2D Theory of Elasticity

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Summary

- Field equations
- Boundary conditions
- Problem formulation
- Plane strain
- Plane stress
- Airy's stress function
- Axially loaded bar
- Pure bending of beam

Introduction

- In principle all practical problems – 3D problems – very complex – difficult to handle
- Reasonable assumptions – bring down the complexity in the problem – reduces a dimension => 2D problem
- 2D problems – two independent variables – static problem – three independent variables – dynamic problems
- Field variables, $\phi = \phi(x, y, t)$

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Introduction

- Field variable is independent of third dimension => $\frac{\partial \phi}{\partial z} = 0$
- Continuum approach – the following laws hold good
  - Conservation of mass
  - Principle of momentum
  - Principle of moment of momentum
  - First law of thermodynamics
  - Second law of thermodynamics - inequality

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Introduction

- No inertia effects – quasi-static / static
- Small deformation and rotations
- Material => isotropic – same properties in all the direction
- Homogeneous – material properties are uniform over the domain
- Linear elastic – Hooke's law holds good => Hookean material

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Field equations

- Strain-displacement equations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{i,j}\right)$$

Six equation

- Compatibility equations

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0$$

Six equations – three independent

Field equations

- Equilibrium equations –

$$\sigma_{ij,j} + B_i = 0$$

Three equations

- Constitutive law – Linear elastic – Hooke's law

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2 \mu \varepsilon_{ij}$$

$$\varepsilon_{ij} = \frac{1+\upsilon}{E} \sigma_{ij} - \frac{\upsilon}{E} \sigma_{kk} \delta_{ij}$$

Six equations

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Field equations

- Total – 15 equations = 3 Equilibrium + 6 strain displacement + 6 constitutive laws
- Unknowns => 3 Displacements, 6 stress components and 6 strain components
- Boundary value problem casted as

$$\Im\left(\underbrace{u_i, \varepsilon_{ij}, \sigma_{ij}}_{\text{Unknowns}}, \underbrace{\lambda, \mu}_{\text{Lame's constants}}, B_i\right) = 0$$

Body load

Not easy to solve analytically 15 equations to find out 15 unknowns

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Boundary conditions

- Solution of a differential (partial) equations – boundary conditions required
- Boundary conditions influence the solution – field variable in the domain
- BCs play a very essential role for properly formulating and solving elasticity problems
- In elasticity – boundary conditions – displacements, tractions and combination

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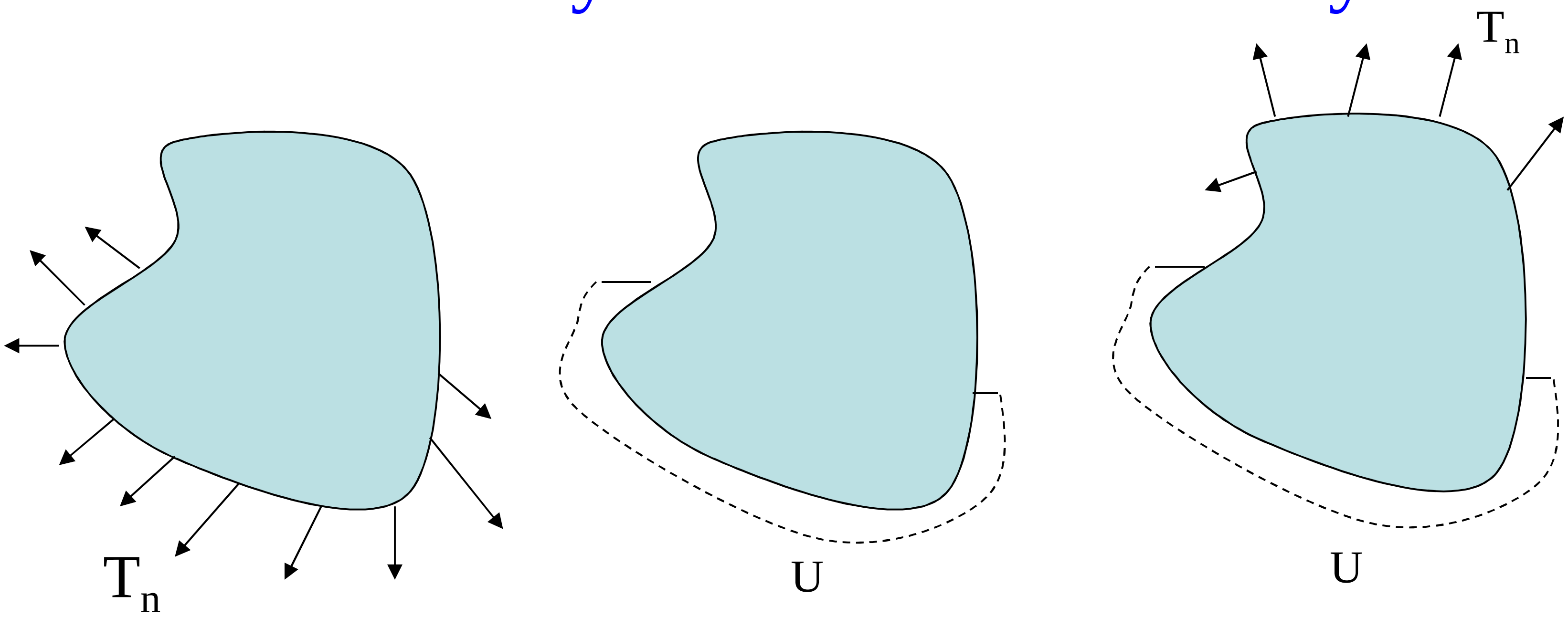
Boundary conditions

Various boundary conditions in elasticity –



Traction boundary conditions

Displacement boundary conditions

Mixed boundary conditions

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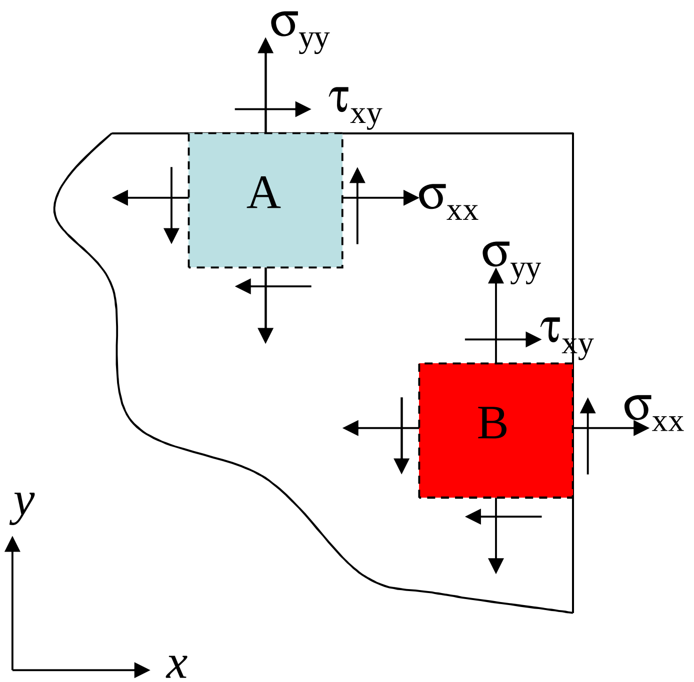
Boundary conditions

R&DE (Engineers), DRDO

- Tractions –



$$T_x = l\sigma_{xx} + m\tau_{xy}$$

$$T_y = l\tau_{xy} + m\sigma_{yy}$$

At ‘A’ – Direction cosines of area normal => (0, 1, 0)

σ_{xx} is inside the boundary – this need not be zero

$T_x = \tau_{xy}$; $T_y = \sigma_{yy}$

At ‘B’ – Direction cosines of area normal => (1, 0, 0)

$T_x = \sigma_{xx}$; $T_y = \tau_{xy}$

σ_{yy} is inside the boundary – this need not be zero

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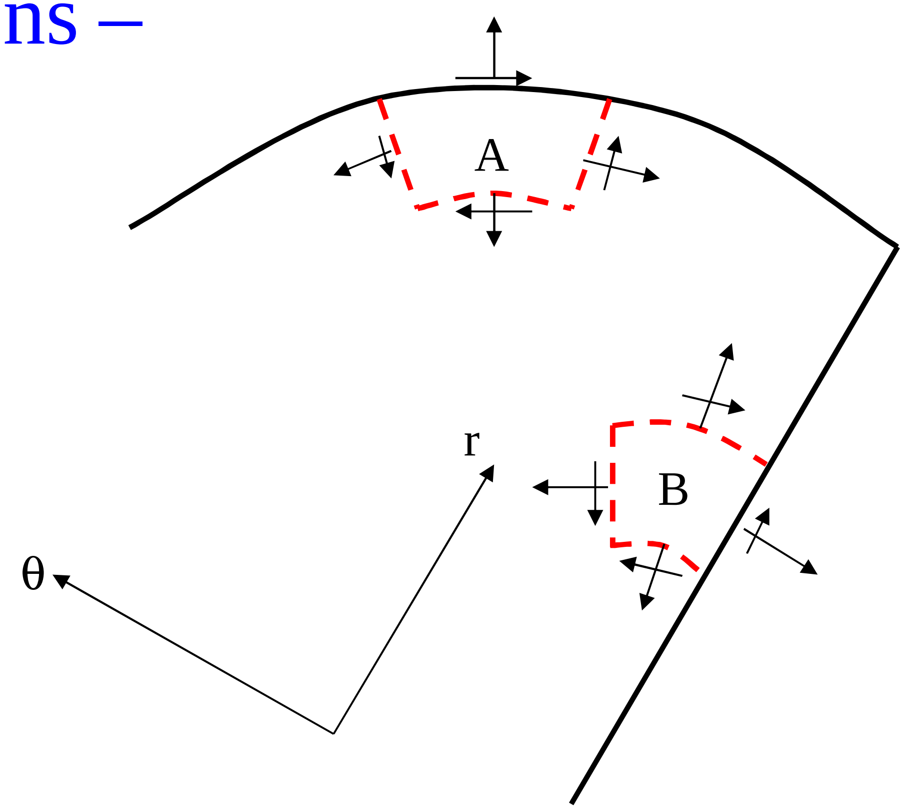
Boundary conditions

- Tractions –



Tractions in Polar co-ordinate system

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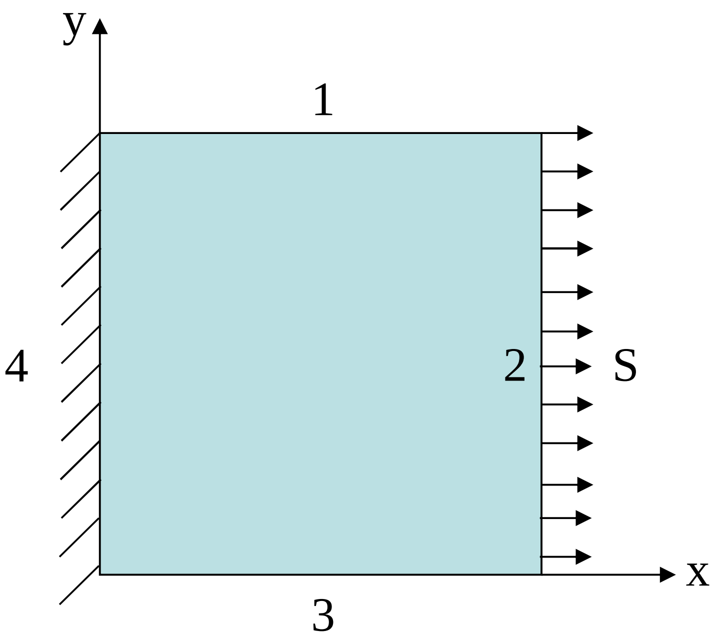
Boundary conditions

- Tractions –



Boundary – 4

$u = 0; v = 0$

Boundary – 1 – traction free

DC => (0, 1, 0)

$T_x = 0.\,\sigma_{xx} + 1.\,\tau_{xy} = 0$

$\Rightarrow T_x = \tau_{xy} = 0$

$\Rightarrow T_y = 0.\,\tau_{xy} + 1.\,\sigma_{yy} = 0 \Rightarrow \sigma_{yy} = 0$

Boundary – 2 – DCs (1, 0, 0)

$T_x = 1.\,\sigma_{xx} + 0.\,\tau_{xy} = S = \sigma_{xx}$

$T_y = 1.\,\tau_{xy} + 0.\,\sigma_{yy} = 0 = \tau_{xy}$

Boundary – 3 – DCs (0, -1, 0)

$T_x = 0.\,\sigma_{xx} - 1.\,\tau_{xy} = 0$

$T_x = \tau_{xy} = 0$

$T_y = 0.\,\tau_{xy} - 1.\,\sigma_{yy} = 0 \Rightarrow \sigma_{yy} = 0$

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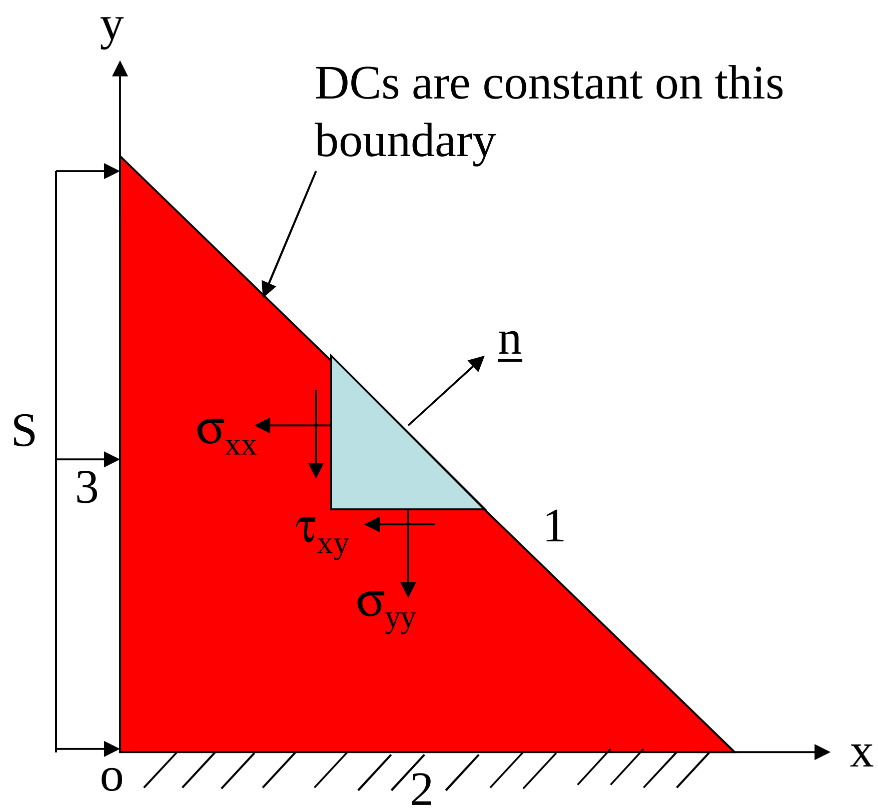
Boundary conditions

R&DE (Engineers), DRDO

- Tractions



Individual stress components are not zero on boundary - 1

Boundary - 1 – DCs *(l, m, 0)*

This is a traction free boundary

$$T_x = l\sigma_{xx} + m\,\tau_{xy} = 0$$

$$T_y = m\sigma_{xx} + l\,\tau_{xy} = 0$$

Boundary – 2 – Constrained boundary – U, V = 0

Boundary – 3- Traction is acting

DCs (-1, 0, 0)

$$T_x = -1\sigma_{xx} + m\,\tau_{xy} = S \Rightarrow \sigma_{xx} = -S$$

$$T_y = m\sigma_{xx} + l\,\tau_{xy} = 0 \Rightarrow \tau_{xy} = 0$$

Problem formulation

- 3D elasticity 15 equations and 15 unknowns - 2D elasticity – 8 equations and 8 unknowns
- Reformulating elasticity problems – mathematically convenient way
- Two approaches –
  - Displacement based
  - Stress based formulations
- Displacement based – express filed equation in terms of displacements, u_i – Navier equation
- Stress based – express filed equation in terms of stresses, σ_{ij}

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Problem formulation

- In displacement based formulation
  - Primary variable – displacement – solution is obtained for displacements – stresses and strains secondary / derived variables
  - Strains – derivation of displacements
  - Stresses – use constitutive law
- In stress based formulation
  - Primary variable – stresses – solution is obtained for stresses – strains and displacements secondary variables
  - Strain compatibility equations

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Displacement formulation

- Governing equation expressed in displacements
- Field equations –

$$\text{Equilibrium} \quad \sigma_{ij,j} + B_i = 0 \quad \text{- (1)}$$

$$\text{Strain - displacement} \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{- (2)}$$

$$\text{Constitutive law} \quad \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \quad \text{- (3)}$$

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Displacement formulation

- Differentiate (3) wrt *'j'*

$$\sigma_{ij,j} = \lambda \varepsilon_{kk,j}\delta_{ij} + 2\mu\varepsilon_{ij,j} \qquad -(4)$$

Plug this in (1)

$$\sigma_{ij,j} + B_i = \lambda \varepsilon_{kk,j}\delta_{ij} + 2\mu\varepsilon_{ij,j} + B_i = 0 \quad -(5)$$

Express all strain components in (5) in terms of displacements – Use equation (2)

$$\lambda u_{k,kj}\delta_{ij} + \mu\left(u_{i,jj} + u_{j,ij}\right) + B_i = 0$$

$$\lambda u_{k,ki} + \mu u_{j,ji} + \mu u_{i,jj} + B_i = 0$$

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Displacement formulation

$$\lambda u_{k,ki} + \mu u_{j,ji} + \mu u_{i,jj} + B_i = 0$$

$$(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} + B_i = 0$$

This gives three equations for i = 1, 2 and 3

For $i = 1$ => x - direction

$$(\lambda + \mu)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + \mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)u_1 + B_1 = 0$$

Combining all three equations into a single equation

$$(\lambda + \mu)\ \nabla\left(\nabla . \underset{\sim}{U}\right) + \mu \nabla^2 \underset{\sim}{U} + \underset{\sim}{B} = 0$$

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Displacement formulation

- Governing equations in terms of displacements – three equations - reduced from '15' equations

$$(\lambda+\mu)u_{j,ji}+\mu u_{i,jj}+B_i=0 \quad \text{-(U)}$$

$$\Im\left(u_i,\lambda,\ \mu,\ B_i\right)=0 \quad \text{Navier equation}$$

Equations (U) known as Navier's or Lame's equations

Boundary conditions => u_i = U_i on 'S' boundary $u_i = u_i(x_1,\ x_2,\ x_3)$

Solve second order PDEs – get displacements

Use strain – displacement relations for calculating strains

Constitutive relations for stresses

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Stress formulation

- In displacement formulation compatibility equations play no role – displacement obtained are single valued and continuous
- In stress formulation express governing equation in terms of stresses
- Stresses – primary variables – obtain secondary variables displacements from strain-displacement relations
- Integrate ε-U relations – compatibility equations come into play

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Stress formulation

- Boundary conditions – only tractions
- From the filed equations eliminate strains and displacements
- Compatibility equation

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \quad - \quad (1)$$

For $k = l$ first three compatibility equations are obtained

Constitutive law $\varepsilon_{ij} = \frac{1+\upsilon}{E}\sigma_{ij} - \frac{\upsilon}{E}\sigma_{kk}\delta_{ij}$

Differentiate this wrt k and l, plug in (1)

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Stress formulation

- Three compatibility equations for $k=l = 1, 2$ and 3

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$(i = j = 1, k = l = 2)$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$(i = j = 2, k = l = 3)$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}$$

$$(i = j = 3, \; k = l = 1)$$

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Stress formulation

This gives –

$$\sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} =$$

$$\frac{\upsilon}{1+\upsilon}\left(\sigma_{mm,kk}\delta_{ij} + \sigma_{mm,ij}\delta_{kk} - \sigma_{mm,jk}\delta_{ik} - \sigma_{mm,ik}\delta_{jk}\right)$$

$$\delta_{kk} = 3$$

$$\text{equilibrium eqn.} \quad \sigma_{ij,j} + B_i = 0$$

$$\sigma_{ij,j} = -B_i$$

Make use of equilibrium equation to reduce further

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Stress formulation

Governing equation in terms of stress –

$$\sigma_{ij,kk} + \frac{1}{1+\upsilon}\sigma_{kk,ij} = \frac{\upsilon}{1+\upsilon}\sigma_{mm,kk}\delta_{ij} - B_{i,j} - B_{j,i}$$

$$\Im\left(\sigma_{ij}, \lambda,\ \mu,\ B_i\right) = 0$$

Compatibility equation in terms of stresses, known as

"Beltrami-Michell" compatibility equation

Six equations can be obtained for $i, j = 1, 2, 3$

Only three are independent – similar to strain compatibility equations

Three more equations are required – complemented by three equilibrium equations - Six unknowns – six equations

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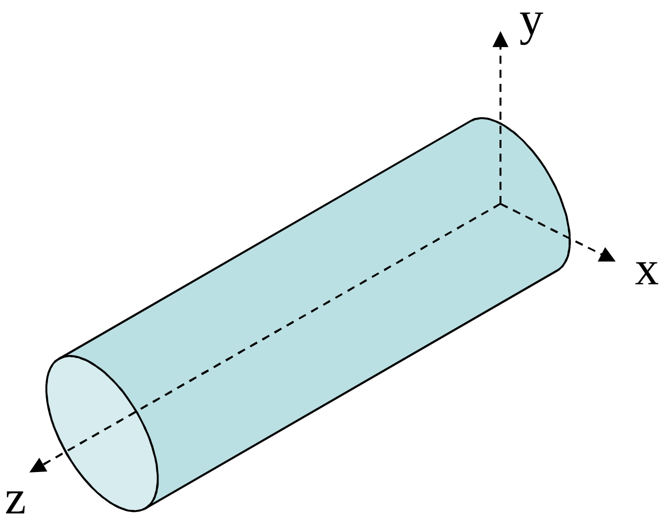
Plane strain

- Out-of-plane dimensions are large compared to in-plane dimensions



Reduce a 3D problem to 2D problem

All loads and boundary conditions independent of 'z' co-ordinate

The deformation in 'z' direction assumed to be zero. $w = 0$

Deformations in 'x' and 'y' directions depend on (x, y) only

$u = u(x, y)$ and $v = v(x, y)$

All cross-sections have same displacements

Out of plane strains vanish

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Plane strain

- In plane strain $\varepsilon_{zz} = 0$, $\gamma_{yz} = 0$ and $\gamma_{zx} = 0$
- Linear elastic material – Hooke's law

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\sigma_{xx} = \lambda \left(\varepsilon_{xx} + \varepsilon_{yy} \right) + 2\mu \varepsilon_{xx}$$

$$\sigma_{yy} = \lambda \left(\varepsilon_{xx} + \varepsilon_{yy} \right) + 2\mu \varepsilon_{yy}$$

$$\sigma_{zz} = \lambda \left(\varepsilon_{xx} + \varepsilon_{yy} \right)$$

$$\tau_{xy} = 2\mu \varepsilon_{xy} = \mu \gamma_{xy}$$

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Plane strain

- Equilibrium equations –

$$\sigma_{ij,j} + B_i = 0$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0 \qquad - \ (1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{xx}}{\partial y} + B_y = 0 \qquad - \ (2)$$

Use constitutive law and strain-displacement equations in (1)

$$\frac{\partial}{\partial x}\left(\lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + 2\mu\varepsilon_{xx}\right) + \frac{\partial}{\partial y}\left(\mu\gamma_{xy}\right) + B_i = 0$$

$$\frac{\partial}{\partial x}\left[\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + B_x = 0$$

Plane strain

Governing equation in x – direction

$$\mu\nabla^2 u + (\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + B_x = 0$$

Use second equilibrium equation – convert to displacements

$$\mu\nabla^2 v + (\lambda + \mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + B_y = 0$$

In each of the above two equations, equilibrium, constitutive law and displacement – strain equations are inbuilt - ***Displacement formulation***

Eight equations => reduced to two equations in displacements

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Plane strain

- Stress based formulation -

In 2D case, compatibility equation $$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Use constitutive law – convert into stresses

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\upsilon}{E}\left(\sigma_{yy} + \sigma_{zz}\right)$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\upsilon}{E}\left(\sigma_{xx} + \sigma_{zz}\right)$$

$$\varepsilon_{zz} = 0 = \frac{\sigma_{zz}}{E} - \frac{\upsilon}{E}\left(\sigma_{xx} + \sigma_{yy}\right) => \sigma_{zz} = \upsilon\left(\sigma_{xx} + \sigma_{yy}\right)$$

$$\gamma_{xy} = \frac{2(1+\upsilon)}{E}\tau_{xy}$$

Plane strain

- Compatibility equation in terms of stresses

$$\frac{\partial^2}{\partial x^2}\left[(1-\upsilon)\sigma_{yy}-\upsilon\sigma_{xx}\right]+\frac{\partial^2}{\partial y^2}\left[(1-\upsilon)\sigma_{xx}-\upsilon\sigma_{yy}\right]=2\frac{\partial^2\tau_{xy}}{\partial x\partial y}$$

Above equation is obtained from compatibility, constitutive equations

Use equilibrium equations to eliminate shear stress from above equation

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+B_x=0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+B_x\right)=0$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{xx}}{\partial y}+B_y=0 \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+B_y\right)=0$$

Add

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Plane strain

This is obtained as –

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

$$\frac{\partial^2}{\partial x^2}\left[(1-\upsilon)\sigma_{yy} - \upsilon\sigma_{xx}\right] + \frac{\partial^2}{\partial y^2}\left[(1-\upsilon)\sigma_{xx} - \upsilon\sigma_{yy}\right] = -\left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

Simplifying this expression

$$\nabla^2\left(\sigma_{xx} + \sigma_{yy}\right) = -\frac{1}{1-\upsilon}\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)$$

Single governing equation for stress based formulation of plane strain problems

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Plane strain

- Governing equation

$$\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right)=-\frac{1}{1-\upsilon}\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)$$

In this equation, equilibrium, constitutive and compatibility equations are inbuilt.

Three stress components in 2D – σ_{xx}, σ_{yy}, τ_{xx}

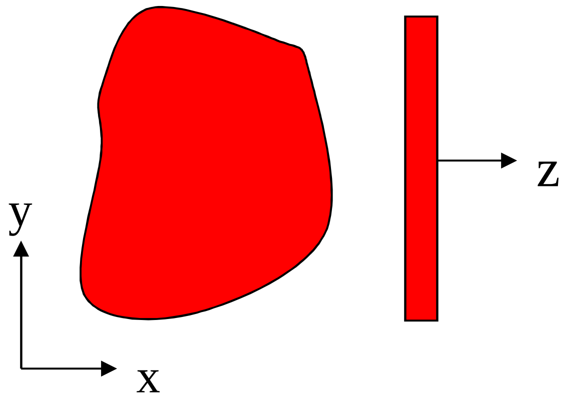
Only one governing equation – three unknowns

In 2D two stress equilibrium equations exist – use these two with above governing equation

Boundary conditions given in terms of tractions. Relate them to stresses

Plane stress

- Third dimension – much smaller than in-plane dimensions



Thin plates – structures

Reducing a 3D problem to a 2D problem

Out of plane stresses σ_{zz}, τ_{xz} and τ_{yz} negligible

Body can have out of plane deformation

ε_{zz} is non-zero, γ_{xz} and $\gamma_{yz} = 0$

Stresses are functions of in-plane displacements

$$\sigma_{xx} = \sigma_{xx}(x, y),$$
$$\sigma_{yy} = \sigma_{yy}(x, y),$$
$$\tau_{xy} = \tau_{xy}(x, y)$$

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Plane stress

- Formulation of plane stress problem – two approaches
  - Displacement based
  - Stress based approach
- Formulation procedure – same as in plane strain
- Formulation takes care of non-zero out of normal strain and zero normal stress
- Use field equations

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Plane stress

- Displacement based approach –

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Plug in stress equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0; \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{xx}}{\partial y} + B_y = 0$$

$$\frac{\partial}{\partial x}\left[\frac{E}{1-\upsilon^2}\varepsilon_{xx} + \frac{E\upsilon}{1-\upsilon^2}\varepsilon_{yy}\right] + \frac{\partial}{\partial y}\left[\frac{E}{2(1+\upsilon)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + B_x = 0$$

$$\frac{\partial}{\partial y}\left[\frac{E}{1-\upsilon^2}\varepsilon_{yy} + \frac{E\upsilon}{1-\upsilon^2}\varepsilon_{xx}\right] + \frac{\partial}{\partial x}\left[\frac{E}{2(1+\upsilon)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + B_y = 0$$

Plane stress

- Use strain displacement equations to convert strains into displacements
- Governing equations in terms of displacements

$$\mu\nabla^2 u + \frac{E}{2(1-\upsilon)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+B_x = 0$$

$$\mu\nabla^2 v + \frac{E}{2(1-\upsilon)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+B_y = 0$$

In the above two equations – equilibrium, constitutive laws and strain-displacements equations

Plane stress

- Stress based approach – field equations in plane stress problems

Constitutive laws

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\upsilon}{E}\sigma_{yy}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\upsilon}{E}\sigma_{xx}$$

$$\gamma_{xy} = \frac{2(1+\upsilon)}{E}\tau_{xy}$$

Equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{xx}}{\partial y} + B_y = 0$$

Compatibility equation

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

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Plane stress

- Use constitutive laws in compatibility equation – eliminate strains in compatibility using constitutive law
- Compatibility equation in terms of stress – stress compatibility equation
- Eliminate shear stress in stress compatibility equation using two equilibrium equations
- Differentiate x – direction equilibrium equation wrt 'x' – y – direction eqn. wrt 'y', add these two get shear stress
- Plug this in stress compatibility equation

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Plane stress

- The governing equation –

$$\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right)=-(1+\upsilon)\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_x}{\partial x}\right)$$

This is derived making use of strain compatibility equation, constitutive laws and equilibrium equations

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Summary – Plane stress and strain

Formulation	Plane strain	Plane stress
Displacement	$\mu\nabla^2 u+(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+B_x=0$ $\mu\nabla^2 v+(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+B_y=0$	$\mu\nabla^2 u+\frac{E}{2(1-\upsilon)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+B_x=0$ $\mu\nabla^2 v+\frac{E}{2(1-\upsilon)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+B_y=0$
Stress	$\nabla^2(\sigma_{xx}+\sigma_{yy})=-\frac{1}{1-\upsilon}\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)$	$\nabla^2(\sigma_{xx}+\sigma_{yy})=-(1+\upsilon)\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_x}{\partial x}\right)$

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Summary – Plane stress and strain

- Basic difference – coefficients involving elastic material constants
- Solving one type of plane problem gives solution to other type also using simple transformation of elastic coefficients
- In absence of body loads – in stress formulation – governing equations of plane stress and strain are same
  - No elastic constants appear in governing equation
- Solving these equations – difficult – some text book problems

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Airy’s stress function

- In stress based formulation – one equation in terms of normal stresses and two equilibrium equations for finding out the complete stress state
- Reduce the complexity of equations
- Introduce a new filed variable ‘Airy’s stress function’ (ϕ)
- Aim – reduce the governing equations from three to one
- Find the distribution of single variable ‘ϕ’ – get stresses from that

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Airy's stress function

- Airy's stress function (ASF) defined for stress based formulations – both plane stress and plane strain
- Basic stress equilibrium equations –

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{xx}}{\partial y} + B_y = 0$$

Body loads can be derived from potential function 'V'

Defining potential

$$\underset{\sim}{B} = -\nabla V \Rightarrow B_i = V_{,i}$$

$$B_x = -\frac{\partial V}{\partial x}, \quad B_y = -\frac{\partial V}{\partial y}$$

Substitute in equilibrium equations

Airy's stress function

Equilibrium equations become

$$\frac{\partial}{\partial x}\left(\sigma_{xx} - V\right) + \frac{\partial \tau_{xy}}{\partial y} = 0; \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial}{\partial y}\left(\sigma_{yy} - V\right) = 0$$

Defining ASF as

$$\sigma_{xx} - V = \frac{\partial^2 \phi}{\partial y^2}; \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}; \qquad \sigma_{yy} - V = \frac{\partial^2 \phi}{\partial x^2}$$

ASF, $\phi = \phi(x, y)$

ASF satisfies both equilibrium equations.

In stress based formulation – it has to satisfy one more governing equation

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Airy's stress function

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- In plane strain – stress based formulation – governing equation

$$\nabla^2(\sigma_{xx}+\sigma_{yy}) = -\frac{1}{1-\upsilon}\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)$$

$$\nabla^2\left(\left(V+\frac{\partial^2\phi}{\partial y}\right)+\left(V+\frac{\partial^2\phi}{\partial x}\right)\right) = -\frac{1}{1-\upsilon}\left(\frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial V}{\partial y}\right)\right)$$

$$\nabla^2\left(\frac{\partial^2\phi}{\partial x}+\frac{\partial^2\phi}{\partial y}\right)+2\nabla^2 V = \frac{1}{1-\upsilon}\nabla^2 V$$

$$\nabla^2.\nabla^2\phi = \left(\frac{1}{1-\upsilon}-2\right)\nabla^2 V$$

Bi-harmonic operator $\nabla^4\phi = -\left(\frac{1-2\upsilon}{1-\upsilon}\right)\nabla^2 V$

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Airy's stress function

- In plane stress – stress based formulation – governing equation

$$\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right)=-(1+\upsilon)\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}\right)$$

$$\nabla^2\left(\left(V+\frac{\partial^2\phi}{\partial y}\right)+\left(V+\frac{\partial^2\phi}{\partial x}\right)\right)=-(1+\upsilon)\left(\frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial V}{\partial y}\right)\right)$$

$$\nabla^2\left(\frac{\partial^2\phi}{\partial x}+\frac{\partial^2\phi}{\partial y}\right)+2\nabla^2 V=(1+\upsilon)\nabla^2 V$$

$$\nabla^2.\nabla^2\phi=(1+\upsilon-2)\nabla^2 V$$

$$\nabla^4\phi=-(1-\upsilon)\nabla^2 V$$

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Airy's stress function

- Governing equations in terms of ASF

$$\nabla^4 \phi = -\left(\frac{1-2\upsilon}{1-\upsilon}\right)\nabla^2 V \qquad \text{Plane strain}$$

$$\nabla^4 \phi = -(1-\upsilon)\nabla^2 V \qquad \text{Plane stress}$$

If body forces are neglected – V = 0 – In both cases the governing equation is same

$$\nabla^4 \phi = 0 \Rightarrow \frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$$

Bi-harmonic equation – solution to this equation => ASF

Problem of elasticity reduced to a single equation – find ϕ in solution domain and boundaries – BCs - Tractions

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Stress formulation

- This formulation is appropriate for use with traction boundary conditions
- Use Hooke's law for calculating strains from stresses
- Compute displacements from strain-displacement relations
- Since compatibility equations are used in formulating governing equations, the displacements obtained from integration of strain-displacement equations yield single valued, continuous filed
- All equations are used for solving the problem
- Closed form analytical solutions for elasticity problems – difficult – rely on numerical methods

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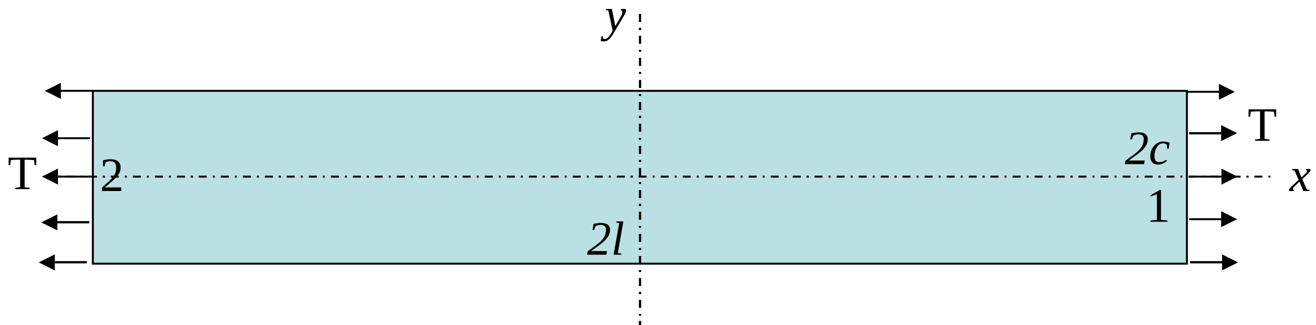
Uniaxial tension – ASF approach

- A bar subjected to tension



Boundary – 1

DCs => (1, 0, 0)

$T_x = T = l\sigma_{xx} + m\ \tau_{xy} = \sigma_{xx} = T$

$T_y = 0 = m\sigma_{yy} + l\ \tau_{xy} => \tau_{xy} = 0$

Boundary – 2

DCs => (-1, 0, 0)

$T_x = -T = l\sigma_{xx} + m\ \tau_{xy} = \sigma_{xx} = T$

$T_y = 0 = m\sigma_{yy} + l\ \tau_{xy} => \tau_{xy} = 0$

Remaining boundaries – stress free.

Stress in x – direction = T = constant

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Uniaxial tension – ASF approach

Stress in x – direction given by ASF

$$\sigma_{xx} = \text{constant} = \frac{\partial^2 \phi}{\partial y^2} = T \Rightarrow \phi = \frac{1}{2} T y^2 + c_1 y + c_2$$

At $x = \pm l$, σ_{yy} and $\tau_{xy} = 0$

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\upsilon}{E}\sigma_y = \frac{\partial u}{\partial x} = \frac{T}{E} \quad - \quad (1)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \frac{\upsilon}{E}\sigma_x = \frac{\partial v}{\partial y} = -\frac{\upsilon T}{E} \quad - \quad (2)$$

Integrate (1) and (2)

$$u = \frac{T}{E} x + f(y)$$

$$v = -\frac{\upsilon T}{E} y + g(x)$$

f(y) and *g(x)* – arbitrary functions

Uniaxial tension – ASF approach

Shear strain,

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} = 0$$

$$=> \; f'(y) + g'(x) = 0$$

$$=> -f'(y) = g'(x) = \omega_o$$

$$=> \frac{\partial f}{\partial y} = -\omega_o \; => \; f = -\omega_o y + u_o$$

$$\frac{\partial g}{\partial x} = \omega_o \; => \; g = \omega_o x + v_o$$

ω_o, u_o, v_o – arbitrary constants of integration.

Uniaxial tension – ASF approach

- Functions '*f*' and '*g*' represent rigid body motions
- Rigid body rotation - ω_o and translations in '*x*' and '*y*' directions => u_o and v_o
- Terms related to rigid body motion result from strain-displacement relations
- Displacements are determined from strain fields only up to an arbitrary rigid motion
- For complete determination of displacement filed – additional boundary conditions required – evaluate ω_o, u_o and v_o
- If rod has no rigid body motion => ω_o, u_o and $v_o = 0$

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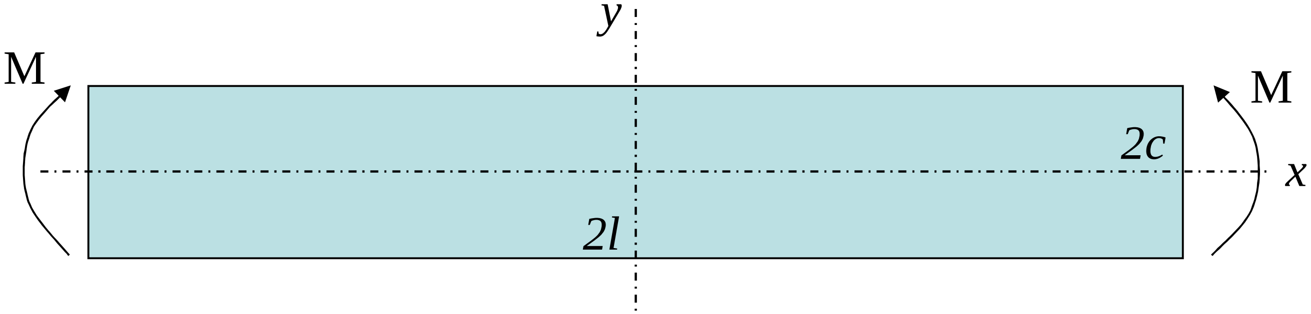
Pure bending of beam – ASF approach

R&DE (Engineers), DRDO

- Beam subjected to pure bending –



Boundary

Conditions

$$\sigma_{yy}(x, \pm c) = 0$$

$$\tau_{xy}(x, \pm c) = 0$$

$$\tau_{xy}(\pm l, y) = 0$$

$$F_x = \int_{-c}^{+c} \sigma_{xx}(\pm l, y)\, dy = 0$$

Statically equivalent boundary condition →

$$-M = \int_{-c}^{+c} \sigma_{xx}(\pm l, y)\, y dy$$

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Pure bending of beam – ASF approach

- Key point in ASF approach – selection of an appropriate stress function - ϕ
- It has to satisfy all boundary conditions – capture physics of the problem
- Some knowledge about structure's behavior helps in selection of 'ϕ'
  - In bending, variation of bending stress is linear along 'y' co-ordinate
  - 'ϕ' should be a cubic polynomial

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Linear variation of bending stress

$$\sigma_{xx} = Ay = \frac{\partial^2 \phi}{\partial y^2}$$ Integrating the function

$$\phi = \frac{A}{6} y^3 + C_1 y + C_2$$ ASF satisfies all BCs

$$\int_{-c}^{+c} \sigma_{xx}(\pm l, y) y dy = -M$$

$$\int_{-c}^{+c} A y^2 dy = -M \Rightarrow -A\left(\frac{y^3}{3}\right)_{-c}^{+c} = M \Rightarrow A = -\frac{3}{2}\frac{M}{c^3}$$

$$\phi = -\frac{3}{2}\frac{M}{c^3}\frac{y^3}{6} + C_1 y + C_2 = -\frac{M}{4c^3} y^3 + C_1 y + C_2$$

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Pure bending of beam – ASF approach

- Stresses and strains

$$\phi = -\frac{M}{4c^3}y^3 + C_1 y + C_2 \Rightarrow \sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = -\frac{3M}{2c^3}y$$

$$\sigma_{yy} = 0, \quad \tau_{xy} = 0$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \upsilon\frac{\sigma_{yy}}{E} \Rightarrow \frac{\partial u}{\partial x} = -\frac{3}{2}\frac{M}{Ec^3}y$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \upsilon\frac{\sigma_{xx}}{E} = \frac{3}{2}\frac{M\upsilon}{Ec^3}y$$

Integrate strain – displacement equations

$$u = -\frac{3M}{2Ec^3}xy + f(y); \qquad v = \frac{3M\upsilon}{4Ec^3}y^2 + g(x)$$

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Pure bending of beam – ASF approach

Shear strain

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \Rightarrow -\frac{3M}{2Ec^3}x + f'(y) + g'(x) = 0$$

$$\Rightarrow -\frac{3M}{2Ec^3}x + g'(x) = -f'(y) = \omega_o \qquad \text{Integrate}$$

$$g(x) = \frac{3M}{2Ec^3}\frac{x^2}{2} + \omega_o x + v_o$$

$$f(y) = -\omega_o y + u_o$$

Functions g(x) and f(y) represent rigid body motions

To evaluate constants, ω_o, u_o, v_o – constraints required

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