thus, $\alpha^i \underset{\sim}{e}_i$ is a nontrivial solution of $\underset{\sim}{T}\underset{\sim}{v} = \underset{\sim}{0}$.

Conversely, suppose $\exists\ \underset{\sim}{u} \neq \underset{\sim}{0} \ni \underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{0}$. The singleton $\{\underset{\sim}{u}\}$ is linearly independent by Exercise A2.2.1. Then by Theorem A2.2.9, $\exists$ a basis $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ for $\mathcal{V}_n$ $\ni \underset{\sim}{e}_1 = \underset{\sim}{u}$. Then by definition (Theorem A3.8.1),

$$\det \underset{\sim}{T} = \frac{\Delta(\underset{\sim}{0}, \underset{\sim}{T}\underset{\sim}{e}_2, \underset{\sim}{T}\underset{\sim}{e}_3, \cdots, \underset{\sim}{T}\underset{\sim}{e}_n)}{\Delta(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n)} = 0$$

by the linearity of Δ. Thus, $\underset{\sim}{T}$ is singular. $\square$

Combining the definition of singular with Theorems A.3.8.3, A2.7.4, and A2.2.8, we get the following useful result.

Theorem A3.8.8. Let $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n$ be nonsingular. Then if $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ is a basis for $\mathcal{V}_n$ so is its image $\{\underset{\sim}{T}\underset{\sim}{e}_1, \underset{\sim}{T}\underset{\sim}{e}_2, \cdots, \underset{\sim}{T}\underset{\sim}{e}_n\}$ under $\underset{\sim}{T}$.

I.e., nonsingular tensors preserve bases.

Supplementary Reading

Same as for § A3.5

A3.9. The Inverse of a Second-Order Tensor

Since a second-order tensor $\underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}$ is simply a linear function on the linear space $\mathcal{V}$ to the same space $\mathcal{V}$, it would be natural to say that $\underset{\sim}{T}$ is *invertible* if the function $\underset{\sim}{T}$ has an inverse function $\underset{\sim}{T}^{-1}$ in the sense of Theorem A1.2.2; i.e., if $\underset{\sim}{T} : \mathcal{V} \to \mathcal{V}$ is one-to-one. However, it is desirable to have $\underset{\sim}{T}^{-1} \in \mathrm{Lin}\, \mathcal{V}$ also; and, as will become clear later, in order to obtain this feature, we require that invertible tensors be onto as well as being one-to-one. Formally, we have

Definition A3.9.1. *A second-order tensor $\underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}$ is invertible*[1] *if*

(i) *the function $\underset{\sim}{T} : \mathcal{V} \to \mathcal{V}$ is one-to-one*

and

(ii) *the function $\underset{\sim}{T} : \mathcal{V} \to \mathcal{V}$ is onto.*

From the definitions of one-to-one and onto, we get

Theorem A3.9.1. *A second-order tensor $\underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}$ is invertible*

---

[1] When the elements of $\mathrm{Lin}\, \mathcal{V}$ are called endomorphisms, then the invertible elements are called *automorphisms*. More generally, the one-to-one and onto elements of $\mathrm{Lin}\, (\mathcal{U}, \mathcal{V})$ are often called *isomorphisms* (cf. Definition A2.2.5).

iff the equation

$$\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v}$$

has a unique solution $\underset{\sim}{u} \in \mathcal{V}$ for each $\underset{\sim}{v} \in \mathcal{V}$.

Proof. First suppose that $\underset{\sim}{T}$ is invertible. Then $\underset{\sim}{T}$ must be onto, which by Definition A1.2.2 $\Rightarrow$

$$\underset{\sim}{T}(\mathcal{V}) = \{ \underset{\sim}{T}\underset{\sim}{u} : \underset{\sim}{u} \in \mathcal{V} \} = \mathcal{V} .$$

In particular, $\mathcal{V} \subset \underset{\sim}{T}(\mathcal{V})$, which $\Rightarrow$

$$\text{given any } \underset{\sim}{v} \in \mathcal{V},\ \exists\ \underset{\sim}{u} \in \mathcal{V} \ni \underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v} .$$

In other words, the equation $\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v}$ has a solution $\underset{\sim}{u}$ for each $\underset{\sim}{v} \in \mathcal{V}$. Suppose $\underset{\sim}{u}'$ is another solution; i.e., $\underset{\sim}{T}\underset{\sim}{u}' = \underset{\sim}{v}$, or

$$\underset{\sim}{T}\underset{\sim}{u}' = \underset{\sim}{T}\underset{\sim}{u} .$$

But being invertible, $\underset{\sim}{T}$ must be one-to-one; and then in accordance with Definition A1.2.3, $\underset{\sim}{T}\underset{\sim}{u}' = \underset{\sim}{T}\underset{\sim}{u} \Rightarrow \underset{\sim}{u}' = \underset{\sim}{u}$. Hence, the solutions of $\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v}$ are unique.

Conversely, suppose that the equation $\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v}$ has a unique solution $\underset{\sim}{u}$ for each $\underset{\sim}{v} \in \mathcal{V}$. Thus, given $\underset{\sim}{v} \in \mathcal{V}$ $\exists$ $\underset{\sim}{u} \in \mathcal{V}$ $\ni$ $\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v}$; i.e., $\mathcal{V} \subset \underset{\sim}{T}(\mathcal{V})$. By definition, $\underset{\sim}{T}(\mathcal{V}) \subset \mathcal{V}$; and $\therefore$ $\underset{\sim}{T}(\mathcal{V}) = \mathcal{V}$, which is to say that $\underset{\sim}{T}$ is onto.

Next, let $\underset{\sim}{u}, \underset{\sim}{u}' \in \mathcal{V}$ and consider $\underset{\sim}{T}\underset{\sim}{u}' = \underset{\sim}{T}\underset{\sim}{u} =: \underset{\sim}{v}$. This says that $\underset{\sim}{u}$ and $\underset{\sim}{u}'$ are both solutions of the equation $\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v}$. Then the assumed uniqueness of solutions $\Rightarrow \underset{\sim}{u}' = \underset{\sim}{u}$; i.e., $\underset{\sim}{T}$ is one-to-one. $\square$

Actually, there is nothing in Definition A3.9.1 or Theorem A3.9.1 that uses the linearity of $\underset{\sim}{T}$. In fact, the definition and theorem stand for a general function whose domain and codomain are arbitrary sets. From this point forward, we utilize the fact that $\mathcal{V}$ is a linear space and $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}$.

Exercise A3.9.1. The *null space*[1] of $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}$ is defined to be the set

$$\mathcal{N}(\underset{\sim}{T}) = \{ \underset{\sim}{u} \in \mathcal{V} : \underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{0} \}.$$

Prove:

(i) $\mathcal{N}(\underset{\sim}{T})$ is a linear subspace of $\mathcal{V}$;

(ii) $\mathcal{N}(\underset{\sim}{T}) = \{\underset{\sim}{0}\} \Leftrightarrow \underset{\sim}{T}$ is one-to-one.

Next we have the important result that the inverse of an <u>invertible</u> tensor is also a tensor on the same space.

Theorem A3.9.2. <u>Let</u> $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}$ <u>be invertible</u>. <u>Then</u> $\underset{\sim}{T}^{-1} \in \mathrm{Lin}\,\mathcal{V}$.

Proof. It is clear from Theorem A1.2.2 that $\underset{\sim}{T}^{-1}$ is a function on $\mathcal{V}$ to $\mathcal{V}$. Accordingly, all we need to show is that $\underset{\sim}{T}^{-1}$ is linear.[2]

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[1] Often, *kernel*.

[2] Here, of course, $\underset{\sim}{T}^{-1}$ means the inverse of the function $\underset{\sim}{T}: \mathcal{V} \to \mathcal{V}$ in the sense of Theorem A1.2.2.

To this end, let $\underset{\sim}{v}, \underset{\sim}{v}' \in \mathcal{V}$ and write

$$\underset{\sim}{T}^{-1}\underset{\sim}{v} = \underset{\sim}{u} \quad \text{and} \quad \underset{\sim}{T}^{-1}\underset{\sim}{v}' = \underset{\sim}{u}'.$$

These equations are respectively equivalent to

$$\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v} \quad \text{and} \quad \underset{\sim}{T}\underset{\sim}{u}' = \underset{\sim}{v}',$$

Let $\alpha, \alpha' \in \mathbb{R}$. By the linearity of $\underset{\sim}{T}$,

$$\underset{\sim}{T}(\alpha\underset{\sim}{u} + \alpha'\underset{\sim}{u}') = \alpha\underset{\sim}{T}\underset{\sim}{u} + \alpha'\underset{\sim}{T}\underset{\sim}{u}' = \alpha\underset{\sim}{v} + \alpha'\underset{\sim}{v}'.$$

$\therefore$

$$\underset{\sim}{T}^{-1}(\alpha\underset{\sim}{v} + \alpha'\underset{\sim}{v}') = \alpha\underset{\sim}{u} + \alpha'\underset{\sim}{u}' = \alpha\underset{\sim}{T}^{-1}\underset{\sim}{v} + \alpha'\underset{\sim}{T}^{-1}\underset{\sim}{v}',$$

and by Theorem A2.4.2, $\underset{\sim}{T}^{-1}$ is linear. $\square$

If we had left <u>onto</u> out of the definition of invertible, then in the above proof we would have had $\underset{\sim}{T}^{-1}$ being a function on the range $\underset{\sim}{T}(\mathcal{V})$ to $\mathcal{V}$, but $\underset{\sim}{T}^{-1}$ would come out to be linear in the same way. Of course, we would start with $\underset{\sim}{v}, \underset{\sim}{v}' \in \underset{\sim}{T}(\mathcal{V})$. Thus, if $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}$ is simply one-to-one, then $\underset{\sim}{T}^{-1} \in \mathrm{Lin}(\underset{\sim}{T}(\mathcal{V}), \mathcal{V})$. We leave it as *Exercise* A3.9.2 to show that $\underset{\sim}{T}(\mathcal{V})$ is a linear subspace of $\mathcal{V}$, so that we have a right to write $\mathrm{Lin}(\underset{\sim}{T}(\mathcal{V}), \mathcal{V})$.[1]

The next theorem shows that for an invertible tensor $\underset{\sim}{T}$, $\underset{\sim}{T}^{-1}$ is an inverse in the multiplicative sense as well as in the functional sense.

[1] Here, $\underset{\sim}{T}$ is not necessarily invertible.

Theorem A3.9.3. If $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}$ is invertible, then

$$\underset{\sim}{T}^{-1}\underset{\sim}{T} = \underset{\sim}{T}\underset{\sim}{T}^{-1} = \underset{\sim}{I}\,.\ )^{1}$$

Proof. For $\underset{\sim}{u} \in \mathcal{V}$, write $\underset{\sim}{v} = \underset{\sim}{T}\underset{\sim}{u}$. Then $\underset{\sim}{u} = \underset{\sim}{T}^{-1}\underset{\sim}{v}$, and

$$(\underset{\sim}{T}^{-1}\underset{\sim}{T})\underset{\sim}{u} = \underset{\sim}{T}^{-1}(\underset{\sim}{T}\underset{\sim}{u}) = \underset{\sim}{T}^{-1}\underset{\sim}{v} = \underset{\sim}{u}\,.$$

Since this holds $\forall\, \underset{\sim}{u} \in \mathcal{V}$, $\underset{\sim}{T}^{-1}\underset{\sim}{T} = \underset{\sim}{I}$. Similarly, for any $\underset{\sim}{v} \in \mathcal{V}$)[2]

$$(\underset{\sim}{T}\underset{\sim}{T}^{-1})\underset{\sim}{v} = \underset{\sim}{T}(\underset{\sim}{T}^{-1}\underset{\sim}{v}) = \underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v} \Rightarrow \underset{\sim}{T}\underset{\sim}{T}^{-1} = \underset{\sim}{I}\,. \quad \square$$

In our approach to invertible tensors in terms of the inverse of the linear function, no question about the uniqueness of the inverse function arises. However, had we adopted the multiplicative approach and used the result of Theorem A3.9.3 as the definition of an inverse, then uniqueness would have been an issue. It is resolved by the next theorem.

Theorem A3.9.4. Let $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}$. Suppose $\exists\ \underset{\sim}{A}, \underset{\sim}{B} \in \mathrm{Lin}\,\mathcal{V} \ni$

$$\underset{\sim}{A}\underset{\sim}{T} = \underset{\sim}{T}\underset{\sim}{A} = \underset{\sim}{I} \quad \text{and} \quad \underset{\sim}{B}\underset{\sim}{T} = \underset{\sim}{T}\underset{\sim}{B} = \underset{\sim}{I}\,.$$

Then

$$\underset{\sim}{A} = \underset{\sim}{B}\,.$$

Proof. Multiply $\underset{\sim}{T}\underset{\sim}{B} = \underset{\sim}{I}$ on the left by $\underset{\sim}{A}$ to get

---

[1] Here, of course, $\underset{\sim}{I}$ is the identity tensor on $\mathcal{V}$.

[2] Here, again, we need that $\underset{\sim}{T}$ is onto $\mathcal{V}$.

$$A(TB) = AI$$

or

$$\begin{aligned} A &= A(TB) && \text{(Theorem A3.5.1 (iv))} \\ &= (AT)B && \text{(associativity of tensor multiplication)} \\ &= IB && \text{(substitution)} \\ &= B && \text{(Theorem A3.5.1 (iv))}. \quad \square \end{aligned}$$

The following theorem addresses the significance of requiring both $T^{-1}T = I$ and $TT^{-1} = I$ in the multiplicative approach to inverses.

Theorem A3.9.5. Let $T \in \operatorname{Lin} V$. Suppose $\exists\, S \in \operatorname{Lin} V \ni ST = I$ but $TS \neq I$;[1] then T is not invertible. Similarly, suppose $\exists\, U \in \operatorname{Lin} V \ni TU = I$ but $UT \neq I$; then T is not invertible.

Proof. The proof is by contradiction. Suppose then that T is invertible. Multiply $ST = I$ on the right by T^{-1} to get

$$(ST)T^{-1} = IT^{-1}$$

or

$$\begin{aligned} T^{-1} &= (ST)T^{-1} && \text{(Theorem A3.5.1 (iv))} \\ &= S(TT^{-1}) && \text{(associativity of tensor multiplication)} \\ &= SI && \text{(Theorem A3.9.3)} \end{aligned}$$

---

[1] Thinking multiplicatively, we would say that S is a left inverse of T, but not a right inverse. We shall see in Theorem A3.9.11 that such behavior is ruled out when V is finite-dimensional.

$$= \underset{\sim}{S} \qquad \text{(Theorem A3.5.1(iv))}.$$

Multiplication on the left by $\underset{\sim}{T}$ yields

$$\begin{aligned}\underset{\sim}{T}\underset{\sim}{S} &= \underset{\sim}{T}\underset{\sim}{T}^{-1} \\ &= \underset{\sim}{I} \qquad \text{(Theorem A3.9.3)},\end{aligned}$$

which contradicts $\underset{\sim}{T}\underset{\sim}{S} \neq \underset{\sim}{I}$. $\therefore$ $\underset{\sim}{T}$ is not invertible. $\square$

Our next result is more or less the converse of Theorem A3.9.3. It shows how the multiplicative view can be useful in finding inverses.

Theorem A3.9.6. Let $\underset{\sim}{T} \in \mathrm{Lin}\, V$. If $\exists\ \underset{\sim}{S}, \underset{\sim}{U} \in \mathrm{Lin}\, V \ni$ both

$$\underset{\sim}{S}\underset{\sim}{T} = \underset{\sim}{I} \quad \text{and} \quad \underset{\sim}{T}\underset{\sim}{U} = \underset{\sim}{I},$$

then $\underset{\sim}{T}$ is invertible and $\underset{\sim}{S} = \underset{\sim}{U} = \underset{\sim}{T}^{-1}$.)[1]

Proof. We will use Theorem A3.9.1. Accordingly, we must show that for each $\underset{\sim}{v} \in V$ $\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v}$ has a unique solution $\underset{\sim}{u}$.

Uniqueness: Suppose $\exists\ \underset{\sim}{u}_1, \underset{\sim}{u}_2 \in V \ni$

---

[1] When V is finite-dimensional, we shall see in Theorem A3.9.11 that $\underset{\sim}{T}$ is invertible if either $\underset{\sim}{S}\underset{\sim}{T} = \underset{\sim}{I}$ or $\underset{\sim}{T}\underset{\sim}{U} = \underset{\sim}{I}$.

$$\mathbf{T}\mathbf{u}_1 = \mathbf{T}\mathbf{u}_2 = \mathbf{v} \, ,$$

Then

$$\mathbf{S}(\mathbf{T}\mathbf{u}_1) = \mathbf{S}(\mathbf{T}\mathbf{u}_2) \qquad \text{(substitution)}$$

$$\Rightarrow$$

$$(\mathbf{S}\mathbf{T})\mathbf{u}_1 = (\mathbf{S}\mathbf{T})\mathbf{u}_2 \qquad \text{(definition of tensor multiplication)}$$

$$\Rightarrow$$

$$\mathbf{I}\mathbf{u}_1 = \mathbf{I}\mathbf{u}_2 \qquad \text{(substitution)}$$

$$\Rightarrow$$

$$\mathbf{u}_1 = \mathbf{u}_2 \qquad \text{(Definition of identity tensor).}$$

Existence: Let $\mathbf{v} \in \mathcal{V}$, and write $\mathbf{U}\mathbf{v} = \mathbf{u}$. Then

$$\begin{aligned}
\mathbf{T}\mathbf{u} &= \mathbf{T}(\mathbf{U}\mathbf{v}) && \text{(substitution)} \\
&= (\mathbf{T}\mathbf{U})\mathbf{v} && \text{(definition of tensor multiplication)} \\
&= \mathbf{I}\mathbf{v} && \text{(substitution)} \\
&= \mathbf{v} && \text{(definition of identity tensor),}
\end{aligned}$$

$\therefore$ the equation $\mathbf{T}\mathbf{u} = \mathbf{v}$ has the solution $\mathbf{u} = \mathbf{U}\mathbf{v}$.

Now, knowing that $\mathbf{T}$ is invertible, we can multiply $\mathbf{S}\mathbf{T} = \mathbf{I}$ and $\mathbf{T}\mathbf{U} = \mathbf{I}$ on the appropriate sides by $\mathbf{T}^{-1}$ to get

$$\mathbf{S} = \mathbf{T}^{-1} \quad \text{and} \quad \mathbf{U} = \mathbf{T}^{-1}. \quad \square$$

Thinking multiplicatively, Theorem A3.9.6 asserts that if $\mathbf{T} \in \operatorname{Lin} \mathcal{V}$ has both a left inverse and a right inverse, then it is invertible, and its inverse equals its left and right inverses, which are necessarily equal.

Easy but important corollaries of Theorem A3.9.6 are gathered in

Theorem A3.9.7. Let $\underset{\sim}{S}, \underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}$ be invertible. Then

(i) $\underset{\sim}{T}^{-1}$ is invertible and

$$(\underset{\sim}{T}^{-1})^{-1} = \underset{\sim}{T};$$

(ii) if $\alpha \in \mathbb{R}$ and $\alpha \neq 0$, then $\alpha \underset{\sim}{T}$ is invertible and

$$(\alpha \underset{\sim}{T})^{-1} = \alpha^{-1} \underset{\sim}{T}^{-1};$$

(iii) $\underset{\sim}{S}\underset{\sim}{T}$ is invertible and

$$(\underset{\sim}{S}\underset{\sim}{T})^{-1} = \underset{\sim}{T}^{-1}\underset{\sim}{S}^{-1};$$

(iv) the identity tensor $\underset{\sim}{I}$ is invertible and

$$\underset{\sim}{I}^{-1} = \underset{\sim}{I}.$$

Proof. The proofs of (i), (iii), and (iv) are left as Exercises A3.9.3 – 5, respectively. The following proof of (ii) serves as a model for these demonstrations.

Consider

$$(\alpha^{-1}\underset{\sim}{T}^{-1})(\alpha \underset{\sim}{T}) = (\alpha^{-1}\alpha)(\underset{\sim}{T}^{-1}\underset{\sim}{T}) \quad \text{(homogeneity of tensor multiplication)}$$

$$= \underset{\sim}{T}^{-1}\underset{\sim}{T} \quad \text{(reciprocals in } \mathbb{R}\text{)}$$

$$= \underset{\sim}{I} \quad \text{(Theorem A3.9.3)}.$$

Similarly,

$$(\alpha \underset{\sim}{T})(\alpha^{-1}\underset{\sim}{T}^{-1}) = \underset{\sim}{I} \, ;$$

and we conclude from Theorem A3.9.6 that $\alpha \underset{\sim}{T}$ is invertible and

$$(\alpha \underset{\sim}{T})^{-1} = \alpha^{-1}\underset{\sim}{T}^{-1}. \quad \square$$

According to Theorem A3.9.7 (iii), the product of invertible tensors is itself an invertible tensor; i.e., multiplication of invertible tensors is <u>closed</u>. In order to formally state this group structure, it is convenient to introduce the following notation.

The various parts of Theorem A3.9.7 have convenient and familiar sounding (because of the corresponding results for matrices) word statements. For example, (iii) asserts the product of invertible tensors is invertible and its inverse is the product of the inverses of the factors in reverse order.

Definition A3.9.2. Let V be a linear space. Then

$$\text{Inv}\,V = \{ \underset{\sim}{T} \in \text{Lin}\,V : \underset{\sim}{T} \text{ is invertible} \}.$$

Thus, Inv V is simply the set of all invertible second-order tensors on V.

Theorem A3.9.8. Inv V is a group w.r.t. tensor multiplication.

Proof. Exercise A3.9.6. □

Exercise A3.9.7. Is Inv V a linear subspace of Lin V?

When the underlying linear space is finite-dimensional, the notion of the determinant of a tensor is available; and we are able to obtain somewhat sharper results concerning inverses. For the remainder of this section, V_n denotes an n-dimensional inner product space.

Theorem A3.9.9. A second-order tensor $\underset{\sim}{T} \in \text{Lin}\,V_n$ is invertible iff it is nonsingular; i.e., iff $\det \underset{\sim}{T} \neq 0$.

Proof. Suppose $\underset{\sim}{T}$ is invertible. Then by Theorem A3.9.3, $\underset{\sim}{T}^{-1}\underset{\sim}{T} = \underset{\sim}{I}$. Taking the determinant of both sides and using Theorems A3.8.4 and A3.8.2 (ii), we get

$$(\det \underset{\sim}{T}^{-1})(\det \underset{\sim}{T}) = 1.$$

$\therefore \det \underset{\sim}{T} \neq 0$.

Conversely, suppose that T is nonsingular. We shall show that $\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v}$ has a unique solution $\underset{\sim}{u}$ for each $\underset{\sim}{v} \in \mathcal{V}_n$ and thereby conclude from Theorem A3.9.1 that $\underset{\sim}{T}$ is invertible.

Uniqueness: Suppose $\exists\, \underset{\sim}{u}_1, \underset{\sim}{u}_2 \in \mathcal{V}_n \ni$

$$\underset{\sim}{T}\underset{\sim}{u}_1 = \underset{\sim}{T}\underset{\sim}{u}_2 = \underset{\sim}{v} .$$

Then by the linearity of $\underset{\sim}{T}$,

$$\underset{\sim}{T}(\underset{\sim}{u}_1 - \underset{\sim}{u}_2) = \underset{\sim}{0} ;$$

and we conclude from Theorem A3.8.7 that $\underset{\sim}{u}_1 - \underset{\sim}{u}_2 = \underset{\sim}{0}$.

Existence: Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ be a basis for $\mathcal{V}_n$. Then by Theorem A3.8.8, $\{\underset{\sim}{T}\underset{\sim}{e}_1, \underset{\sim}{T}\underset{\sim}{e}_2, \cdots, \underset{\sim}{T}\underset{\sim}{e}_n\}$ is also a basis for $\mathcal{V}_n$. Let $\bar{v}^1, \bar{v}^2, \cdots, \bar{v}^n$ denote the components of $\underset{\sim}{v}$ relative to $\{\underset{\sim}{T}\underset{\sim}{e}_1, \underset{\sim}{T}\underset{\sim}{e}_2, \cdots, \underset{\sim}{T}\underset{\sim}{e}_n\}$; i.e.,

$$\begin{aligned} \underset{\sim}{v} &= \bar{v}^i (\underset{\sim}{T}\underset{\sim}{e}_i) \\ &= \underset{\sim}{T}(\bar{v}^i \underset{\sim}{e}_i) \qquad (\text{linearity of } \underset{\sim}{T}). \end{aligned}$$

This says that $\underset{\sim}{u} = \bar{v}^i \underset{\sim}{e}_i$ is a solution of $\underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{v}$. $\square$

The first part of the above proof yields

Theorem A3.9.10. Let $\underset{\sim}{T} \in \operatorname{Inv} \mathcal{V}_n$. Then

$$\det(\underset{\sim}{T}^{-1}) = (\det \underset{\sim}{T})^{-1}.$$

The finite-dimensional analog of Theorem A3.9.6 is

Theorem A3.9.11. Let $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$. If $\exists\ \underset{\sim}{S} \in \operatorname{Lin} \mathcal{V}_n \ni$

$$\underset{\sim}{S}\underset{\sim}{T} = \underset{\sim}{I},$$

then $\underset{\sim}{T}$ is invertible and $\underset{\sim}{S} = \underset{\sim}{T}^{-1}$. Similarly, if $\exists\ \underset{\sim}{U} \in \operatorname{Lin} \mathcal{V}_n \ni$

$$\underset{\sim}{T}\underset{\sim}{U} = \underset{\sim}{I},$$

then $\underset{\sim}{T}$ is invertible and $\underset{\sim}{U} = \underset{\sim}{T}^{-1}$.

Proof. Suppose that $\underset{\sim}{S}\underset{\sim}{T} = \underset{\sim}{I}$. Then taking the determinant of both sides and using Theorems A3.8.4 and A3.8.2 (ii), we see that $\underset{\sim}{T}$ is nonsingular and hence invertible by Theorem A3.9.9. Multiplication of $\underset{\sim}{S}\underset{\sim}{T} = \underset{\sim}{I}$ on the right by $\underset{\sim}{T}^{-1}$ and use of Theorem A3.9.3 yield $\underset{\sim}{S} = \underset{\sim}{T}^{-1}$. $\square$

Theorem A3.9.11 can be used to establish

Theorem A3.9.12. Let $\underset{\sim}{T} \in \operatorname{Inv} \mathcal{V}_n$. Then $\underset{\sim}{T}^{T} \in \operatorname{Inv} \mathcal{V}_n$ and

$$(\underset{\sim}{T}^{T})^{-1} = (T^{-1})^{T}.$$

Proof. Exercise A3.9.8. $\square$

An obvious corollary of Theorem A3.9.12 is

Theorem A3.9.13. Let $\underset{\sim}{T} \in Inv\, \mathcal{V}_n$. Then if $\underset{\sim}{T}$ is symmetric, $\underset{\sim}{T}^{-1}$ is symmetric.

As might be expected, the matrices of components of the inverse are related to the inverses of the matrices of components.

Theorem A3.9.14. Let $\underset{\sim}{T} \in Inv\, \mathcal{V}_n$. Then

$$[(\underset{\sim}{T}^{-1})^{i}_{\cdot j}] = [T^{i}_{\cdot j}]^{-1},$$

$$[(\underset{\sim}{T}^{-1})_{i}^{\cdot j}] = [T_{i}^{\cdot j}]^{-1},$$

$$[(\underset{\sim}{T}^{-1})^{ij}] = [T_{ij}]^{-1},$$

$$[(\underset{\sim}{T})^{-1}_{ij}] = [T^{ij}]^{-1}.$$

Proof. By Theorem A3.9.3, $\underset{\sim}{T}^{-1}\underset{\sim}{T} = \underset{\sim}{I}$. By Theorems A3.5.4 and A3.2.8 this is equivalent to

$$(\underset{\sim}{T}^{-1})^{i}_{\cdot j}\, T^{j}_{\cdot k} = \delta^{i}_{k}, \quad i,k \in \mathrm{I\!I}_n .$$

Thus, the matrix $[(\underset{\sim}{T}^{-1})^{i}_{\cdot j}]$ is the inverse of the matrix $[T^{i}_{\cdot j}]$. The remaining relations are proved in the same manner. $\square$

It is important to realize that <u>in general a matrix of components of the inverse equals the inverse of the corresponding matrix of components of the original tensor only for mixed components</u>. Of course, when the underlying basis for $\mathcal{V}_n$ is orthonormal, there is no chance of confusion.

Supplementary Reading

Same as for §A3.5

A3.10. Orthogonal Second-Order Tensors

Orthogonal tensors are ubiquitous in continuum mechanics. The notion can be introduced in several ways; some, but not all, of the approaches are given in

Theorem A3.10.1. Let $\underset{\sim}{Q} \in Lin\, \mathcal{V}$, where $\mathcal{V}$ is an inner product space. Then the following three statements are equivalent.

(i) $\underset{\sim}{Q}$ preserves inner products; i.e.,

$$(\underset{\sim}{Q}\underset{\sim}{u}) \cdot (\underset{\sim}{Q}\underset{\sim}{v}) = \underset{\sim}{u} \cdot \underset{\sim}{v} \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}.$$

(ii) $\underset{\sim}{Q}$ preserves magnitudes; i.e.,

$$|\underset{\sim}{Q}\underset{\sim}{u}| = |\underset{\sim}{u}| \quad \forall\, \underset{\sim}{u} \in \mathcal{V}.$$

(iii) $\underset{\sim}{Q}$ preserves metrics[1];

$$|\underset{\sim}{Q}(\underset{\sim}{u} - \underset{\sim}{v})| = |\underset{\sim}{u} - \underset{\sim}{v}| \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}.$$

A second-order tensor $\underset{\sim}{Q} \in Lin\, \mathcal{V}$ which meets any (and hence all) of these is said to be orthogonal[2].

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[1] Cf. Theorem A5.1.1.

[2] Often, isometric or an isometry.

Proof. Obviously, all of the equivalencies will be proven if we can establish that

$$\begin{tikzcd} & (i) \arrow[dr, Rightarrow] & \\ (iii) \arrow[ur, Rightarrow] & & (ii) \arrow[ll, Rightarrow] \end{tikzcd}$$

(i) $\Rightarrow$ (ii): In (i) take $\mathbf{v} = \mathbf{u}$ to get

$$(\mathbf{Q}\mathbf{u}) \cdot (\mathbf{Q}\mathbf{u}) = \mathbf{u} \cdot \mathbf{u} ;$$

i.e.,

$$|\mathbf{Q}\mathbf{u}|^2 = |\mathbf{u}|^2 \Rightarrow |\mathbf{Q}\mathbf{u}| = |\mathbf{u}| .$$

(ii) $\Rightarrow$ (iii): In (ii) replace $\mathbf{u}$ by $\mathbf{u} - \mathbf{v}$ to get

$$|\mathbf{Q}(\mathbf{u} - \mathbf{v})| = |\mathbf{u} - \mathbf{v}| .$$

(iii) $\Rightarrow$ (i): Take $\mathbf{v} = \mathbf{0}$ in (iii) to get

$$|\mathbf{Q}\mathbf{u}| = |\mathbf{u}| \Rightarrow \mathbf{Q}\mathbf{u} \cdot \mathbf{Q}\mathbf{u} = \mathbf{u} \cdot \mathbf{u} .$$

Next, expand (iii) to get

$$|\mathbf{Q}\mathbf{u} - \mathbf{Q}\mathbf{v}| = |\mathbf{u} - \mathbf{v}|$$

$$\Rightarrow$$

$$(\mathbf{Q}\mathbf{u} - \mathbf{Q}\mathbf{v}) \cdot (\mathbf{Q}\mathbf{u} - \mathbf{Q}\mathbf{v}) = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$$

$$\Rightarrow$$

$$\mathbf{Q}\mathbf{u} \cdot \mathbf{Q}\mathbf{u} - 2\mathbf{Q}\mathbf{u} \cdot \mathbf{Q}\mathbf{v} + \mathbf{Q}\mathbf{v} \cdot \mathbf{Q}\mathbf{v} = \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} ,$$

which in view of the above result $\Rightarrow \mathbf{Q}\mathbf{u} \cdot \mathbf{Q}\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$. $\square$

The symbol $\underset{\sim}{Q}$ is a favorite name for orthogonal tensors in continuum mechanics.

For the remainder of the section, V_n denotes an n-dimensional inner product space.

A variant of the following theorem will play an important role in characterizing rigid deformations when we get to continuum mechanics.

Theorem A3.10.2. Let the function $\underset{\sim}{f} : V_n \to V_n$ preserve inner products; i.e.,

$$\underset{\sim}{f}(\underset{\sim}{u}) \cdot \underset{\sim}{f}(\underset{\sim}{v}) = \underset{\sim}{u} \cdot \underset{\sim}{v} \quad \forall\ \underset{\sim}{u}, \underset{\sim}{v} \in V_n .$$

Then $\underset{\sim}{f}$ is linear; hence, $\underset{\sim}{f}$ is an orthogonal tensor.

Proof. Let $\{\underset{\sim}{e}_{\langle 1 \rangle}, \underset{\sim}{e}_{\langle 2 \rangle}, \cdots, \underset{\sim}{e}_{\langle n \rangle}\}$ be an orthonormal basis for V_n. Then $\forall\ i, j \in \mathbb{I}_n$

$$\underset{\sim}{f}(\underset{\sim}{e}_{\langle i \rangle}) \cdot \underset{\sim}{f}(\underset{\sim}{e}_{\langle j \rangle}) = \underset{\sim}{e}_{\langle i \rangle} \cdot \underset{\sim}{e}_{\langle j \rangle} = \delta_{ij} .$$

Thus, the set $\{\underset{\sim}{f}(\underset{\sim}{e}_{\langle 1 \rangle}), \underset{\sim}{f}(\underset{\sim}{e}_{\langle 2 \rangle}), \cdots, \underset{\sim}{f}(\underset{\sim}{e}_{\langle n \rangle})\}$ is also orthonormal; consequently, by Theorems A2.3.14 and A2.2.8, it is also a basis for V_n. Then by Theorem A2.3.16, for $\underset{\sim}{u} \in V_n$

$$\underset{\sim}{f}(\underset{\sim}{u}) = \sum_{i=1}^{n} \left[\underset{\sim}{f}(\underset{\sim}{u}) \cdot \underset{\sim}{f}(\underset{\sim}{e}_{\langle i \rangle}) \right] \underset{\sim}{f}(\underset{\sim}{e}_{\langle i \rangle})$$

$$= \sum_{i=1}^{n} (\underset{\sim}{u} \cdot \underset{\sim}{e}_{\langle i \rangle}) \, \underset{\sim}{f}(\underset{\sim}{e}_i) \; ;$$

and $\underset{\sim}{f}$ is linear because of the linearity of the inner product. $\square$

The following theorem occasionally serves as the definition of orthogonal tensors on $\mathcal{V}_n$.

Theorem A3.10.3. A second-order tensor $\underset{\sim}{Q} \in \mathrm{Lin}\,\mathcal{V}_n$ is orthogonal iff it is invertible and

$$\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^{T}.$$

Proof. By Theorem A3.6.4, any $\underset{\sim}{Q} \in \mathrm{Lin}\,\mathcal{V}_n$ has a transpose, which by Theorem A3.6.1 is characterized by

$$\underset{\sim}{u} \cdot \underset{\sim}{Q}\underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{Q}^{T}\underset{\sim}{u} \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_n .$$

In particular, this must hold if $\underset{\sim}{u}$ is replaced by $\underset{\sim}{Q}\underset{\sim}{u}$; hence,

$$\underset{\sim}{Q}\underset{\sim}{u} \cdot \underset{\sim}{Q}\underset{\sim}{v} = (\underset{\sim}{Q}^{T}\underset{\sim}{Q}\underset{\sim}{u}) \cdot \underset{\sim}{v} \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_n . \qquad (*)$$

Suppose that $\underset{\sim}{Q}$ is invertible and $\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^{T}$. Then by Theorem A3.9.3, $\underset{\sim}{Q}^{T}\underset{\sim}{Q} = \underset{\sim}{I}$; and $(*)$ becomes

$$\underset{\sim}{Q}\underset{\sim}{u} \cdot \underset{\sim}{Q}\underset{\sim}{v} = \underset{\sim}{u} \cdot \underset{\sim}{v} \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_n ,$$

Thus, $\underset{\sim}{Q}$ is orthogonal.

Conversely, suppose that $\underset{\sim}{Q}$ is orthogonal. Then (*) $\Rightarrow$

$$(\underset{\sim}{Q}^T \underset{\sim}{Q}\, \underset{\sim}{u}) \cdot \underset{\sim}{v} = \underset{\sim}{u} \cdot \underset{\sim}{v} \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_n.$$

By Theorem A2.3.4, this $\Rightarrow$

$$\underset{\sim}{Q}^T \underset{\sim}{Q}\, \underset{\sim}{u} = \underset{\sim}{u} \quad \forall\, \underset{\sim}{u} \in \mathcal{V}_n ;$$

and then by Theorem A3.1.5,

$$\underset{\sim}{Q}^T \underset{\sim}{Q} = \underset{\sim}{I}.$$

Consequently, by Theorem A3.9.11, $\underset{\sim}{Q}$ is invertible and

$$\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^T. \quad \square$$

Combining Theorems A3.10.3 and A3.9.11, we are lead to

Theorem A3.10.4. *A second-order tensor $\underset{\sim}{Q} \in \mathrm{Lin}\, \mathcal{V}_n$ is orthogonal iff*

$$\underset{\sim}{Q}^T \underset{\sim}{Q} = \underset{\sim}{I}$$

or, alternatively, iff

$$\underset{\sim}{Q}\, \underset{\sim}{Q}^T = \underset{\sim}{I}.$$

Proof. Exercise A3.10.1. $\square$

When the "orthogonality conditions" $\underset{\sim}{Q}^T \underset{\sim}{Q} = \underset{\sim}{I}$ and $\underset{\sim}{Q}\, \underset{\sim}{Q}^T = \underset{\sim}{I}$ are expressed in terms of components, we get

Theorem A3.10.5. Let $\underset{\sim}{Q} \in \mathrm{Lin}\, \mathcal{V}_n$. If $\underset{\sim}{Q}$ is orthogonal, then its components must meet

$$Q^{i}_{\cdot k} Q^{jk} = Q_k^{\cdot i} Q^{kj} = Q^{ik} Q^{j}_{\cdot k} = Q^{ki} Q_k^{\cdot j} = g^{ij},$$

$$Q_i^{\cdot k} Q_{jk} = Q^k_{\cdot i} Q_{kj} = Q_{ik} Q_j^{\cdot k} = Q_{ki} Q^k_{\cdot j} = g_{ij},$$

$$Q_i^{\cdot k} Q^j_{\cdot k} = Q^k_{\cdot i} Q_k^{\cdot j} = Q_{ik} Q^{jk} = Q_{ki} Q^{kj} = \delta_i^j.$$

Conversely, if the components of $\underset{\sim}{Q}$ meet any one of the above 12 equations[1], then $\underset{\sim}{Q}$ is orthogonal.

Proof. Exercise A3.10.2. □

Theorem A3.10.4 has several important corollaries, which are gathered in

Theorem A3.10.6. Let $\underset{\sim}{Q}, \underset{\sim}{R} \in \mathrm{Lin}\, \mathcal{V}_n$ be orthogonal. Then

(i) $\underset{\sim}{Q}$ is invertible and $\underset{\sim}{Q}^{-1}$ is orthogonal;

(ii) $\underset{\sim}{Q}\underset{\sim}{R}$ is orthogonal;

(iii) $\det \underset{\sim}{Q} = \pm 1$;

---

[1] We count as follows: Each of the three lines contains four distinct equations which all have the same g symbol on the r.h.s.

(iv) $\underset{\sim}{I}$ is orthogonal.

Proof. (i): By Theorem A3.10.4, $\underset{\sim}{Q}^T\underset{\sim}{Q} = \underset{\sim}{I}$. Then by Theorem A3.9.11, $\underset{\sim}{Q}$ is invertible and $\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^T$. By Theorem A3.9.12, $\underset{\sim}{Q}^T$ is invertible and $(\underset{\sim}{Q}^T)^{-1} = (\underset{\sim}{Q}^{-1})^T$. By Theorem A3.9.7, $\underset{\sim}{Q}^{-1}$ is invertible. Using these results to take the inverse of each side of $\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^T$, we get

$$(\underset{\sim}{Q}^{-1})^{-1} = (\underset{\sim}{Q}^T)^{-1} = (\underset{\sim}{Q}^{-1})^T.$$

Thus, $\underset{\sim}{Q}^{-1}$ is orthogonal by Theorem A3.10.3.

The proofs of (ii), (iii), and (iv) are left as *Exercises* A.3.10.3–5. $\square$

According to Theorem A3.10.6 (ii), the product of orthogonal tensors is itself an orthogonal tensor; i.e., multiplication of orthogonal tensors is closed. The following notation will be useful in formally stating this group structure.

Definition A3.10.1. Let $\mathcal{V}_n$ be an n-dimensional inner product space. Then

$$\text{Orth}\,\mathcal{V}_n = \{\underset{\sim}{Q} \in \text{Lin}\,\mathcal{V}_n : \underset{\sim}{Q} \text{ is orthogonal}\}^{\,1}$$

and

---

[1] Often, $O(\mathcal{V}_n)$, $\mathcal{O}(\mathcal{V}_n)$.

$$\mathrm{Orth}^+\mathcal{V}_n = \{ \underset{\sim}{Q} \in \mathrm{Orth}\,\mathcal{V}_n : \det \underset{\sim}{Q} = 1 \} \,. \qquad)[1]$$

Theorem A3.10.7. $\mathrm{Orth}\,\mathcal{V}_n$ and $\mathrm{Orth}^+\mathcal{V}_n$ are groups w.r.t. tensor multiplication. $\mathrm{Orth}^+\mathcal{V}_n$ is a proper subgroup of $\mathrm{Orth}\,\mathcal{V}_n$. $\mathrm{Orth}\,\mathcal{V}_n$ and $\mathrm{Orth}^+\mathcal{V}_n$ are called the orthogonal group and the proper orthogonal group[2], respectively.

Proof. Exercise A3.10.6. □

Exercise A3.10.7. Is $\mathrm{Orth}\,\mathcal{V}_n$ a linear subspace of $\mathrm{Lin}\,\mathcal{V}_n$?

The next theorem shows that orthonormal bases for $\mathcal{V}_n$ are related by orthogonal tensors.

Theorem A3.10.8. Let $\{ \underset{\sim}{e}_{\langle 1\rangle}, \underset{\sim}{e}_{\langle 2\rangle}, \cdots, \underset{\sim}{e}_{\langle n\rangle} \}$ be an orthonormal basis for $\mathcal{V}_n$. Then $\{ \underset{\sim}{\bar{e}}_{\langle 1\rangle}, \underset{\sim}{\bar{e}}_{\langle 2\rangle}, \cdots, \underset{\sim}{\bar{e}}_{\langle n\rangle} \}$ defined by

$$\underset{\sim}{\bar{e}}_{\langle i\rangle} = \underset{\sim}{Q}\, \underset{\sim}{e}_{\langle i\rangle} \,, \quad \underset{\sim}{Q} \in \mathrm{Orth}\,\mathcal{V}_n$$

is also an orthonormal basis for $\mathcal{V}_n$.

Conversely, if $\{ \underset{\sim}{e}_{\langle 1\rangle}, \underset{\sim}{e}_{\langle 2\rangle}, \cdots, \underset{\sim}{e}_{\langle n\rangle} \}$ and $\{ \underset{\sim}{\bar{e}}_{\langle 1\rangle}, \underset{\sim}{\bar{e}}_{\langle 2\rangle}, \cdots, \underset{\sim}{\bar{e}}_{\langle n\rangle} \}$

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[1] Often, $O^+(\mathcal{V}_n)$, $\mathcal{O}^+(\mathcal{V}_n)$, $SO(\mathcal{V}_n)$.

[2] Often, rotation group, special orthogonal group. The elements of $\mathrm{Orth}^+\mathcal{V}_n$ are often called rotations.

are two orthonormal bases for $\mathcal{V}_n$, then $\exists$ a unique $\underset{\sim}{Q} \in \mathrm{Orth}\,\mathcal{V}_n$ $\ni$ $\bar{\underset{\sim}{e}}_{\langle i\rangle} = \underset{\sim}{Q}\,\underset{\sim}{e}_{\langle i\rangle}$, $i \in \mathbb{I}_n$. In fact, the components of $\underset{\sim}{Q}$ relative to $\{\underset{\sim}{e}_{\langle 1\rangle}, \underset{\sim}{e}_{\langle 2\rangle}, \cdots, \underset{\sim}{e}_{\langle n\rangle}\}$ are given by

$$Q_{\langle ij\rangle} = \underset{\sim}{e}_{\langle i\rangle} \cdot \bar{\underset{\sim}{e}}_{\langle j\rangle} = \cos(\underset{\sim}{e}_{\langle i\rangle}, \bar{\underset{\sim}{e}}_{\langle j\rangle}),\ i,j \in \mathbb{I}_n\,.\,)^1$$

The bases $\{\underset{\sim}{e}_{\langle 1\rangle}, \underset{\sim}{e}_{\langle 2\rangle}, \cdots, \underset{\sim}{e}_{\langle n\rangle}\}$ and $\{\bar{\underset{\sim}{e}}_{\langle 1\rangle}, \bar{\underset{\sim}{e}}_{\langle 2\rangle}, \cdots, \bar{\underset{\sim}{e}}_{\langle n\rangle}\}$ are like-handed[2] iff $\underset{\sim}{Q} \in \mathrm{Orth}^+\mathcal{V}_n$.

Proof. Exercise A3.10.8. □

An orthogonal tensor on a finite-dimensional space is necessarily bounded (Theorem A3.4.2). In fact, we easily have the following more general result.

Theorem A3.10.9. Let $\underset{\sim}{Q} \in \mathrm{Orth}\,\mathcal{V}$. Then in terms of the inner product norm, $\underset{\sim}{Q}$ is bounded and $\|\underset{\sim}{Q}\| = 1$.

Proof. Exercise A3.10.9. □

Supplementary Reading

Same as for § A3.5

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[1] $\cos(\underset{\sim}{e}_{\langle i\rangle}, \bar{\underset{\sim}{e}}_{\langle j\rangle})$ means the cosine of the angle between $\underset{\sim}{e}_{\langle i\rangle}$ and $\bar{\underset{\sim}{e}}_{\langle j\rangle}$

[2] See Definition A2.7.6.

A3.11. Eigenvalues, Eigenvectors, and Eigenspaces of a Second-Order Tensor

A second-order tensor, of course, maps a linear space into itself.[1] Nonzero elements that are mapped into parallel elements by a given tensor play a special role in the study of that tensor.

Definition A3.11.1. Let $\underset{\sim}{T} \in \text{Lin}\,\mathcal{V}$. Then $\lambda \in \mathbb{R}$ is an *eigenvalue*[2] of $\underset{\sim}{T}$ if $\exists\ \underset{\sim}{e} \in \mathcal{V} \ni \underset{\sim}{e} \neq \underset{\sim}{0}$ and

$$\underset{\sim}{T}\underset{\sim}{e} = \lambda \underset{\sim}{e}\,.$$

In this case, $\underset{\sim}{e}$ is called an *eigenvector*[3] of $\underset{\sim}{T}$ corresponding to the eigenvalue λ; and $(\lambda, \underset{\sim}{e})$ is called an *eigenpair*[4] of $\underset{\sim}{T}$.

Insert material on p. A3.11.1a

As an immediate consequence of the linearity of $\underset{\sim}{T}$, we have

Theorem A3.11.2. Let $(\lambda, \underset{\sim}{e})$ be an eigenpair of $\underset{\sim}{T} \in \text{Lin}\,\mathcal{V}$. Then $(\lambda, \alpha\underset{\sim}{e})$ where $\alpha \neq 0$ is also an eigenpair of $\underset{\sim}{T}$.

Thus, if $\underset{\sim}{e}$ is an eigenvector corresponding to a given eigenvalue,

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[1] See Definition A2.3.6.

[2] Often, *principal value*, *characteristic number*, *proper number*.

[3] Often, *principal direction*, *characteristic vector*; *proper vector*.

[4] This useful bit of nomenclature is not very common.

INSERT for p.3.11.1

Clearly, the definition permits an eigenvalue to have more than one corresponding eigenvector, and we shall elucidate this feature below. However, <u>an eigenvector can correspond to but one eigenvalue</u>; in fact, we have the following slightly more general result.

Theorem A3.11.1. <u>Let</u> $(\lambda_1, \underset{\sim}{e}_1)$ <u>and</u> $(\lambda_2, \underset{\sim}{e}_2)$ <u>be eigenpairs of</u> $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}$. <u>If</u> $\underset{\sim}{e}_1$ <u>and</u> $\underset{\sim}{e}_2$ <u>are parallel</u>, <u>then</u> $\lambda_1 = \lambda_2$.

Proof. *Exercise* A.3.11.1. $\square$

Return to p. A3.11.1

then any nonzero vector parallel to $\underset{\sim}{e}$ is also an eigenvector corresponding to the same eigenvalue. To remove this trivial nonuniqueness, it is quite common to normalize the eigenvectors with the requirement that they be unit vectors; i.e., $\|\underset{\sim}{e}\| = 1$. Of course, this requires the underlying linear space to be a normed linear space. This normalization still leaves $(\lambda, -\underset{\sim}{e})$ as an eigenpair whenever $(\lambda, \underset{\sim}{e})$ is an eigenpair. We shall not require any normalization.

Sometimes the totality of eigenvectors associated with a given eigenvalue consists of more than a set of parallel elements. In this case, the following definition is especially useful.

Definition A3.11.2. Let λ be an eigenvalue of $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}$. Then the *eigenspace*[1] of $\underset{\sim}{T}$ corresponding to λ is the set

$$\mathcal{E}_\lambda = \{ \underset{\sim}{u} \in \mathcal{V} : \underset{\sim}{T}\underset{\sim}{u} = \lambda \underset{\sim}{u} \} = \mathcal{N}(\underset{\sim}{T} - \lambda \underset{\sim}{I}) \, . \,)^2$$

It is important to realize that the eigenspace of $\underset{\sim}{T}$ corresponding to the eigenvalue λ contains the zero element, and consequently it is more than the totality of eigenvectors associated with λ. On the other hand, clearly the zero element is the only member of an eigenspace besides the eigenvectors.

The observation above that the characteristic space $\mathcal{E}_\lambda$ is the null space of $\underset{\sim}{T} - \lambda \underset{\sim}{I}$ follows from writing the equation

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[1] Often, *characteristic space*. [2] Cf. Exercise A3.9.1.

in the homogeneous form

$$(\underset{\sim}{T} - \lambda \underset{\sim}{I})\underset{\sim}{u} = \underset{\sim}{0}.$$

This also allows us to read off the following result from Exercise A3.9.1.

Theorem A3.11.3. An eigenspace of a second-order tensor $\underset{\sim}{T} \in \text{Lin}\, V$ is a linear subspace of V.

The next result is easily established by mathematical induction.

Theorem A3.11.4. Let $(\lambda, \underset{\sim}{e})$ be an eigenpair of $\underset{\sim}{T} \in \text{Lin}\, V$. Then for any $m \in \{1, 2, \cdots\}$, $(\lambda^m, \underset{\sim}{e})$ is an eigenpair of $\underset{\sim}{T}^m$.

Proof. Exercise A3.11.2. □

Another easy consequence of the linearity of $\underset{\sim}{T}$ is

Theorem A3.11.5. Let $(\lambda, \underset{\sim}{e})$ be an eigenpair of $\underset{\sim}{T} \in \text{Inv}\, V$. Then

(i) $\lambda \neq 0$

and

(ii) $(\lambda^{-1}, \underset{\sim}{e})$ is an eigenpair of $\underset{\sim}{T}^{-1}$.

Proof. Exercise A3.11.3. □

A key result is that eigenvectors which correspond to distinct eigenvalues of a symmetric tensor on an inner product space are orthogonal. Formally, we have

Theorem A3.11.6. Let $\mathbf{S} \in \operatorname{Sym} \mathcal{V}$, where $\mathcal{V}$ is an inner product space. Let $(\lambda_1, \mathbf{e}_1)$ and $(\lambda_2, \mathbf{e}_2)$ be eigenpairs of $\mathbf{S}$. Then

$$\lambda_1 \neq \lambda_2 \implies \mathbf{e}_1 \cdot \mathbf{e}_2 = 0.$$

Proof. By hypothesis,

$$\mathbf{S}\mathbf{e}_1 = \lambda_1 \mathbf{e}_1 \quad \text{and} \quad \mathbf{S}\mathbf{u}_2 = \lambda_2 \mathbf{e}_2.$$

By the definition of the transpose,

$$\mathbf{e}_1 \cdot \mathbf{S}\mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{S}^T \mathbf{e}_1$$

$$= \mathbf{e}_2 \cdot \mathbf{S}\mathbf{e}_1 \qquad (\mathbf{S} \in \operatorname{Sym} \mathcal{V}).$$

$\therefore$

$$\mathbf{e}_1 \cdot (\lambda_2 \mathbf{e}_2) = \mathbf{e}_2 \cdot (\lambda_1 \mathbf{e}_1)$$

or

$$(\lambda_2 - \lambda_1)\, \mathbf{e}_1 \cdot \mathbf{e}_2 = 0. \quad \square$$

As usual, when the underlying linear space is finite-dimensional, we can easily establish quite explicit results. For the remainder of this section, $\mathcal{V}_n$ denotes an n-dimensional inner product space.

Theorem A3.11.7. $\lambda \in \mathbb{R}$ is an eigenvalue of $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$ iff

$$\det(\underset{\sim}{T} - \lambda \underset{\sim}{I}) = 0.$$

Proof. First, write $\underset{\sim}{T}\underset{\sim}{e} = \lambda \underset{\sim}{e}$ in the homogeneous form

$$(\underset{\sim}{T} - \lambda \underset{\sim}{I})\underset{\sim}{e} = \underset{\sim}{0},$$

and then note from Theorem A3.8.7 that this equation has a nontrivial solution iff $\det(\underset{\sim}{T} - \lambda \underset{\sim}{I}) = 0$. □

Theorem A3.11.8. Given $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$,

$$\det(\underset{\sim}{T} - \lambda \underset{\sim}{I})$$

is an n^{th} degree polynomial in λ. It is called the *characteristic polynomial*[1] of $\underset{\sim}{T}$, and $\det(\underset{\sim}{T} - \lambda \underset{\sim}{I}) = 0$ is called the *characteristic*

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[1] For the sake of consistency, we would prefer the terminology *eigenpolynomial*, but it appears no one else would.

equation of $\underset{\sim}{T}$. In particular, for

$$n=3, \quad \det(\underset{\sim}{T}-\lambda \underset{\approx}{I}) = -\lambda^3 + \iota_1(\underset{\sim}{T})\lambda^2 - \iota_2(\underset{\sim}{T})\lambda + \iota_3(\underset{\sim}{T}),$$

where

$$\iota_1(\underset{\sim}{T}) = \operatorname{tr} \underset{\sim}{T},$$

$$\iota_2(\underset{\sim}{T}) = \tfrac{1}{2}\left[(\operatorname{tr}\underset{\sim}{T})^2 - \operatorname{tr}(\underset{\sim}{T}^2)\right],$$

$$\iota_3(\underset{\sim}{T}) = \det \underset{\sim}{T};$$

and for

$$n=2, \quad \det(\underset{\sim}{T}-\lambda \underset{\approx}{I}) = \lambda^2 - \iota_1(\underset{\sim}{T}) + \iota_2(\underset{\sim}{T}),$$

where

$$\iota_1(\underset{\sim}{T}) = \operatorname{tr} \underset{\sim}{T},$$

$$\iota_2(\underset{\sim}{T}) = \det \underset{\sim}{T}.$$

The $\iota(\underset{\sim}{T})$'s are called the *fundamental invariants*[1] of $\underset{\sim}{T}$.

Proof. By Theorems A3.8.5, A3.2.5, and A3.2.7,

$$\det(\underset{\sim}{T}-\lambda \underset{\approx}{I}) = \det\left[T^{i}_{\cdot j} - \lambda \delta^{i}_{j}\right];$$

and the result follows on expanding the determinant of the matrix.

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[1] Often, *principal invariants*; and often denoted by $I_i(\underset{\sim}{T})$ or $I(T)$, $II(T)$, $III(T)$. The use of this terminology will be justified in §A8.4.

The details for the case $n = 3$ are left as Exercise A3.11.4. $\square$

Of course, specific formulas for the fundamental invariants can be written down also in the n-dimensional case (see TRUESDELL and NOLL's The Non-Linear Field Theories of Mechanics).

The next theorem begins to hint at the significance of eigenvalues and eigenvectors.

Theorem A3.11.9. If $\{\underset{\sim}{e}_{\langle 1\rangle}, \underset{\sim}{e}_{\langle 2\rangle}, \cdots, \underset{\sim}{e}_{\langle n\rangle}\}$ is an orthonormal basis for $\mathcal{V}_n \ni$ the matrix of components[1] of $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n$ relative to $\{\underset{\sim}{e}_{\langle 1\rangle}, \underset{\sim}{e}_{\langle 2\rangle}, \cdots, \underset{\sim}{e}_{\langle n\rangle}\}$ is diagonal; i.e.,

$$[T^{\langle ij\rangle}] = \begin{bmatrix} \lambda_1 & 0 & \cdots & & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & & \cdots & 0 & \lambda_n \end{bmatrix} =: \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

or equivalently[2] $\ni \underset{\sim}{T}$ admits the representation

$$\underset{\sim}{T} = \lambda_1 \underset{\sim}{e}_{\langle 1\rangle} \otimes \underset{\sim}{e}_{\langle 1\rangle} + \lambda_2 \underset{\sim}{e}_{\langle 2\rangle} \otimes \underset{\sim}{e}_{\langle 2\rangle} + \cdots + \lambda_n \underset{\sim}{e}_{\langle n\rangle} \otimes \underset{\sim}{e}_{\langle n\rangle},$$

then $\underset{\sim}{T} \in \mathrm{Sym}\,\mathcal{V}_n$ and for each $i \in \mathrm{I\!I}_n$ $(\lambda_i, \underset{\sim}{e}_{\langle i\rangle})$ is an eigenpair of $\underset{\sim}{T}$.

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[1] Cf. Theorem A3.2.4.

[2] Cf. Theorem A3.2.2.

Proof. That $\underset{\sim}{T}^T = \underset{\sim}{T}$ follows from either Theorem A3.6.3 or Theorem A3.6.4. By Theorem A3.2.2,

$$\underset{\sim}{T} = T^{\langle ij\rangle} \underset{\sim}{e}_{\langle i\rangle} \otimes \underset{\sim}{e}_{\langle j\rangle} ,$$

and $\therefore$

$$\begin{aligned}
\underset{\sim}{T}\underset{\sim}{e}_{\langle k\rangle} &= \left(T^{\langle ij\rangle} \underset{\sim}{e}_{\langle i\rangle} \otimes \underset{\sim}{e}_{\langle j\rangle}\right) \underset{\sim}{e}_{\langle k\rangle} && \text{(substitution)} \\
&= T^{\langle ij\rangle} \left(\underset{\sim}{e}_{\langle i\rangle} \otimes \underset{\sim}{e}_{\langle j\rangle}\right) \underset{\sim}{e}_{\langle k\rangle} && \text{(defns. of addition and scalar multiplication of tensors)} \\
&= T^{\langle ij\rangle} \left(\underset{\sim}{e}_{\langle j\rangle} \cdot \underset{\sim}{e}_{\langle k\rangle}\right) \underset{\sim}{e}_{\langle i\rangle} && \text{(defn. of tensor product)} \\
&= T^{\langle ij\rangle} \delta_{jk} \underset{\sim}{e}_{\langle i\rangle} && \text{(orthonormality)} \\
&= T^{\langle ik\rangle} \underset{\sim}{e}_{\langle i\rangle} && \text{(property of Kronecker delta)} \\
&= \lambda_k \underset{\sim}{e}_{\langle k\rangle} && \text{(given structure of } [T^{\langle ik\rangle}]\text{)}. \quad \square
\end{aligned}$$

Exercise A3.11.5. Let $\underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}_n$. Prove that if $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ and $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^n\}$ are a basis and its dual for $\mathcal{V}_n$ $\ni$

$$[T^{i}_{\cdot j}] = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

then for each $i \in \mathrm{II}_n$ $(\lambda_i, \underset{\sim}{e}_i)$ is an eigenpair of $\underset{\sim}{T}$.

For any $\underset{\sim}{T} \in \mathrm{Sym}\, \mathcal{V}_n$, we will be able to establish the existence of a set of n eigenvectors which comprise an orthonormal basis for $\mathcal{V}_n$. This is worked out in § A3.13. In § A3.12, we address some notions prerequisite to this.

We end this section with some easy results on the eigenvalue problem for skew tensors.

Theorem A3.11.10. Let $\mathcal{V}$ be an inner product space, and let $\underset{\sim}{W} \in \text{Skw}\,\mathcal{V}$ (meaning $\underset{\sim}{W}$ has a transpose and $\underset{\sim}{W}^T = -\underset{\sim}{W}$). Then any eigenvalues of $\underset{\sim}{W}$ are 0.

Proof. Suppose λ is an eigenvalue with corresponding eigenvector $\underset{\sim}{e}$. Then

$$\underset{\sim}{W}\underset{\sim}{e} = \lambda \underset{\sim}{e}.$$

$\Rightarrow$

$$\lambda\, \underset{\sim}{e}\cdot\underset{\sim}{e} = \underset{\sim}{e}\cdot\underset{\sim}{W}\underset{\sim}{e} \qquad \text{(take inner product with } \underset{\sim}{e}\text{)}$$

$$= \underset{\sim}{e}\cdot\underset{\sim}{W}^T\underset{\sim}{e} \qquad \text{(definition of transpose)}$$

$$= -\underset{\sim}{e}\cdot\underset{\sim}{W}e \qquad (\underset{\sim}{W} \in \text{Skw}\,\mathcal{V} \text{ and linearity of inner prod}$$

$\therefore\ \lambda\, \underset{\sim}{e}\cdot\underset{\sim}{e} = 0 \Rightarrow \lambda = 0$ since $\underset{\sim}{e} \neq \underset{\sim}{0}$. $\square$

Of course, Theorem A3.11.10 does not gurantee the existence of any eigenvalues for a skew tensor. However, when the underlying space is finite-dimensional with odd dimension, a skew tensor will always have an eigenvalue.

Theorem A3.11.11. Let $\underset{\sim}{W} \in \text{Skw}\,\mathcal{V}_n$, where n is odd. Then 0 is an eigenvalue of $\underset{\sim}{W}$.

A3.12. Direct Sum Decompositions. Invariant Linear Subspaces

Theorem A3.12.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be linear subspaces of a linear space $\mathcal{W}$. Then the set*

$$\mathcal{U} + \mathcal{V} := \{ \underset{\sim}{u} + \underset{\sim}{v} : \underset{\sim}{u} \in \mathcal{U}, \underset{\sim}{v} \in \mathcal{V} \}$$

is a linear subspace of $\mathcal{W}$. It is called the **sum**[1] *of the linear subspaces $\mathcal{U}$ and $\mathcal{V}$.*

Proof. **Exercise** A.3.12.1. □

We already have considerable experience with sums of linear subspaces, eventhough we have not heretofore bothered to formally introduce the concept. E.g., suppose that $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2\}$ is a basis for the two-dimensional linear space $\mathcal{V}_2$. Then any $\underset{\sim}{u} \in \mathcal{V}_2$ admits the representation

$$\underset{\sim}{u} = u^1 \underset{\sim}{e}_1 + u^2 \underset{\sim}{e}_2 ,$$

where $u^1, u^2 \in \mathbb{R}$. But elements of the form $u^1 \underset{\sim}{e}_1$ comprise the linear subspace $Lsp\{\underset{\sim}{e}_1\}$; thus,

$$\mathcal{V}_2 = Lsp\{\underset{\sim}{e}_1\} + Lsp\{\underset{\sim}{e}_2\} .$$

In fact, since components w.r.t. a given basis are unique, the

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[1] Sometimes, **inner sum**.

decomposition of an element of $\mathcal{V}_2$ into the sum of elements from $Lsp\{\underset{\sim}{e}_1\}$ and $Lsp\{\underset{\sim}{e}_2\}$ can be done in only one way. This leads us to

Definition A3.12.1. Let $\mathcal{U}$ and $\mathcal{V}$ be linear subspaces of a linear space $\mathcal{W}$ $\ni$

$$\mathcal{W} = \mathcal{U} + \mathcal{V}.$$

Then $\mathcal{W}$ is the *direct sum*[1] of the linear subspaces $\mathcal{U}$ and $\mathcal{V}$, and we write

$$\mathcal{W} = \mathcal{U} \oplus \mathcal{V},$$

if the decomposition is unique; i.e., if each $\underset{\sim}{w} \in \mathcal{W}$ can be expressed as

$$\underset{\sim}{w} = \underset{\sim}{u} + \underset{\sim}{v} \quad \text{with } \underset{\sim}{u} \in \mathcal{U} \text{ and } \underset{\sim}{v} \in \mathcal{V}$$

in one and only one way.

Exercise A 3.12.2. Let $\mathcal{V}_n$ be an n-dimensional inner product space. Show that

$$Lin\,\mathcal{V}_n = Sym\,\mathcal{V}_n \oplus Skew\,\mathcal{V}_n.$$

The following theorem provides a useful characterization of

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[1] It also is possible to put together a meaningful notion of direct sum when $\mathcal{U}$ and $\mathcal{V}$ are not linear subspaces of the same parent space. See, e.g., the references by HALMOS or by NAYLOR and SELL.

direct sums.

<u>Theorem</u> A3.12.2. <u>Let $\mathcal{U}$ and $\mathcal{V}$ be linear subspaces of a linear space</u> $\mathcal{W}$. <u>Then</u> $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ <u>iff</u>

$$(i) \quad \mathcal{W} = \mathcal{U} + \mathcal{V}$$

<u>and</u>

$$(ii) \quad \mathcal{U} \cap \mathcal{V} = \{\underset{\sim}{0}\}.$$

<u>Proof</u>. Suppose $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$. Then every $\underset{\sim}{w} \in \mathcal{W}$ can be written in the form

$$\underset{\sim}{w} = \underset{\sim}{u} + \underset{\sim}{v} \quad \text{with } \underset{\sim}{u} \in \mathcal{U},\ \underset{\sim}{v} \in \mathcal{V}$$

uniquely. Clearly, $\mathcal{W} = \mathcal{U} + \mathcal{V}$. To show that $\mathcal{U} \cap \mathcal{V} = \{\underset{\sim}{0}\}$, let $\underset{\sim}{x} \in \mathcal{U} \cap \mathcal{V}$. Then $\underset{\sim}{x} \in \mathcal{U}$ and $\underset{\sim}{x} \in \mathcal{V}$. We always can write $\underset{\sim}{x}$ as

$$\underset{\sim}{x} = \underset{\sim}{x} + \underset{\sim}{0} \quad , \ \underset{\sim}{x} \in \mathcal{U},\ \underset{\sim}{0} \in \mathcal{V}$$

and also as

$$\underset{\sim}{x} = \underset{\sim}{0} + \underset{\sim}{x} \quad , \ \underset{\sim}{0} \in \mathcal{U},\ \underset{\sim}{x} \in \mathcal{V}.$$

Now $\underset{\sim}{x}$ is necessarily an element of the underlying linear space $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$; and then by the uniqueness of the decomposition, $\underset{\sim}{x} = \underset{\sim}{0}$. Consequently, $\mathcal{U} \cap \mathcal{V} \subset \{\underset{\sim}{0}\}$; obviously, $\{\underset{\sim}{0}\} \subset \mathcal{U} \cap \mathcal{V}$. $\therefore \mathcal{U} \cap \mathcal{V} = \{\underset{\sim}{0}\}$.

Conversely, suppose that $\mathcal{W} = \mathcal{U} + \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \{\underset{\sim}{0}\}$. $\mathcal{W} = \mathcal{U} + \mathcal{V}$ means that any $\underset{\sim}{w} \in \mathcal{W}$ admits the decomposition

$$\underset{\sim}{w} = \underset{\sim}{u} + \underset{\sim}{v} \quad \text{with } \underset{\sim}{u} \in \mathcal{U},\ \underset{\sim}{v} \in \mathcal{V}.$$

To show that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$, we must prove that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ in this decomposition are unique. To this end, suppose that

$$\underset{\sim}{w} = \underset{\sim}{u}' + \underset{\sim}{v}' \text{ with } \underset{\sim}{u}' \in \mathcal{U},\ \underset{\sim}{v}' \in \mathcal{V}$$

is another such decomposition. Then

$$\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{u}' + \underset{\sim}{v}'$$

$$\Longrightarrow$$

$$\underset{\sim}{u} - \underset{\sim}{u}' = \underset{\sim}{v}' - \underset{\sim}{v} .$$

Now, since $\mathcal{U}$ and $\mathcal{V}$ are linear subspaces, $\underset{\sim}{u} - \underset{\sim}{u}' \in \mathcal{U}$ and $\underset{\sim}{v}' - \underset{\sim}{v} \in \mathcal{V}$; and since $\underset{\sim}{u} - \underset{\sim}{u}' = \underset{\sim}{v}' - \underset{\sim}{v}$, both

$$\underset{\sim}{u} - \underset{\sim}{u}' \in \mathcal{U} \cap \mathcal{V} \text{ and } \underset{\sim}{v}' - \underset{\sim}{v} \in \mathcal{U} \cap \mathcal{V}.$$

Then $\mathcal{U} \cap \mathcal{V} = \{\underset{\sim}{0}\} \Longrightarrow \underset{\sim}{u} - \underset{\sim}{u}' = \underset{\sim}{0}$ and $\underset{\sim}{v}' - \underset{\sim}{v} = \underset{\sim}{0}$, or

$$\underset{\sim}{u}' = \underset{\sim}{u} \text{ and } \underset{\sim}{v}' = \underset{\sim}{v}. \quad \square$$

In the finite-dimensional case, we have the following important result.

Theorem A3.12.3. Let $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ where $\mathcal{U}$ and $\mathcal{V}$ are finite-dimensional linear subspaces of $\mathcal{W}$. Then $\mathcal{W}$ is finite-dimensional, and

$$\dim \mathcal{W} = \dim \mathcal{U} + \dim \mathcal{V}.$$

Proof. Let $\mathcal{U}$ be m-dimensional with a basis $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_m\}$, and let $\mathcal{V}$ be n-dimensional with a basis $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_n\}$. We will show that the subset $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_m, \underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_n\} \subset \mathcal{W}$ is a basis for $\mathcal{W}$.

Accordingly, consider any $\underset{\sim}{w} \in \mathcal{W} = \mathcal{U} \oplus \mathcal{V}$. Then $\underset{\sim}{w} = \underset{\sim}{u} + \underset{\sim}{v}$ with $\underset{\sim}{u} \in \mathcal{U}$, $\underset{\sim}{v} \in \mathcal{V}$, so that

$$\underset{\sim}{w} = (u^1\underset{\sim}{u}_1 + u^2\underset{\sim}{u}_2 + \cdots + u^m\underset{\sim}{u}_m) + (v^1\underset{\sim}{v}_1 + v^2\underset{\sim}{v}_2 + \cdots + v^n\underset{\sim}{v}_n).$$

Thus, $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_m, \underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_n\}$ spans $\mathcal{W}$.

To show that $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_m, \underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_n\}$ is linearly independent, consider

$$\alpha_1\underset{\sim}{u}_1 + \alpha_2\underset{\sim}{u}_2 + \cdots + \alpha_m\underset{\sim}{u}_m + \beta_1\underset{\sim}{v}_1 + \beta_2\underset{\sim}{v}_2 + \cdots + \beta_n\underset{\sim}{v}_n = \underset{\sim}{0}.$$

Now, $\alpha_1\underset{\sim}{u}_1 + \alpha_2\underset{\sim}{u}_2 + \cdots + \alpha_m\underset{\sim}{u}_m \in \mathcal{U}$ and $\beta_1\underset{\sim}{v}_1 + \beta_2\underset{\sim}{v}_2 + \cdots + \beta_n\underset{\sim}{v}_n \in \mathcal{V}$, but we always can write $\underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{0}$; and $\therefore$ by the uniqueness of such representations,

$$\alpha_1\underset{\sim}{u}_1 + \alpha_2\underset{\sim}{u}_2 + \cdots + \alpha_m\underset{\sim}{u}_m = \underset{\sim}{0} \text{ and } \beta_1\underset{\sim}{v}_1 + \beta_2\underset{\sim}{v}_2 + \cdots + \beta_n\underset{\sim}{v}_n = \underset{\sim}{0}.$$

It then follows from the linear independence of $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_m\}$ and of $\{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \cdots, \underset{\sim}{v}_n\}$ that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_m = \beta_1 = \beta_2 = \cdots = \beta_n = 0.$$

Hence, the set $\{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \dots, \underset{\sim}{u}_m, \underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, \underset{\sim}{v}_n\}$ is linearly independent. □

Consider a linear subspace $\mathcal{S}$ of an inner product space $\mathcal{V}$. An associated linear subspace of special importance is the set of all elements of $\mathcal{V}$ that are orthogonal to all elements of $\mathcal{S}$.

Theorem A3.12.4. Let $\mathcal{S}$ be a linear subspace of an inner product space $\mathcal{V}$. Then the subset

$$\mathcal{S}^{\perp} := \{\underset{\sim}{u} \in \mathcal{V} : \underset{\sim}{u} \cdot \underset{\sim}{s} = 0 \quad \forall\, \underset{\sim}{s} \in \mathcal{S}\} \qquad)^{1}$$

also is a linear subspace of $\mathcal{V}$. It is called the orthogonal complement of $\mathcal{S}$.

Proof. Exercise A 3.12.3. □

Exercise A3.12.4. Let $\{\underset{\sim}{e}_{<1>}, \underset{\sim}{e}_{<2>}, \underset{\sim}{e}_{<3>}\}$ be an orthonormal basis for a 3-dimensional inner product space $\mathcal{V}_3$. Let

$$\mathcal{S} = Lsp\{\underset{\sim}{e}_{<1>}, \underset{\sim}{e}_{<2>}\} \quad \dots \text{ "the 1-2 plane".}$$

Show that

$$\mathcal{S}^{\perp} = Lsp\{\underset{\sim}{e}_{<3>}\} \quad \dots \text{ "the 3-axis".}$$

The following important result is a famous part of functional analysis. Of course, in the infinite-dimensional case, additional

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[1] Read as "$\mathcal{S}$ perp".

hypotheses are needed.

Theorem A3.12.5. (*Projection Theorem*) <u>Let $\mathcal{S}$ be a finite-dimensional subspace of an inner product space $\mathcal{V}$. Then</u>

$$\mathcal{V} = \mathcal{S} \oplus \mathcal{S}^{\perp}.$$

Proof. First, suppose $\mathcal{S} \neq \{\mathbf{0}\}$. Let $\dim \mathcal{S} = m$ and let $\{\mathbf{e}_{\langle 1\rangle}, \mathbf{e}_{\langle 2\rangle}, \ldots, \mathbf{e}_{\langle m\rangle}\}$ be an orthonormal basis for $\mathcal{S}$. Define $\mathbf{P} \in \operatorname{Lin} \mathcal{V}$ through

$$\mathbf{P}\mathbf{u} = (\mathbf{e}_{\langle 1\rangle} \cdot \mathbf{u})\mathbf{e}_{\langle 1\rangle} + (\mathbf{e}_{\langle 2\rangle} \cdot \mathbf{u})\mathbf{e}_{\langle 2\rangle} + \cdots + (\mathbf{e}_{\langle m\rangle} \cdot \mathbf{u})\mathbf{e}_{\langle m\rangle}, \quad \mathbf{u} \in \mathcal{V}.$$

The linearity of $\mathbf{P}$ follows from the linearity of the inner product. Since the range of $\mathbf{P}$ is necessarily in $\mathcal{S}$, it might be said that $\mathbf{P}$ "projects" $\mathcal{V}$ into $\mathcal{S}$.)[1] By the definition of $\mathbf{P}$,

$$(\mathbf{u} - \mathbf{P}\mathbf{u}) \cdot \mathbf{e}_{\langle i\rangle} = 0, \quad i \in \mathcal{I}_m, \ \mathbf{u} \in \mathcal{V}.$$

Thus, for any $\mathbf{s} \in \mathcal{S}$,

$$(\mathbf{u} - \mathbf{P}\mathbf{u}) \cdot \mathbf{s} = 0;$$

i.e., $(\mathbf{u} - \mathbf{P}\mathbf{u}) \in \mathcal{S}^{\perp}$.)[2] Then on writing

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[1] Generally, a second-order tensor $\mathbf{T}$ is called a *projection operator* or a *projector* if it is *idempotent*; i.e. if $\mathbf{T}^2 = \mathbf{T}$. Show that $\mathbf{P}^2 = \mathbf{P}$ as *Exercise* A3.12.5.

$$\underset{\sim}{u} = \underset{\sim}{P}\underset{\sim}{u} + (\underset{\sim}{u} - \underset{\sim}{P}\underset{\sim}{u}),$$

we see that $\mathcal{V} = \mathcal{S} + \mathcal{S}^{\perp}$.

By Theorem A3.12.2, the proof will be complete if we show that $\mathcal{S} \cap \mathcal{S}^{\perp} = \{\underset{\sim}{0}\}$. Suppose $\underset{\sim}{u} \in \mathcal{S} \cap \mathcal{S}^{\perp}$. Then $\underset{\sim}{u} \in \mathcal{S}$ and $\underset{\sim}{u} \in \mathcal{S}^{\perp}$; consequently $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0 \Rightarrow \underset{\sim}{u} = \underset{\sim}{0}$. $\therefore \mathcal{S} \cap \mathcal{S}^{\perp} \subset \{\underset{\sim}{0}\}$. Obviously, $\{\underset{\sim}{0}\} \subset \mathcal{S} \cap \mathcal{S}^{\perp}$; whence $\mathcal{S} \cap \mathcal{S}^{\perp} = \{\underset{\sim}{0}\}$.

Finally, suppose $\mathcal{S} = \{\underset{\sim}{0}\}$. Then

$$\mathcal{S}^{\perp} = \{\underset{\sim}{u} \in \mathcal{V} : \underset{\sim}{u} \cdot \underset{\sim}{0} = 0\} = \mathcal{V},$$

since all elements are orthogonal to the zero element. It is also clear that

$$\mathcal{V} = \mathcal{V} + \{\underset{\sim}{0}\}$$

and that

$$\mathcal{V} \cap \{\underset{\sim}{0}\} = \{\underset{\sim}{0}\}.$$

$\therefore \quad \mathcal{V} = \{\underset{\sim}{0}\} \oplus \mathcal{V}$ or $\mathcal{V} = \mathcal{S} \oplus \mathcal{S}^{\perp}$. $\square$

(cont. from p. A3.12.7)

2 In the example of Exercise A3.12.4, $\mathcal{S} = \mathrm{Lsp}\{\underset{\sim}{e}_{\langle 1\rangle}, \underset{\sim}{e}_{\langle 2\rangle}\}$ so $\underset{\sim}{P}\underset{\sim}{u} = (\underset{\sim}{e}_{\langle 1\rangle} \cdot \underset{\sim}{u})\underset{\sim}{e}_{\langle 1\rangle} + (\underset{\sim}{e}_{\langle 2\rangle} \cdot \underset{\sim}{u})\underset{\sim}{e}_{\langle 2\rangle} = u_{\langle 1\rangle}\underset{\sim}{e}_{\langle 1\rangle} + u_{\langle 2\rangle}\underset{\sim}{e}_{\langle 2\rangle}$. Clearly, $\underset{\sim}{P}$ projects $\mathcal{V}_3$ into $\mathcal{S}$, in fact, <u>onto</u> $\mathcal{S}$. Note also that

$$\underset{\sim}{u} - \underset{\sim}{P}\underset{\sim}{u} = u_{\langle 1\rangle}\underset{\sim}{e}_{\langle 1\rangle} + u_{\langle 2\rangle}\underset{\sim}{e}_{\langle 2\rangle} + u_{\langle 3\rangle}\underset{\sim}{e}_{\langle 3\rangle} - \left(u_{\langle 1\rangle}\underset{\sim}{e}_{\langle 1\rangle} + u_{\langle 2\rangle}\underset{\sim}{e}_{\langle 2\rangle}\right)$$

$$= u_{\langle 3\rangle}\underset{\sim}{e}_{\langle 3\rangle} \in \mathrm{Lsp}\{\underset{\sim}{e}_{\langle 3\rangle}\} = \mathcal{S}^{\perp}.$$

Now we return to our study of second-order tensors.

Definition A3.12.2. Let $\underset{\sim}{T} \in \text{Lin}\, \mathcal{V}$. A linear subspace $\mathcal{S}$ of $\mathcal{V}$ is an invariant linear subspace under $\underset{\sim}{T}$ if

$$\underset{\sim}{u} \in \mathcal{S} \Rightarrow \underset{\sim}{T}\underset{\sim}{u} \in \mathcal{S}.$$

Thus, invariant linear subspaces have the property that they are mapped into themselves (by the tensor under consideration) as the name suggests. The immediate usefulness of this notion lies in the fact that the one-dimensional linear subspace spanned by an eigenvector of $\underset{\sim}{T} \in \text{Lin}\, \mathcal{V}$ is an invariant linear subspace under $\underset{\sim}{T}$.

Theorem A3.12.6. Let $\mathcal{V}$ be an inner product space, and let $\underset{\sim}{T} \in \text{Lin}\, \mathcal{V}$ have a transpose. Then if $\mathcal{S}$ is an invariant linear subspace under $\underset{\sim}{T}$, its orthogonal complement $\mathcal{S}^\perp$ is an invariant linear subspace under the transpose $\underset{\sim}{T}^T$. In particular, if $\mathcal{S}$ is an invariant linear subspace under $\underset{\sim}{T} \in \text{Sym}\, \mathcal{V}$, then $\mathcal{S}^\perp$ also is an invariant linear subspace under $\underset{\sim}{T}$.

Proof. First, we note from Theorem A3.12.4 that $\mathcal{S}^\perp$ is a linear subspace of $\mathcal{V}$. From the definition of the transpose,

$$\underset{\sim}{u} \cdot \underset{\sim}{T}^T \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{T}\underset{\sim}{u} \quad \forall\, \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}.$$

Take $\underset{\sim}{u} \in \mathcal{V}$ and $\underset{\sim}{v} \in \mathcal{S}^\perp$. Then since $\mathcal{S}$ is invariant under $\underset{\sim}{T}$, $\underset{\sim}{T}\underset{\sim}{u} \in \mathcal{S}$; and $\underset{\sim}{v} \cdot \underset{\sim}{T}\underset{\sim}{u} = 0$ because $\underset{\sim}{v} \in \mathcal{S}^\perp$. Consequently,

$$\underset{\sim}{v} \in \mathcal{S}^\perp \Rightarrow (\underset{\sim}{T}^T \underset{\sim}{v}) \cdot \underset{\sim}{u} = 0 \ \forall \ \underset{\sim}{u} \in \mathcal{S} \Rightarrow \underset{\sim}{T}^T \underset{\sim}{v} \in \mathcal{S}^\perp ;$$

i.e., $\mathcal{S}^\perp$ is an invariant linear subspace under $\underset{\sim}{T}^T$. □

Supplementary Reading

Same as for § A3.5

A3.13. The Spectral Theorem for Symmetric Tensors on Finite-Dimensional Spaces

Roughly speaking, the spectral theorem is the converse of Theorem A3.11.9. The proof to be given here rests on the following lemma. As usual, $\mathcal{V}_n$ denotes an n-dimensional inner product space throughout this section.

Lemma A3.13.1. Let $\underset{\sim}{S} \in \operatorname{Sym} \mathcal{V}_n$. Then $\exists\ \underset{\sim}{e} \in \mathcal{V}_n$ with $|\underset{\sim}{e}| = 1 \ni (\underset{\sim}{e} \cdot \underset{\sim}{S}\underset{\sim}{e}, \underset{\sim}{e})$ is an eigenpair of $\underset{\sim}{S}$.

Proof. Consider the function

$$\vartheta : \mathcal{V}_n - \{\underset{\sim}{0}\} \to \mathbb{R}$$

defined by

$$\vartheta(\underset{\sim}{u}) = \frac{\underset{\sim}{u} \cdot \underset{\sim}{S}\underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}} .$$

Since ϑ is continuous[1] and since the unit sphere

$$\Omega := \{ \underset{\sim}{u} \in \mathcal{V}_n : |\underset{\sim}{u}| = 1 \}$$

is compact[2], ϑ assumes its minimum value on Ω; i.e., $\exists\ \underset{\sim}{e} \in \Omega \ni$

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[1] Prove this as Exercise A3.13.1.

[2] Prove this as Exercise A3.13.2. Since $\Omega \subset \mathcal{V}_n$, by the HEINE-BOREL Theorem it suffices to show that Ω is closed and bounded.

$$\vartheta(\underset{\sim}{u}) \geq \vartheta(\underset{\sim}{e}) \quad \forall \ \underset{\sim}{u} \in \Omega .$$

Now, any nonzero $\underset{\sim}{u} \in \mathcal{V}_n$ can be written as $\underset{\sim}{u} = |\underset{\sim}{u}|\,\underset{\sim}{w}$ where $|\underset{\sim}{w}| = 1$; and then from the structure of ϑ, we see that $\vartheta(\underset{\sim}{u}) = \vartheta(\underset{\sim}{w})$. $\therefore$

$$\vartheta(\underset{\sim}{u}) \geq \vartheta(\underset{\sim}{e}) \quad \forall \ \underset{\sim}{u} \in (\mathcal{V}_n - \{\underset{\sim}{0}\}) .$$

In order to capitalize on this property, let $\underset{\sim}{v}$ be an arbitrary element of $\mathcal{V}_n$ and define

$$\varphi : \mathbb{R} \to \mathbb{R}$$

through

$$\varphi(t) = \vartheta(\underset{\sim}{e} + t\underset{\sim}{v}) .$$

By the above property of ϑ, φ has a minimum at $t=0$. By the construction of ϑ, φ has a derivative[1]; $\therefore$ since 0 is an interior point of $\mathbb{R}$,

$$\varphi'(0) = 0 .$$

We leave it as *Exercise* A3.13.4 to show that

$$\varphi'(0) = 2(\underset{\sim}{S}\underset{\sim}{e} - \lambda \underset{\sim}{e}) \cdot \underset{\sim}{v} ,$$

where $\lambda = \underset{\sim}{e} \cdot \underset{\sim}{S}\underset{\sim}{e}$. (This is where the symmetry of $\underset{\sim}{S}$ comes in.)

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[1] Prove this as *Exercise* A3.13.3.

Thus,

$$(\underset{\sim}{S}\underset{\sim}{e} - \lambda \underset{\sim}{e}) \cdot \underset{\sim}{v} = 0 \; ;$$

and since this must hold $\forall \; \underset{\sim}{v} \in \mathcal{V}_n$,

$$\underset{\sim}{S}\underset{\sim}{e} - \lambda \underset{\sim}{e} = \underset{\sim}{0} \, .$$

I.e., $(\lambda, \underset{\sim}{e})$ is an eigenpair of $\underset{\sim}{S}$. $\square$

Now we are in a position to establish

Theorem A3.13.1. (*Spectral Theorem*) Let $\underset{\sim}{S} \in \mathrm{Sym}\,\mathcal{V}_n$. Then $\underset{\sim}{S}$ has a set of n eigenvectors $\{\overset{*}{\underset{\sim}{e}}_{\langle 1 \rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2 \rangle}, \cdots, \overset{*}{\underset{\sim}{e}}_{\langle n \rangle}\}$ which form an orthonormal basis for $\mathcal{V}_n$. $\{\overset{*}{\underset{\sim}{e}}_{\langle i \rangle}\}$ is called an *eigenvector basis*[1] of $\underset{\sim}{S}$. Moreover, if the components of $\underset{\sim}{S}$ relative to $\{\overset{*}{\underset{\sim}{e}}_{\langle i \rangle}\}$ are denoted by $\overset{*}{S}{}^{\langle ij \rangle}$, then

$$[\, \overset{*}{S}{}^{\langle ij \rangle} \,] = \mathrm{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_n) \, ,$$

where $\lambda_i = \overset{*}{\underset{\sim}{e}}_{\langle i \rangle} \cdot \underset{\sim}{S}\, \overset{*}{\underset{\sim}{e}}_{\langle i \rangle}$ is the eigenvalue of $\underset{\sim}{S}$ associated with the eigenvector $\overset{*}{\underset{\sim}{e}}_{\langle i \rangle}$; equivalently, $\underset{\sim}{S}$ admits the *spectral resolution*

$$\underset{\sim}{S} = \lambda_1 \overset{*}{\underset{\sim}{e}}_{\langle 1 \rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1 \rangle} + \lambda_2 \overset{*}{\underset{\sim}{e}}_{\langle 2 \rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2 \rangle} + \cdots + \lambda_n \overset{*}{\underset{\sim}{e}}_{\langle n \rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n \rangle} \, .$$

Proof. By Lemma A3.13.1, $\underset{\sim}{S}$ has an eigenpair, which we

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[1] Often, *principal basis*.

denote by $(\lambda_1, \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle})$. Also, $|\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}| = 1$. Let

$$\mathcal{E}_1 = Lsp\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}\}.$$

Then for $\underset{\sim}{u} \in \mathcal{E}_1$ $\exists\, \alpha \in \mathbb{R} \ni \underset{\sim}{u} = \alpha \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}$ and

$$\underset{\sim}{S}\underset{\sim}{u} = \underset{\sim}{S}(\alpha \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}) = \alpha\, \underset{\sim}{S}\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} = \alpha\lambda \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \in \mathcal{E}_1 .$$

Hence, $\mathcal{E}_1$ is an invariant linear subspace under $\underset{\sim}{S}$; and by Theorem A3.12.6, its orthogonal complement $\mathcal{E}_1^{\perp}$ also is invariant under $\underset{\sim}{S}$. By Exercise A2.2.1, and Definitions A2.2.2 and A2.2.3, $\mathcal{E}_1$ is 1-dimensional; then by Theorems A3.12.5 and A3.12.3, $\mathcal{E}_1^{\perp}$ is $(n-1)$-dimensional;

Next, let $\underset{\sim}{S}_1$ be the restriction of $\underset{\sim}{S}$ to $\mathcal{E}_1^{\perp}$.)[1] Since $\mathcal{E}_1^{\perp}$ is invariant under $\underset{\sim}{S}$, $\underset{\sim}{S}_1 \in Lin\, \mathcal{E}_1^{\perp}$. Obviously, the symmetry of $\underset{\sim}{S}$ carries over to $\underset{\sim}{S}_1$, so that $\underset{\sim}{S}_1 \in Sym\, \mathcal{E}_1^{\perp}$. Application of Lemma A3.13.1 to $\underset{\sim}{S}_1$ yields the existence of an eigenvector $\overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \in \mathcal{E}_1^{\perp}$. Again, $|\overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}| = 1$. Since $\overset{*}{\underset{\sim}{e}}_2 \in \mathcal{E}_1^{\perp}$ and $\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \in \mathcal{E}_1$, $\overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \cdot \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} = 0$. Of course, an eigenvector of $\underset{\sim}{S}_1$ is necessarily an eigenvector of $\underset{\sim}{S}$. Let $\mathcal{E}_2$ be the 2-dimensional linear subspace

$$\mathcal{E}_2 = Lsp\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}\}.$$

$\mathcal{E}_2$ is an invariant linear subspace under $\underset{\sim}{S}$; consequently, $\mathcal{E}_2^{\perp}$

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[1] I.e., $\underset{\sim}{S}_1$ is the function $\underset{\sim}{S}_1 : \mathcal{E}_1^{\perp} \to \mathcal{V}_n$ defined by $\underset{\sim}{S}_1 \underset{\sim}{u} = \underset{\sim}{S}\underset{\sim}{u}$.

is an $(n-2)$-dimensional invariant linear subspace under $\underset{\sim}{S}$.

We continue this procedure. Let $\underset{\sim}{S}_2$ be the restriction of $\underset{\sim}{S}$ to $\mathcal{E}_2^{\perp}$. Then $\underset{\sim}{S}_2 \in \operatorname{Sym} \mathcal{E}_2^{\perp}$, and $\underset{\sim}{S}_2$ has an unit eigenvector $\overset{*}{\underset{\sim}{e}}_{\langle 3\rangle} \in \mathcal{E}_2^{\perp}$. As an element of $\mathcal{E}_2^{\perp}$, $\overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}$ is orthogonal to all elements of $\mathcal{E}_2 = \operatorname{Lsp}\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}\}$; in particular, $\overset{*}{\underset{\sim}{e}}_{\langle 3\rangle} \cdot \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} = \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle} \cdot \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} = 0$.

After n such steps, we have an orthonormal set of eigenvectors $\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}, \ldots, \overset{*}{\underset{\sim}{e}}_{\langle n\rangle}\}$. By Theorems A2.2.8 and A2.3.14, $\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \ldots, \overset{*}{\underset{\sim}{e}}_{\langle n\rangle}\}$ is a basis for $\mathcal{V}_n$.

The components of $\underset{\sim}{S}$ relative to $\{\overset{*}{\underset{\sim}{e}}_{\langle i\rangle}\}$ are given by Theorem A3.2.2 as

$$\overset{*}{S}{}^{\langle ij\rangle} = \overset{*}{\underset{\sim}{e}}_{\langle i\rangle} \cdot \underset{\sim}{S}\, \overset{*}{\underset{\sim}{e}}_{\langle j\rangle} = \overset{*}{\underset{\sim}{e}}_{\langle i\rangle} \cdot (\lambda_j \overset{*}{\underset{\sim}{e}}_{\langle j\rangle}) = \lambda_j \delta_{ij} .$$

Thus,

$$[\overset{*}{S}{}^{\langle ij\rangle}] = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) . \quad \square$$

The terminology "spectral" theorem has its roots in

<u>Definition</u> A3.13.1. <u>The spectrum</u>[1] <u>of a second-order tensor</u> $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}$ <u>is the set of all eigenvalues of</u> $\underset{\sim}{T}$.

Since all of the eigenvalues of $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_n$ must be roots of the nth degree characteristic polynomial,[2] there is

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[1] Sometimes, point spectrum or discrete spectrum.

[2] Cf. Theorems A3.11.7 and A3.11.8.

no point in searching for more than n of them. Sometimes there will not even be n distinct eigenvalues because the characteristic polynomial could have multiple roots. Also it can happen that an eigenvalue will have several associated (nonparallel) eigenvectors, so there can be more than n eigenvectors. The important facts stated in the spectral theorem are that for a symmetric tensor on a finite-dimensional space

(i) $\exists$ an orthonormal basis of eigenvectors

and

(ii) all of the roots of the characteristic polynomial are real.

In view of the proof of the spectral theorem, the eigenvalues of $\underset{\sim}{S} \in \mathrm{Sym}\, \mathcal{V}_n$ enjoy certain extremal properties. More precisely, we have

Theorem A3.13.2. Order the n-eigenvalues[1] of $\underset{\sim}{S} \in \mathrm{Sym}\, \mathcal{V}_n$ according to

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n .$$

Then

$$\lambda_1 = \min_{|\underset{\sim}{u}|=1} (\underset{\sim}{u} \cdot \underset{\sim}{S}\underset{\sim}{u}) \quad \text{and} \quad \lambda_n = \max_{|\underset{\sim}{u}|=1} (\underset{\sim}{u} \cdot \underset{\sim}{S}\underset{\sim}{u}) .$$

Proof. From the proofs of Theorem A3.13.1 and Lemma

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[1] Of course, to get n eigenvalues, any multiple roots of the characteristic polynomial must be repeated an appropriate number of times.

A3.13.1, each eigenvalue is characterized by

$$\lambda = \min_{\mathcal{D}} (\underset{\sim}{u} \cdot \underset{\sim}{S} \underset{\sim}{u}),$$

but the domain $\mathcal{D}$ is cut down at each step. Thus, for λ_1, $\mathcal{D} = \Omega = \{\underset{\sim}{u} \in \mathcal{V}_n : |\underset{\sim}{u}| = 1\}$; for λ_2, $\mathcal{D} = \Omega \cap \mathcal{E}_1^{\perp}$; for λ_3, $\mathcal{D} = \Omega \cap \mathcal{E}_2^{\perp}$, etc. Hence, $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ and

$$\lambda_1 = \min_{|\underset{\sim}{u}|=1} (\underset{\sim}{u} \cdot \underset{\sim}{S} \underset{\sim}{u}).$$

To get

$$\lambda_n = \max_{|\underset{\sim}{u}|=1} (\underset{\sim}{u} \cdot \underset{\sim}{S} \underset{\sim}{u}),$$

observe that we could just as well have characterized the eigenvalues by the corresponding maximum problem. □

Since $S^{<ii>} = \underset{\sim}{e}_{<i>} \cdot \underset{\sim}{S}\, \underset{\sim}{e}_{<i>}$, Theorem A3.13.2 can be interpreted as asserting that λ_1 and λ_n are, respectively, the minimum and the maximum possible diagonal components of $\underset{\sim}{S}$ relative to all orthonormal bases.

The extremal characterization of the eigenvalues is sufficiently reminiscent of the operator norm[1] that the following result is not surprising.

[1] See Definition A3.4.4 and, especially, Theorem A3.4.6.

Theorem A3.13.3. Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ denote the spectrum of $\underset{\sim}{S} \in \mathrm{Sym}\, \mathcal{V}_n$. Then

$$\|\underset{\sim}{S}\| = \max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}.$$

Proof. By Theorem A3.4.6 (iii),

$$\|\underset{\sim}{S}\| = \underset{|\underset{\sim}{u}|=1}{\mathrm{lub}}\, |\underset{\sim}{S}\underset{\sim}{u}|,$$

which can be written as

$$\|\underset{\sim}{S}\| = \max_{|\underset{\sim}{u}|=1} |\underset{\sim}{S}\underset{\sim}{u}|$$

since the unit sphere in $\mathcal{V}_n$ is compact. Now

$$\begin{aligned} |\underset{\sim}{S}\underset{\sim}{u}|^2 &= (\underset{\sim}{S}\underset{\sim}{u})\cdot(\underset{\sim}{S}\underset{\sim}{u}) && (\text{definition of } |\cdot| \text{ in } \mathcal{V}_n) \\ &= \underset{\sim}{u}\cdot\underset{\sim}{S}^{T}(\underset{\sim}{S}\underset{\sim}{u}) && (\text{definition of transpose}) \\ &= \underset{\sim}{u}\cdot\underset{\sim}{S}(\underset{\sim}{S}\underset{\sim}{u}) && (\text{symmetry of } \underset{\sim}{S}) \\ &= \underset{\sim}{u}\cdot\underset{\sim}{S}^{2}\underset{\sim}{u} && (\text{definition of tensor multiplication}). \end{aligned}$$

Hence,

$$\begin{aligned} \max_{|\underset{\sim}{u}|=1} |\underset{\sim}{S}\underset{\sim}{u}| &= \max_{|\underset{\sim}{u}|=1} \sqrt{\underset{\sim}{u}\cdot\underset{\sim}{S}^2\underset{\sim}{u}} && (\text{substitution}) \\ &= \sqrt{\max_{|\underset{\sim}{u}|=1} \underset{\sim}{u}\cdot\underset{\sim}{S}^2\underset{\sim}{u}} && (\text{monotonicity of } \sqrt{\cdot}\,) \end{aligned}$$

$$= \sqrt{\mu_n} \quad , \qquad \text{(Theorem A3.13.2)}$$

where $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ is the ordered spectrum of $\underset{\sim}{S}^2$. By Theorem A3.11.4 and the fact that a symmetric tensor on $\mathcal{V}_n$ has no more than n distinct eigenvalues,

$$\{\mu_1, \mu_2, \cdots, \mu_n\} = \{\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2\}.$$

Since some or all of the eigenvalues of $\underset{\sim}{S}$ can be negative, the largest eigenvalue of $\underset{\sim}{S}^2$ need not correspond to the (algebraically) largest eigenvalue of $\underset{\sim}{S}$; however, we can say that

$$\sqrt{\mu_n} = \max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}.$$

Hence,

$$\|\underset{\sim}{S}\| = \max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}. \quad \square$$

The remainder of this section is restricted to the important 3-dimensional case.

Theorem A3.13.4. The fundamental invariants of $\underset{\sim}{S} \in \text{Sym}\,\mathcal{V}_3$ can be expressed in terms of its eigenvalues $\lambda_1, \lambda_2, \lambda_3$)[1] as

$$\iota_1(\underset{\sim}{S}) = \lambda_1 + \lambda_2 + \lambda_3, \quad \iota_2(\underset{\sim}{S}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad \iota_3(\underset{\sim}{S}) = \lambda_1\lambda_2\lambda_3 .$$

Proof. Exercise A3.13.5. □

The next theorem delineates the extent to which eigenvector bases of symmetric tensors on $\mathcal{V}_3$ are unique.

Theorem A3.13.5. Let λ_1, λ_2, and λ_3)[1] denote the eigenvalues of $\underset{\sim}{S} \in \text{Sym}\,\mathcal{V}_3$, and let $\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}\}$ be an associated eigenvector basis with $(\lambda_i, \overset{*}{\underset{\sim}{e}}_{\langle i\rangle})$, $i \in \mathbb{I}_3$ being eigenpairs.

(i) (Distinct Eigenvalues) If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, then (for each $i \in \mathbb{I}_3$) there are only two normalized eigenvectors corresponding to λ_i, namely, $\overset{*}{\underset{\sim}{e}}_{\langle i\rangle}$ and $-\overset{*}{\underset{\sim}{e}}_{\langle i\rangle}$; the corresponding eigenspaces are the three mutually perpendicular "lines" $\mathcal{E}_i = \text{Lsp}\,\{\overset{*}{\underset{\sim}{e}}_{\langle i\rangle}\}$; and the eigenvector basis $\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}\}$ is unique except for the sense of the vectors.

(ii) (Exactly Two Distinct Eigenvalues) If, say, $\lambda_1 \neq \lambda_2 = \lambda_3$, then there are only two normalized eigenvectors corresponding to λ_1, namely, $\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}$ and $-\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}$; all unit vectors perpendicular to $\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}$ are normalized eigenvectors corresponding to λ_2; the

[1] Cf. the footnote on p. A3.13.6.

corresponding eigenspaces are the line $\mathcal{E}_1 = \text{Lsp}\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}\}$ and the plane $\mathcal{E}_2 = \mathcal{E}_1^{\perp} = \text{Lsp}\{\overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}\}$; any orthonormal basis containing $\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}$ or $-\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}$ is an eigenvector basis; and $\underset{\sim}{S}$ has the form

$$\underset{\sim}{S} = \lambda_1 \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \lambda_2 \left(\underset{\sim}{I} - \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \right).$$

(iii) (*All Eigenvalues Equal*) If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then every nonzero vector is an eigenvector corresponding to λ; the corresponding eigenspace is the "whole space" $\mathcal{V}_3$; any orthonormal basis for $\mathcal{V}_3$ is an eigenvector basis; and $\underset{\sim}{S}$ has the *isotropic* form[1]

$$\underset{\sim}{S} = \lambda \underset{\sim}{I}.$$

Proof. (i): Suppose $\{\hat{\underset{\sim}{e}}_{\langle 1\rangle}, \hat{\underset{\sim}{e}}_{\langle 2\rangle}, \hat{\underset{\sim}{e}}_{\langle 3\rangle}\}$ is another eigenvector basis of $\underset{\sim}{S}$. Then by Theorem A3.11.5,

$$\hat{\underset{\sim}{e}}_{\langle i\rangle} \cdot \overset{*}{\underset{\sim}{e}}_{\langle j\rangle} = 0 \quad \text{for } i \neq j.$$

E.g., $\hat{\underset{\sim}{e}}_{\langle 1\rangle}$ is orthogonal to $\overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}$ and $\overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}$. Then $\hat{\underset{\sim}{e}}_{\langle 1\rangle}$ and $\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}$ are both unit normals to the plane of $\overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}$ and $\overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}$. Consequently, by Theorem A2.8.3, $\hat{\underset{\sim}{e}}_{\langle 1\rangle} = \pm \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}$. In the same manner, $\hat{\underset{\sim}{e}}_{\langle 2\rangle} = \pm \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}$ and $\hat{\underset{\sim}{e}}_{\langle 3\rangle} = \pm \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}$.

The eigenspace corresponding to λ_1 is

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[1] Often, *spherical* form.

$$\mathcal{E}_1 = \{ \underset{\sim}{u} \in \mathcal{V}_n : \underset{\sim}{S}\underset{\sim}{u} = \lambda_1 \underset{\sim}{u} \} .$$

Thus, for $\underset{\sim}{u} \in \mathcal{E}_1$ $\exists$ two possibilities :

$\underset{\sim}{u} = \underset{\sim}{0}$, and $\underset{\sim}{u} \in Lsp\{\overset{*}{\underset{\sim}{e}}_{<1>}\}$;

$\underset{\sim}{u} \neq \underset{\sim}{0}$, and $\underset{\sim}{u}$ is an eigenvector corresponding to λ_1 .

In the later case,

$$\underset{\sim}{u} = \overset{*}{u}{}^{<i>} \overset{*}{\underset{\sim}{e}}_{<i>} \quad (\{\overset{*}{\underset{\sim}{e}}_{<i>}\} \text{ is a basis for } \mathcal{V}_3)$$

$$= \sum_{i=1}^{3} \left(\overset{*}{\underset{\sim}{e}}_{<i>} \cdot \underset{\sim}{u} \right) \overset{*}{\underset{\sim}{e}}_{<i>} \quad (\text{Theorem A2.3.16})$$

$$= \left(\overset{*}{\underset{\sim}{e}}_{<1>} \cdot \underset{\sim}{u} \right) \overset{*}{\underset{\sim}{e}}_{<1>} \quad \left(\overset{*}{\underset{\sim}{e}}_{<2>} \cdot \underset{\sim}{u} = \overset{*}{\underset{\sim}{e}}_{<3>} \cdot \underset{\sim}{u} = 0 \text{ because eigenvectors corresponding to distinct eigenvalues are orthogonal (Theorem A3.11.6)} \right) ;$$

and $\underset{\sim}{u} \in Lsp\{\overset{*}{\underset{\sim}{e}}_{<1>}\}$. $\therefore$ $\mathcal{E}_1 \subset Lsp\{\overset{*}{\underset{\sim}{e}}_{<1>}\}$. Trivially, $Lsp\{\overset{*}{\underset{\sim}{e}}_{<1>}\} \subset \mathcal{E}_1$; so we have

$$\mathcal{E}_1 = Lsp\{\overset{*}{\underset{\sim}{e}}_{<1>}\} .$$

Similarly, $\mathcal{E}_i = \{ \underset{\sim}{u} \in \mathcal{V}_n : \underset{\sim}{S}\underset{\sim}{u} = \lambda_i \underset{\sim}{u} \} = Lsp\{\overset{*}{\underset{\sim}{e}}_{<i>}\}$, $i = 2, 3$.

(ii): By the proof of (i), $\overset{*}{\underset{\sim}{e}}_{<1>}$ is unique to within sense and $\mathcal{E}_1 = Lsp\{\overset{*}{\underset{\sim}{e}}_{<1>}\}$.

Since $\lambda_2 = \lambda_3$, the spectral resolution becomes

$$\underset{\sim}{S} = \lambda_1 \overset{*}{\underset{\sim}{e}}_{<1>} \otimes \overset{*}{\underset{\sim}{e}}_{<1>} + \lambda_2 \left(\overset{*}{\underset{\sim}{e}}_{<2>} \otimes \overset{*}{\underset{\sim}{e}}_{<2>} + \overset{*}{\underset{\sim}{e}}_{<3>} \otimes \overset{*}{\underset{\sim}{e}}_{<3>} \right).$$

To reduce this further, we note that since $(1, \underset{\sim}{u})$ is an eigenpair of $\underset{\sim}{I}$ for any $\underset{\sim}{u} \neq \underset{\sim}{0}$, the identity tensor has the spectral resolution

$$\underset{\sim}{I} = \overset{*}{\underset{\sim}{e}}_{<1>} \otimes \overset{*}{\underset{\sim}{e}}_{<1>} + \overset{*}{\underset{\sim}{e}}_{<2>} \otimes \overset{*}{\underset{\sim}{e}}_{<2>} + \overset{*}{\underset{\sim}{e}}_{<3>} \otimes \overset{*}{\underset{\sim}{e}}_{<3>} \ .)^1$$

$\therefore$

$$\overset{*}{\underset{\sim}{e}}_{<2>} \otimes \overset{*}{\underset{\sim}{e}}_{<2>} + \overset{*}{\underset{\sim}{e}}_{<3>} \otimes \overset{*}{\underset{\sim}{e}}_{<3>} = \underset{\sim}{I} - \overset{*}{\underset{\sim}{e}}_{<1>} \otimes \overset{*}{\underset{\sim}{e}}_{<1>},$$

and

$$\underset{\sim}{S} = \lambda_1 \overset{*}{\underset{\sim}{e}}_{<1>} \otimes \overset{*}{\underset{\sim}{e}}_{<1>} + \lambda_2 \left(\underset{\sim}{I} - \overset{*}{\underset{\sim}{e}}_{<1>} \otimes \overset{*}{\underset{\sim}{e}}_{<1>} \right).$$

Then for any $\underset{\sim}{u} \in \mathcal{V}_3 \ni \underset{\sim}{u} \cdot \overset{*}{\underset{\sim}{e}}_{<1>} = 0$ (i.e., for any $\underset{\sim}{u} \in \mathcal{E}_1^{\perp}$),

$$\begin{aligned}
\underset{\sim}{S}\underset{\sim}{u} &= \left(\lambda_1 \overset{*}{\underset{\sim}{e}}_{<1>} \otimes \overset{*}{\underset{\sim}{e}}_{<1>} + \lambda_2 \left(\underset{\sim}{I} - \overset{*}{\underset{\sim}{e}}_{<1>} \otimes \overset{*}{\underset{\sim}{e}}_{<1>} \right) \right) \underset{\sim}{u} \\
&= \lambda_1 (\underset{\sim}{u} \cdot \overset{*}{\underset{\sim}{e}}_{<1>}) \overset{*}{\underset{\sim}{e}}_{<1>} + \lambda_2 \underset{\sim}{u} - \lambda_2 (\underset{\sim}{u} \cdot \overset{*}{\underset{\sim}{e}}_{<1>}) \underset{\sim}{u} \quad (\text{defns. of } \underset{\sim}{v} \otimes \underset{\sim}{w} \text{ and } \underset{\sim}{I}) \\
&= \lambda_2 \underset{\sim}{u} \quad (\underset{\sim}{u} \in \mathcal{E}_1^{\perp}).
\end{aligned}$$

In particular, any nonzero vector orthogonal to $\overset{*}{\underset{\sim}{e}}_{<1>}$ is an eigenvector corresponding to λ_2, and the eigenspace corresponding to λ_2 is $\mathcal{E}_2 = \mathcal{E}_1^{\perp}$. The demonstration that

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[1] This important fact can also be read off from Theorem A3.2.7.

$$\mathcal{E}_1^{\perp} = \left(Lsp\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}\}\right)^{\perp} = Lsp\{\overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}\}$$

is left as *Exercise* A3.13.6.

(iii): In this case the spectral resolution collapses to

$$\underset{\sim}{S} = \lambda\left(\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}\right)$$

$$= \lambda \underset{\sim}{I}\,.$$

Then for any $\underset{\sim}{u} \in \mathcal{V}_3$,

$$\underset{\sim}{S}\underset{\sim}{u} = \lambda \underset{\sim}{u}\,,$$

In particular, any nonzero vector is an eigenvector corresponding to λ, and the eigenspace corresponding to λ is $\mathcal{V}_3$. $\square$

A useful analog of Theorem A3.13.5 is:

Theorem A3.13.6. Let $\underset{\sim}{S} \in Sym\,\mathcal{V}_3$ have spectrum $\{\lambda_1, \lambda_2, \lambda_3\}$ and denote the corresponding eigenspaces of $\underset{\sim}{S}$ by $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$. Then we have the following decomposition of the underlying linear space:

(i) if $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, then $\mathcal{V}_3 = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3$;)[1]

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[1] The repeated direct sum is defined by the natural extension of Definition A3.12.1.

(ii) *if* $\lambda_1 \neq \lambda_2 = \lambda_3$, *then* $\mathcal{V}_3 = \mathcal{E}_1 \oplus \mathcal{E}_2$;

(iii) *if* $\lambda_1 = \lambda_2 = \lambda_3$, *then* $\mathcal{V}_3 = \mathcal{E}_1$.

Proof. (i) $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$. In this case, it follows from Theorem A3.13.5 that $\mathcal{E}_i = Lsp\{\overset{*}{\underset{\sim}{e}}_{\langle i\rangle}\}$, where $\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}\}$ is an orthonormal basis of eigenvectors of $\underset{\sim}{S}$. Let $\underset{\sim}{u} \in \mathcal{V}_3$. Then since $\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}\}$ is a basis for $\mathcal{V}_3$, $\exists$ unique reals $\overset{*}{u}_{\langle i\rangle} \ni$

$$\underset{\sim}{u} = \overset{*}{u}_{\langle 1\rangle}\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \overset{*}{u}_{\langle 2\rangle}\overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \overset{*}{u}_{\langle 3\rangle}\overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}. \text{ [1]}$$

Then since $\overset{*}{u}_{\langle i\rangle}\overset{*}{\underset{\sim}{e}}_{\langle i\rangle} \in Lsp\{\overset{*}{\underset{\sim}{e}}_{\langle i\rangle}\} = \mathcal{E}_i$, $\underset{\sim}{u} \in \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3$. Thus, $\mathcal{V}_3 \subset \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3$, but necessarily $\mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3 \subset \mathcal{V}_3$; $\therefore$ $\mathcal{V}_3 = \mathcal{E}_1 \oplus \mathcal{E}_2 \oplus \mathcal{E}_3$.

The proofs for cases (ii) and (iii) are left as *Exercise* A3.13.7. $\square$

Supplementary Reading

Same as in $\S$ A3.5 plus

COURANT and HILBERT, Methods of Mathematical Physics, Vol. I

INDRITZ, Methods in Analysis.

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[1] See Theorem A2.2.11.

A3.14. The Hamilton - Cayley Theorem

As usual, $\mathcal{V}_n$ denotes an n-dimensional inner product space throughout this section. In view of Theorems A3.11.4 and 5, the following two corollaries of the spectral theorem are not surprising.

Theorem A3.14.1. Let $\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}, \cdots, \overset{*}{\underset{\sim}{e}}_{\langle n\rangle}\}$ be an eigenvector basis of $\underset{\sim}{S} \in \mathrm{Sym}\,\mathcal{V}_n$, and let $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the associated eigenvalues. Then for any $m \in \{1, 2, \cdots\}$

$$\underset{\sim}{S}^m = \lambda_1^m \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \lambda_2^m \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \cdots + \lambda_n^m \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle}.$$

Proof. The proof is by induction on m. The theorem is obviously true for $m=1$; because that case is simply the spectral resolution

$$\underset{\sim}{S} = \lambda_1 \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \lambda_2 \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \cdots + \lambda_n \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle}.$$

Suppose that the theorem holds for m. Then

$$\underset{\sim}{S}^{m+1} = \underset{\sim}{S}\,\underset{\sim}{S}^m = \left(\lambda_1 \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \cdots + \lambda_n \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \underset{\sim}{e}_{\langle n\rangle}\right)\left(\lambda_1^m \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \cdots + \lambda_n^m \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle}\right)$$

$$= \lambda_1^{m+1} \left(\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}\right)\left(\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}\right) + \lambda_1 \lambda_2^m \left(\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}\right)\left(\overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}\right)$$

$$+ \cdots + \lambda_n^{m+1} \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \qquad \text{(Theorem A3.5.1)}$$

$$= \lambda_1^{m+1} \left(\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \cdot \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}\right) \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \lambda_1 \lambda_2^m \left(\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \cdot \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}\right) \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}$$

$$+ \cdots + \lambda_n^{m+1} \left(\overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \cdot \overset{*}{\underset{\sim}{e}}_{\langle n\rangle}\right) \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \qquad \text{(Theorem A3.5.3)}$$

$$= \lambda_1^{m+1} \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \cdots + \lambda_n^{m+1} \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \quad (\text{orthonormality of } \{\overset{*}{\underset{\sim}{e}}_{\langle i\rangle}\}).$$

Thus, if the theorem holds for m, it holds $m+1$. □

Theorem A3.14.2. Let $\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}, \ldots, \overset{*}{\underset{\sim}{e}}_{\langle n\rangle}\}$ be an eigenvector basis of $\underset{\sim}{S} \in \mathrm{Sym}\,\mathcal{V}_n$, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the associated eigenvalues. If $\underset{\sim}{S}$ is invertible, then $\lambda_i \neq 0 \; \forall \; i \in \mathbb{I}_n$ and

$$\underset{\sim}{S}^{-1} = \lambda_1^{-1} \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \lambda_2^{-1} \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \cdots + \lambda_n^{-1} \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle}.$$

Proof. Exercise A3.14.1. □

We are now in position to give an easy proof to a limited version of the Hamilton-Cayley theorem, which is a major technical tool of modern continuum mechanics.

Theorem A3.14.3. (Hamilton-Cayley Theorem) A second-order tensor $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n$ satisfies its own characteristic equation; e.g., when $n=3$,

$$-\underset{\sim}{T}^3 + \iota_1(\underset{\sim}{T})\,\underset{\sim}{T}^2 - \iota_2(\underset{\sim}{T})\,\underset{\sim}{T} + \iota_3(\underset{\sim}{T})\,\underset{\sim}{I} = \underset{\sim}{O},$$

Proof. The proof given here is for symmetric tensors on $\mathcal{V}_3$, but the theorem is valid as stated.

By Theorem A3.14.1, for $\{\overset{*}{\underset{\sim}{e}}_{\langle 1\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle}, \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle}\}$ an eigenvector basis of $\underset{\sim}{T} \in \mathrm{Sym}\,\mathcal{V}_3$ with $\lambda_1, \lambda_2, \lambda_3$ the corresponding eigenvalues,

$$\underset{\sim}{T} = \lambda_1 \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \lambda_2 \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \lambda_3 \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle},$$

$$\underset{\sim}{T}^2 = \lambda_1^2 \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \lambda_2^2 \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \lambda_3^2 \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle},$$

$$\underset{\sim}{T}^3 = \lambda_1^3 \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \lambda_2^3 \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \lambda_3^3 \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle};$$

and, of course,

$$\underset{\sim}{I} = \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 3\rangle},$$

$\therefore$ using the linear space properties of $\mathrm{Sym}\,\mathcal{V}_3$, we calculate

$$-\underset{\sim}{T}^3 + \iota_1(\underset{\sim}{T})\underset{\sim}{T}^2 - \iota_2(\underset{\sim}{T})\underset{\sim}{T} + \iota_3(\underset{\sim}{T})\underset{\sim}{I} =$$

$$= \sum_{i=1}^{3} \left(-\lambda_i^3 + \iota_1(\underset{\sim}{T})\lambda_i^2 - \iota_2(\underset{\sim}{T})\lambda_i + \iota_3(\underset{\sim}{T})\right) \overset{*}{\underset{\sim}{e}}_{\langle i\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle i\rangle}$$

$= 0$ (each eigenvalue satisfies the characteristic equation[1]). $\square$

The following occasionally useful result is a corollary of the Hamilton–Cayley theorem. As a matter of mathematical integrity, one should keep in mind that even though this result is stated for elements of $\mathrm{Lin}\,\mathcal{V}_3$, our proof of the Hamilton–Cayley theorem was restricted to elements of $\mathrm{Sym}\,\mathcal{V}_3$.

Theorem A3.14.4. Let $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_3$. Then

$$\iota_3(\underset{\sim}{T}) = \frac{1}{6}\left[(\mathrm{tr}\,\underset{\sim}{T})^3 - 3(\mathrm{tr}\,\underset{\sim}{T})\,\mathrm{tr}(\underset{\sim}{T}^2) + 2\,\mathrm{tr}(\underset{\sim}{T}^3)\right].$$

---

[1] Theorem A3.11.8.

In addition, if $\underset{\sim}{T}$ is invertible, then

$$\underset{\sim}{T}^{-1} = \frac{1}{\det \underset{\sim}{T}} \left[\underset{\sim}{T}^2 - \iota_1(\underset{\sim}{T})\underset{\sim}{T} + \iota_2(\underset{\sim}{T})\, \underset{=}{I} \right]$$

and

$$\iota_2(T) = \operatorname{tr}\left[(\det \underset{\sim}{T})\, \underset{\sim}{T}^{-1} \right].$$

Proof. Exercise 3.14.2. $\square$

One consequence of the above theorem is that we now have expressions for the fundamental invariants of $\underset{\sim}{T} \in \operatorname{Lin} \mathcal{V}_3$ as polynomials in traces of powers of $\underset{\sim}{T}$.

Supplementary Reading

Same as for § 3.5

A3.15. Positive Definite Tensors. Tensor Square Roots. The Polar Decomposition Theorem

The polar decomposition theorem is a major tool of modern continuum mechanics. In order to state and prove it, we need the notion of positive definite tensors and their square roots.

Definition A3.15.1. Let $\mathcal{V}$ be an inner product space. Then a symmetric tensor $\underset{\sim}{S} \in \mathrm{Sym}\,\mathcal{V}$ is positive definite if

$$\underset{\sim}{u} \cdot \underset{\sim}{S}\underset{\sim}{u} \geq 0 \quad \forall \underset{\sim}{u} \in \mathcal{V} \text{ with } \underset{\sim}{u} \cdot \underset{\sim}{S}\underset{\sim}{u} = 0 \text{ iff } \underset{\sim}{u} = \underset{\sim}{0}\,.)^1$$

The reason that the definition of positive definite is restricted to symmetric tensors is explained by the following exercise.

Exercise A3.15.1. Let $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}$, where $\mathcal{V}$ is an inner product space, have a transpose. Show that $\forall\, \underset{\sim}{u} \in \mathcal{V}$

$$\underset{\sim}{u} \cdot \underset{\sim}{T}\underset{\sim}{u} = \underset{\sim}{u} \cdot (\mathrm{sym}\,\underset{\sim}{T})\underset{\sim}{u}\,.$$

The following notation is often convenient.

Definition A3.15.2. Let $\mathcal{V}$ be an inner product space.

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[1] Of course, trivially, $\underset{\sim}{u} = \underset{\sim}{0} \Rightarrow \underset{\sim}{u} \cdot \underset{\sim}{S}\underset{\sim}{u} = 0$.

Then

$$\text{Psym } \mathcal{V} = \{ \underset{\sim}{S} \in \text{Sym } \mathcal{V} : \underset{\sim}{S} \text{ is positive definite} \}.$$

Exercise A3.15.2. Is Psym $\mathcal{V}$ a linear subspace of Sym $\mathcal{V}$?

Most of the rest of our deliberations in this section are limited to the finite-dimensional case. As usual, $\mathcal{V}_n$ denotes an n-dimensional inner product space for the remainder of this section.

Theorem A3.15.1. A symmetric tensor on a finite-dimensional inner product space is positive definite iff each of its eigenvalues is strictly positive (i.e., > 0).

Proof. Suppose $\underset{\sim}{S} \in \text{Psym } \mathcal{V}_n$. Let $(\lambda, \underset{\sim}{u})$ be an eigenpair of $\underset{\sim}{S}$. Then $\underset{\sim}{u} \neq \underset{\sim}{0}$ and $\underset{\sim}{S}\underset{\sim}{u} = \lambda \underset{\sim}{u} \Rightarrow$

$$\lambda = \frac{\underset{\sim}{u} \cdot \underset{\sim}{S}\underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}} > 0. \quad)^1$$

Conversely, suppose that each eigenvalue of $\underset{\sim}{S} \in \text{Sym } \mathcal{V}_n$ is strictly positive. By the spectral theorem (Theorem A3.13.1), $\underset{\sim}{S}$ has the spectral resolution

$$\underset{\sim}{S} = \lambda_1 \overset{*}{\underset{\sim}{e}}_{\langle 1 \rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1 \rangle} + \lambda_2 \overset{*}{\underset{\sim}{e}}_{\langle 2 \rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2 \rangle} + \cdots + \lambda_n \overset{*}{\underset{\sim}{e}}_{\langle n \rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n \rangle}.$$

Then

$$\underset{\sim}{S}\underset{\sim}{u} = \sum_{i=1}^{n} \lambda_i (\overset{*}{\underset{\sim}{e}}_{\langle i \rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle i \rangle}) \underset{\sim}{u}$$ (defns. of sums and scalar multiples of second-order tensors)

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[1] In this direction, we do not need the restriction to finite dimensions.

$$= \sum_{i=1}^{n} \lambda_i \left(\underset{\sim}{u} \cdot \overset{*}{\underset{\sim}{e}}_{\langle i \rangle} \right) \overset{*}{\underset{\sim}{e}}_{\langle i \rangle} \qquad (\text{defn. of } \underset{\smile}{v} \otimes \underset{\sim}{w}),$$

and

$$\underset{\sim}{u} \cdot \underset{\sim}{S}\, \underset{\sim}{u} = \sum_{i=1}^{n} \lambda_i \left(\underset{\sim}{u} \cdot \overset{*}{\underset{\sim}{e}}_{\langle i \rangle} \right)^2 \quad (\text{linearity and symmetry of inner product})$$

$$\geq 0 \quad \forall\, \underset{\sim}{u} \in \mathcal{V}_n \quad (\lambda_i > 0 \ \forall\, i \in \mathbb{I}_n).$$

Moreover,

$$\underset{\sim}{u} \cdot \underset{\sim}{S}\, \underset{\sim}{u} = 0 \Rightarrow \underset{\sim}{u} \cdot \overset{*}{\underset{\sim}{e}}{}^{\langle i \rangle} = 0 \quad \forall\, i \in \mathbb{I}_n$$
$$\Rightarrow \underset{\sim}{u} = \underset{\sim}{0} \quad (\text{Theorems A2.3.16 and A2.2.12}). \square$$

A useful corollary of Theorem A3.15.1 is

Theorem A3.15.2. <u>If a symmetric tensor on a finite-dimensional inner product space is positive definite, then it is invertible.</u>

Proof. Exercise A3.15.3. (Hint: $\det \underset{\sim}{S} = \lambda_1 \lambda_2 \cdots \lambda_n$). $\square$

Positive definite tensors are like positive numbers in that they have square roots. More precisely, we have

Theorem A3.15.3. <u>Let</u> $\underset{\sim}{S} \in \mathrm{Psym}\, \mathcal{V}_n$. <u>Then</u> $\exists$ <u>a unique</u> $\underset{\sim}{U} \in \mathrm{Psym}\, \mathcal{V}_n \ni$

$$\underset{\sim}{U}^2 = \underset{\sim}{S}.$$

<u>$\underset{\sim}{U}$ is called the</u> positive definite square root of $\underset{\sim}{S}$, <u>and we write</u>

$$\underset{\sim}{U} = \sqrt{\underset{\sim}{S}}\ .$$

Proof. Suppose that $\underset{\sim}{U}$ is a positive definite square root of $\underset{\sim}{S}$; i.e., $\underset{\sim}{U} \in \operatorname{Psym} \mathcal{V}_n$ and $\underset{\sim}{U}^2 = \underset{\sim}{S}$. Let $(\lambda, \underset{\sim}{e})$ be an eigenpair of $\underset{\sim}{S}$. Then

$$\underset{\sim}{0} = (\underset{\sim}{S} - \lambda \underset{\sim}{I})\underset{\sim}{e} \quad \text{(definition of eigenpair)}$$

$$= (\underset{\sim}{U}^2 - \lambda \underset{\sim}{I})\underset{\sim}{e} \quad \text{(substitution)}$$

$$= (\underset{\sim}{U} + \sqrt{\lambda}\, \underset{\sim}{I})(\underset{\sim}{U} - \sqrt{\lambda}\, \underset{\sim}{I})\underset{\sim}{e} \quad \text{(standard properties of tensor multiplication)}.$$[1]

Call

$$(\underset{\sim}{U} - \sqrt{\lambda}\, \underset{\sim}{I})\underset{\sim}{e} = \hat{\underset{\sim}{e}}$$

to get

$$(\underset{\sim}{U} + \sqrt{\lambda}\, \underset{\sim}{I})\hat{\underset{\sim}{e}} = \underset{\sim}{0}.$$

Then we see that $\hat{\underset{\sim}{e}} = \underset{\sim}{0}$; for if not, $(-\sqrt{\lambda}, \hat{\underset{\sim}{e}})$ is an eigenpair of $\underset{\sim}{U}$ — contradicting the fact that all the eigenvalues of $\underset{\sim}{U}$ must be strictly positive. $\therefore$ we are left with the conclusion that

$$(\underset{\sim}{U} - \sqrt{\lambda}\, \underset{\sim}{I})\underset{\sim}{e} = \underset{\sim}{0};$$

i.e., $(\sqrt{\lambda}, \underset{\sim}{e})$ is an eigenpair of $\underset{\sim}{U}$.

Consequently, if $\{\overset{*}{\underset{\sim}{e}}_{\langle i \rangle}\}$ is an eigenvector basis of $\underset{\sim}{S}$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then $\{\overset{*}{\underset{\sim}{e}}_{\langle i \rangle}\}$ is an eigenvector basis of $\underset{\sim}{U}$ corresponding to the eigenvalues $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n}$; and we have the spectral resolutions

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[1] $\sqrt{\lambda}$ makes sense since $\lambda > 0$ by Theorem A3.15.1.

$$\underset{\sim}{S} = \lambda_1 \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \lambda_2 \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \cdots + \lambda_n \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle}$$

and

$$\underset{\sim}{U} = \sqrt{\lambda_1}\, \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \sqrt{\lambda_2}\, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \cdots + \sqrt{\lambda_n}\, \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} .$$

Thus, if $\underset{\sim}{U}$ is a positive definite square root of $\underset{\sim}{S}$, it must have the above form. This establishes the uniqueness. That there is some arbitrariness in the eigenvector basis $\{\overset{*}{\underset{\sim}{e}}_{\langle i\rangle}\}$ of $\underset{\sim}{S}$ is not a concern here. If $\hat{\underset{\sim}{U}}$ were another positive definite square root of $\underset{\sim}{S}$, the above argument would lead us to

$$\hat{\underset{\sim}{U}} = \sqrt{\lambda_1}\, \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \sqrt{\lambda_2}\, \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \cdots + \sqrt{\lambda_n}\, \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} ;$$

i.e., $\hat{\underset{\sim}{U}} = \underset{\sim}{U}$.

It remains to show that the $\underset{\sim}{U}$ deduced above has the requisite properties. First of all, $\underset{\sim}{U}$ is symmetric by Theorem A3.6.3; and by Theorem A3.15.1, it is positive definite. Finally, by Theorem A3.14.1,

$$\underset{\sim}{U}^2 = \lambda_1 \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 1\rangle} + \lambda_2 \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle 2\rangle} + \cdots + \lambda_n \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} \otimes \overset{*}{\underset{\sim}{e}}_{\langle n\rangle} = \underset{\sim}{S} . \quad \square$$

This completes our preparation for the polar decomposition theorem. We have chosen to state it in symbols that are traditional in the continuum mechanics context in which it arises.

Theorem A3.15.4. (Polar Decomposition Theorem) Let $\underset{\sim}{F} \in \text{Lin}\,\mathcal{V}_n$ with $\det \underset{\sim}{F} \neq 0$. Then $\exists$ unique $\underset{\sim}{U}, \underset{\sim}{V} \in \text{Psym}\,\mathcal{V}_n$ and a unique $\underset{\sim}{R} \in \text{Orth}\,\mathcal{V}_n \ni \underset{\sim}{F}$ admits the polar decompositions

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} = \underset{\sim}{V}\,\underset{\sim}{R}.$$

Moreover, $\det \underset{\sim}{R} = +1$ or -1 according as $\det \underset{\sim}{F} > 0$ or $\det \underset{\sim}{F} < 0$.

Proof. Suppose that $\exists\ \underset{\sim}{U} \in \text{Psym}\,\mathcal{V}_n$ and $\underset{\sim}{R} \in \text{Orth}\,\mathcal{V}_n \ni \underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U}$. Then

$$\begin{aligned}\underset{\sim}{F}^T &= \underset{\sim}{U}^T \underset{\sim}{R}^T \quad (\text{Theorem A3.6.3 (iv)})\\ &= \underset{\sim}{U}\, R^T \quad (\underset{\sim}{U} \in \text{Psym}\,\mathcal{V}_n),\end{aligned}$$

and

$$\begin{aligned}\underset{\sim}{F}^T \underset{\sim}{F} &= (\underset{\sim}{U}\,\underset{\sim}{R}^T)(\underset{\sim}{R}\,\underset{\sim}{U}) \quad (\text{substitution})\\ &= \underset{\sim}{U}(\underset{\sim}{R}^T\underset{\sim}{R})\underset{\sim}{U} \quad (\text{associativity of tensor multiplication})\\ &= \underset{\sim}{U}\,\underset{\sim}{I}\,\underset{\sim}{U} \quad (\underset{\sim}{R} \in \text{Orth}\,\mathcal{V}_n)\\ &= \underset{\sim}{U}^2 \quad (\underset{\sim}{I} \text{ is neutral element for multiplication}).\end{aligned}$$

Then if $\underset{\sim}{F}^T\underset{\sim}{F} \in \text{Psym}\,\mathcal{V}_n$, $\underset{\sim}{U} = \sqrt{\underset{\sim}{F}^T\underset{\sim}{F}}$; and $\underset{\sim}{U}$ is unique. Then $\underset{\sim}{F} = R\underset{\sim}{U} \Rightarrow \underset{\sim}{R} = \underset{\sim}{F}\underset{\sim}{U}^{-1}$, so $\underset{\sim}{R}$ is unique; and $\underset{\sim}{F} = \underset{\sim}{V}\,\underset{\sim}{R} \Rightarrow \underset{\sim}{V} = F\underset{\sim}{R}^{-1}$, so $\underset{\sim}{V}$ is unique. These observations suggest the following development.

Define the tensor $\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}$. Then

$$\begin{aligned}\underset{\sim}{C}^T &= \underset{\sim}{F}^T(\underset{\sim}{F}^T)^T \quad (\text{Theorem A3.6.3 (iv)})\\ &= \underset{\sim}{F}^T\underset{\sim}{F} \quad (\text{Theorem A3.6.3 (i)}).\end{aligned}$$

Thus, $\underset{\sim}{C}$ is symmetric. It is also positive definite as can be seen by considering

$$\underset{\sim}{u}\cdot\underset{\sim}{C}\underset{\sim}{u} = \underset{\sim}{u}\cdot[(\underset{\sim}{F}^T\underset{\sim}{F})\underset{\sim}{u}] \qquad \text{(substitution)}$$

$$= \underset{\sim}{u}\cdot[\underset{\sim}{F}^T(\underset{\sim}{F}\underset{\sim}{u})] \qquad \text{(definition of tensor multiplication)}$$

$$= (\underset{\sim}{F}\underset{\sim}{u})\cdot(\underset{\sim}{F}^T)^T\underset{\sim}{u} \qquad \text{(definition of transpose)}$$

$$= (\underset{\sim}{F}\underset{\sim}{u})\cdot(\underset{\sim}{F}\underset{\sim}{u}) \qquad \text{(} (\underset{\sim}{F}^T)^T = \underset{\sim}{F} \text{ by Theorem A3.6.3)}$$

$$\geq 0 \text{ with } = 0 \text{ iff } \underset{\sim}{F}\underset{\sim}{u} = \underset{\sim}{0} \qquad \text{(positive definiteness of inner product)}.$$

$\therefore$

$$\underset{\sim}{u}\cdot\underset{\sim}{C}\underset{\sim}{u} \geq 0 \quad \forall\, \underset{\sim}{u}\in V_n \;,$$

and

$$\underset{\sim}{u}\cdot\underset{\sim}{C}\underset{\sim}{u} = 0 \Rightarrow \underset{\sim}{F}\underset{\sim}{u} = \underset{\sim}{0} \Rightarrow \underset{\sim}{u} = \underset{\sim}{0}$$

since $\underset{\sim}{F}$ is nonsingular (Theorem A3.8,7). Thus, $\underset{\sim}{C} \in \mathrm{Psym}\, V_n$.

By Theorem A3.15,3, $\underset{\sim}{C}$ has a unique positive definite symmetric square root $\underset{\sim}{U} = \sqrt{\underset{\sim}{C}}$. By Theorem A3.15,2, $\underset{\sim}{U}$ is invertible, and we define

$$\underset{\sim}{R} = \underset{\sim}{F}\underset{\sim}{U}^{-1},$$

so that $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U}$. To see that $\underset{\sim}{R}$ is orthogonal, consider

$$\underset{\sim}{R}^T\underset{\sim}{R} = (\underset{\sim}{F}\underset{\sim}{U}^{-1})^T(\underset{\sim}{F}\underset{\sim}{U}^{-1})$$

$$= ((\underset{\sim}{U}^{-1})^T\underset{\sim}{F}^T)(\underset{\sim}{F}\underset{\sim}{U}^{-1}) \quad \text{(Theorem A3.6.3 (iv))}$$

$$= (\underset{\sim}{U}^{-1}\underset{\sim}{F}^T)(\underset{\sim}{F}\underset{\sim}{U}^{-1}) \quad \text{(Theorem A3.9.13)}$$

$$= \underset{\sim}{U}^{-1}(\underset{\sim}{F}^T\underset{\sim}{F})\underset{\sim}{U}^{-1} \quad \text{(associativity of tensor multiplication)}$$

$$= \underset{\sim}{U}^{-1}\underset{\sim}{C}\underset{\sim}{U}^{-1} \quad \text{(substitution)}$$

$$= \underset{\sim}{U}^{-1}\underset{\sim}{U}^2\underset{\sim}{U}^{-1} \quad (\underset{\sim}{U} = \sqrt{\underset{\sim}{C}})$$

$$= \underset{\sim}{U}^{-1}\underset{\sim}{U}\underset{\sim}{U}\underset{\sim}{U}^{-1} \quad \text{(definition of } \underset{\sim}{U}^2\text{)}$$

$$= (\underset{\sim}{U}^{-1}\underset{\sim}{U})(\underset{\sim}{U}\underset{\sim}{U}^{-1}) \quad \text{(associativity)}$$

$$= \underset{\sim}{I}\,\underset{\sim}{I} \quad \text{(Theorem A3.9.3)}$$

$$= \underset{\sim}{I} \quad \text{(Theorem A3.5.1)}.$$

$\therefore$ $\underset{\sim}{R} \in \text{Orth } V_n$ by Theorem A3.10.4.

Also

$$\det \underset{\sim}{R} = \det(\underset{\sim}{F}\underset{\sim}{U}^{-1}) \quad \text{(substitution)}$$

$$= (\det \underset{\sim}{F})(\det \underset{\sim}{U}^{-1}) \quad \text{(Theorem A3.8.4)}$$

$$= (\det \underset{\sim}{F})(\det \underset{\sim}{U})^{-1} \quad \text{(Theorem A3.9.10)},$$

Since $\underset{\sim}{U} \in \mathrm{Psym}\, \mathcal{V}_n$, its eigenvalues, say $\lambda_1, \lambda_2, \cdots, \lambda_n$, are all strictly positive (Theorem A3.15.1); ∴

$$\det \underset{\sim}{U} = \lambda_1 \lambda_2 \cdots \lambda_n > 0 \,.$$ [1]

Consequently, $\det \underset{\sim}{R}$ and $\det \underset{\sim}{F}$ have the same sign; and by Theorem A3.10.6 (iii), $\det \underset{\sim}{R} = +1$ or -1 according as $\det \underset{\sim}{F} > 0$ or $\det \underset{\sim}{F} < 0$.

Next define

$$\underset{\sim}{V} = \underset{\sim}{F}\,\underset{\sim}{R}^{-1},$$

so that $\underset{\sim}{F} = \underset{\sim}{V}\,\underset{\sim}{R}$. To see that $\underset{\sim}{V} \in \mathrm{Psym}\, \mathcal{V}_n$, first consider

$$\begin{aligned} \underset{\sim}{V} &= \underset{\sim}{F}\,\underset{\sim}{R}^{-1} \\ &= \underset{\sim}{F}\,\underset{\sim}{R}^{T} && (\text{Theorem A3.10.3}) \\ &= (\underset{\sim}{R}\,\underset{\sim}{U})\,R^{T} && (\text{substitution}). \end{aligned}$$

Then

$$\begin{aligned} \underset{\sim}{V}^{T} &= (\underset{\sim}{R}^{T})^{T} (\underset{\sim}{R}\,\underset{\sim}{U})^{T} && (\text{Theorem A3.6.3 (iv)}) \\ &= \underset{\sim}{R}\,(R\underset{\sim}{U})^{T} && (\text{Theorem A3.6.3 (i)}) \\ &= \underset{\sim}{R}\,(\underset{\sim}{U}^{T}\underset{\sim}{R}^{T}) && (\text{Theorem A3.6.3 (iv)}) \\ &= \underset{\sim}{R}\,(\underset{\sim}{U}\,\underset{\sim}{R}^{T}) && (\underset{\sim}{U} \in \mathrm{Sym}\, \mathcal{V}_n) \\ &= (\underset{\sim}{R}\,\underset{\sim}{U})\,\underset{\sim}{R}^{T} && (\text{associativity}) \\ &= \underset{\sim}{V} && (\text{substitution}). \end{aligned}$$

∴ $\underset{\sim}{V} \in \mathrm{Sym}\, \mathcal{V}_n$. Second, consider

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[1] Cf. the spectral theorem (Theorem A3.13.1) and Theorem A3.8.5.

$$\underset{\sim}{u} \cdot \underset{\sim}{V} \underset{\sim}{u} = \underset{\sim}{u} \cdot [(\underset{\sim}{R}\underset{\sim}{U})\underset{\sim}{R}^T] \underset{\sim}{u}$$

$= \underset{\sim}{u} \cdot (\underset{\sim}{R}\underset{\sim}{U})\underset{\sim}{R}^T\underset{\sim}{u}$ (definition of tensor multiplication)
$= (\underset{\sim}{R}^T\underset{\sim}{u}) \cdot (\underset{\sim}{R}\underset{\sim}{U})^T\underset{\sim}{u}$ (definition of transpose)
$= (\underset{\sim}{R}^T\underset{\sim}{u}) \cdot (\underset{\sim}{U}^T\underset{\sim}{R}^T)\underset{\sim}{u}$ (Theorem A3.6.3 (iv))
$= (\underset{\sim}{R}^T\underset{\sim}{u}) \cdot \underset{\sim}{U}^T(\underset{\sim}{R}^T\underset{\sim}{u})$ (definition of tensor multiplication)
$= (\underset{\sim}{R}^T\underset{\sim}{u}) \cdot \underset{\sim}{U}(\underset{\sim}{R}^T\underset{\sim}{u})$ ($\underset{\sim}{U} \in \mathrm{Sym}\, \mathcal{V}_n$)
≥ 0 with $= 0$ iff $\underset{\sim}{R}^T\underset{\sim}{u} = \underset{\sim}{0}$ ($\underset{\sim}{U} \in \mathrm{Psym}\, \mathcal{V}_n$).

$\therefore$

$$\underset{\sim}{u} \cdot \underset{\sim}{V} \underset{\sim}{u} \geq 0 \quad \forall\ \underset{\sim}{u} \in \mathcal{V}_n,$$

and

$$\underset{\sim}{u} \cdot \underset{\sim}{V} \underset{\sim}{u} = 0 \Rightarrow \underset{\sim}{R}^T\underset{\sim}{u} = \underset{\sim}{0} \Rightarrow \underset{\sim}{u} = \underset{\sim}{0}$$

since $\underset{\sim}{R}^T$ is nonsingular[1]. Thus, $\underset{\sim}{V} \in \mathrm{Psym}\, \mathcal{V}_n$.

The uniqueness of $\underset{\sim}{U}$, $\underset{\sim}{R}$, and $\underset{\sim}{V}$ was established at the beginning of the proof. □

In the above proof, we showed that $\underset{\sim}{U} \in \mathrm{Psym}\, \mathcal{V}_n \Rightarrow \underset{\sim}{V} = \underset{\sim}{R}\,\underset{\sim}{U}\,\underset{\sim}{R}^T \in \mathrm{Psym}\, \mathcal{V}_n$ by working directly with the definition of $\mathrm{Psym}\, \mathcal{V}_n$ — an approach for which we need not apologize. However, we could have proven that $\underset{\sim}{U}$ and $\underset{\sim}{V}$ have the same eigenvalues and then invoked Theorem A3.15.1. The next two exercises provide alternative ways of doing this.

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[1] Cf. Theorems A3.8.6 and A3.10.6 (iii).

Exercise A3.15.4. Let $\underset{\sim}{Q} \in \text{Orth}\, \mathcal{V}_n$. Show that $\underset{\sim}{T} \in \text{Lin}\, \mathcal{V}_n$ and $\underset{\sim}{S} = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T$ have the same characteristic polynomial. Hence, conclude that $\underset{\sim}{T}$ and $\underset{\sim}{S}$ have the same spectra.

Exercise A3.15.5. Let $\underset{\sim}{Q} \in \text{Orth}\, \mathcal{V}_n$. Use the spectral resolution of $\underset{\sim}{T} \in \text{Sym}\, \mathcal{V}_n$ and the identity

$$\underset{\sim}{Q}\,(\underset{\sim}{u} \otimes \underset{\sim}{u})\,\underset{\sim}{Q}^T = (\underset{\sim}{Q}\underset{\sim}{u}) \otimes (\underset{\sim}{Q}\underset{\sim}{u}) \quad \forall\, \underset{\sim}{u} \in \mathcal{V}_n$$

(cf: Exercise A3.6.6) to obtain the spectral resolution of $\underset{\sim}{S} = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T$. Hence, conclude that $\underset{\sim}{T}$ and $\underset{\sim}{S}$ have the same spectra and that if $\{\overset{*}{\underset{\sim}{e}}_{\langle i \rangle}\}$ is an eigenvector basis of $\underset{\sim}{T}$, the $\{\underset{\sim}{Q}\overset{*}{\underset{\sim}{e}}_{\langle i \rangle}\}$ is an eigenvector basis of $\underset{\sim}{S}$.

Until recently[1], schemes for calculating the factors in the polar decomposition have rested on the computation of the positive definite square root of $\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}$ as in the proof of Theorem A3.15.3. The nonuniqueness of eigenvector bases for $\underset{\sim}{C}$ presented numerical difficulties, and, of course, the calculation of eigenvectors is never an elevating experience. In the following results, eigenvalues are needed — but not eigenvectors.

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[1] Cf. Hoger and Carlson, "Determination of the stretch and rotation in the polar decomposition of the deformation gradient", Quart. Appl. Math. **42** (1984) 113–117, Ting, "Determination of $\underset{\sim}{C}^{1/2}$, $\underset{\sim}{C}^{-1/2}$ and more general isotropic tensor functions of $\underset{\sim}{C}$", J. Elasticity 15 (1985) 319–323, Marsden and Hughes, Mathematical Foundations of Elasticity, Prentice-Hall (1983), Sawyers, "Comments on the paper 'Determination (cont.)

Theorem A3.15.5. Let $\underset{\sim}{F} \in \mathrm{Lin}\, \mathcal{V}_3$ with $\det \underset{\sim}{F} \neq 0$. Then the factors in the polar decompositions

$$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U} = \underset{\sim}{V}\,\underset{\sim}{R}$$

are given by

$$\underset{\sim}{U} = \left[\iota_1(\underset{\sim}{U})\,\iota_2(\underset{\sim}{U}) - \iota_3(\underset{\sim}{U})\right]^{-1} \left\{ \iota_1(\underset{\sim}{U})\,\iota_3(\underset{\sim}{U})\,\underset{\sim}{I} + \left[\iota_1(\underset{\sim}{U})^2 - \iota_2(\underset{\sim}{U})\right]\underset{\sim}{C} - \underset{\sim}{C}^2 \right\},$$

$$\underset{\sim}{U}^{-1} = \iota_3(\underset{\sim}{U})^{-1}\left[\iota_1(\underset{\sim}{U})\,\iota_2(\underset{\sim}{U}) - \iota_3(\underset{\sim}{U})\right]^{-1} \Big\{ \left[\iota_1(\underset{\sim}{U})\,\iota_2(\underset{\sim}{U})^2 - \iota_3(\underset{\sim}{U})\left(\iota_1(\underset{\sim}{U})^2 + \iota_2(\underset{\sim}{U})\right)\right]\underset{\sim}{I}$$

$$- \left[\iota_3(\underset{\sim}{U}) + \iota_1(\underset{\sim}{U})\left(\iota_1(\underset{\sim}{U})^2 - \iota_2(\underset{\sim}{U})\right)\right]\underset{\sim}{C} + \iota_1(\underset{\sim}{U})\,\underset{\sim}{C}^2 \Big\},$$

$$\underset{\sim}{R} = \underset{\sim}{F}\,\underset{\sim}{U}^{-1}, \quad \underset{\sim}{V} = \underset{\sim}{F}\,\underset{\sim}{U}^{-1}\underset{\sim}{F}^T,$$

where $\underset{\sim}{C} = \underset{\sim}{F}^T\underset{\sim}{F}$. Moreover, the fundamental invariants of $\underset{\sim}{U}$ are given by

$$\iota_1(\underset{\sim}{U}) = \lambda_1 + \lambda_2 + \lambda_3, \quad \iota_2(\underset{\sim}{U}) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad \iota_3(\underset{\sim}{U}) = \lambda_1\lambda_2\lambda_3,$$

where $\lambda_i > 0$ with λ_i^2 $(i \in \mathbb{I}_3)$ being the eigenvalues of $\underset{\sim}{C}$.

Proof. By the Hamilton-Cayley theorem,

$$\underset{\sim}{U}^3 - \iota_1(\underset{\sim}{U})\,\underset{\sim}{U}^2 + \iota_2(\underset{\sim}{U})\,\underset{\sim}{U} - \iota_3(\underset{\sim}{U})\,\underset{\sim}{I} = \underset{\sim}{O}. \qquad (1)$$

Multiplication by $\underset{\sim}{U}$ yields

of the stretch and rotation in the polar decomposition of the deformation gradient" by A. Hoger and D.E. Carlson, "Quart. Appl. Math. 44 (1986) 309-311. Here we follow Ting.

$$\underset{\sim}{U}^4 - \iota_1(\underset{\sim}{U})\underset{\sim}{U}^3 + \iota_2(\underset{\sim}{U})\underset{\sim}{U}^2 - \iota_3(\underset{\sim}{U})\underset{\sim}{U} = \underset{\sim}{O} . \qquad (2)$$

Next use (1) to replace $\underset{\sim}{U}^3$ in (2):

$$\underset{\sim}{U}^4 - [\iota_1(\underset{\sim}{U})^2 - \iota_2(\underset{\sim}{U})]\underset{\sim}{U}^2 + [\iota_1(\underset{\sim}{U})\iota_2(\underset{\sim}{U}) - \iota_3(\underset{\sim}{U})]\underset{\sim}{U} - \iota_1(\underset{\sim}{U})\iota_3(\underset{\sim}{U})\underset{\sim}{I} = \underset{\sim}{O} .$$

On noting that $\underset{\sim}{U}^2 = \underset{\sim}{C}$ and $\underset{\sim}{U}^4 = \underset{\sim}{C}^2$,)[1] we get the claimed result, provided that

$$\iota_1(\underset{\sim}{U})\iota_2(\underset{\sim}{U}) - \iota_3(\underset{\sim}{U}) \neq 0 . \qquad (3)$$

The establishment of (3) is left as Exercise A3.15.6. (Hint: Express the l.h.s. of (3) in terms of eigenvalues.)

To get $\underset{\sim}{U}^{-1}$, multiply (1) by $\underset{\sim}{U}^{-1}$:

$$\underset{\sim}{U}^2 - \iota_1(\underset{\sim}{U})\underset{\sim}{U} + \iota_2(\underset{\sim}{U})\underset{\sim}{I} - \iota_3(\underset{\sim}{U})\underset{\sim}{U}^{-1} = \underset{\sim}{O} . \qquad (4)$$

Then in (4), replace $\underset{\sim}{U}$ from the result already proved, and replace $\underset{\sim}{U}^2$ by $\underset{\sim}{C}$.

The formulas for $\underset{\sim}{R}$ and $\underset{\sim}{V}$ follow directly from the polar decomposition itself. Work this out as Exercise A3.15.7.

The fact that the eigenvalues of $\underset{\sim}{U} = \sqrt{\underset{\sim}{C}}$ are given by $\sqrt{\lambda_i^2}$ is clear from the proof of Theorem A3.15.3. □

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[1] See the proof of the polar decomposition theorem.

A3.15.14

Supplementary Reading

Same as for § 3.5

Chapter A4

Higher-Order Tensors

A4.1. The Direct Approach

Throughout this section, $\mathcal{V}_n$ denotes an n-dimensional inner product space[1]. To extend our direct or basis-free approach[2] to higher-order tensors, we proceed inductively and adopt

Definition A4.1.1. A tensor of order m[3] on $\mathcal{V}_n$ is a linear function on $\mathcal{V}_n$ that maps elements of $\mathcal{V}_n$ into tensors of order $(m-1)$.

To get this started in a way consistent with what has gone before, we take tensors of order 0 to be scalars; i.e. elements of $\mathbb{R}$, regarded as a linear space[4]. Then tensors of order 1 are linear functions on $\mathcal{V}_n$ to $\mathbb{R}$; i.e. linear functionals[5]. But then tensors of

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[1] Cf. Theorem A2.3.2.

[2] See the last paragraph of §A3.3.6.

[3] Often, tensor of rank m or m^{th}-order tensor.

[4] See Exercise A2.1.2.

[5] See Definition A2.5.1.

order 2 are linear functions on $\mathcal{V}_n$ to the dual space $\mathcal{V}_n^*$, which is not in accord with Definition A3.1.1. To get around this difficulty, we note that by the representation theorem for linear functionals (Theorem A2.5.1) to each linear functional f on $\mathcal{V}_n$ there corresponds a unique element $\underset{\sim}{f}$ of $\mathcal{V}_n \ni$

$$f(\underset{\sim}{u}) = \underset{\sim}{f} \cdot \underset{\sim}{u} \quad \forall \underset{\sim}{u} \in \mathcal{V}_n. \qquad (*)$$

Conversely, to each element $\underset{\sim}{f} \in \mathcal{V}_n$ there corresponds a unique linear functional f on $\mathcal{V}_n$ defined by the evaluation rule $(*)$. Accordingly, <u>for the purposes of Definition A4.1.1, we agree to identify a linear functional f on $\mathcal{V}_n$ with the element $\underset{\sim}{f} \in \mathcal{V}_n$ given by the representation $(*)$</u>. With this agreement, <u>tensors of order 1 are simply the elements of the underlying linear space</u> $\mathcal{V}_n$, and there is no problem with the definition of 2nd order tensors.

There is no uniformity of notation for higher-order tensors. We shall denote the typical tensor of order m by $\overset{(m)}{\underset{\sim}{T}}$, and then for $\underset{\sim}{u} \in \mathcal{V}_n$, we write

$$\overset{(m)}{\underset{\sim}{T}} \underset{\sim}{u} = \overset{(m-1)}{\underset{\sim}{T}}.$$

<u>Addition</u>, <u>scalar multiplication</u>, <u>equality</u>, and <u>zero element</u> for tensors of order m are defined just as for linear transformations[1] in general. The details were

[1] See §A2.4

reiterated for second-order tensors in §A3.1, and they need not be repeated again here. With this understanding, it follows from Theorem A2.4.5 that the set of all tensors of order m on $\mathcal{V}_n$ is a linear space. Indeed, if

$$\overset{(m)}{\mathcal{T}_n} = \text{linear space of all tensors of order } m \text{ on } \mathcal{V}_n,$$

then

$$\overset{(m)}{\mathcal{T}_n} = \operatorname{Lin}(\mathcal{V}_n, \overset{(m-1)}{\mathcal{T}_n}).$$

We can proceed to bases, components, transformation rules, etc. just as we did for second-order tensors by introducing higher-order tensor products. We sketch the details for third-order tensors.

Theorem A4.1.2. The third-order tensor product, $\underset{\sim}{u} \otimes \underset{\sim}{v} \otimes \underset{\sim}{w}$, of elements $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathcal{V}_n$, defined as a function on $\mathcal{V}_n$ to $\operatorname{Lin}\mathcal{V}_n$ by the evaluation rule

$$(\underset{\sim}{u} \otimes \underset{\sim}{v} \otimes \underset{\sim}{w})\underset{\sim}{x} = (\underset{\sim}{w} \cdot \underset{\sim}{x})(\underset{\sim}{u} \otimes \underset{\sim}{v}),$$

is a third-order tensor on $\mathcal{V}_n$.

Proof. Exercise A4.1.1. □

Exercise A4.1.2. For third-order tensor products, formulate and prove the analog of Theorem A3.1.7.

To get a representation for a third-order tensor $\overset{(3)}{\underset{\sim}{T}}$ in terms of a basis for $\overset{(3)}{\mathcal{T}}_n$, let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_n\}$ and $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \ldots, \underset{\sim}{e}^n\}$ be a basis and its dual for $\mathcal{V}_n$ and consider for any $\underset{\sim}{u} \in \mathcal{V}_n$

$$\overset{(3)}{\underset{\sim}{T}}\underset{\sim}{u} = (\overset{(3)}{T}\underset{\sim}{u})^{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \qquad \text{(Theorem A3.2.2)}$$

$$= \underset{\sim}{e}^i \cdot [(\overset{(3)}{\underset{\sim}{T}}\underset{\sim}{u})\underset{\sim}{e}^j]\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \qquad \text{(Theorem A3.2.2)}$$

$$= \underset{\sim}{e}^i \cdot \{[\overset{(3)}{\underset{\sim}{T}}(u_k \underset{\sim}{e}^k)]\underset{\sim}{e}^j\}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \qquad \text{(Theorem A2.5.8)}$$

$$= \underset{\sim}{e}^i \cdot \{[\overset{(3)}{\underset{\sim}{T}}((\underset{\sim}{e}_k \cdot \underset{\sim}{u})\underset{\sim}{e}^k)]\underset{\sim}{e}^j\}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \qquad \text{(Theorem A1.5.8)}$$

$$= \underset{\sim}{e}^i \cdot [(\overset{(3)}{\underset{\sim}{T}}\underset{\sim}{e}^k)\underset{\sim}{e}^j]\,(\underset{\sim}{e}_k \cdot \underset{\sim}{u})\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \qquad \text{(homogeneity)}$$

$$= \underset{\sim}{e}^i \cdot [(\overset{(3)}{\underset{\sim}{T}}\underset{\sim}{e}^k)\underset{\sim}{e}^j]\,[(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k)\underset{\sim}{u}] \qquad \text{(Definition A4.1.2)}.$$

Since $\underset{\sim}{u}$ is arbitrary, we have established

Theorem A4.1.3. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_n\}$ and $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \ldots, \underset{\sim}{e}^n\}$ be a basis and its dual for an n-dimensional inner product space $\mathcal{V}_n$. Then any third-order tensor $\overset{(3)}{\underset{\sim}{T}}$ on $\mathcal{V}_n$ admits the representation

$$\overset{(3)}{\underset{\sim}{T}} = T^{ijk}\, \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k ,$$

where

$$T^{ijk} = \underset{\sim}{e}^i \cdot [(\overset{(3)}{\underset{\sim}{T}}\underset{\sim}{e}^k)\underset{\sim}{e}^j].$$

Of course, the T^{ijk} above are called the **contravariant components** of $\overset{(3)}{\underset{\sim}{T}}$ w.r.t. the basis system $\{\underset{\sim}{e}_i\}$, $\{\underset{\sim}{e}^i\}$ for $\mathcal{V}_n$. It should be clear how one gets the covariant and various mixed components of $\overset{(3)}{\underset{\sim}{T}}$; if not, proceed to

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Exercise A4.1.3. Refer to Theorem A4.1.3. Work out the representation of $\overset{(3)}{\underset{\sim}{T}}$ in terms of covariant components.

We now have enough machinery assembled to determine the dimension of the linear space of third-order tensors.

Theorem A4.1.4. *Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$ be a basis for an n-dimensional inner product space $\mathcal{V}_n$. Then the set of n^3 third-order tensors $\{\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k\}$ is a basis for $\overset{(3)}{\mathcal{T}}_n$, and, consequently, $\overset{(3)}{\mathcal{T}}_n$ is n^3-dimensional.*

Proof. **Exercise** A4.1.4. □

The components of the third-order tensor product come out as one would expect from the second-order case.

Theorem A4.1.5. *Let $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathcal{V}_n$. Then the contravariant components of $\underset{\sim}{u} \otimes \underset{\sim}{v} \otimes \underset{\sim}{w}$ are given by*

$$(\underset{\sim}{u} \otimes \underset{\sim}{v} \otimes \underset{\sim}{w})^{ijk} = u^i v^j w^k.$$

Proof. Exercise A4.1.5. □

There, of course, is an analog of Theorem A3.2.5 for third-order tensors that can be read off from Theorem A2.2.12. And the tensor equation $\overset{(3)}{\underset{\sim}{T}}\underset{\sim}{u} = \overset{(2)}{\underset{\sim}{T}} = \underset{\sim}{T}$ is equivalent to various component equations a la Theorem A3.2.9. Also, the g-symbols can be used to raise and lower indices on third-order tensor components in the manner of Theorems A2.5.11 and A3.2.8. On the other hand, the notion of identity tensor makes sense only for second-order tensors.

The transformation rules for the components of third-order tensors are easily worked out. For the sake of variety, we take a different attack than in the proofs of Theorems A2.6.2 and A3.3.1. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$, $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^n\}$ and $\{\underset{\sim}{\bar{e}}_1, \underset{\sim}{\bar{e}}_2, \cdots, \underset{\sim}{\bar{e}}_n\}$, $\{\underset{\sim}{\bar{e}}^1, \underset{\sim}{\bar{e}}^2, \cdots, \underset{\sim}{\bar{e}}^n\}$ be two basis systems for an n-dimensional inner product space $\mathcal{V}_n$. By Theorem A4.1.3, $\overset{(3)}{\underset{\sim}{T}} \in \overset{(3)}{\mathcal{T}}_n$ admits the representations

$$\overset{(3)}{\underset{\sim}{T}} = T^{ijk}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k = \bar{T}^{ijk}\underset{\sim}{\bar{e}}_i \otimes \underset{\sim}{\bar{e}}_j \otimes \underset{\sim}{\bar{e}}_k$$

in terms of contravariant components. By Theorem A2.6.1,

$$\underset{\sim}{\bar{e}}^i = \alpha^i_j \underset{\sim}{e}^j \;;$$

and ∴ by Theorem A4.1.3,

$$\overline{T}^{ijk} = \overline{\underset{\sim}{e}}^i \cdot \left[\left(\overset{(3)}{\underset{\sim}{T}} \overline{\underset{\sim}{e}}^k \right) \overline{\underset{\sim}{e}}^j \right]$$

$$= \alpha^i_l \underset{\sim}{e}^l \cdot \left[\left(\overset{(3)}{\underset{\sim}{T}} \alpha^k_n \underset{\sim}{e}^n \right) \alpha^j_m \underset{\sim}{e}^m \right] \qquad \text{(substitution)}$$

$$= \alpha^i_l \alpha^j_m \alpha^k_n \underset{\sim}{e}^l \cdot \left[\left(\overset{(3)}{\underset{\sim}{T}} \underset{\sim}{e}^n \right) \underset{\sim}{e}^m \right] \qquad \text{(linearity)}$$

$$= \alpha^i_l \alpha^j_m \alpha^k_n T^{lmn} \qquad \text{(Theorem A4.1.3)}.$$

Thus, we are led to

Theorem A4.1.6. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$, $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^n\}$ and $\{\overline{\underset{\sim}{e}}_1, \overline{\underset{\sim}{e}}_2, \cdots, \overline{\underset{\sim}{e}}_n\}$, $\{\overline{\underset{\sim}{e}}^1, \overline{\underset{\sim}{e}}^2, \cdots, \overline{\underset{\sim}{e}}^n\}$ be two bases and their duals for an n-dimensional inner-product-space $\mathcal{V}_n$. Define

$$\alpha^j_i = \overline{\underset{\sim}{e}}^j \cdot \underset{\sim}{e}_i \quad , \quad \beta^i_j = \overline{\underset{\sim}{e}}_j \cdot \underset{\sim}{e}^i \quad (i,j \in \mathbb{I}_n).$$

Let $\overset{(3)}{\underset{\sim}{T}} \in \overset{(3)}{\mathcal{L}}_n$ so that

$$\overset{(3)}{\underset{\sim}{T}} = T^{ijk} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k = \overline{T}^{ijk} \overline{\underset{\sim}{e}}_i \otimes \overline{\underset{\sim}{e}}_j \otimes \overline{\underset{\sim}{e}}_k .$$

Then, for $i,j \in \mathbb{I}_n$,

$$\overline{T}^{ijk} = \alpha^i_l \alpha^j_m \alpha^k_n T^{lmn} \quad \text{... transformation rule for contravariant components}$$

and

$$T^{ijk} = \beta^i_l \beta^j_m \beta^k_n \overline{T}^{lmn} \quad \text{--- inverse transformation rule.}$$

Proof. All that remains to be proved is the inverse transformation rule, and thus is left as Exercise A4.1.6. □

The transformation rules for covariant components and for the various mixed components can be worked out in the same way.

Of course, if we have a list of n^3 real numbers, then we can generate a third-order tensor by taking the numbers to be the components of a third-order tensor w.r.t. some basis. As an example of this procedure, let us take the ordinary 3-dimensional alternating symbol (ε_{ijk})[1] to define the covariant components of a third-order tensor w.r.t. the basis system $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}, \{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \underset{\sim}{e}^3\}$:

$$\overset{(3)}{\underset{\sim}{E}} := e_{ijk}\, \underset{\sim}{e}^i \otimes \underset{\sim}{e}^j \otimes \underset{\sim}{e}^k, \quad e_{ijk} \equiv \varepsilon_{ijk}.$$

To get other components w.r.t. this basis system, we can raise and lower indices via the g-symbols. E.g.,

$$e^{ijk} = g^{il} g^{jm} g^{kn} e_{lmn} = g^{il} g^{jm} g^{kn} \varepsilon_{lmn}$$
$$= \det[g^{pq}]\, \varepsilon_{ijk}$$

by a standard theorem on determinants of matrices.[2] To

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[1] See Definition A2.7.2.

[2] See, e.g., HOHN's Elementary Matrix Algebra.

get components w.r.t. another basis system, we use the transformation rules. E.g.,

$$\bar{\varepsilon}_{ijk} = \beta^l_i \beta^m_j \beta^n_k e_{lmn} = \beta^l_i \beta^m_j \beta^n_k \varepsilon_{lmn}$$
$$= \det[\beta^p_q]\, \varepsilon_{ijk}.$$

More interesting is when we have lists of components w.r.t. each basis of a potential tensor given to us by some natural scheme. The third-order analog of Theorems A2.6.3 and A3.2.2 is

Theorem A4.1.7. Let the hypotheses of Theorem A4.1.6 hold. Let there be lists of n^3 real numbers associated with each basis for $\mathcal{V}_n$:

$$T^{ijk} \text{ with } \{\underset{\sim}{e}_i\},\quad \bar{T}^{ijk} \text{ with } \{\bar{\underset{\sim}{e}}_i\},\ \text{etc.}$$

These numbers are the contravariant[1] components of a third-order tensor

$$\overset{(3)}{\underset{\sim}{T}} = T^{ijk}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j \otimes \underset{\sim}{e}_k = \bar{T}^{ijk}\bar{\underset{\sim}{e}}_i \otimes \bar{\underset{\sim}{e}}_j \otimes \bar{\underset{\sim}{e}}_k = \cdots$$

on $\mathcal{V}_n$ iff they satisfy the transformation rules

$$\bar{T}^{ijk} = \alpha^i_l \alpha^j_m \alpha^k_n T^{lmn}, \quad T^{ijk} = \beta^i_l \beta^j_m \beta^k_n \bar{T}^{lmn}, \quad \text{etc.}$$

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[1] Of course, there are strictly analogous results for the covariant and mixed components.

Proof. *Exercise* A 4.1.7. □

Higher-order tensors are handled similarly. The main result is

Theorem A4.1.8. Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \dots, \underset{\sim}{e}_n\}$ and $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \dots, \underset{\sim}{e}^n\}$ be a basis and its dual for an n-dimensional inner product space $\mathcal{V}_n$. Then any m^{th}-order tensor $\overset{(m)}{\underset{\sim}{T}}$ on $\mathcal{V}_n$ admits the representation

$$\overset{(m)}{\underset{\sim}{T}} = T^{i_1 i_2 \cdots i_m}\, \underset{\sim}{e}_{i_1} \otimes \underset{\sim}{e}_{i_2} \otimes \cdots \otimes \underset{\sim}{e}_{i_n},$$

where

$$T^{i_1 i_2 \cdots i_m} = \underset{\sim}{e}^{i_1} \cdot \left[\left(\cdots \left(\left(\overset{(m)}{\underset{\sim}{T}} \underset{\sim}{e}^{i_m} \right) \underset{\sim}{e}^{i_{m-1}} \right) \cdots \right) \underset{\sim}{e}^{i_2} \right].$$

Proof. *Exercise* A4.1.8. (Hint: use mathematical induction on m.) □

Supplementary Reading

Same as for § A3.1 plus

PRAGER, Introduction to Mechanics of Continua

A4.2. The Component-Transformation Rule Approach

While the direct approach to tensors of the previous section is the most useful viewpoint from the perspective of general continuum mechanics, it is not widely used outside of this community. Most engineers and physical scientists prefer to identify tensors through their components. There are at least two reasons for this: components are real numbers — a familiar notion; specific practical problems generally are solved by resorting to components. The disadvantage with components is that they depend on bases (or more generally coordinate systems), and "good" physical results should be independent of bases. In the component-transformation rule approach, the dependence on bases is taken care of by the transformation rules. A particular set of components does, indeed, depend on a particular basis, but the corresponding components w.r.t. any other basis can be calculated via the appropriate transformation rule. Taking the direct approach to be definitive, we saw how this works for vectors (first-order tensors) in §A2.6, for second-order tensors in §A3.3, and for third-order tensors in §A4.1.

Let $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$, $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^n\}$ and $\{\underset{\sim}{\bar{e}}_1, \underset{\sim}{\bar{e}}_2, \cdots, \underset{\sim}{\bar{e}}_n\}$, $\{\underset{\sim}{\bar{e}}^1, \underset{\sim}{\bar{e}}^2, \cdots, \underset{\sim}{\bar{e}}^n\}$ be two bases and their duals for an n-dimensional inner product space[1] $\mathcal{V}_n$, and define

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[1] Cf. Theorem A2.3.2.

$$\alpha^j_i = \bar{\underset{\sim}{e}}^j \cdot \underset{\sim}{e}_i \; ; \; \beta^i_j = \bar{\underset{\sim}{e}}_j \cdot \underset{\sim}{e}^i \quad (i, j \in \mathbb{I}_n).$$

For the sake of concreteness in this rather messy business, we consider the particularly important case of second-order tensors. Guided by Theorems A3.3.1 and A3.3.2, we see that the definition of a second-order tensor in the component-transformation rule approach should be as follows:

An entity represented in each basis system for $\mathcal{V}_n$ by 4 sets of n^2 real numbers,

$$\left.\begin{array}{l} T^{ij} \text{ ... contravariant components in } \{\underset{\sim}{e}_i\}, \{\underset{\sim}{e}^i\} \\ T_{ij} \text{ ... covariant components in } \{\underset{\sim}{e}_i\}, \{\underset{\sim}{e}^i\} \\ \left.\begin{array}{l} T^i_{\cdot j} \\ T_i^{\cdot j} \end{array}\right\} \text{ ... mixed components in } \{\underset{\sim}{e}_i\}, \{\underset{\sim}{e}^i\} \end{array}\right. ,$$

$$\left.\begin{array}{l} \bar{T}^{ij} \text{ ... contravariant components in } \{\bar{\underset{\sim}{e}}_i\}, \{\bar{\underset{\sim}{e}}^i\} \\ \bar{T}_{ij} \text{ ... covariant components in } \{\bar{\underset{\sim}{e}}_i\}, \{\bar{\underset{\sim}{e}}^i\} \\ \left.\begin{array}{l} \bar{T}^i_{\cdot j} \\ \bar{T}_i^{\cdot j} \end{array}\right\} \text{ ... mixed components in } \{\bar{\underset{\sim}{e}}_i\}, \{\bar{\underset{\sim}{e}}^i\} \end{array}\right. ,$$

etc.,

is a second-order tensor on $\mathcal{V}_n$ if the components obey the transformation rules

$\bar{T}^{ij} = \alpha^i_k \alpha^j_l T^{kl}$ --- transformation rule for contravariant components,

$\bar{T}_{ij} = \beta^k_i \beta^l_j T_{kl}$ --- transformation rule for covariant components,

$$\left.\begin{aligned} \bar{T}^{i}_{\cdot j} &= \alpha^i_k \beta^l_j T^{k}_{\cdot l} \\ \bar{T}_{i}^{\cdot j} &= \beta^k_i \alpha^j_l T_{k}^{\cdot l} \end{aligned}\right\}$$ --- transformation rules for mixed components.

The inverse transformation rules can be derived (see Exercise A3.3.1) and need not be included in the definition.

Commonly, especially in "pure" mathematical treatments, one sees partial definitions such as an entity represented in each basis system for V_n by n^2 real numbers,

T^{ij} ... contravariant components in $\{\underline{e}_i\}, \{\underline{e}^i\}$,
$\bar{T}^{ij}$ ... contravariant components in $\{\bar{\underline{e}}_i\}, \{\bar{\underline{e}}^i\}$,
etc.,

is a second-order contravariant tensor on V_n if the components obey the transformation rule

$$\bar{T}^{ij} = \alpha^i_k \alpha^j_l T^{kl}.$$

This is not adequate for our interpretation (based on the direct approach) that all of the possible components represent one and same thing; namely, the second-order tensor

$$\underset{\sim}{T} = T^{ij}\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = T_{ij}\underset{\sim}{e}^i \otimes \underset{\sim}{e}^j = T^i_{\cdot j}\underset{\sim}{e}_i \otimes \underset{\sim}{e}^j = T_i^{\cdot j}\underset{\sim}{e}^i \otimes \underset{\sim}{e}_j$$

$$= \bar{T}^{ij}\bar{\underset{\sim}{e}}_i \otimes \bar{\underset{\sim}{e}}_j = \bar{T}_{ij}\bar{\underset{\sim}{e}}^i \otimes \bar{\underset{\sim}{e}}^j = \bar{T}^i_{\cdot j}\bar{\underset{\sim}{e}}_i \otimes \bar{\underset{\sim}{e}}^j = \bar{T}_i^{\cdot j}\bar{\underset{\sim}{e}}^i \otimes \bar{\underset{\sim}{e}}_j$$

$$\cdots,$$

wherein the various components w.r.t. a given basis system are related through the g-symbols, e.g.,

$$T^{ij} = g^{im} g^{jn} T_{mn}\ .\ )^1$$

The component-transformation rule definition for higher-order tensors is readily inferred from the preceding development. However, the increasingly numerous possibilities for mixed components make a general definition difficult. In this regard, the "pure" mathematical treatments often disregard the ordering of the mixed indices. Again, this is inadequate for our interpretation as can be seen by results such as

$$T^i_{\cdot j} = \underset{\sim}{e}^i \cdot (\underset{\sim}{T}\underset{\sim}{e}_j) \text{ but } T_j^{\cdot i} = \underset{\sim}{e}_j \cdot (\underset{\sim}{T}\underset{\sim}{e}^i).\ )^2$$

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[1] See Theorem A3.2.8.

[2] See Theorem A3.2.2.

Supplementary Reading

Same as for § A3.1 plus

BISHOP and GOLDBERG, Tensor Analysis on Manifolds

SOKOLNIKOFF, Tensor Analysis

THOMAS, Concepts from Tensor Analysis and Differential Geometry

A4.3. The Multilinear Functional Approach

Just as the direct and the component-transformation rule approaches have their adherents, mathematicians generally prefer to define tensors as real-valued multilinear functions of vectors (i.e., multilinear functionals[1]), the order of the tensor corresponding to the number of vector arguments. We shall call this the multilinear functional approach. Actually, the mathematicians often draw some of the vectors from the dual space, but that is not done in the variant outlined below.

Throughout this section, $\mathcal{V}_n$ denotes an n-dimensional inner product space[2]; $\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \cdots, \underset{\sim}{e}_n\}$, $\{\underset{\sim}{e}^1, \underset{\sim}{e}^2, \cdots, \underset{\sim}{e}^n\}$ and $\{\bar{\underset{\sim}{e}}_1, \bar{\underset{\sim}{e}}_2, \cdots, \bar{\underset{\sim}{e}}_n\}$, $\{\bar{\underset{\sim}{e}}^1, \bar{\underset{\sim}{e}}^2, \cdots, \bar{\underset{\sim}{e}}^n\}$ are bases and their duals for $\mathcal{V}_n$; and

$$\alpha_i^j = \bar{\underset{\sim}{e}}^j \cdot \underset{\sim}{e}_i \;;\; \beta_j^i = \bar{\underset{\sim}{e}}_j \cdot \underset{\sim}{e}^i .$$

Then in the multilinear functional approach, the definition of an m^{th}-order tensor on $\mathcal{V}_n$ is

An m^{th}-order tensor on $\mathcal{V}_n$ is a multilinear functional

$$\overset{(m)}{T} : \underbrace{\mathcal{V}_n \times \mathcal{V}_n \times \cdots \times \mathcal{V}_n}_{m \text{ times}} \to R .$$

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[1] Cf. Definition A2.5.1.

[2] Cf. Theorem A2.3.2.

Let us consider a first-order tensor $\overset{(1)}{T}$ from this point of view. $\overset{(1)}{T}: \mathcal{V}_n \to \mathbb{R}$ is simply a linear functional, and by the representation theorem (Theorem A2.5.1), $\exists$ a unique $\underset{\sim}{t} \in \mathcal{V}_n \ni$

$$\overset{(1)}{T}(\underset{\sim}{u}) = \underset{\sim}{t} \cdot \underset{\sim}{u} \quad \forall \underset{\sim}{u} \in \mathcal{V}_n. \qquad (*)$$

Clearly, $(*)$ sets up a one-to-one correspondence between first-order tensors $\overset{(1)}{T}$ from the multilinear functional approach and first-order tensors $\underset{\sim}{t} \in \mathcal{V}_n$ from the direct approach.

Evaluation of $\overset{(1)}{T}$ on basis elements gives

$$\overset{(1)}{T}(\underset{\sim}{e}_i) = \underset{\sim}{t} \cdot \underset{\sim}{e}_i = t_i$$

and

$$\overset{(1)}{T}(\underset{\sim}{e}^i) = \underset{\sim}{t} \cdot \underset{\sim}{e}^i = t^i.$$

by Theorem A2.5.8. Thus, if we call $\overset{(1)}{T}(\underset{\sim}{e}_i)$ and $\overset{(1)}{T}(\underset{\sim}{e}^i)$ the covariant and contravariant components of $\overset{(1)}{T}$ w.r.t. the basis system $\{\underset{\sim}{e}_i\}, \{\underset{\sim}{e}^i\}$, they are exactly the components of the corresponding element $\underset{\sim}{t} \in \mathcal{V}_n$. Then necessarily the components from the two different approaches follow the same transformation rules. E.g.,

$$\overset{(1)}{T}(\bar{\underset{\sim}{e}}^i) = \underset{\sim}{t} \cdot \bar{\underset{\sim}{e}}^i = \bar{t}^i = \alpha^i_j t^j = \alpha^i_j \overset{(1)}{T}(\underset{\sim}{e}^j)$$

by Theorem A2.6.2.

One can get the transformation rules directly from the multilinearity without going the through the identification of $\overset{(1)}{T}$ with $\underset{\sim}{t}$. E.g., by Theorem A2.6.1,

$$\overset{(1)}{T}(\bar{\underset{\sim}{e}}^i) = \overset{(1)}{T}(\alpha^i_j \underset{\sim}{e}^j) = \alpha^i_j \overset{(1)}{T}(\underset{\sim}{e}^j).$$

Thus, <u>the components of the first-order tensor $\overset{(1)}{T}$ in the multilinear-functional approach satisfy the defining transformation rules in the component-transformation rule approach.</u>

Since we take the direct approach to be fundamental, perhaps the best way to view $\overset{(1)}{T}$ is that its components are the components of

$$\underset{\sim}{t} = \overset{(1)}{T}(\underset{\sim}{e}_i)\underset{\sim}{e}^i = \overset{(1)}{T}(\underset{\sim}{e}^i)\underset{\sim}{e}_i$$

$$= \overset{(1)}{T}(\bar{\underset{\sim}{e}}_i)\bar{\underset{\sim}{e}}^i = \overset{(1)}{T}(\bar{\underset{\sim}{e}}^i)\bar{\underset{\sim}{e}}_i$$

$$\cdots$$

— a first-order tensor from the direct point of view.

From the multilinear functional point of view, a second-order tensor $\overset{(2)}{T}$ is a bilinear functional $\overset{(2)}{T}: \mathcal{V}_n \times \mathcal{V}_n \to R$. We have to work a bit harder to identify directly $\overset{(2)}{T}$ with a second-order tensor from the direct point of view. For each fixed $\underset{\sim}{v} \in \mathcal{V}_n$, $\overset{(2)}{T}(\underset{\sim}{u}, \underset{\sim}{v})$ is a linear functional in $\underset{\sim}{u}$; more precisely,

$$\overset{(2)}{T}(\cdot, \underset{\sim}{v}) \in \mathcal{V}_n^*.$$

$\therefore$ by the representation theorem, $\exists$ a unique element of $\mathcal{V}_n$ which we denote by $\underset{\sim}{f}(\underset{\sim}{v})$ to denote its dependence on $\underset{\sim}{v}$ $\ni$

$$\overset{(2)}{T}(\underset{\sim}{u}, \underset{\sim}{v}) = \underset{\sim}{f}(\underset{\sim}{v}) \cdot \underset{\sim}{u} \quad \forall \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_n.$$

But again by the linearity, $\underset{\sim}{f}$ is a linear function of $\underset{\sim}{v}$; the detailed proof of which we leave as Exercise A4.3.1. Thus, $\underset{\sim}{f}$ is a second-order tensor from the direct viewpoint of Chapter A3, and we write

$$\underset{\sim}{f}(\underset{\sim}{v}) = \underset{\sim}{T}\underset{\sim}{v}.$$

Then

$$\overset{(2)}{T}(\underset{\sim}{u}, \underset{\sim}{v}) = \underset{\sim}{u} \cdot \underset{\sim}{T}\underset{\sim}{v}. \quad \forall \underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}_n. \quad (**)$$

Again, $(**)$ <u>sets up a one-to-one correspondence between second-order tensors $\overset{(2)}{T}$ from the multilinear functional approach and second-order tensors $\underset{\sim}{T} \in \mathrm{Lin}\,\mathcal{V}_n$ from the direct approach</u>.

Evaluation of $\overset{(2)}{T}$ on basis elements gives, e.g.,

$$\overset{(2)}{T}(\underset{\sim}{e}_i, \underset{\sim}{e}^j) = \underset{\sim}{e}_i \cdot \underset{\sim}{T}\,\underset{\sim}{e}^j = T_i^{\bullet j}$$

by Theorem A3.2.2. In parallel with the first-order case, we call $\overset{(2)}{T}(\underset{\sim}{e}_i, \underset{\sim}{e}^j)$ a set of mixed components of $\overset{(2)}{T}$ w.r.t. the basis system $\{\underset{\sim}{e}_i\}$, $\{\underset{\sim}{e}^i\}$, and note that they are exactly the same set of mixed components of the corresponding element in $\underset{\sim}{T} \in \mathrm{Lin}\, \mathcal{V}_n$. Of course, the three other sets of components are handled in the same manner.

Again, from either the transformation properties of the components of $\underset{\sim}{T}$ or the multilinearity of $\overset{(2)}{T}$, we get transformation rules such as

$$\overset{(2)}{T}(\bar{\underset{\sim}{e}}_i, \bar{\underset{\sim}{e}}^j) = \beta^k_i \alpha^j_l \overset{(2)}{T}(\underset{\sim}{e}_k, \underset{\sim}{e}^l).$$

Thus, the components of the second-order tensor $\overset{(2)}{T}$ in the multilinear functional approach satisfy the defining transformation rules in the component-transformation rule approach.

As in the first-order case, we look at $\overset{(2)}{T}$ from the viewpoint that its components are the components of

$$\underset{\sim}{T} = \overset{(2)}{T}(\underset{\sim}{e}^i, \underset{\sim}{e}^j)\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j = \overset{(2)}{T}(\underset{\sim}{e}_i, \underset{\sim}{e}_j)\underset{\sim}{e}^i \otimes \underset{\sim}{e}^j = \overset{(2)}{T}(\underset{\sim}{e}^i, \underset{\sim}{e}_j)\underset{\sim}{e}_i \otimes \underset{\sim}{e}^j = \overset{(2)}{T}(\underset{\sim}{e}_i, \underset{\sim}{e}^j)\underset{\sim}{e}^i \otimes \underset{\sim}{e}_j$$

$$= \overset{(2)}{T}(\bar{\underset{\sim}{e}}^i, \bar{\underset{\sim}{e}}^j)\bar{\underset{\sim}{e}}_i \otimes \bar{\underset{\sim}{e}}_j = \overset{(2)}{T}(\bar{\underset{\sim}{e}}_i, \bar{\underset{\sim}{e}}_j)\bar{\underset{\sim}{e}}^i \otimes \bar{\underset{\sim}{e}}^j = \overset{(2)}{T}(\bar{\underset{\sim}{e}}^i, \bar{\underset{\sim}{e}}_j)\bar{\underset{\sim}{e}}_i \otimes \bar{\underset{\sim}{e}}^j = \overset{(2)}{T}(\bar{\underset{\sim}{e}}_i, \bar{\underset{\sim}{e}}^j)\bar{\underset{\sim}{e}}^i \otimes \bar{\underset{\sim}{e}}_j$$

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— a second-order tensor from the direct point of view.

In order to get a fix on the pattern here, we next consider the less intuitive third-order case. From the multilinear functional point of view, a third-order tensor $\overset{(3)}{T}$ is a trilinear functional $\overset{(3)}{T}: \mathcal{V}_n \times \mathcal{V}_n \times \mathcal{V}_n \to \mathbb{R}$. For each fixed $\underset{\sim}{w} \in \mathcal{V}_n$, $\overset{(3)}{\underset{\sim}{T}}(\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w})$ is a bilinear functional in $\underset{\sim}{u}$ and $\underset{\sim}{v}$; i.e.,

$$\overset{(3)}{T}(\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}) = \overset{(2)}{T}_{w}(\underset{\sim}{u}, \underset{\sim}{v}) \quad \forall \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathcal{V}_n ,$$

where we have called the bilinear functional $\overset{(2)}{T}_{\underset{\sim}{w}}$ to emphasize it's dependence on $\underset{\sim}{w}$. From the analysis of the second-order case, the bilinear functional $\overset{(2)}{T}_{\underset{\sim}{w}}$ corresponds to a second-order tensor $\overset{(2)}{\underset{\sim}{T}}_{\underset{\sim}{w}}$ (from the direct point approach) in such a way that

$$\overset{(2)}{T}_{\underset{\sim}{w}}(\underset{\sim}{u}, \underset{\sim}{v}) = \underset{\sim}{u} \cdot \overset{(2)}{\underset{\sim}{T}}_{\underset{\sim}{w}} \underset{\sim}{v} .$$

Then

$$\overset{(3)}{\underset{\sim}{T}}(\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}) = \underset{\sim}{u} \cdot \overset{(2)}{\underset{\sim}{T}}_{\underset{\sim}{w}} \underset{\sim}{v} \quad \forall \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathcal{V}_n .$$

By the linearity, the second-order tensor $\overset{(2)}{\underset{\sim}{T}}_{\underset{\sim}{w}}$ (direct approach) depends linearly on $\underset{\sim}{w}$ (Exercise A 4.3.2). Thus, by Definition A4.1.1, $\overset{(2)}{\underset{\sim}{T}}_{(\cdot)}$ is a third-order tensor from the direct point of view; i.e.,

$$\overset{(2)}{\underset{\sim}{T}}_{\underset{\sim}{w}} = \overset{(3)}{\underset{\sim}{T}} \underset{\sim}{w} \qquad \forall \underset{\sim}{w} \in \mathcal{V}_n .$$

Then

$$\overset{(3)}{T}(\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}) = \underset{\sim}{u} \cdot \left[(\overset{(3)}{\underset{\sim}{T}} \underset{\sim}{w}) \underset{\sim}{v} \right] \quad \forall \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathcal{V}_n . \ (***)$$

Once again, we see (Exercise A4.3.3) that (***) sets up a one-to-one correspondence between third-order tensors $\overset{(3)}{T}$ from the multilinear functional approach and third-order tensors $\underset{\sim}{\overset{(3)}{T}} \in \overset{(3)}{\mathcal{T}}_n$ from the direct approach.

Evaluation of $\overset{(3)}{T}$ on basis elements yields, e.g.,

$$\overset{(3)}{T}(\underset{\sim}{e}^i, \underset{\sim}{e}^j, \underset{\sim}{e}^k) = \underset{\sim}{e}^i \cdot \left[(\underset{\sim}{\overset{(2)}{T}} \underset{\sim}{e}^k) \underset{\sim}{e}^j \right] = \overset{(2)}{T}{}^{ijk}$$

by Theorem A4.1.3. We call $\overset{(3)}{T}(\underset{\sim}{e}^i, \underset{\sim}{e}^j, \underset{\sim}{e}^k)$ the contravariant components of $\overset{(3)}{T}$ w.r.t. the basis system $\{\underset{\sim}{e}_i\}, \{\underset{\sim}{e}^i\}$, and note that they are exactly the same as the contravariant components of the corresponding element $\underset{\sim}{\overset{(3)}{T}} \in \overset{(3)}{\mathcal{T}}_n$. Other components are defined similarly.

Once again, from either the transformation properties of the components of $\underset{\sim}{\overset{(3)}{T}}$ or the multilinearity of $\overset{(3)}{T}$, we get transformation rules such as

$$\overset{(3)}{T}(\bar{\underset{\sim}{e}}^i, \bar{\underset{\sim}{e}}^j, \bar{\underset{\sim}{e}}^k) = \alpha^i_{\ell} \alpha^j_m \alpha^k_n \overset{(3)}{T}(\underset{\sim}{e}^{\ell}, \underset{\sim}{e}^m, \underset{\sim}{e}^n).$$

Hence, the components of the third-order tensor $\overset{(3)}{T}$ in the multilinear functional approach satisfy the defining transformation rules in the component-transformation rule approach.

As in the lower-order cases, we look at $\overset{(3)}{T}$ from the viewpoint that its components are the components of

$$\overset{(3)}{\mathbf{T}} = \overset{(3)}{T}(\mathbf{e}^i, \mathbf{e}^j, \mathbf{e}^k)\,\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \overset{(3)}{T}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)\,\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k = \overset{(3)}{T}(\mathbf{e}^i, \mathbf{e}_j, \mathbf{e}_k)\,\mathbf{e}_i \otimes \mathbf{e}^j \otimes \mathbf{e}^k = \cdots$$

$$= \overset{(3)}{T}(\bar{\mathbf{e}}^i, \bar{\mathbf{e}}^j, \bar{\mathbf{e}}^k)\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}_j \otimes \bar{\mathbf{e}}_k = \overset{(3)}{T}(\bar{\mathbf{e}}_i, \bar{\mathbf{e}}_j, \bar{\mathbf{e}}_k)\,\bar{\mathbf{e}}^i \otimes \bar{\mathbf{e}}^j \otimes \bar{\mathbf{e}}^k = \overset{(3)}{T}(\bar{\mathbf{e}}^i, \bar{\mathbf{e}}_j, \bar{\mathbf{e}}_k)\,\bar{\mathbf{e}}_i \otimes \bar{\mathbf{e}}^j \otimes \bar{\mathbf{e}}^k = \cdots$$

$$\cdots$$

— a third-order tensor from the direct point of view.

The foregoing should provide enough background for working out the details for any order. The connection between an mth-order tensor $\overset{(m)}{T}$ in the multilinear functional approach and an mth-order tensor $\overset{(m)}{\mathbf{T}}$ in the direct approach is provided by

$$\overset{(m)}{T}(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m) = \mathbf{u}_1 \cdot \left[\left(\cdots\left(\left(\overset{(m)}{\mathbf{T}}\mathbf{u}_m\right)\mathbf{u}_{m-1}\right)\cdots\right)\mathbf{u}_2\right] \quad \forall\, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m \in \mathcal{V}_n,$$

which can be established by mathematical induction.

We close this section with the observation that while the multilinear functional approach to tensors is basis-free, it lacks the "directness" of the direct approach, wherein tensors act on vectors to deliver tensors (one order lower) which are usually of direct interest. E.g., in rigid body dynamics, the inertia tensor in the direct approach acts on the angular velocity vector to give the angular momentum vector, whereas in the multilinear functional approach, the interia tensor acts on base vectors to give components, and we do not get to more interesting physical quantities without introducing additional operations.

[1] Cf. with the components components of $\overset{(m)}{\mathbf{T}}$ in Theorem A4.1.8.

Supplementary Reading

Same as for § A3.1 plus

BISHOP and GOLDBERG, Tensor Analysis on Manifolds

JAUNZEMIS, Continuum Mechanics

WILLS, Vector Analysis with an Introduction to Tensor Analysis