

A PHENOMENOLOGICAL THEORY FOR PLASTIC DEFORMATION OF POLYCRYSTALLINE METALS*

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A consistent phenomenological theory is developed for the plastic deformation properties of a restricted class of polycrystalline metals. The theory introduces the concept of a "hardness state" that can be related to the dislocation "structure" generated in the course of plastic deformation. The conditions that must be satisfied if a plastic equation of state is to hold are investigated, and it is concluded that it is likely that such an equation of state exists. The implications for current microscopic theories are explored.

Some of the current conclusions that are drawn from experimental data are criticized. In particular, it is shown that the neglect of the influences of anelasticity and of grain boundary sliding has led to incorrect interpretations of grain matrix flow properties. Furthermore, the existence of a steady state deformation régime is challenged.

THEORIE PHENOMENOLOGIQUE DE LA DEFORMATION PLASTIQUE DES METAUX POLYCRISTALLINS

L'auteur présente une théorie phénoménologique cohérente pour les propriétés de la déformation plastique d'une catégorie restreinte de métaux polycristallins. La théorie introduit le concept d'un "état de dureté" qui peut être relié à la "structure" des dislocations engendrée au cours de la déformation plastique. L'auteur étudie les conditions à satisfaire pour qu'une équation d'état soit valable dans ce cas, et il conclut qu'il est vraisemblable qu'une telle équation existe. Il recherche les conséquences sur les théories courantes macroscopiques.

L'auteur critique certaines conclusions courantes qui sont tirées des résultats expérimentaux. En particulier il montre qu'on arrive à des interprétations incorrectes des propriétés plastiques de la matrice des grains si on néglige l'influence de l'anélasticité et du glissement des joints de grains. En outre, l'existence d'un régime de déformation stationnaire est discutée.

EINE PHÄNOMENOLOGISCHE THEORIE DER PLASTISCHEN VERFORMUNG POLYKRISTALLINER METALLE

Eine konsistente phänomenologische Theorie der plastischen Verformung einer beschränkten Gruppe polykristalliner Metalle wird entwickelt. Die Theorie führt einen "Härtzustand" ein, der mit der im Verlauf der Verformung erzeugten "Versetzungsstruktur" in Zusammenhang gebracht werden kann. Bedingungen, die erfüllt sein müssen, wenn eine plastische Zustandsgleichung gelten soll, werden untersucht und es wird geschlossen, daß wahrscheinlich eine solche Zustandsgleichung existiert. Die Folgerungen für vorhandene mikroskopische Theorien werden behandelt.

Einige der bisherigen Schlußfolgerungen aus experimentellen Daten werden kritisiert. Insbesondere wird gezeigt, daß die Vernachlässigung des Einflusses der Anelastizität und der Korngrenzengleitung zur falschen Interpretation der Fließeigenschaften der Kornmatrix geführt hat. Außerdem wird die Existenz eines statischen Verformungsbereichs in Frage gestellt.

1. INTRODUCTION

The customary description of the plastic deformation behavior of metals consists largely of collections of tensile stress-strain curves for several extension rates and creep curves for a variety of loading conditions. This mode of description generally serves to emphasize that the flow rate at any given time under some applied load depends on the prior deformation history. The question naturally arises whether it is possible to summarize the prior history at any instant in terms of some readily measurable property of the material so that future deformation response can then be predicted independently of any more detailed knowledge of the prior history. If such a procedure is to be really effectual, the entire description should be possible in terms of well defined deformation variables.

It should be noted that, if such a description is not possible, the phenomenon of plastic deformation of

metals will necessarily appear to be rather capricious, and attempts at theoretical elucidation of microscopic mechanisms then will lack clearly defined goals. In other words what is desired is a reasonably comprehensive phenomenology for the facts of deformation behavior as distinct from the present empiricism.

The purpose of the present paper is to examine the current state of this problem and to propose a tentative solution for a restricted, but nevertheless broad, class of materials and phenomena. It will become evident, as the theory is developed below, that some of the conclusions we shall make about the nature of plastic deformation are in disagreement with commonly held views. It will be concluded, for example, that there is no clear evidence for a steady state deformation régime. Clearly, such a conclusion is of considerable importance from a theoretical point of view, and we emphasize it here simply to underline the magnitude of the contrast that is possible between data considered only empirically and the same data organized into a consistent phenomenology.

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The plan of the paper is as follows: In Section 2, after briefly considering the relationship of the plastic properties to the nature of the test data, the current view will be described and criticized. Section 3 will define the restrictions on the proposed description and will introduce a definition of "hardness." Section 4 will be concerned with the change of hardness during deformation and consider the possible relationships between the strain hardening and steady state behavior. Section 5 will show how predictions are to be made with a theory of this type by means of specific examples. In Section 6 we consider the nature of the experimental data needed for a crucial test of the theory and examine the conclusions that have been drawn from current load change experiments. Section 7 will be devoted to discussion of the relationship of the phenomenology to theories of deformation based on microscopic considerations.

2. THE PROBLEM

The application of a tensile stress σ to a metal specimen results in a total tensile strain ϵ_t that is in general time dependent. The accumulated total strain at any time is the sum of three identifiable components: elastic strain ϵ_e , anelastic strain ϵ_a and plastic strain ϵ_p .

$$\epsilon_t = \epsilon_e + \epsilon_a + \epsilon_p.$$

The time dependence of ϵ_e is determined only by inertia; that of ϵ_a and ϵ_p is determined by relaxation and dissipation and is different in character from that of ϵ_e . The elastic and anelastic strain is recoverable after release of the stress σ , the plastic strain is not recoverable by such load release. These characteristics distinguish the three strain components uniquely.

The laws that are satisfied by the recoverable strain components are relatively simple and are fairly well understood. They are characterized by the fact that the strain components ϵ_e and ϵ_a are good deformation *state variables* of the material. There is a natural unstrained state relative to which the amount of recoverable strain can be measured.

The situation is rather different for the plastic strain. The plastic strain components need not even be compatible with a single valued displacement field in general. Even when an arbitrary reference state is chosen, the mechanical state of a specimen after some further plastic deformation depends on the actual deformation path rather than only on the final state of strain relative to the reference state. Nevertheless, previous attempts at a rational description of plastic properties, in particular those by Ludwik⁽¹⁾ and by Zener and Hollomon,⁽²⁾ have employed ϵ_p as a prominent state variable. Measurements by Dorn *et al.*⁽³⁾

showed that the treatment of Zener and Hollomon would not hold for aluminium if temperature changes are allowed. It can probably be argued that, if all straining is done at one temperature, the Zener-Hollomon treatment *almost* works, and, in fact, common practice in describing deformation properties proceeds as though it did work. In other words, it is customary to describe the state of strain hardening of a specimen in terms of the amount of strain it has undergone. This view is in contradiction with the equally commonly held point of view that there is a steady state régime, for certainly, if a "steady state" is reached, then, by definition of steady state, no significant change will accompany further deformation under either the same stress or strain rate. However, subsequent deformation at a different strain rate must result in an eventual new steady state. That latter steady state can have been reached then by different amounts of strain, depending on how much strain was introduced at the interim steady state.

It is the purpose of the present treatment to show that, although the state of strain hardening of a specimen cannot be described solely in terms of the accumulated strain, there may exist a "plastic equation of state" in somewhat more general terms and that such an equation of state may be rather simple. The existence of such an equation of state is, furthermore, independent of whether or not there exists a steady state deformation régime.

3. THE EQUATION OF STATE

The theory to be presented here will be restricted to deformation histories at a single temperature and to polycrystalline specimens of relatively pure f.c.c. metals. We exclude from consideration discontinuous and inhomogeneous flow processes. Furthermore we purport to describe the behavior of such specimens in a range of strain rates for which grain boundary sliding plays a negligible role. Of considerable importance is the requirement that we will consider relationships only between the stress and the *plastic strain*. The modifications that depend on the anelastic strain components will be considered separately below. Since experimental observations measure all strain components together, the description in the present section may seem to the reader to be occasionally unfamiliar. Such problems, however, are easily settled when the recoverable strain components are added later.

We proceed then to the consideration of a hypothetical, purely plastic material whose properties we shall then show are not in disagreement with what is thus far known for the class of restricted materials

mentioned above. The description will be made in terms of plastic tensile strain ϵ (we now drop the subscript p for simplicity), strain rate $\dot{\epsilon}$, and tensile stress σ . The extension to multi-axial deformation will be reversed for a later publication.

Of the three deformation variables σ , $\dot{\epsilon}$ and ϵ , the two that are uniquely indicative of the mechanical state of the specimen are the first two, σ and $\dot{\epsilon}$. If the imposed strain rate $\dot{\epsilon}$ is stated, the corresponding stress σ to produce $\dot{\epsilon}$ is a measure of the current state of mechanical strength of the specimen. In fact the mechanical state of the specimen at any instant is fully specified by the set of all possible pairs of values of σ and $\dot{\epsilon}$ of which it is immediately capable. We shall define a property that we shall call the *hardness* (we employ the term in the sense that one speaks of strain hardening, and, of course, our term has nothing to do with a hardness indenter index) and shall assign it the parameter symbol y . At this point we shall not give it a definite number value, since as will be evident later, there is a variety of assignments that can be made, and it will seldom be necessary to use it explicitly as a variable for computation. Its property is illustrated by the *state equation* that at any current value of the hardness,

$$\sigma = \sigma(y, \dot{\epsilon}) \quad (1)$$

The parameter y then orders a one-parameter family of σ - $\dot{\epsilon}$ relations of which one is always the characteristic σ - $\dot{\epsilon}$ relation for a specimen at its current hardness. In Fig. 1 we show schematically a few such curves for different states of hardness of a single material. For definiteness, and because the limited data available makes the assumption reasonable, we represent them as parallel curves with slope ν on a plot of $\log \sigma$ vs. $\log \dot{\epsilon}$. The meaning of the curves is that if a specimen, when loaded at the stress σ_1 , deforms at the rate $\dot{\epsilon}_1$,

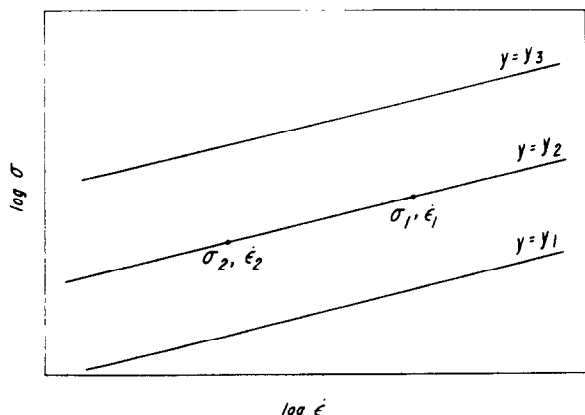
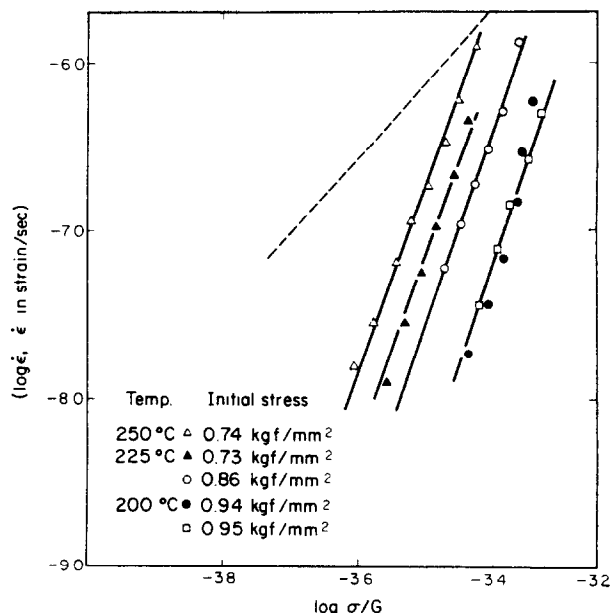


FIG. 1. Several lines of constant hardness are shown schematically. For illustrative purposes they are represented as parallel, straight lines with a common slope ν .



Influence of stress, σ , on steady-state creep rate, $\dot{\epsilon}$, of aluminium. For each plotted point the stress is normalized by dividing by the modulus at the temperature in question.

Solid lines: present work. Each line refers to one specimen only, and the plotted points on each line give the succession of steady creep rates as stress reductions were made.

Broken line: previous work. Each specimen was subjected to creep at a different constant load at 250°C.

FIG. 2. Data plot from the work of Mitra and McLean, Ref. 4.

then, if the stress is changed suddenly to σ_2 , the new plastic rate $\dot{\epsilon}_2$ will be on the line that contains σ_1 , $\dot{\epsilon}_1$. There must be negligible plastic strain during the time of the measurement or else the specimen will have strain hardened further so that it lies on a higher neighboring line.

Tests of this sort were carried out on Ni and Al in an important study by Mitra and McLean⁽⁴⁾ in which they observed a series of relatively steady strain rates following successive step reductions of load in creep. They observed in addition, at each load change, a short time relaxation process which we shall show below is most likely an anelastic strain recovery proportional to the stress change. Their observed values of $\dot{\epsilon}$ vs. σ for Al are shown in Fig. 2. These are substantially the type of curve that we are characterizing as curves of constant hardness. If a plastic equation of state of this form is to hold, it is necessary that, if two samples that have undergone different strain-strain rate histories exhibit the datum point σ_1 , $\dot{\epsilon}_1$, then both must, when the stress is changed to σ_2 , strain at the same rate $\dot{\epsilon}_2$. In other words it must not be possible for two constant hardness curves to intersect. All curves for a given material must be parallel. There is little data available to test this

requirement. Mitra and McLean did produce curves corresponding to more than one initial hardness at several temperatures and obtained curves that were essentially parallel. It is important that their curves are not parallel to the locus of points corresponding to the usual data of stress vs. minimum creep rate. In fact the slopes differ by as much as a factor of two. We shall discuss this work further in a later section.

To return to equation (1) we see now that the proposal of Ludwik and of Zener and Hollomon combines the required behavior we have described above, and the additional assumption that the parameter y that distinguishes one line from another is a function only of the accumulated strain ϵ . The point of the present work is that, if this latter assumption is replaced by a more realistic kinetic equation, there is a fair likelihood that the resultant description will be successful.

4. THE RIGOROUS PHENOMENOLOGY

In this section we proceed to a discussion of the strain hardening process. This can be done in two ways: (1) by describing the rate of change of the current flow stress as a function of strain and strain rate increments (or, where it is pertinent, the rate of change of $\dot{\epsilon}$ as a function of σ and strain increment), or (2) by describing the rate of change of a suitably selected hardness y as a function of any of the controlled deformation variables. Both procedures are equivalent, and both are useful.

We shall first develop the necessary background for both procedures.

All the deformation histories in which we are interested can be described incrementally by the differential form

$$d \ln \sigma = \gamma d\epsilon + \nu d \ln \dot{\epsilon} \quad (2)$$

This form simply describes the smooth variation of any of the deformation variables in terms of the others. In general, the coefficients γ and ν at any stage in the deformation process depend on the current values of at least two of the variables σ , ϵ and $\dot{\epsilon}$ and also on the previous deformation history. We choose $\ln \sigma$ and $\ln \dot{\epsilon}$ as variables rather than σ and $\dot{\epsilon}$ because of practical simplicity. This description is the same as that employed by Hart⁽⁵⁾ in an earlier study with the difference that what was there called m is here called ν . The reason for this change in notation is that, as we shall see below, there is more than one important rate sensitivity coefficient and the common use of the letters m or n does not make this distinction. For any particular deformation path, γ and ν can be identified

as the expressions

$$\gamma \equiv (\partial \ln \sigma / \partial \epsilon) \quad (3)$$

and

$$\nu \equiv (\partial \ln \sigma / \partial \ln \dot{\epsilon})_{\epsilon} \quad (4)$$

It will be seen immediately that, if a plastic equation of state of the form described in the previous section exists, then ν is the same as the slope at each point of the constant hardness lines of Fig. 1, and, furthermore, ν is then independent of history and depends only on the current value of σ and $\dot{\epsilon}$.

The conditions for the existence of a plastic equation of state and a clue to its explicit form can be obtained by a discussion of the *integrability* of the differential form of equation (2).

With a view to the desired form of the ultimate relationship we rewrite equation (2) in the form

$$d\epsilon = \alpha d \ln \sigma + \beta d \ln \dot{\epsilon} \quad (5)$$

where

$$\alpha \equiv 1/\gamma \quad (6)$$

$$\beta \equiv -\nu/\gamma \quad (7)$$

We now seek the conditions under which equation (5) possesses integral solutions *that are independent of the path of integration*. If such solutions exist the differential form is a perfect differential.

The *first condition* that must be satisfied is that α and β (and therefore γ and ν) be unique functions of σ and $\dot{\epsilon}$. In other words γ and ν must be of the form

$$\gamma = \gamma(\sigma, \dot{\epsilon}) \quad (8)$$

$$\nu = \nu(\sigma, \dot{\epsilon}) \quad (9)$$

If equations (8) and (9) are satisfied equation (5) is known as a *Pfaffian form* and it is directly integrable if and only if

$$(\partial \alpha / \partial \ln \dot{\epsilon})_{\sigma} = (\partial \beta / \partial \ln \sigma)_{\dot{\epsilon}} \quad (10)$$

Then if equation (10) holds there will exist a state relation of the form

$$\epsilon = \epsilon(\sigma, \dot{\epsilon}) \quad (11)$$

or equivalently

$$\sigma = \sigma(\epsilon, \dot{\epsilon}) \quad (12)$$

This is the form of the Ludwik and Zener and Hollomon equation of state, and again we see that it involves *two* explicit requirements, viz. equations (8) and (9) and equation (10).

Now, since equation (5) is a form in only two independent variables, the theory of Pfaffian forms⁽⁶⁾ shows that, so long as at least equations (8) and (9) hold, and *even if equation (10) does not hold*, there always exists an integrating factor of the form $F(\sigma, \dot{\epsilon})$ such

that

$$Fd\epsilon = \alpha F d \ln \sigma + \beta F d \ln \dot{\epsilon} \quad (13)$$

is a perfect differential. The original Pfaffian form is then said to be integrable, and we can then write

$$\begin{aligned} dy &= F d\epsilon \\ &= \alpha F d \ln \sigma + \beta F d \ln \dot{\epsilon} \end{aligned} \quad (14)$$

And so there exists a family of integral curves

$$y = y(\sigma, \dot{\epsilon}) \quad (15)$$

where y can now be identified as what we called the hardness parameter. Furthermore, there is an infinite number of integrating factors and so the specific choice of y need not be unique although the curves are unique. Finally for each fixed value of y the corresponding curve given by equation (15) is a solution of the Pfaffian equation

$$dy = 0$$

or, equivalently,

$$d\epsilon = 0 \quad (16)$$

and so all points along each integral curve are connected only by paths for which $d\epsilon = 0$ in consonance with our prescription in Section 3. Thus the integral curves satisfying equation (16) are our curves of constant hardness.

We can now summarize the outcome of this analysis as follows: A necessary and sufficient condition for the existence of a plastic equation of state is that γ and ν be functions only of σ and $\dot{\epsilon}$ (at a single temperature). If the strain is to be a good state variable it is further necessary that equation (5) be a perfect differential.

We shall now discuss the two procedures stated at the beginning of this section.

The first procedure follows immediately from the satisfaction of the integrability requirement, equations (8) and (9). It is first necessary to determine by experiment whether those equations hold and then to determine their explicit form. Given that this has been accomplished, equation (2) becomes

$$d \ln \sigma = \gamma(\sigma, \dot{\epsilon}) d\epsilon + \nu(\sigma, \dot{\epsilon}) d \ln \dot{\epsilon} \quad (2')$$

and this differential equation can then be integrated over any prescribed deformation path to predict the mechanical behavior of the material. We shall do that in Section 5 for a variety of familiar test situations and for an assumed form of γ and ν . It can be seen that, even when ϵ is not a state variable, it can still be handled as a path variable with no difficulty.

It is occasionally convenient to use the hardness y as an explicit state variable. First, however, it is

necessary to select, among the many possible equivalent choices for y , one choice that is well suited for both theoretical and practical purposes. The choice that we shall make is arrived at in the following way. If in equation (8) γ is assigned a constant value, a locus of σ - $\dot{\epsilon}$ points is defined at each of which γ has the same value. In particular if we choose the locus $\gamma = 1$, we have determined a curve at each point of which

$$(\partial \sigma / \partial \dot{\epsilon})_{\gamma} = \sigma \quad (17)$$

The point at which equation (17) is satisfied is of course very nearly the point of plastic instability in tensile deformation. Furthermore in the constant extension rate test it is substantially the point of load maximum, and in the constant load creep test it is the point of minimum strain rate. It is therefore a readily obtained and commonly available set of data, and it defines a $\log \sigma$ - $\log \dot{\epsilon}$ curve that is substantially a straight line. It is the slope of that line that is commonly taken as the strain rate sensitivity. Furthermore, it is, in much of the current literature, loosely characterized as the steady state locus. The actual steady state locus, if one exists, is of course the line corresponding to $\gamma = 0$. We shall have more to say on this last point later. We have plotted in Fig. 3 the curve $\gamma = 1$ together with several curves of constant hardness. We assign the symbol μ to the slope of the $\gamma = 1$ curve. In general $\mu > \nu$. Clearly, Fig. 3 is in complete agreement with the data of Mitra and McLean shown in Fig. 2. The additional curves shown parallel to the $\gamma = 1$ line are related to

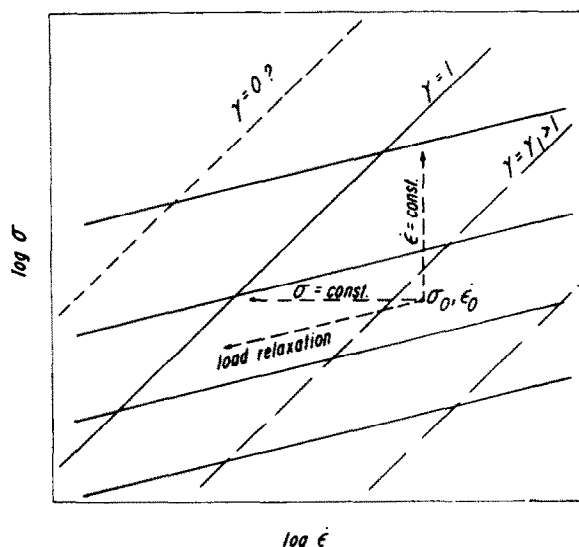


FIG. 3. Schematic "equation of state" plot showing lines of constant γ intersecting lines of constant $\dot{\epsilon}$. The labelled dashed lines issuing from the point $\sigma_0, \dot{\epsilon}_0$ are the loci for the three deformation histories described in the text.

the assumed complete model we shall discuss below and are lines for other values of γ . Since every constant hardness line intersects the $\gamma = 1$ line we now choose the value of $\dot{\epsilon}$ at that intersection as the value of y . We could equally well have chosen the value of σ at the intersection, but the choice given here is mathematically convenient. Note that the existence of such intersections means that a specimen at any hardness level whatsoever always can be made to exhibit a value of $\gamma = 1$ by choosing the correct value of $\dot{\epsilon}$. In other words there is in principle always a test that can be made to measure the current value of y , i.e. it has an operational meaning.

We may now reduce the content of Fig. 3 to analytic form. If $\sigma_1(\dot{\epsilon})$ is the functional form of the line $\gamma = 1$, we have

$$\sigma_1 = \alpha \dot{\epsilon}^\mu \quad (18)$$

where α is a constant at any one temperature. It is easily verified that the equation for the line corresponding to any given value of y is

$$\sigma = \alpha y^{\mu-r} \dot{\epsilon}^r \quad (19)$$

We may note also that from its definition

$$\gamma = (\mu - r)(\partial \ln y / \partial \ln \dot{\epsilon})_{\epsilon} \quad (20)$$

Since y necessarily has the same value over any single integral curve,

$$(\partial \ln y / \partial \ln \dot{\epsilon})_{\epsilon} \equiv 0$$

and so the derivative in equation (20) can be replaced by a total derivative. Then

$$\gamma = (\mu - r)(d \ln y / d \epsilon) \quad (21)$$

and so as soon as we know how γ depends on σ and $\dot{\epsilon}$ we know how y changes with strain increments.

In the foregoing discussion we have maintained the temperature T fixed. We now explicitly add the assumption that y is unchanged at T_1 by an excursion to another temperature T_2 followed by a return to T_1 so long as $d\epsilon = 0$ during the process. Then T will appear simply as a parameter and γ , μ , r and α will in general depend on T parametrically. Because of our mode of measuring y it will, even for a fixed state of hardness, depend on T . It is possible, however, to generalize our treatment so that T is treated explicitly as a state variable. When that is done it is possible to define the hardness state variable y in such a way that it is not temperature dependent just as it at present is not strain rate dependent. That procedure, however, requires the knowledge of the quantity

$$Q \equiv [\partial \ln \sigma / \partial (1/kT)]_{\dot{\epsilon}, \epsilon}$$

as a complete function of σ , $\dot{\epsilon}$ and T , and this is not available from the existing data. We, therefore, adhere in the present paper to the parametric approach. The more general treatment, just described, will be explored further in a subsequent publication.

5. A SPECIFIC MODEL

In this section we shall discuss the manner in which test results are predicted within the framework of the theory set forth above. In principle, all deformation histories can be obtained from integration of equation (2') once the functional forms of γ and r are known. For illustrative purposes we shall make an explicit assumption about the nature of those functions. This assumption, of course, affects only the special form of the solutions in the examples considered below, whereas the general theory will admit any functional forms for γ and r that are able to be determined experimentally or that can be predicted by microscopic theory. Nevertheless, the author believes that the one used here is in fact appropriate to the simple f.c.c. metals.

We postulate, then, that

$$r = \text{constant} \quad (22)$$

and

$$\gamma = (\sigma_1 / \sigma)^{1/\lambda} \quad (23)$$

where σ_1 is given by equation (18), and λ is a constant with a value generally around $\frac{1}{5}$. For clarity we recapitulate the other equations that are part of our model.

$$d \ln \sigma = \gamma d \epsilon + r d \ln \dot{\epsilon} \quad (2)$$

$$\sigma_1 = \alpha \dot{\epsilon}^\mu \quad (18)$$

There are for any given material four constant parameters in the model, viz. α , μ , r and λ .

We shall now compute the resulting behavior for tests at constant stress, constant strain rate and stress relaxation at fixed extension.

I. Constant strain rate

This case, which is not exactly the same as constant extension rate, yields, of course, the ideal tensile stress-strain curve. Since $\dot{\epsilon}$ is constant, equation (2) becomes

$$d \ln \sigma = \gamma d \epsilon \quad (24)$$

and σ_1 has the constant value given by equation (18). Then

$$d \ln \sigma = (\sigma_1 / \sigma)^{1/\lambda} d \epsilon \quad (25)$$

We must still specify what the state of hardness of the specimen was when the straining commenced. At the value of $\dot{\epsilon}$ we employ this can be done by specifying the initial value of σ or of γ . However, if we wish to compare identical specimens at several strain rates we

must know the corresponding values at each strain rate. This is most easily done by determining or specifying the initial value of y , which is common to all the specimens, and employing equation (19) to then determine the initial value of σ at each value of $\dot{\epsilon}$. We shall designate the values of σ , γ and y when $\epsilon = 0$ as σ_0 , γ_0 and y_0 .

We can readily integrate equation (25) to obtain

$$\sigma = \sigma_1[(\sigma_0/\sigma_1)^{1/\lambda} + (\epsilon/\lambda)]^\lambda \quad (26)$$

It is readily verified that equations (18) and (19) yield the result

$$\sigma_0/\sigma_1 = (y_0/\dot{\epsilon})^{\mu-\nu} \quad (27)$$

and so we may write equation (26) in the explicit form

$$\sigma = \alpha \dot{\epsilon}^\mu [(y_0/\dot{\epsilon})^{(\mu-\nu)/\lambda} + (\epsilon/\lambda)]^\lambda \quad (28)$$

Substitution of equation (28) into equation (23) also gives the result

$$\gamma = [(y_0/\dot{\epsilon})^{(\mu-\nu)/\lambda} + (\epsilon/\lambda)]^{-1} \quad (29)$$

In order to make the result more concrete, we have plotted equation (28) for assumed values $\lambda = 2\mu = 4\nu = 0.2$ for a few strain rates in Fig. 4. These values

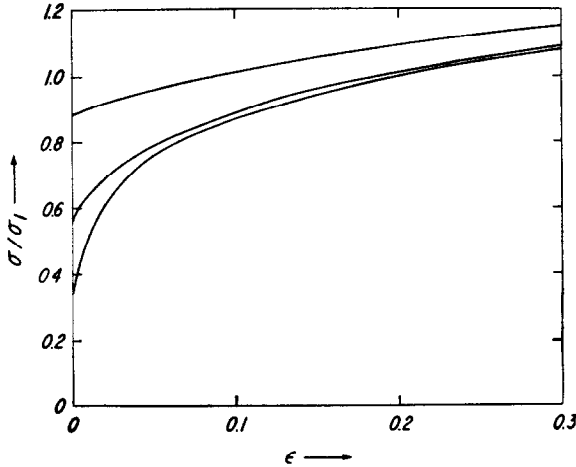


FIG. 4. Constant strain rate tensile curves as computed from equation (28). The strain rate ratio between adjacent curves is 10^{-4} . The uppermost curve is for the lowest strain rate. Note that each curve is plotted relative to the value of σ at $\gamma = 1$ as unity.

are not inconsistent with the data available for Al at moderate temperatures. The reader is cautioned at this point to remember that the curve deduced here is not to be compared immediately with an experimental curve at constant extension rate. The reason is that the anelastic strain has not been included. That component affects only the first few per cent of strain since the plastic strain rates does not come up to its full value until the anelastic strain rate contribution

drops to a low value, and this effect simulates a spurious, high initial hardening rate.

The principal features of the stress-strain curve we have computed is that it is a power law with a displaced origin for the strain scale that depends strongly on the strain rate. Note that the curve for very low strain rate is a typical "high temperature" curve with very low strain hardening rate. Despite the ostensible qualitative differences in the curves, all of them obey the same law, and none of them will reach a steady state.

II. Constant stress

For this case $d \ln \sigma = 0$, and equation (2) becomes

$$0 = \gamma d\epsilon + \nu d \ln \dot{\epsilon} \quad (30)$$

Now from equation (23)

$$\begin{aligned} \gamma &= (\sigma_1/\sigma)^{1/\lambda} \\ &= (\alpha \dot{\epsilon}^\mu / \sigma)^{1/\lambda} \end{aligned} \quad (31)$$

If we express the applied stress σ in terms of $\dot{\epsilon}_1$, the strain rate that will be reached when $\gamma = 1$ at the stress σ , equation (31) can be written in the simpler form

$$\gamma = (\dot{\epsilon}/\dot{\epsilon}_1)^{\mu/\lambda} \quad (32)$$

where $\dot{\epsilon}_1$ is simply a measure of the applied stress and is given by

$$\sigma \equiv \alpha \dot{\epsilon}_1^\mu \quad (33)$$

Now equation (30) can be written in a simple form for integration as

$$d(\epsilon/\nu) = -(\dot{\epsilon}/\dot{\epsilon}_1)^{-1-(\mu/\lambda)} d(\dot{\epsilon}/\dot{\epsilon}_1) \quad (34)$$

If $\dot{\epsilon}_0$ is the initial value of the strain rate, this integrates readily to

$$(\dot{\epsilon}_1/\dot{\epsilon})^{\mu/\lambda} = (\dot{\epsilon}_1/\dot{\epsilon}_0)^{\mu/\lambda} + (\mu/\lambda\nu)\epsilon \quad (35)$$

where $\dot{\epsilon}_1$ is given by equation (33) and, if y_0 is the initial hardness, $\dot{\epsilon}_0$ is given by

$$\sigma = \alpha y_0^{\mu-\nu} \dot{\epsilon}_0^\nu \quad (36)$$

The apparent shape of the curve of $\dot{\epsilon}$ vs. ϵ is determined by the value of the additive term $(\dot{\epsilon}_1/\dot{\epsilon}_0)^{\mu/\lambda}$. For a given specimen (i.e. for fixed y_0) the value of that term depends strongly on the value of σ . Since $\mu > \nu$, the additive term is small for large σ , and *vice versa*. The two different extremes simulate the presence or absence of sizable amounts of "primary creep".

Since $\dot{\epsilon} \equiv d\epsilon/dt$, equation (35) can be integrated once more to obtain ϵ as a function of t . We present the result here for completeness.

$$\begin{aligned} (\mu/\lambda\nu)\dot{\epsilon}_1 t &= [1 + (\lambda/\mu)]^{-1} \{ [(\dot{\epsilon}_1/\dot{\epsilon}_0)^{\mu/\lambda} \\ &\quad + (\mu/\lambda\nu)\epsilon]^{1+(\lambda/\mu)} - (\dot{\epsilon}_1/\dot{\epsilon}_0)^{1+(\mu/\lambda)} \} \end{aligned} \quad (37)$$

Although this expression is somewhat messy, it is actually quite simple. It has the interesting feature that at sufficiently large values of t

$$\varepsilon \sim t^{1/[1 + (\lambda/\mu)]} \quad (38)$$

which yields $t^{1/3}$ when $\lambda = 2\mu$.

III. Stress relaxation test

In the stress relaxation test the cross-head of a tensile testing machine is held fixed after a prior interval of constant extension velocity. The specimen then continues to undergo plastic straining under the influence of the stress due to the residual elastic strain in the specimen and the load measuring cell. As the specimen extends plastically, the stress drops due to the relaxation of the elastic strain in such a way that

$$\dot{\sigma} = -\kappa \dot{\varepsilon} \quad (39)$$

where κ is the appropriate elastic coefficient for the specimen and machine. If κ is large (a stiff machine) the strain accomplished during the subsequent history will be very small and so the specimen will experience a σ - $\dot{\varepsilon}$ history that essentially follows a line of constant hardness. This test has been described by several authors, and a detailed description can be found in Ref. 5.

We shall now relate the test to the parameters of our theory.

From equation (39) we obtain directly the two relationships

$$d\sigma = -\kappa d\varepsilon \quad (40)$$

and

$$d \ln \sigma = d \ln \dot{\varepsilon} \quad (41)$$

We substitute these mechanical constraint relationships in equation (2) to obtain

$$\begin{aligned} d \ln \sigma &= -(\gamma/\kappa) d\sigma + \nu d \ln \dot{\sigma} \\ &= -(\gamma\sigma/\kappa) d \ln \sigma + \nu d \ln \dot{\sigma} \end{aligned} \quad (42)$$

or, solving for ν ,

$$\nu = (d \ln \sigma / d \ln \dot{\sigma}) [1 + (\gamma\sigma/\kappa)] \quad (43)$$

In a hard machine $\kappa \gg \sigma$, and, furthermore, as $\dot{\varepsilon}$ decreases rapidly, γ also becomes quite small. Therefore, to a very good approximation

$$\nu = d \ln \sigma / d \ln \dot{\sigma} \quad (44)$$

The stress relaxation test is then a very good test for tracing out constant hardness curves and accomplishes by a continuous variation what was attempted by the discrete load change tests.

The calculation presented here does not include a

correction of the early part of the relaxation curve for anelasticity.

If desired, of course, the relaxation data can be reduced to an explicit plot of σ vs. $\dot{\varepsilon}$ at constant hardness by using equation (39) and a measurement of κ .

Although a detailed analysis of the fit to experimental data has not yet been carried out, the author believes that the temperature dependence of the parameters of the theory may be relatively simple. He anticipates that the temperature dependence of μ , ν and λ may be only slight, and that the main variation with temperature will be found for α and for the value of γ for any given state of a specimen.

6. RELATIONSHIP TO EXPERIMENTAL DATA

In order to discuss the relationship of the proposed theory to the results of experiments it is first necessary to consider in a little more detail the effect of anelastic phenomena in the experiments. Our concern here is not to examine the detailed nature of anelasticity but rather to show the way in which it can mask some of the facts of plastic deformation. The phenomenon itself is discussed fully in the excellent monograph by Zener.⁽⁷⁾

Anelastic strain ε_a attendant upon the application of a stress σ to a specimen can generally be characterized by laws that are linear in ε_a , σ and their time derivatives. The progress of ε_a with time is generally (a) exponential with at least one time constant, (b) proportional to σ at full relaxation, and (c) is reversible upon reduction of σ . The relaxation time does not depend on σ . The physical basis for anelastic relaxation can be as diverse as viscous grain boundary relaxation, stress induced ordering of solutes, and short distance displacement or bowing of dislocations. Since the effects can be produced by small displacements of dislocation arrays in cell walls that result from plastic deformation, the magnitude of the effect can depend to some extent on prior plastic deformation as well, but at any one stage it still retains its basic properties outlined above and has no direct relationship to the nature of the plastic straining process. It is essential, therefore, that the anelastic strain component be identified before one attempts to make detailed interpretation of the essentially irreversible plastic component.

For illustrative purposes we shall show that the recovery time measured by Mitra and McLean⁽⁴⁾ can be explained quantitatively, entirely on the basis of an anelastic relaxation and *no recovery of strain hardening*. As related above, they deformed a tensile specimen in creep for an initial time interval and then dropped the stress σ_1 by an amount $\Delta\sigma$. The

strain rate then dropped abruptly to a very low value after which it gradually rose to a new relatively steady value $\dot{\epsilon}_2$ that was less than the value $\dot{\epsilon}_1$ at the former stress σ_1 . They measured a time Δt that was the interval from the time of stress change to the intersection of the time axis by a straight line drawn through the strain-time points when the strain rate had become steady. In other words, if $t = 0$ when the stress was changed, the strain as a function of t ultimately become asymptotic to

$$\epsilon(t) = \dot{\epsilon}_2(t - \Delta t) \quad (45)$$

Now, if it is assumed that, upon the change $\Delta\sigma$, the plastic strain rate immediately becomes $\dot{\epsilon}_2$, and the anelastic recovery is $\Delta\epsilon_a(t)$, the incremental strain after $t = 0$ would at all values of t be

$$\epsilon(t) = \dot{\epsilon}_2 t + \Delta\epsilon_a(t) \quad (46)$$

Furthermore $\Delta\epsilon_a$ will after a relaxation time reach the value

$$\Delta\epsilon_a = -(\alpha G)^{-1} \Delta\sigma \quad (47)$$

where α is a constant of the order of unity since relaxed anelastic moduli are comparable to the elastic moduli. Then, taking the value of equation (46) at large times and comparing it with equation (45) we see that Mitra and McLean's Δt can be expressed as

$$\begin{aligned} \Delta t &= -\Delta\epsilon_a / \dot{\epsilon}_2 \\ &= \Delta\sigma / \alpha G \dot{\epsilon}_2 \end{aligned} \quad (48)$$

They then formed the quantity r that they termed the recovery rate as

$$r/G \equiv \Delta\sigma / G \Delta t \quad (49)$$

With our interpretation this is computed to be

$$r/G = \alpha \dot{\epsilon}_2 \quad (50)$$

This last result agrees quantitatively with their measurements.

We see, therefore, that the Δt so determined has nothing to do with either stress recovery or even with the anelastic relaxation time. It is instead a measure of the ratio of the total anelastic recovery to the final strain rate.

This point has been discussed in detail because it has strong bearing on the interpretation of all experiments of this type. It should be easily possible in such experiments to measure the anelastic strain in the absence of appreciable plastic straining by interrupting the test by reverting to considerably lower stress levels. It should then be possible to observe the successive introduction and recovery of anelastic

strain by repeated cycling between two stress holding levels. This in fact is what is occasionally done monotonically in the course of creep tests and is known as creep recovery. The large magnitude of the effect is discussed in the chapter on "anelastic creep" in the book by Lubahn and Felgar.⁽⁸⁾ They show the dramatic change in the appearance of creep curves when the anelastic, or recoverable, strain is measured and subtracted from the total strain. The important modification occurs, of course, in the early part of the curve and makes the shape flatter.

Substantially, the same phenomenon is present in the constant extension rate tensile test. Part of the strain during the initial loading is anelastic and so the plastic strain rate does not rise to the value of the imposed strain rate until most of the anelastic straining is concluded. The same phenomenon is probably responsible for the hysteretic behavior attendant on interrupted tensile testing in which straining is interrupted and the load reduced to zero followed by resumption of straining (cf. Ref. 8, chapter on Bauschinger effect).

In order to be able to sort out the recoverable strain from the total strain it is necessary to collect data that will determine the nature of the anelastic behavior for each system with which one deals. The data from conventional tests can then be analyzed to determine the character of the purely plastic strain behavior. It is, of course, the plastic behavior that the author believes will be susceptible to description by a theory of the form that has been developed above.

We do not claim that there cannot be additional short time relaxation effects associated with the change of the plastic strain rate accompanying a stress change, but only that the effects associated with recoverable strain must first be accounted for before the remainder has been identified. In the case of the measurements of Mitra and McLean there does not seem to be any remainder. We shall discuss other possible relaxations a bit further in Section 7.

It is of further importance that the data that is intended to elucidate grain matrix deformation properties, with which we are here concerned, should not have sizable contributions from grain boundary sliding. There is, after all, no reason to believe that a simple behavior or a single theoretical description will characterize data that represents composite phenomena. The marked effects that can be produced by grain boundary sliding were clearly exhibited by Servi and Grant⁽⁹⁾ and are even more dramatically evident in the current literature on "superplasticity". Nevertheless, most of the high temperature data that is freely interpreted in terms of purely matrix flow

mechanisms lie wholly in the grain boundary sliding régime or in the transition range. This has obscured the understanding of the matrix deformation processes at elevated temperatures. It is in fact probable that those processes are not appreciably different from low temperature behavior even up to an appreciable fraction of the melting temperature.

The data with which we are concerned, then, is for coarse grained specimens and generally at rather high strain rates or at moderate temperatures.

There is reason to believe that, if we can adequately describe the matrix flow properties and the grain boundary sliding properties separately, then it will be possible to produce a theory that describes the combined operation of both mechanisms. An attempt at such a theory has been made by Hart⁽¹⁰⁾ for the presumed "steady state" behavior. A similar treatment should be possible for the entire strain hardening régime.

We shall conclude this section with a brief discussion of the problem of the "steady state". Considering the importance of the implications that arise from the existence of a steady state, it is amazing that its existence has been rather generally accepted with such slim experimental evidence. In tensile tests the specimen becomes unstable against necking as soon as $\gamma = 1$. Certainly, no conclusion of a crucial nature can be drawn concerning the strain hardening rate beyond that point. It is not usually recognized that the constant nominal stress creep test is no better in this regard than the constant load test, and it does not have the built in indicator of the creep rate minimum to signal that point as the latter test does. Such tests in fact enter the range of instability quite as early as the constant load test and the apparent flatness of behavior at larger strains is misleading. Cavity formation at grain boundaries can produce further spurious strength reduction at large strains that has nothing to do with the matrix flow behavior. Finally, the putative steady state is generally identified by a guess that it has "almost been reached" rather than by a quantitative criterion.

A simple reduction in strain hardening rate to some low value is not a sufficient characterization of the approach to steady state behavior as we have shown by our model above. Most theories that describe a steady state property imply a rate of approach to the steady state that is exponential in time or in the deformation path variable. In fact it is very hard to construct a theory that does not have such a prediction. There is absolutely no evidence of such behavior in the data for metal deformation. Even the tests of Taylor and Quinney⁽¹¹⁾ in compression of copper

showed considerable additional hardening beyond the point that is usually taken as the steady state and showed no cessation of hardening as far as the test was carried. Such behavior more nearly resembles the power law exhibited above rather than an exponential one.

We claim, then, that there is no convincing evidence in the literature for the existence of a steady state regime. This point is not a trivial one since it has a profound effect on the nature of both phenomenological and microscopic theoretical description.

Despite the strong wording of the foregoing paragraphs it is possible, as is shown in the next section, that many of the current theoretical ideas concerning the "steady state" creep may nevertheless have application in a broader context.

We also note, parenthetically, that the existence of a steady state is not at all in contradiction with the general phenomenology of Section 4. If for example $\gamma(\sigma, \dot{\epsilon})$ is of the form

$$\gamma = \lambda_1[(\sigma_s/\sigma)^{1/\lambda_2} - 1] \quad (51)$$

where σ_s is the steady state stress given by

$$\sigma_s = \alpha_s \dot{\epsilon}^\mu \quad (52)$$

and λ_1 , λ_2 and α_s are constants, steady state behavior will be described. We do not pursue this further.

7. RELATIONSHIP TO MICROSCOPIC THEORY

As claimed in the Introduction, a phenomenological theory can provide a proper framework within which to construct and test theories based on microscopic mechanisms. We shall now show that our theory implies some restrictions on theories that have not generally been recognized. In particular it has strong implications concerning the effect of the dislocation substructure that is developed during plastic deformation.

In order to make the discussion concrete we shall present it in terms of the specific model of Section 5. However, it should be recognized that the features we are concerned with are actually properties of the general theory and do not depend on any specific functional form of γ .

We first rewrite equation (19) in a form that is more usual for microscopic theories as

$$\dot{\epsilon} = \alpha^{-1/r} \sigma^{1/r} y^{-(\mu-r)/r} \quad (53)$$

Furthermore, from equations (18), (21) and (23) we may write the kinetic equation satisfied by y as

$$d \ln y / d\epsilon = (\mu - r)^{-1} (\dot{\epsilon} / y)^{(\mu-r)/\lambda} \quad (54)$$

Now any one value of y at a single temperature T

represents a fixed hardness state of the specimen, and from the fundamental standpoint it represents a particular configuration of dislocation substructure. As noted above, the parameter y has a value that depends on T even if the hardness state is unchanged. This is only because of our development of the theory in a special form that treated T as a parameter and of our particular choice for a measure of y . Thus, if we had been more detailed and had sorted out T as a state variable, we would have found a hardness variable that was independent of T and $\dot{\epsilon}$ just as y is independent of $\dot{\epsilon}$. Now the point is that the existence of y shows the existence of a well defined deformation state property that is the same as what is commonly called "structure". The resulting strain rate $\dot{\epsilon}$ then depends on the stress σ and the structure y , and on T through α and y .

In the theory of steady state creep by dislocation climb of Weertman,⁽¹²⁾ the structure parameter is the dislocation source density. Since that parameter is treated as fixed, the theory is addressed in principle to describing the constant structure or constant hardness lines and cannot be applicable to lines of constant y . The steady state theory for motion of jogged screw dislocations of Barrett and Nix⁽¹³⁾ employs as structure parameter the "density of mobile dislocations". That parameter is in fact prescribed to have different values along the "steady state" line, and its value is determined by separate measurements. The theory however does not contain a kinetic law relating that structure parameter to deformation history. Furthermore, the stress exponent contributions from the mobility factor and from the structure factor do not separately agree with the exponent differences between constant structure and constant y lines as measured by Mitra and McLean. The "dislocation dynamics" theory of Johnston and Gilman,⁽¹⁴⁾ which is the prototype for the Barrett and Nix theory, also employs the mobile dislocation density as a structure parameter and is the only theory that explicitly prescribes a history dependence for the parameter.

It is clear, in addition, that the only steady state feature of the theories that are so characterized is that which has to do with the dislocation flow rate at any *fixed* or *prescribed* structure, and so the theories actually neither predict nor require steady state behavior in the sense of saturation of work hardening as we discussed in the preceding section. The possible additional short time relaxations attendant upon stress or temperature changes, over and above those due to anelastic strain recovery, could in principle be computed from either of the "steady state"

theories, but they would clearly have nothing to do with structural strain hardening as such.

Thus the phenomenological theory places stronger bounds on the form of the microscopic theory than those determined only empirically.

We conclude, therefore, that, while any of the proposed rate limiting mechanisms may be in some way operative, the theories that have been constructed on them are incomplete and probably too simple, and, in addition, they have frequently been addressed to the wrong data.

In general terms we may then state the form or shape that a theory of metal matrix deformation must take if it is to satisfy the requirements of the phenomenological theory. It is that

$$\dot{\epsilon} = \dot{\epsilon}(\sigma, Z, T) \quad (55)$$

and

$$dZ/d\epsilon = f(\dot{\epsilon}, Z, T) \quad (56)$$

where Z is some dimension or moment of the dislocation structure that figures as a sensitive element of the theory and as a well defined state variable. Any such theory can then readily be put into the form we have discussed above and will be capable of predicting all pertinent deformation histories.

8. CONCLUSIONS

We have concluded that there probably exists a plastic equation of state for the stress and plastic strain rate for the crystal matrix in polycrystalline specimens of a restricted class of materials. The necessary and sufficient conditions for the existence of a hardness state variable have been deduced. When such a state variable exists it can be identified with some strength determining property of the dislocation substructure that is developed in the course of plastic deformation. The kinetic law followed by the hardness state variable has been derived and its relationship to strain hardening rates has been shown.

We have further concluded that the current microscopic theories differ in the way they relate to this phenomenological analysis. In some way all of them are incomplete and some are not fully consistent with the restrictions imposed by the phenomenology.

We have also concluded that the concept of steady state deformation is presently misapplied to the experimental data, and that the concept, in the form in which it is currently understood, is irrelevant to the theories.

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