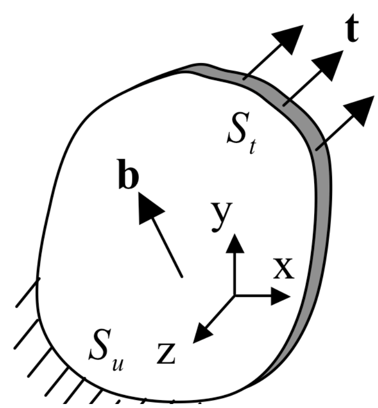
2. Plane Elasticity Problems

Main Reference: *Theory of Elasticity*, by S.P. Timoshenko and J.N. Goodier, McGraw-Hill, New York. Chapters 2,3,4.

2.1 The plane-stress problem

A thin sheet of an isotropic material is subject to loads in the plane of the sheet. The sheet lies in the (x, y) plane. Both top and the bottom surfaces of the sheet are traction-free. The edge of the sheet may have two kinds of the boundary conditions: displacement prescribed, or traction prescribed. In the latter case, we write



$$\sigma_{xx} n_x + \tau_{xy} n_y = t_x$$
$$\tau_{xy} n_x + \sigma_{yy} n_y = t_y$$

where t_x and t_y are components of the traction vector prescribed on the edge of the sheet, and n_x and n_y are the components of the unit vector normal to the edge of the sheet. The above two equations provide two conditions for the components of the stress tensor along the edge.

Semi-inverse method. We next go into the interior of the sheet. We already have obtained a full set of governing equations for linear elasticity problems. No general approach exists to solve these partial differential equations analytically, although numerical methods are readily available to solve most elasticity problem. In this introductory course, in order to gain insight into solid mechanics, we will make reasonable guesses of solutions, and see if they satisfy all the governing equations. This trial-and-error approach has a name: it is called the semi-inverse method.

It seems reasonable to guess that the stress field in the sheet only has nonzero components in its plane: $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$, and that the components out of plane vanish:

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0.$$

Furthermore, we guess that the in-plane stress components may vary with *x* and *y*, but are independent of *z*. That is, the stress field in the sheet is described by three functions:

$$\sigma_{xx}(x,y),\ \ \sigma_{yy}(x,y),\ \ \tau_{xy}(x,y).$$

Will these guesses satisfy the governing equations of elasticity? Let us go through the equations one by one.

1. *Equilibrium equations*. Using the guessed stress field, we reduce the three equilibrium equations to two equations:

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0, \quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}=0 .$$

These two equations by themselves are insufficient to determine the three functions.

2. Stress-strain relations. Given the guessed stress field, the 6 components of the strain field are

$$\varepsilon_{xx}=\frac{\sigma_{xx}}{E}-\nu\frac{\sigma_{yy}}{E}, \quad \varepsilon_{yy}=\frac{\sigma_{yy}}{E}-\nu\frac{\sigma_{xx}}{E}, \quad \gamma_{xy}=\frac{2(1+\nu)}{E}\tau_{xy}$$

$$\varepsilon_{zz}=-\frac{\nu}{E}\left(\sigma_{xx}+\sigma_{yy}\right), \ \gamma_{xz}=\gamma_{yz}=0 .$$

3. Strain-displacement relations. Recall the 6 strain-displacement relations:

$$\varepsilon_{xx}=\frac{\partial u}{\partial x},\ \ \varepsilon_{yy}=\frac{\partial v}{\partial y},\ \ \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

$$\varepsilon_{zz}=\frac{\partial w}{\partial z},\ \ \gamma_{xz}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x},\ \ \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} .$$

It seems reasonable to assume that the in-plane displacements u and v vary only with x and y, but not with z. From these guesses, together with the conditions that $\gamma_{xz}=\gamma_{yz}=0$, we find that

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial y}=0 .$$

Thus, w is independent of x and y, and can only be a function of z. If we insist that ε_{zz} be independent of z, and from $\varepsilon_{zz}=\partial w/\partial z$, then ε_{zz} must be a constant, $\varepsilon_{zz}=c$, and $w=cz+b$. On the other hand, we also have $\varepsilon_{zz}=-\nu\left(\sigma_{xx}+\sigma_{yy}\right)/E$, which may not be a constant. This inconsistency shows that our guesses are generally incorrect.

Summary of equations of plane elasticity problems. Instead of abandoning these guesses, we will just call our guesses the plane-stress approximation. If you neglect the inconsistency between $\varepsilon_{zz} = c$ and $\varepsilon_{zz} = -\nu(\sigma_{xx} + \sigma_{yy})/E$, at least the following set of equations is self-consistent:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

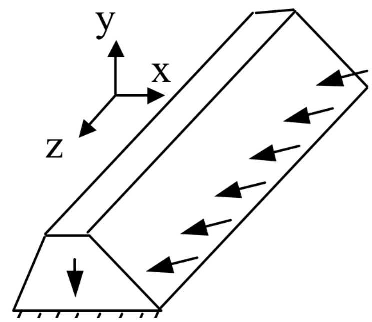
$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \nu \frac{\sigma_{yy}}{E}, \quad \varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \nu \frac{\sigma_{xx}}{E}, \quad \gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

These are 8 equations for 8 functions. We will focus on these 8 equations.

2.2 The plane-strain problem

Consider an infinitely long cylinder with axis in the z-direction and a cross section in the (x,y) plane. We assume that the loading is invariant along the z-direction. Under these conditions, the displacement field takes the form:



$$u(x,y),\ v(x,y),\ w=0\ .$$

From the strain displacement relations, we find that only the three in-plane strains are nonzero: $\varepsilon_{xx}(x,y), \varepsilon_{yy}(x,y), \gamma_{xy}(x,y)$. The three out-of-plane strains vanish: $\varepsilon_{zz}=\gamma_{xz}=\gamma_{yz}=0$.

Because $\gamma_{xz}=\gamma_{yz}=0$, the stress-strain relations imply that $\tau_{xz}=\tau_{yz}=0$. From $\varepsilon_{zz}=0$ and $\varepsilon_{zz}=\left(\sigma_{zz}-\nu\sigma_{xx}-\nu\sigma_{yy}\right)$, we obtain further that

$$\sigma_{zz}=\nu\left(\sigma_{xx}+\sigma_{yy}\right).$$

Furthermore, we have

$$\varepsilon_{xx}=\frac{1}{E}\left(\sigma_{xx}-\nu\sigma_{yy}-\nu\sigma_{zz}\right)=\frac{1-\nu^2}{E}\left(\sigma_{xx}-\frac{\nu}{1-\nu}\sigma_{yy}\right),$$

$$\varepsilon_{yy}=\frac{1}{E}\left(\sigma_{yy}-\nu\sigma_{xx}-\nu\sigma_{zz}\right)=\frac{1-\nu^2}{E}\left(\sigma_{yy}-\frac{\nu}{1-\nu}\sigma_{xx}\right),$$

$$\gamma_{xy}=\frac{2(1+\nu)}{E}\tau_{xy}.$$

These three stress-strain relations look similar to those under the plane-stress conditions, provided we make the following substitutions:

$$\bar{E}=\frac{E}{1-\nu^2},\quad \bar{\nu}=\frac{\nu}{1-\nu}.$$

The quantity $\bar{E}$ is called the plane strain modulus. Finally, we also have the equilibrium equations:

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0, \quad \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}=0 .$$

2.3 Solution of plane problems and the Airy stress function

From the forgoing, it is clear that plane stress and plane strain problems are described by the same equations, as long as one uses the appropriate elastic constants. This also means that the solution technique for both types of problems is the same. We make use of the following calculus theorem:

A theorem in calculus. If two functions $f(x,y)$ and $g(x,y)$ satisfy the following relationship

$$\frac{\partial f}{\partial x}=\frac{\partial g}{\partial y},$$

then, there exists a function $A(x,y)$, such that

$$f=\frac{\partial A}{\partial y}, \quad g=\frac{\partial A}{\partial x}.$$

The Airy stress function. We now apply the above theorem to the equilibrium equations. From the equation

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0,$$

we deduce that there exists a function $A(x,y)$, such that

$$\sigma_{xx}=\frac{\partial A}{\partial y}, \quad \tau_{xy}=-\frac{\partial A}{\partial x}.$$

From the equation

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}=0,$$

we further deduce that that there exists a function $B(x,y)$, such that

$$\sigma_{yy}=\frac{\partial B}{\partial x}, \quad \tau_{xy}=-\frac{\partial B}{\partial y}.$$

Finally, from

$$\frac{\partial A}{\partial x} = \frac{\partial B}{\partial y},$$

we deduce that that there exists a function $\phi(x, y)$, such that

$$A = \frac{\partial \phi}{\partial y}, \quad B = \frac{\partial \phi}{\partial x}.$$

The function $\phi(x, y)$ is known as the ***Airy stress function***. The three components of the stress field can now be represented by the stress function:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \ \ \sigma_{yy} = \frac{\partial^2 \phi}{\partial y^2}, \ \ \tau_{xy} = -\frac{\partial^2 \phi}{\partial y \partial x}.$$

Using the stress-strain relations, we can also express the three components of strain field in terms of the Airy stress function:

$$\varepsilon_{xx} = \frac{1}{E}\left(\frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2}\right), \quad \varepsilon_{yy} = \frac{1}{E}\left(\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2}\right), \ \ \gamma_{xy} = -\frac{2(1+\nu)}{E}\frac{\partial^2 \phi}{\partial x \partial y}.$$

Compatibility equation. Recall the strain-displacement relations.

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}, \ \ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \ \ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

We derived the compatibility equation by eliminating the two displacements in the three strain displacement relations to obtain the compatibility equation

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}.$$

Biharmonic equation. Inserting the expressions of the strains in terms of $\phi(x, y)$ into the compatibility equation, we obtain that

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0.$$

This equations can also be written as

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=0\,.$$

Because of its obvious similarity to the harmonic equation, it is called the biharmonic equation.

Thus, a procedure to solve a plane stress problem is to solve for $\phi(x,y)$ from the above PDE, and then calculate stresses and strains. After the strains are obtained, the displacement field can be obtained by integrating the strain-displacement relations.

Dependence on elastic constants. For a plane problem with traction-prescribed boundary conditions, both the governing equation and the boundary conditions can be expressed in terms of ϕ. All these equations are independent of elastic constants. Consequently, the stress field in such a boundary value problem is independent of the elastic constants. Once we go over specific examples, we will find that the above statement is only correct for boundary value problems in simply connected regions. For multiply connected regions, the above equations in terms of ϕ do not guarantee that the displacement field is continuous. When we insist that displacement field be continuous, elastic constants may enter the stress field.

2.3.1 Solution of 2D problems in Cartesian coordinates: A half space subject to periodic traction on the surface

An elastic material occupies a half space, $x>0$. On the surface of the material, $x=0$, the traction vector is prescribed

$$\sigma_{xx}(0,y,z)=\sigma_0\cos ky,\quad \tau_{xy}(0,y,z)=\tau_{xz}(0,y,z)=0\,.$$

Determine the stress field inside the material.

Solution: The material clearly deforms under the plane strain conditions. It is reasonable to guess that the Airy stress function should take the form

$$\phi(x,y)=f(x)\cos ky\,.$$

The biharmonic equation then becomes

$$\frac{d^4f}{dx^4}-2k^2\frac{d^2f}{dx^2}+k^4f=0\,.$$

This is a homogenous ODE with constant coefficients. The solution must be of the form

$$f(x) = e^{\alpha x}.$$

Insert this form into the ODE, and we obtain that

$$\left(\alpha^2 - k^2\right)^2 = 0.$$

The algebraic equation has double roots of $\alpha = -k$, and double roots of $\alpha = +k$. Consequently, the general solution is of the form

$$f(x) = Ae^{kx} + Be^{-kx} + Cxe^{kx} + Dxe^{-kx},$$

where A, B, C and D are constants of integration.

We expect that the stress field vanish as $x \to +\infty$, so that the stress function should be of the form

$$f(x) = Be^{-kx} + Dxe^{-kx}.$$

We next determine the constants B and D by using the traction boundary conditions. The stress fields are

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = -\left(Be^{-kx} + Dxe^{-kx}\right)k^2 \cos ky$$
$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \left(-Be^{-kx} + \frac{D}{k}e^{-kx} - Dxe^{-kx}\right)k^2 \sin ky$$
$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = \left(Be^{-kx} - 2\frac{D}{k}e^{-kx} + Dxe^{-kx}\right)k^2 \cos ky$$

Recall the boundary conditions

$$\sigma_{xx}(0, y) = \sigma_0 \cos ky, \quad \tau_{xy}(0, y) = 0.$$

Consequently, we find that

$$B = -\sigma_0 / k^2, \quad D = -\sigma_0 / k.$$

The stress field inside the material is

$$\sigma_{xx} = \sigma_0 (1 + kx) e^{-kx} \cos ky$$
$$\tau_{xy} = -\sigma_0 kx e^{-kx} \sin ky$$
$$\sigma_{yy} = \sigma_0 (1 - kx) e^{-kx} \cos ky$$

The stress field decays exponentially.

We have solved the problem where the traction on the boundary of a half space is given by a simple cosine function. Through application of the superposition principle, which is valid for linear elastic materials, it is now straightforward to extend this analysis to any periodic traction distribution. Indeed, a periodic traction distribution can be written as a Fourier series each term of which is of the form found in the previous problem.

Application: de Saint-Venant's principle

When a load is applied in a small region, and the load has a vanishing resultant force and resultant moment, then the stress field is localized. We used this principle in discussing the laminate problem, where we have neglected the edge effects. While Saint-Venant's principle cannot be proved in such a loose form, the foregoing is a nice example of the principle: the traction applied to the boundary is self-balancing and hence the stress field associated with the tractions die out. If we had imposed an additional constant traction term, the stress field would quickly decay to a constant stress.