2.3.2 Solution of 2D problems in polar coordinates

1. Transformation of stress components due to change of coordinates. A material particle is in a state of plane stress. If we represent the material particle by a square in the (x, y) coordinate system, the components of the stress state are $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$. If we represent the same material particle under the same state of stress by a square in the (r, θ) coordinate system, the components of the stress state are $\sigma_{rr}, \sigma_{\theta\theta}, \tau_{r\theta}$. From the transformation rules, we know that the two sets of the stress components are related as

$$\sigma_{rr} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\sigma_{\theta\theta} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$\tau_{r\theta} = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

2. Equations in polar coordinates. The Airy stress function is a function of the polar coordinates, $\phi(r, \theta)$. The stresses are expressed in terms of the Airy stress function:

$$\sigma_{rr} = \frac{\partial^2\phi}{r^2\partial\theta^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial\phi}{r\partial\theta}\right)$$

The biharmonic equation is

$$\left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{r^2\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2} + \frac{\partial\phi}{r\partial r} + \frac{\partial^2\phi}{r^2\partial\theta^2}\right) = 0 .$$

The stress-strain relations in polar coordinates are similar to those in the rectangular coordinate system:

$$\varepsilon_{rr} = \frac{\sigma_{rr}}{E} - \nu\frac{\sigma_{\theta\theta}}{E}, \quad \varepsilon_{\theta\theta} = \frac{\sigma_{\theta\theta}}{E} - \nu\frac{\sigma_{rr}}{E}, \quad \gamma_{r\theta} = \frac{2(1+\nu)}{E}\tau_{r\theta}$$

The strain-displacement relations are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{\partial u_\theta}{r\partial\theta}, \quad \gamma_{r\theta} = \frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} .$$

3. A stress field symmetric about an axis. Let the Airy stress function be $\phi(r)$. The biharmonic equation becomes

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=0 \ .$$

Each term in this equation has the same dimension in the independent variable *r*. Such an ODE is known as an equi-dimensional equation. A solution to an equi-dimensional equation is of the form

$$\phi = r^m \ .$$

Inserting into the biharmonic equation, we obtain that

$$m^2(m-2)^2 \ .$$

The fourth order algebraic equation has a double root of 0 and a double root of 2. Consequently, the general solution to the ODE is

$$\phi(r)= A\log r + Br^2\log r + Cr^2 + D \ .$$

where *A*, *B*, *C* and *D* are constants of integration. The components of the stress field are

$$\sigma_{rr}=\frac{\partial^2\phi}{r^2\partial\theta^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}=\frac{A}{r^2}+B(1+2\log r)+2C \ ,$$

$$\sigma_{\theta\theta}=\frac{\partial^2\phi}{\partial r^2}=-\frac{A}{r^2}+B(3+2\log r)+2C \ ,$$

$$\tau_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{\partial\phi}{r\partial\theta}\right)=0 \ .$$

The stress field is linear in *A*, *B* and *C*.

The contributions due to *A* and *C* are familiar: they are the same as the Lame problem. For example, for a hole of radius *a* in an infinite sheet subject to a remote biaxial stress *S*, the stress field in the sheet is

$$\sigma_{rr} = S\left[1-\left(\frac{a}{r}\right)^2\right],\ \sigma_{\theta\theta} = S\left[1+\left(\frac{a}{r}\right)^2\right].$$

The stress concentration factor of this hole is 2. We may compare this problem with that of a spherical cavity in an infinite elastic solid under remote tension:

$$\sigma_{rr} = S\left[1-\left(\frac{a}{r}\right)^3\right],\ \sigma_{\theta\theta} = S\left[1+\frac{1}{2}\left(\frac{a}{r}\right)^3\right].$$

A cut-and-weld operation. How about the contributions due to B? Let us study the stress field (Timoshenko and Goodier, pp. 77-79)

$$\sigma_{rr} = B(1+2\log r),\quad \sigma_{\theta\theta} = B(3+2\log r),\quad \tau_{r\theta} = 0\,.$$

The strain field is

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) = \frac{B}{E}[(1-3\nu)+2(1-\nu)\log r]$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) = \frac{B}{E}[(3-\nu)+2(1-\nu)\log r]$$

$$\gamma_{r\theta} = 0$$

To obtain the displacement field, recall the strain-displacement relations

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r},\ \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{\partial u_\theta}{r\partial\theta},\ \gamma_{r\theta} = \frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}.$$

Integrating ε_{rr}, we obtain that

$$u_r = \frac{B}{E}[2(1-\nu)r\log r - (1+\nu)r] + f(\theta),$$

where $f(\theta)$ is a function still undetermined. Integrating $\varepsilon_{\theta\theta}$, we obtain that

$$u_\theta = \frac{4Br\theta}{E} - \int f(\theta)d\theta + g(r),$$

where $g(r)$ is another function still undetermined. Inserting the two displacements into the expression

$$\gamma_{r\theta} = \frac{\partial u_r}{r\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = 0,$$

and we obtain that

$$f'(\theta) + \int f(\theta)d\theta = g(r) - rg'(r)$$

In the equation, the left side is a function of θ, and the right side is a function of *r*. Consequently, the both sides must equal a constant independent of *r* and θ, namely,

$$f'(\theta) + \int f(\theta)d\theta = G$$
$$g(r) - rg'(r) = G$$

Solving these equations, we obtain that

$$f(\theta) = H\sin\theta + K\cos\theta$$
$$g(r) = Fr + G$$

Substituting back into the displacement field, we obtain that

$$u_r = \frac{B}{E}\left[2(1-\nu)r\log r - (1+\nu)r\right] + H\sin\theta + K\cos\theta$$
$$u_\theta = \frac{4Br\theta}{E} + Fr + H\cos\theta - K\sin\theta$$

Consequently, *F* represents a rigid-body rotation, and *H* and *K* represent a rigid-body translation.

Now we can give an interpretation of *B*. Imagine a ring, with a wedge of angle α cut off. The ring with the missing wedge was then weld together. This operation requires that after a rotation of a circle, the displacement is

$$v(2\pi) - v(0) = \alpha r$$

This condition gives

$$B = \frac{\alpha E}{8\pi}.$$

This cut-and-weld operation clearly introduces a stress field in the ring. The stress field is axisymmetric, as given above.

4. A circular hole in an infinite sheet under remote shear. Remote from the hole, the sheet is in a state of pure shear:

$$\tau_{xy} = S, \ \ \sigma_{xx} = \sigma_{yy} = 0 .$$

The remote stresses in the polar coordinates are

$$\sigma_{rr} = S \sin 2\theta, \quad \sigma_{\theta\theta} = -S \sin 2\theta, \quad \tau_{r\theta} = S \cos 2\theta .$$

Recall that

$$\sigma_{rr} = \frac{\partial^2 \phi}{r^2 \partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial \phi}{r \partial \theta}\right).$$

We guess that the stress function must be in the form

$$\phi(r,\theta) = f(r) \sin 2\theta .$$

The biharmonic equation becomes

$$\left(\frac{d^2}{dr^2} + \frac{d}{rdr} - \frac{4}{r^2}\right)\left(\frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{r \partial r} - \frac{4f}{r^2}\right) = 0 .$$

A solution to this equi-dimensional ODE takes the form $f(r) = r^m$. Inserting this form into the ODE, we obtain that

$$\left((m-2)^2 - 4\right)\left(m^2 - 4\right) = 0 .$$

The algebraic equation has four roots: 2, -2, 0, 4. Consequently, the stress function is

$$\phi(r,\theta) = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D\right) \sin 2\theta .$$

The stress components inside the sheet are

$$\sigma_{rr} = \frac{\partial^2 \phi}{r^2 \partial \theta^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\sin 2\theta$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial \phi}{r\partial \theta}\right) = \left(-2A - 6Br^2 + \frac{6C}{r^4} + \frac{2D}{r^2}\right)\cos 2\theta .$$

To determine the constants A, B, C, D, we invoke the boundary conditions:

1. Remote from the hole, namely, $r \to \infty$, $\sigma_{rr} = S\sin 2\theta, \tau_{r\theta} = S\cos 2\theta$, giving $A = -S/2, B = 0$.
2. On the surface of the hole, namely, $r = a$, $\sigma_{rr} = 0, \tau_{r\theta} = 0$, giving $D = Sa^2$ and $C = -Sa^4/2$.

The stress field inside the sheet is

$$\sigma_{rr} = S\left[1 + 3\left(\frac{a}{r}\right)^4 - 4\left(\frac{a}{r}\right)^2\right]\sin 2\theta$$

$$\sigma_{\theta\theta} = -S\left[1 + 3\left(\frac{a}{r}\right)^4\right]\sin 2\theta$$

$$\tau_{r\theta} = S\left[1 - 3\left(\frac{a}{r}\right)^4 + 2\left(\frac{a}{r}\right)^2\right]\cos 2\theta$$

5. A hole in an infinite sheet subject to a remote uniaxial stress. Use this as an example to illustrate linear superposition. A state of uniaxial stress is a linear superposition of a state of pure shear and a state of biaxial tension. The latter is the Lame problem. When the sheet is subject to remote tension of magnitude S, the stress field in the sheet is given by

$$\sigma_{rr} = S\left[1 - \left(\frac{a}{r}\right)^2\right], \quad \sigma_{\theta\theta} = S\left[1 + \left(\frac{a}{r}\right)^2\right].$$

Illustrate the superposition in figures. Show that under uniaxial tensile stress, the stress around the hole has a concentration factor of 3. Under uniaxial compression, the material may split in the loading direction.

6. A line force acting on the surface of a half space. A half space of an elastic material is subject to a line force on its surface. Let P be the force per unit length. The half space lies in $x > 0$, and the force points in the direction of x. This problem has no length scale. Linearity and dimensional considerations requires that the stress field take the form

$$\sigma_{ij}(r,\theta) = \frac{P}{r} g_{ij}(\theta),$$

where $g_{ij}(\theta)$ are dimensionless functions of θ. We guess that the stress function takes the form

$$\phi(r) = rPf(\theta),$$

where $f(\theta)$ is a dimensionless function of θ. (A homework problem will show that this guess is not completely correct, but it suffices for the present problem.)

Inserting this form into the biharmonic equation, we obtain an ODE for $f(\theta)$:

$$f + 2\frac{d^2 f}{d\theta^2} + \frac{d^4 f}{d\theta^4} = 0 .$$

The general solution is

$$\phi(r,\theta) = rP(A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta).$$

Observe that $r\sin\theta = y$ and $r\cos\theta = x$ do not contribute to any stress, so we drop these two terms. By the symmetry of the problem, we look for stress field symmetric about $\theta = 0$, so that we will drop the term $\theta\cos\theta$. Consequently, the stress function takes the form

$$\phi(r,\theta) = rPC\theta\sin\theta .$$

We can calculate the components of the stress field:

$$\sigma_{rr} = \frac{2CP\cos\theta}{r},\ \sigma_{\theta\theta} = \tau_{r\theta} = 0 .$$

This field satisfies the traction boundary conditions, $\sigma_{\theta\theta} = \tau_{r\theta} = 0$ at $\theta = 0$ and $\theta = \pi$. To determine *C*, we require that the resultant force acting on a cylindrical surface of radius *r* balance the line force *P*. On each element $rd\theta$ of the surface, the radial stress provides a vertical component of force $\sigma_{rr} \cos\theta r d\theta$. The force balance of the half cylinder requires that

$$P + \int_{-\pi/2}^{\pi/2} \sigma_{rr} \cos\theta r d\theta = 0 .$$

Integrating, we obtain that $C = -1/\pi$.

The stress components in the *x*-*y* coordinates are

$$\sigma_{xx} = -\frac{2P}{\pi x}\cos^4\theta,\ \sigma_{yy} = -\frac{2P}{\pi x}\sin^2\theta\cos^2\theta,\ \tau_{xy} = -\frac{2P}{\pi x}\sin\theta\cos^3\theta$$

The displacement field is

$$u_r = -\frac{2P}{\pi E}\cos\theta\log r - \frac{(1-\nu)P}{\pi E}\theta\sin\theta$$
$$u_\theta = -\frac{2\nu P}{\pi E}\sin\theta + \frac{2P}{\pi E}\sin\theta\log r - \frac{(1-\nu)P}{\pi E}\theta\cos\theta - \frac{(1-\nu)P}{\pi E}\sin\theta$$

7. Separation of variable. One can obtain many solutions by using the procedure of separation of variable, assuming that

$$\phi(r,\theta) = R(r)\Theta(\theta).$$

Formulas for stresses and displacements can be found on p. 205, Deformation of Elastic Solids, by A.K. Mal and S.J. Singh.

A real-life example.

From: S. Ho, C. Hillman, F.F. Lange and Z. Suo, "Surface cracking in layers under biaxial, residual compressive stress," *J. Am. Ceram. Soc.* **78**, 2353-2359 (1995).

In previous treatment of laminates, we have ignored edge effect. However, we also know that edges are often the site for failure to initiate. Here is a phenomenon discovered in the lab of Fred Lange at UCSB. A thin layer of material 1 was sandwiched in two thick blocks of material 2. Material 1 has a smaller coefficient of thermal expansion than material 2, so that, upon cooling, material 1 develops a biaxial compression in the plane of the laminate. The two blocks are nearly stress free. Of course, these statements are only valid at a distance larger than the thickness of the thin layer. It was observed in experiment that the thin layer cracked, as shown in Fig. 1.

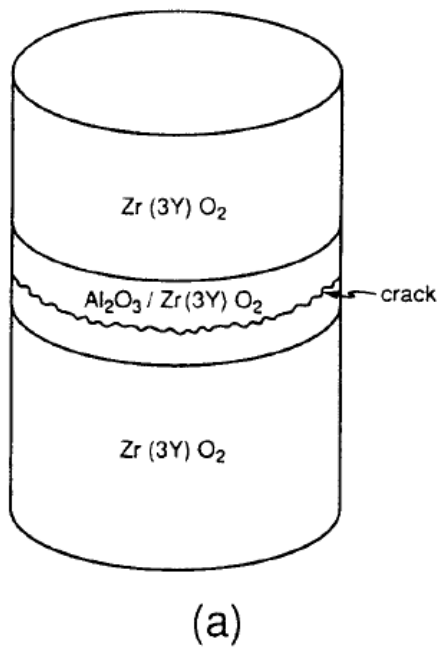

Fig. 1. (a) A thin layer of $Al_2O_3/Zr(Y)O_2$ is bonded between two blocks of $Zr(Y)O_2$. A crack runs parallel to the interfaces, in the $Al_2O_3/Zr(Y)O_2$ layer. (b) An optical micrograph of a crack running in the $Al_2O_3/Zr(Y)O_2$ layer. (c) SEM micrograph of fracture surface showing sequential positions of the crack front (partial dashed lines) extending from the surface near the center of the $Al_2O_3/Zr(Y)O_2$ layer.

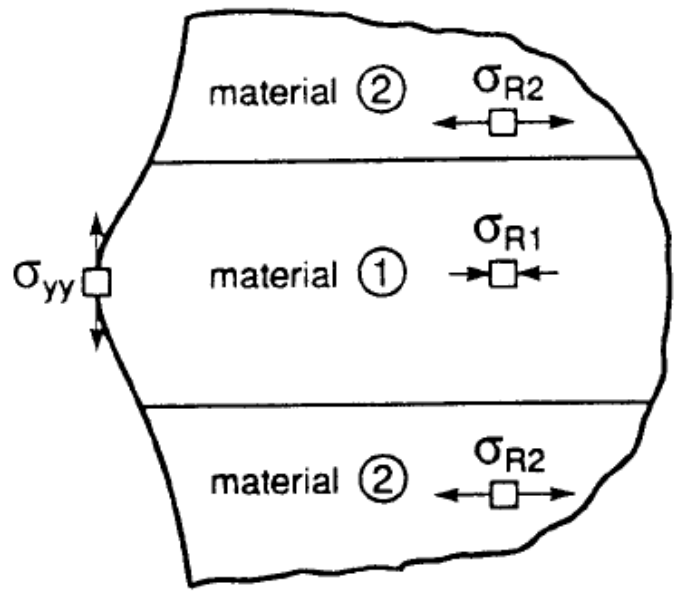

Fig. 2. Far away from the edge, the stress is biaxial in the plane of the laminate, compressive in $Al_2O_3/Zr(Y)O_2$, and tensile in $Zr(Y)O_2$. At the edge, there is a tensile stress normal to the interfaces in $Al_2O_3/Zr(Y)O_2$.

It is clear from Fig. 2 that a tensile stress σ_{yy} can develop near the edge. We would like to know its magnitude, and how fast it decays as we go into the layer.

We analyze this problem by a linier superposition shown in Fig. 3. Let σ_M be the magnitude of the biaxial stress in the thin layer far from the edge. In Problem A, we apply a compressive traction of magnitude σ_M on the edge of the thin layer, so that the stress field in thin layer in Problem A is the uniform biaxial stress in the thin layer, with no other stress components. In problem B, we remover thermal expansion misfit, but applied a tensile traction on the edge of the thin layer. The original problem is the superposition of Problem A and Problem B. Thus, the residual stress field σ_{yy} in the original problem is the same as the stress σ_{yy} in Problem B.

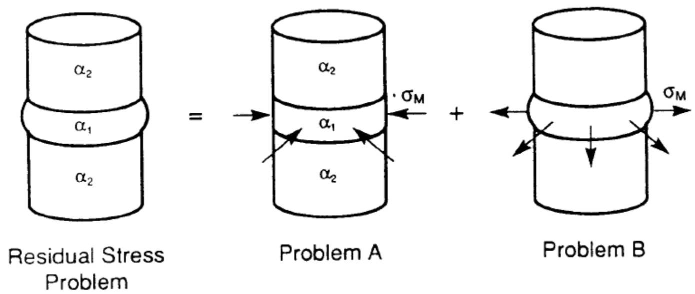

Fig. 3. The residual stress problem is a superposition of the following two problems: (problem A) a band of pressure of magnitude σ_M is applied in addition to the thermal mismatch; (problem B) a band of tensile traction of magnitude σ_M is applied, and there is no thermal mismatch.

With reference to Fig. 4, let us calculate the stress distribution $\sigma_{yy}(x,0)$. Recall that when a half space is subject to a line force *P*, the stress is given by

$$\sigma_{yy} = -\frac{2P}{\pi x}\sin^2\theta\cos^2\theta .$$

We now consider a line-force acting at $y=\eta$. On an element of the edge, $d\eta$, the tensile traction applied the line force $P = -\sigma_M d\eta$. Summing up over all elements, we obtain the stress field in the layer:

$$\sigma_{yy}(x,0) = \int_{-t/2}^{t/2} \frac{2\sigma_M d\eta}{\pi x}\sin^2\theta\cos^2\theta .$$

Note that $\eta = x\tan\theta$, and let $\tan\beta = t/2x$. Consequently,

$$d\eta = \frac{x}{\cos^2\theta}d\theta ,$$

and the integral becomes

$$\sigma_{yy}(x,0) = \frac{2\sigma_M}{\pi}\int_{-\beta}^{\beta}\sin^2\theta d\theta = \frac{2\sigma_M}{\pi}\int_{-\beta}^{\beta}\frac{1-\cos 2\theta}{2}d\theta .$$

Integrating, we obtain that

$$\sigma_{yy}(x,0) = \frac{2\sigma_M}{\pi}\left(\beta - \frac{1}{2}\sin 2\beta\right).$$

At the edge of the layer, $x/t \to 0$ and $\beta = \pi/2$, so that $\sigma_{yy}(0,0) = \sigma_M$. Far from the edge, $t/x \to 0$,

$$\sigma_{yy}(x,0) \to \frac{\sigma_M}{6\pi}\left(\frac{t}{x}\right)^3.$$

Thus, this stress decays as x^{-3}.

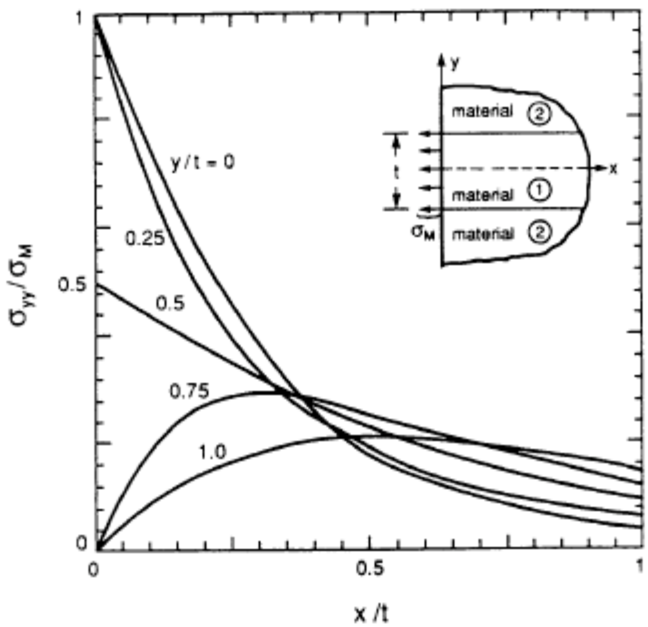

Fig. 4. Distribution of the stress component $\sigma_{yy}(x, y)$ near the edge. The elastic mismatch in this system is assumed to be zero.