

A New Method for Solving Solid Structures

Jiang Ke*

Department of Civil Engineering, Shaanxi University of Technology, Hanzhong, 723001 China

Email: kj5525@sina.com

Abstract A new method for solving an isotropic, linear, elastic solid structure is presented. Based on the simple knowledge of the mechanics of materials, the model considers a new element called the “Ke” element. This element is a truss, and the solid structure is considered to consist of many such “Ke” elements. Thus the solid structure becomes a truss structure, and the internal forces and displacements of the solid structure can be found by calculating this truss structure. The physical sense of this new method is clear, and it is very simple and wonderful. The paper not only provides a new method for solving a solid structure, but also reveals the mechanical mechanism of a solid body.

Key words: solid structure, theory of elasticity, mechanics of materials, Ke element

INTRODUCTION

The theory of elasticity is a mechanics for studying elastic (mainly isotropic, linear, elastic materials) solid structures. For the calculation of linear elastic solid structures using the knowledge of the theory of elasticity, an exact solution based on the analytical method can only be obtained for a few easy problems. Therefore, a numerical method, for example, the finite element method or the finite difference method, is generally used for most problems. These two methods are very complicated [1]. Based on the simple knowledge of the mechanics of materials [2], a simple new method for solving an isotropic, linear, elastic solid structure is presented here.

NEW METHOD

One infinitesimal regular hexahedron element (Fig. 1) is arbitrarily taken from the solid structure, the theory of elasticity being based on the static equilibrium, deformation and constitutive relations of the element. First, a model of a new element, called here the Ke element, is set up. The Ke element includes the Ke-1 element (Fig. 2) and the Ke-2 element (Fig. 4(a)). The Ke-1 element (which is adopted for spatial elastic problems) is a space truss and the Ke-2 element (which is adopted for plane elastic problems) is a plane truss. The Ke-1 element (Fig. 2) and the regular hexahedron element (Fig. 1) are the same in shape and dimension, the moduli of elasticity of their material are the same, and the deformations of the two elements under equivalent external forces are the same. Moreover, the deformation of the regular hexahedron element under an external force obeys the generalized Hooke’s law for an isotropic elastic material. According to this condition, the cross-sectional area of each bar of the Ke-1 element can be determined, then Ke-1 element model is set up. The methods for setting up the Ke-2 element model and the Ke-1 element model are similar. Then the solid structure can be divided into many Ke elements, the neighboring Ke elements being connected through their common nodes. In this way the solid structure becomes a truss structure, and its internal forces and displacements can be obtained by calculating this truss structure.

1. Ke-1 Element The Ke-1 element (Fig. 2) consists of 32 bars, and it is a symmetric model. Nodes 1–8 are all hinge nodes, but node 9 is a special constraint, namely, it is also a hinge node when the Ke-1

element is under a uniaxial loading state (Fig. 2(b-d)). But the axial forces of bars 19, 29, 39, 49, 59, 69, 79, 89 are all zero when the Ke-1 element is in a pure shear state (Fig. 2(e-g)). Thus the Ke-1 element is a special space truss, its bars including seven types with different cross-sectional areas:

$$\begin{aligned} A_1 &= A_{14} = A_{23} = A_{67} = A_{58}, & A_2 &= A_{12} = A_{56} = A_{87} = A_{43}, & A_3 &= A_{15} = A_{48} = A_{37} = A_{26}, \\ A_4 &= A_{13} = A_{24} = A_{57} = A_{68}, & A_5 &= A_{18} = A_{54} = A_{27} = A_{63}, & A_6 &= A_{16} = A_{25} = A_{47} = A_{38}, \\ A_7 &= A_{19} = A_{29} = A_{39} = A_{49} = A_{59} = A_{69} = A_{79} = A_{89}, \end{aligned}$$

where A is the cross-sectional area of the bar. For example, A_{14} is the cross-sectional area of bar 14, 1 and 4 being its two endpoints. The dimensions in the x , y , and z directions of the Ke-1 element (Fig. 2) and the regular hexahedron element (Fig. 1) are respectively the same:

$$l_1 = l_{PA} = l_{14} = l_{23} = l_{67} = l_{58}, \quad l_2 = l_{PB} = l_{12} = l_{56} = l_{87} = l_{43}, \quad l_3 = l_{PC} = l_{15} = l_{48} = l_{37} = l_{26},$$

where l is the length. For example, l_{14} is the length of bar 14, 1 and 4 being its two endpoints. The moduli of elasticity of all the bars and the regular hexahedron element are the same.

The seven stress states of the regular hexahedron element shown in Fig. 1(a-g) and the seven loading states of the Ke-1 element shown in Fig. 2(a-g) correspond to each other. According to the superposition principle, the general stress state of the regular hexahedron element (Fig. 1(a)) can be regarded as the superposition of the three uniaxial normal stress states (Fig. 1(b-d)) and the three pure shear stress states (Fig. 1(e-g)). The general loading state of the Ke-1 element (Fig. 2(a)) can be regarded as the superposition of the three uniaxial loading states (Fig. 2(b-d)) and the three pure shear states (Fig. 2(e-g)):

$$\begin{aligned} P_1 &= F_1 + F_5 + F_6, & P_2 &= F_2 + F_4 + F_9, & P_3 &= F_3 + F_7 + F_8, & P_4 &= F_1 - F_5 + F_6, \\ P_5 &= F_2 - F_4 - F_9, & P_6 &= F_3 + F_7 - F_8, & P_7 &= F_1 + F_5 - F_6, & P_8 &= F_2 + F_4 - F_9, \\ P_9 &= F_3 - F_7 - F_8, & P_{10} &= F_1 - F_5 - F_6, & P_{11} &= F_2 - F_4 + F_9, & P_{12} &= F_3 - F_7 + F_8, \end{aligned}$$

where P and F are the external forces acting on the Ke-1 element.

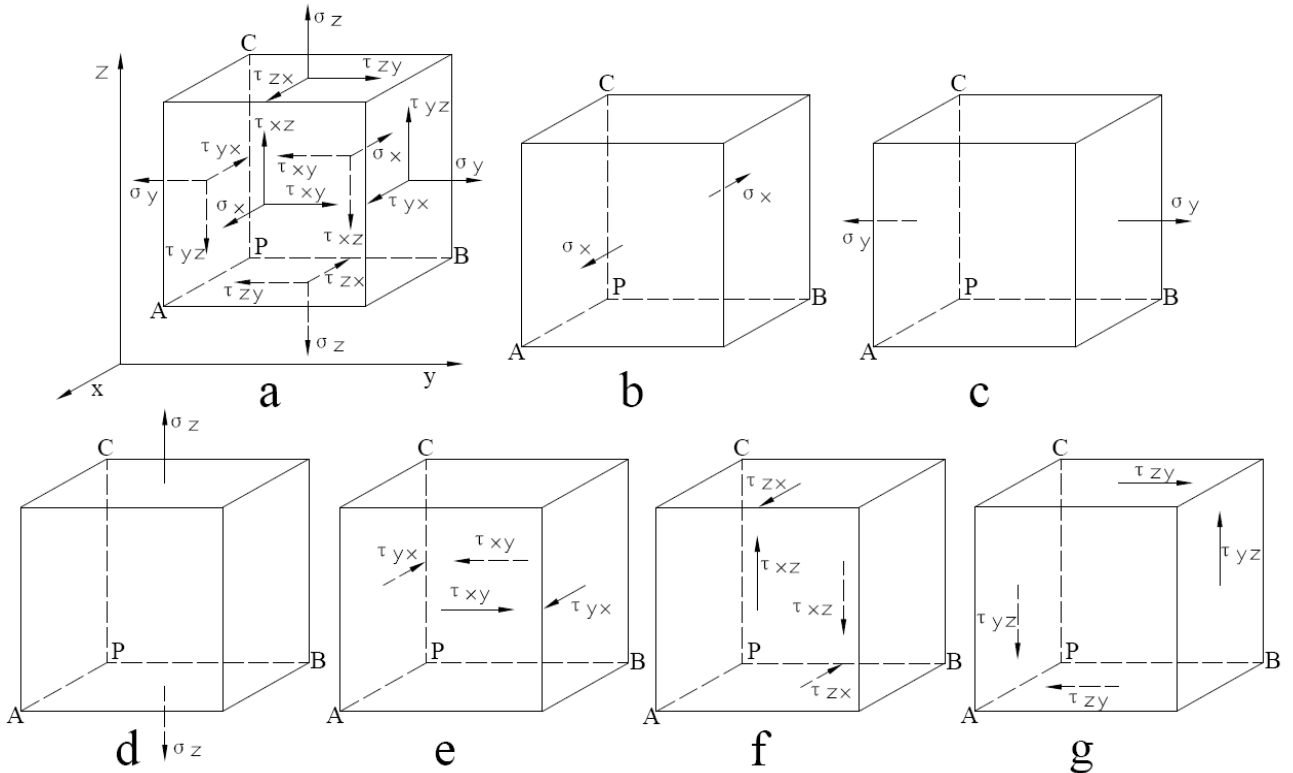


Figure 1: (a) The general stress state, (b-d) three uniaxial normal stress states, (e-g) three pure shear stress states of the regular hexahedron element

The external force acting on the Ke-1 element (Fig. 2(b-g)) and the stress acting on the regular hexahedron element (Fig. 1(b-g)) are equivalent, namely, the corresponding relation of the equivalent external forces is that the total stress acting on any one surface of the regular hexahedron element is assigned averagely to the corresponding four nodes of the Ke-1 element, as follows

$$F_1 = \frac{1}{4} \sigma_x l_2 l_3, \quad F_2 = \frac{1}{4} \sigma_y l_1 l_3, \quad F_3 = \frac{1}{4} \sigma_z l_1 l_2, \quad F_4 = \frac{1}{4} \tau_{xy} l_2 l_3, \quad F_5 = \frac{1}{4} \tau_{yx} l_1 l_3, \\ F_6 = \frac{1}{4} \tau_{xz} l_1 l_2, \quad F_7 = \frac{1}{4} \tau_{zx} l_2 l_3, \quad F_8 = \frac{1}{4} \tau_{yz} l_1 l_3, \quad F_9 = \frac{1}{4} \tau_{zy} l_1 l_2,$$

where $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zx}, \tau_{yz}, \tau_{zy}$ are the stresses acting on the regular hexahedron element.

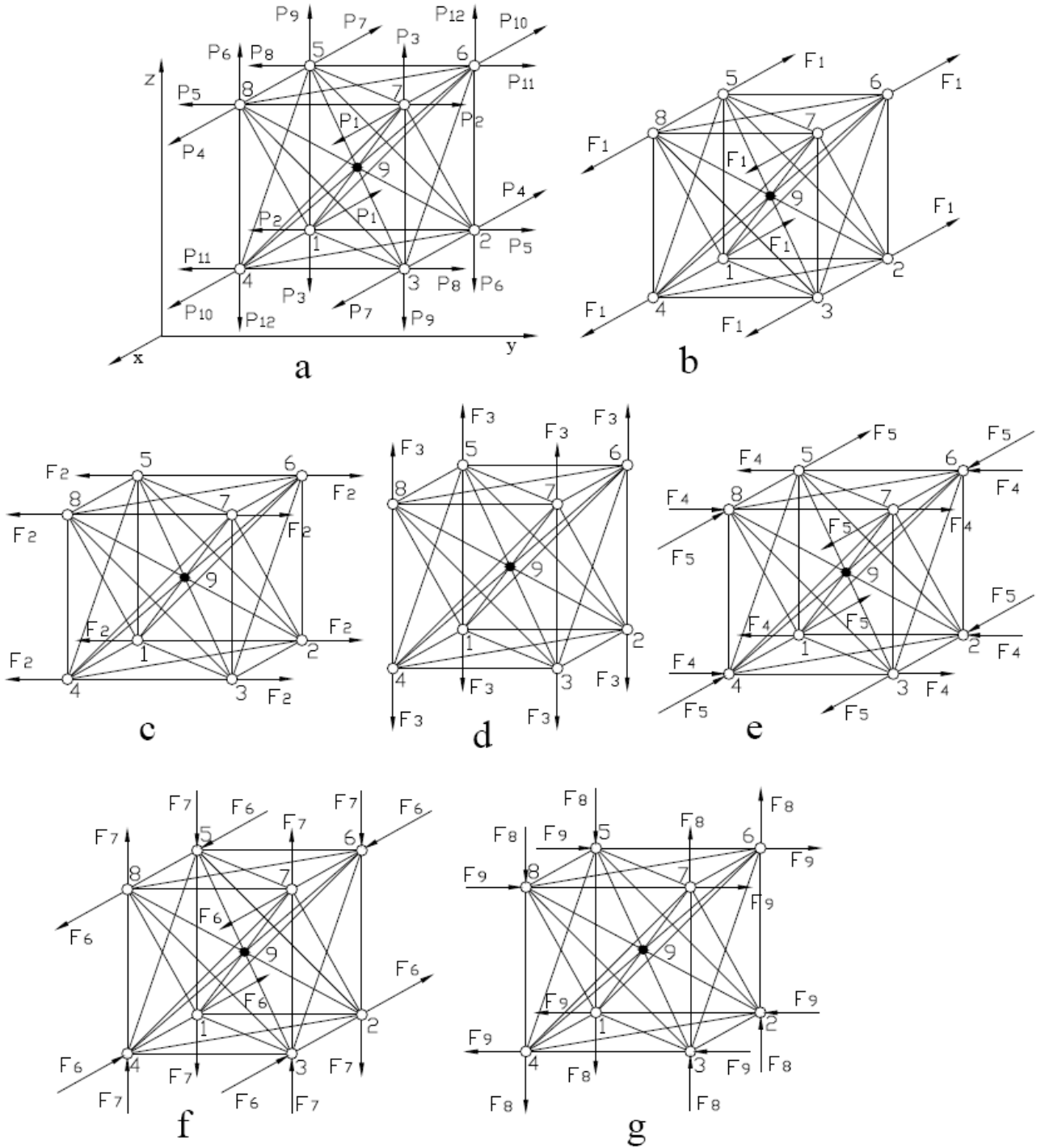


Figure 2: (a) The general loading state, (b-d) three uniaxial loading states, (e-g) three pure shear states of the Ke-1 element

If the deformations of the Ke-1 element under each simple loading state (Fig. 2(b-g)) and the deformations of the regular hexahedron element under the corresponding simple stress state (Fig. 1(b-g)) are all the same, then according to the superposition principle, we know that the deformation of the Ke-1 element under the general loading state (Fig. 2(a)) and the deformation of the regular hexahedron element under the general stress state (Fig. 1(a)) are also the same. Thus the force-deformation relations of the Ke-1 element and the regular hexahedron element are completely equivalent.

Since the deformations of the Ke-1 element and the regular hexahedron element under equivalent external forces are the same, and moreover, since the deformation of the regular hexahedron element under an external force obeys the generalized Hooke's law for isotropic elastic materials, considering the equilibrium of forces on node 7 for each simple loading state, then twelve equilibrium equations can be written. If the equilibrium of the forces on nodes 1, 2, 3, 4, 5, 6, 8, respectively, is considered, these equilibrium equations are exactly the same as the equilibrium equations on node 7.

$$A_1 + \frac{(l_1^2 - \nu l_2^2)l_1}{(l_1^2 + l_2^2)^{1.5}} A_4 + \frac{(l_1^2 - \nu l_3^2)l_1}{(l_1^2 + l_3^2)^{1.5}} A_5 + \frac{(l_1^2 - \nu l_2^2 - \nu l_3^2)l_1}{(l_1^2 + l_2^2 + l_3^2)^{1.5}} A_7 = \frac{1}{4} l_2 l_3 \quad (1)$$

$$- \nu A_2 + \frac{(l_1^2 - \nu l_2^2)l_2}{(l_1^2 + l_2^2)^{1.5}} A_4 - \frac{\nu l_2}{(l_2^2 + l_3^2)^{0.5}} A_6 + \frac{(l_1^2 - \nu l_2^2 - \nu l_3^2)l_2}{(l_1^2 + l_2^2 + l_3^2)^{1.5}} A_7 = 0 \quad (2)$$

$$- \nu A_3 + \frac{(l_1^2 - \nu l_3^2)l_3}{(l_1^2 + l_3^2)^{1.5}} A_5 - \frac{\nu l_3}{(l_2^2 + l_3^2)^{0.5}} A_6 + \frac{(l_1^2 - \nu l_2^2 - \nu l_3^2)l_3}{(l_1^2 + l_2^2 + l_3^2)^{1.5}} A_7 = 0 \quad (3)$$

$$- \nu A_1 + \frac{(l_2^2 - \nu l_1^2)l_1}{(l_1^2 + l_2^2)^{1.5}} A_4 - \frac{\nu l_1}{(l_1^2 + l_3^2)^{0.5}} A_5 + \frac{(l_2^2 - \nu l_1^2 - \nu l_3^2)l_1}{(l_1^2 + l_2^2 + l_3^2)^{1.5}} A_7 = 0 \quad (4)$$

$$A_2 + \frac{(l_2^2 - \nu l_1^2)l_2}{(l_1^2 + l_2^2)^{1.5}} A_4 + \frac{(l_2^2 - \nu l_3^2)l_2}{(l_2^2 + l_3^2)^{1.5}} A_6 + \frac{(l_2^2 - \nu l_1^2 - \nu l_3^2)l_2}{(l_1^2 + l_2^2 + l_3^2)^{1.5}} A_7 = \frac{1}{4} l_1 l_3 \quad (5)$$

$$- \nu A_3 - \frac{\nu l_3}{(l_1^2 + l_3^2)^{0.5}} A_5 + \frac{(l_2^2 - \nu l_3^2)l_3}{(l_2^2 + l_3^2)^{1.5}} A_6 + \frac{(l_2^2 - \nu l_1^2 - \nu l_3^2)l_3}{(l_1^2 + l_2^2 + l_3^2)^{1.5}} A_7 = 0 \quad (6)$$

$$- \nu A_1 - \frac{\nu l_1}{(l_1^2 + l_2^2)^{0.5}} A_4 + \frac{(l_3^2 - \nu l_1^2)l_1}{(l_1^2 + l_3^2)^{1.5}} A_5 + \frac{(l_3^2 - \nu l_1^2 - \nu l_2^2)l_1}{(l_1^2 + l_2^2 + l_3^2)^{1.5}} A_7 = 0 \quad (7)$$

$$- \nu A_2 - \frac{\nu l_2}{(l_1^2 + l_2^2)^{0.5}} A_4 + \frac{(l_3^2 - \nu l_2^2)l_2}{(l_2^2 + l_3^2)^{1.5}} A_6 + \frac{(l_3^2 - \nu l_1^2 - \nu l_2^2)l_2}{(l_1^2 + l_2^2 + l_3^2)^{1.5}} A_7 = 0 \quad (8)$$

$$A_3 + \frac{(l_3^2 - \nu l_1^2)l_3}{(l_1^2 + l_3^2)^{1.5}} A_5 + \frac{(l_3^2 - \nu l_2^2)l_3}{(l_2^2 + l_3^2)^{1.5}} A_6 + \frac{(l_3^2 - \nu l_1^2 - \nu l_2^2)l_3}{(l_1^2 + l_2^2 + l_3^2)^{1.5}} A_7 = \frac{1}{4} l_1 l_2 \quad (9)$$

$$\frac{8(1+\nu)l_1 l_2}{(l_1^2 + l_2^2)^{1.5}} A_4 = l_3 \quad (10)$$

$$\frac{8(1+\nu)l_1 l_3}{(l_1^2 + l_3^2)^{1.5}} A_5 = l_2 \quad (11)$$

$$\frac{8(1+\nu)l_2 l_3}{(l_2^2 + l_3^2)^{1.5}} A_6 = l_1 \quad (12)$$

where ν is the Poisson's ratio.

Solving Equations (1) to (12) simultaneously, if $\nu = 0.25$, then results are:

$$\begin{aligned}
A_1 &= \frac{3l_2^2 l_3^2 - l_1^2 l_2^2 - l_1^2 l_3^2}{10l_2 l_3}, & A_2 &= \frac{3l_1^2 l_3^2 - l_1^2 l_2^2 - l_2^2 l_3^2}{10l_1 l_3}, & A_3 &= \frac{3l_1^2 l_2^2 - l_1^2 l_3^2 - l_2^2 l_3^2}{10l_1 l_2}, \\
A_4 &= \frac{l_3(l_1^2 + l_2^2)^{1.5}}{10l_1 l_2}, & A_5 &= \frac{l_2(l_1^2 + l_3^2)^{1.5}}{10l_1 l_3}, & A_6 &= \frac{l_1(l_2^2 + l_3^2)^{1.5}}{10l_2 l_3}, & A_7 &= 0
\end{aligned} \quad (13)$$

Therefore, the Ke-1 element becomes a space truss consisting of 24 bars.

Solving Equations (1) to (12) simultaneously, if $l_1 = l_2 = l_3 = l$, then results in

$$A_1 = A_2 = A_3 = \frac{l^2}{8(1+\nu)}, \quad A_4 = A_5 = A_6 = \frac{\sqrt{2}l^2}{4(1+\nu)}, \quad A_7 = \frac{3\sqrt{3}(4\nu-1)l^2}{8(1+\nu)(1-2\nu)} \quad (14)$$

From formula (14), if $\nu > 0.25$, then $A_7 > 0$; if $\nu < 0.25$, then $A_7 < 0$; if $\nu \rightarrow 0.5$, then $A_7 \rightarrow +\infty$; if $\nu = 0$, then $A_7 = -3\sqrt{3}l^2/8$; if $\nu \rightarrow -1$, then $A_7 \rightarrow -\infty$; if $\nu = 0.25$, then $A_1 = A_2 = A_3 = 0.1l^2$, $A_4 = A_5 = A_6 = 0.2\sqrt{2}l^2$, $A_7 = 0$.

Node 9 of the Ke-1 element is a special constraint, which may possibly cause some difficulty in the mathematics when writing the calculating program. Thus, when $l_1 = l_2 = l_3 = l$, the special constraint (Fig. 3(a)) is replaced by the twelve bars (Fig. 3(b)) that are located at the centre of the Ke-1 element (these include bars ab, bc, cd, da, ef, fg, gh, he, ae, bf, cg, dh). The lengths of these bars are all cl (the value of c is small, for example, 0.001), and their cross-sectional areas are all A_7 (or a larger value). The lengths of bars 1a, 2b, 3c, 4d, 5e, 6f, 7g, 8h (Fig. 3(b)) are all $0.5\sqrt{3}(1-c)l$, and their cross-sectional areas are all A_7 . The moduli of elasticity of all the bars are all equal to the modulus of elasticity of the solid structure. Thus the Ke-1 element becomes a new space truss consisting of 44 bars, and from Fig. 3(b), it can obviously carry out the special constraint. The displacements of one Ke-1 element under each simple loading state have been calculated using the finite element software ANSYS, and it is found that the precision of the calculation results is extremely good.

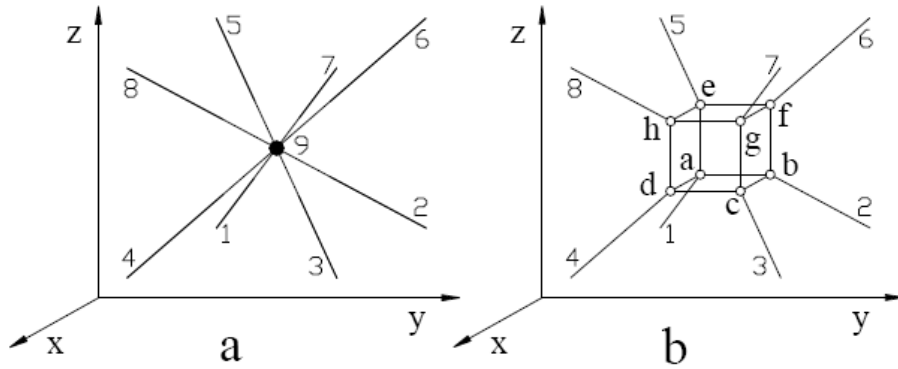


Figure 3: (a) The special constraint 9; (b) Replacement of the special constraint 9

2. Ke-2 Element

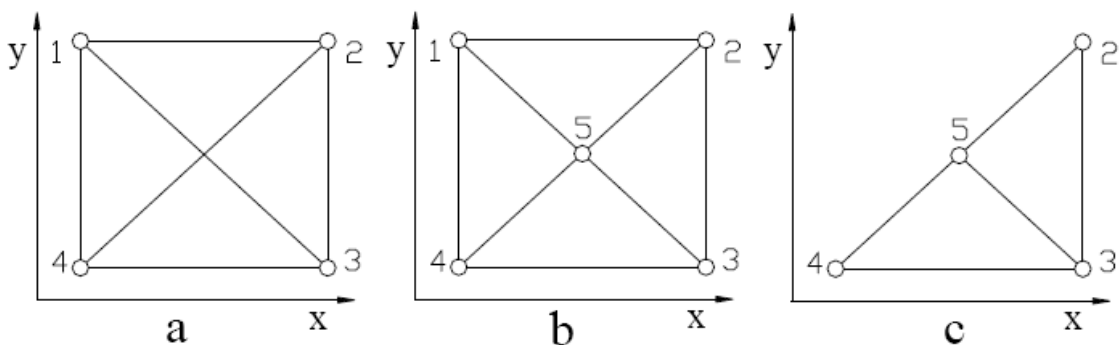


Figure 4: (a) The Ke-2 element; (b, c) Transform of the Ke-2 element

The Ke-2 element (Fig. 4a) is a rectangular truss consisting of six bars, and it is a symmetric model. The dimensions in the x and y directions are $l_1 = l_{12} = l_{43}, l_2 = l_{14} = l_{23}$, where l is the length of the bar. The bars include three types with different cross-sectional areas, i.e. $A_1 = A_{12} = A_{43}$, $A_2 = A_{14} = A_{23}$, $A_3 = A_{13} = A_{24}$, where A is the cross-sectional area of the bar. The moduli of elasticity of all the bars are all equal to the modulus of elasticity of the solid structure. Plane elastic problems include plane stress and plane strain problems, thus the Ke-2 element model is set up for these two problems respectively, assuming that the thickness in the z direction of the plane body is h .

1) *Plane Stress Problems* For plane stress problems, the methods for setting up the Ke-2 element model and the Ke-1 element model are similar. Considering the equilibrium of forces on each node of the Ke-2 element for each simple loading state, five equilibrium equations are written as follows:

$$A_1 + \frac{(l_1^2 - \nu l_2^2)l_1}{(l_1^2 + l_2^2)^{1.5}} A_3 = \frac{1}{2} l_2 h \quad (15)$$

$$- \nu A_2 + \frac{(l_1^2 - \nu l_2^2)l_2}{(l_1^2 + l_2^2)^{1.5}} A_3 = 0 \quad (16)$$

$$- \nu A_1 + \frac{(l_2^2 - \nu l_1^2)l_1}{(l_1^2 + l_2^2)^{1.5}} A_3 = 0 \quad (17)$$

$$A_2 + \frac{(l_2^2 - \nu l_1^2)l_2}{(l_1^2 + l_2^2)^{1.5}} A_3 = \frac{1}{2} l_1 h \quad (18)$$

$$\frac{4(1 + \nu)l_1 l_2}{(l_1^2 + l_2^2)^{1.5}} A_3 = h \quad (19)$$

Solving Equations (15) to (19) simultaneously, if $\nu = 1/3$, then results in

$$A_1 = \frac{9l_2^2 - 3l_1^2}{16l_2} h, \quad A_2 = \frac{9l_1^2 - 3l_2^2}{16l_1} h, \quad A_3 = \frac{3h(l_1^2 + l_2^2)^{1.5}}{16l_1 l_2} \quad (20)$$

If $\nu \neq 1/3$, it is found that there are only three unknowns to satisfy three equations, and the other two equations cannot be satisfied.

2) *Plane Strain Problems* For plane strain and plane stress problems, the methods for setting up the Ke-2 element model are the same. Five equilibrium equations are written as follows:

$$(1 - \nu^2)A_1 + \frac{(1 - \nu^2)l_1^3 - (\nu + \nu^2)l_1 l_2^2}{(l_1^2 + l_2^2)^{1.5}} A_3 = \frac{1}{2} l_2 h \quad (21)$$

$$- \nu A_2 + \frac{(1 - \nu)l_1^2 l_2 - \nu l_2^3}{(l_1^2 + l_2^2)^{1.5}} A_3 = 0 \quad (22)$$

$$- \nu A_1 + \frac{(1 - \nu)l_2^2 l_1 - \nu l_1^3}{(l_1^2 + l_2^2)^{1.5}} A_3 = 0 \quad (23)$$

$$(1 - \nu^2)A_2 + \frac{(1 - \nu^2)l_2^3 - (\nu + \nu^2)l_2 l_1^2}{(l_1^2 + l_2^2)^{1.5}} A_3 = \frac{1}{2} l_1 h \quad (24)$$

$$\frac{4(1 + \nu)l_1 l_2}{(l_1^2 + l_2^2)^{1.5}} A_3 = h \quad (25)$$

Solving equations (21) to (25) simultaneously, if $\nu = 0.25$, then results are

$$A_1 = \frac{3l_2^2 - l_1^2}{5l_2} h, \quad A_2 = \frac{3l_1^2 - l_2^2}{5l_1} h, \quad A_3 = \frac{h(l_1^2 + l_2^2)^{1.5}}{5l_1 l_2} \quad (26)$$

If $\nu \neq 0.25$, it is found that there are only three unknowns to satisfy three equations, and the other two equations cannot be satisfied.

3. Transform of Ke-2 Element If the two intersecting bars of the rectangular truss model shown in Fig. 4(a) are replaced by four intersecting bars, and the cross-sectional areas of the six bars are the same, then the new rectangular truss model shown in Fig. 4(b) is set up. It is obvious that the two models are equivalent. On cutting the rectangular truss model shown in Fig. 4b along the diagonal, the triangular truss model shown in Fig. 4(c) is obtained (the cross-sectional areas of the two bars at the diagonal are half of the original value).

3. Analysis Based on a new method for calculating isotropic, linear, elastic solid structures, the internal forces and displacements of the structure are approximate solutions when the dimensions of the Ke element are finite. The internal forces and displacements of the structure approach the exact solutions when the dimensions of the Ke element approach zero (because the corresponding regular hexahedron element is infinitesimal when the Ke element model is set up). If the neighboring bars of neighboring Ke elements are merged into one bar, namely the cross-sectional areas of these neighboring bars are added algebraically, this would greatly reduce the number of bars of the truss structure, and thus improve the computational efficiency.

The new method is obviously correct, and it may be verified quite easily by analysing a practical problem. Using the new method and the finite element method of the theory of elasticity (i.e., adopting the Ke element and a solid element, respectively), some solid structures have been analysed, and the results of the two methods are in good agreement except the neighborhood of the load application point and the constraint boundary. Any program that can analyse truss structures can use the new method to analyse linear elastic solid structures.

A SIMPLE EXAMPLE

A cantilever beam can be solved using the new method. The cross-sectional dimensions of the cantilever beam are $8\text{cm} \times 8\text{cm}$, its length is 32cm , its modulus of elasticity is $E = 200\text{GPa}$, its Poisson's ratio is $\nu = 0.25$, and a single force of 1000kN is applied to the free end of the beam. First, the beam is divided into Ke-1 elements $1\text{cm} \times 1\text{cm} \times 1\text{cm}$ in size. A truss structure model containing 2048 Ke-1 elements, 2673 nodes and 49152 bars is thus obtained. The modulus of elasticity of all the bars is $E = 200\text{GPa}$. There are two types of bars: there are 24576 bars with a cross-sectional area of 0.1cm^2 , and 24576 bars with a cross-sectional area of $0.2\sqrt{2}\text{cm}^2$. Then solving the truss structure by using the ANSYS finite element software, the vertical displacement of the free end of the truss structure is found to be 1.62cm . If the cantilever beam is solved by using the finite element method of the theory of elasticity (also using the ANSYS finite element software, but adopting a solid element), the vertical displacement of the free end of the cantilever beam is 1.65cm ; thus the results of the two methods agree well.

CONCLUSIONS

(1)The paper provides a new method for solving an isotropic, linear, elastic solid structure. It is very simple and wonderful.

(2)The physical sense of the new method is clear. From the Ke element model, it can be obviously seen that the Ke element profoundly reveals the mechanical mechanism of a solid body. For example, a solid body will contract in the transverse direction when it be elongated in the axial direction.

REFERENCES

1. Lu MW, Luo XF. *Foundation of Elasticity*. Tsinghua University Press, Beijing, China, 1990 (in Chinese).
2. Su YF. *Mechanics of Materials*. Higher Education Press, Beijing, 1984 (in Chinese).