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On modeling failure of rubber-like materials

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ABSTRACT

Strain energy increases unlimitedly with the increase of deformation according to traditional hyperelastic models of materials. This 'growth condition' is evidently unphysical because no real material can sustain large enough deformations without failure. To introduce failure in hyperelasticity we propose to replace the strain energy of any intact material, W, with a modified expression: $\psi = \Phi\{\Gamma(1/m, 0) - \Gamma(1/m, W^m/\Phi^m)\}/m$, where Φ is a material constant which sets a limit for the energy that can be accumulated during deformation, Γ is the upper incomplete gamma function, and m is a material constant controlling the sharpness of the transition to failure. The new formula for the strain energy is used to model failure of natural (NR) and styrene–butadiene (SBR) rubbers under plane stress conditions and the results are compared to the available experiments and other theories. The comparisons show that the proposed approach can be efficient for modeling failure in solids. The new theory allows reassessing the local criteria of material failure.

1. Introduction

Traditional hyperelastic models of materials ignore the fact that no material can sustain large enough deformations without failure. The theories targeting a constitutive description of bulk failure were pioneered by Kachanov (1958) and Rabotnov (1963) and they appear by name of continuum damage mechanics (CDM) in the modern literature. Originally, CDM was invented to analyze the gradual failure accumulation and propagation in creep and fatigue. The need to describe the failure accumulation, i.e. evolution of the material microstructure, explains why CDM is very similar to plasticity theories including (a) the internal damage variable (inelastic strain), (b) the critical threshold condition (yield surface), and (c) the damage evolution equation (flow rule). The subsequent development of the formalism of CDM (Lemaitre and Desmorat, 2005) left its physical origin well behind the mathematical and computational techniques and, eventually, led to the use of CDM for the description of *any* bulk failure (Kachanov, 1994). Unfortunately, it is impossible to measure the damage parameter directly and the experimental calibration should be implicit and include both the damage evolution equation and criticality condition.

A physically motivated alternative to the continuum damage mechanics in the cases of failure related with the bond rupture was introduced by Gao and Klein (1998) and Klein and Gao (1998) who showed how to mix the atomic/molecular and continuum descriptions in order to simulate material failure. Another multiscale model is due to Dal and Kaliske (2009) who included a possible rupture of polymer chains into the constitutive description of rubbers. Of course, a direct appeal to the material structure is very attractive. It should not be missed, however, that the transition from the smaller to larger length scales is always related to an averaging procedure, which presumes, often implicitly, additional assumptions concerning the macroscopic behavior of material that can be revealed in the macroscopic experiments only.[1] Besides, the multiscale link is usually computationally involved what limits its applications.

A very simple alternative to continuum damage mechanics on the one hand and multiscale methods on the other hand was developed by Volokh (2004, 2007, 2008). The basic idea of the new approach was to formulate an expression of the strain energy density, which would include the energy limiter(s) – the failure energy. The energy limiter enforces the limit point on the stress–strain curve separating intact and failure behaviors of bulk material analogously to the bond energy of atomic interactions. Specifically, the following universal formula was proposed and examined in Volokh (2007): $\psi = \Phi\{1 - \exp(-W/\Phi)\}$, where ψ was the strain energy with account of failure, W was the strain energy without failure, and Φ was the energy limiter which showed the maximum or separation energy that an infinitesimal material volume could accumulate without failure. The above formula is useful for a description of

[1] This notion makes us skeptical regarding claims of the 'rigorous derivation' of the continuum constitutive equations from the equations at smaller length scales. There is no doubt, nonetheless, that the smaller-scale considerations can guide the search of the continuum theories.

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smooth failure[2] with a flat limit point on the stress–strain curve, which corresponds to a gradual process of the bond rupture. In the case of more abrupt bond ruptures, however, a much sharper transition to the material instability occurs. To describe such sharp transition to failure the strain energy formula is further modified in the present work: $\psi = \Phi\{\Gamma(1/m,0) - \Gamma(1/m,(W/\Phi)^m)\}/m$, where Γ is the upper incomplete gamma function and m is a new material constant controlling the 'sharpness' of the transition to failure. The new formula for the strain energy is applied to the traditional Yeoh and Ogden models of intact materials describing deformation of natural (NR) and styrene–butadiene (SBR) rubbers. The modified models are examined under plane stress conditions and the results are compared to the available experiments and other theories. The comparisons show that the proposed formula can be valuable for modeling failure in rubber-like solids. The new theory allows reassessing the local criteria of material failure.

2. Elasticity with energy limiters

To motivate the introduction of energy limiters we briefly describe the atomistic-continuum link. Let us consider interaction of two atoms/molecules/particles. The reference distance between them corresponds to zero interaction force and zero stored energy. The interaction passes three stages with the increase of the distance. At the first stage the force increases proportionally to the increasing distance: the linear stage. At the second stage the force–distance relationship deviates from the linear proportionality: the nonlinear stage. At the third stage the force drops with the increasing distance: the separation or failure stage.

In the case of solids composed of many particles two first stages of the particle interaction are described by the linear and nonlinear theories of elasticity correspondingly where the changing distance between particles is averaged by a continuum strain measure and the energy of the particle interaction is averaged by a strain energy function. Surprisingly, the third, failure, stage of the particle interaction is beyond the scope of elasticity theories. However, the failure description can still be introduced in elasticity by analogy with the failure description in the particle interaction. Indeed, the force of the pair interaction decreases with the increase of the interaction distance because the energy that can be stored during separation is limited by the constant of the bond energy. If the energy limiter exists for the pair interaction then it should exist in the multiple interactions. The latter means that we should limit the magnitude of the strain energy in order to describe material failure within the framework of elasticity. A review of some works where elasticity with energy limiters was applied to various problems can be found in Volokh (2008).

Though the choice of the strain energy expression including the energy limiters should generally be material/problem-specific, a general or "try first" formula was introduced in Volokh (2007) to enrich the already existing models of intact materials with a failure description: $\psi(W(\mathbf{C})) = \Phi\{1 - \exp(-W(\mathbf{C})/\Phi)\}$, where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy–Green tensor and $\mathbf{F} = \partial\mathbf{y}/\partial\mathbf{x}$ is the deformation gradient for a generic material particle occupying position $\mathbf{x}$ at the reference state and position $\mathbf{y}(\mathbf{x})$ at the current state of deformation. An example of the experimental calibration of energy limiters can be found in Volokh and Vorp (2008) for the case of the material of Abdominal Aortic Aneurysm (AAA). AAA is rubber-like and its strain energy can be written in the form: $\psi = \Phi\{1 - \exp(-\alpha_1(I_1-3)/\Phi - \alpha_2(I_1-3)^2/\Phi)\}$, where $I_1 = \mathrm{tr}\mathbf{C}$; α_1 and α_2 are the elasticity constants of the material; and Φ is the energy limiter. The results of the uniaxial tension test are shown in Fig. 1, where the model was fitted with the following constants: $\alpha_1 = 10.3\ \mathrm{N/cm^2}$; $\alpha_2 = 18.0\ \mathrm{N/cm^2}$; $\Phi = 40.2\ \mathrm{N/cm^2}$.

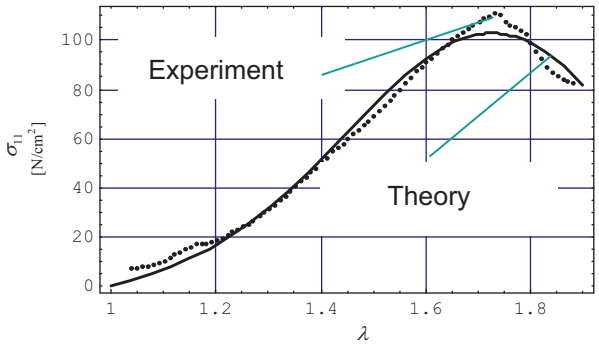

Fig. 1. Theory versus experiment for uniaxial tension of AAA material. σ_{11} and λ are the axial Cauchy stress and stretch accordingly.

It is worth noting that energy limiters set failure energy per unit *volume* contrary to the approach of fracture mechanics, which sets fracture or 'tearing' (Rivlin and Thomas, 1953) energy per unit *area*.

Remark The approach of softening hyperelasticity, which is proposed in this paper, includes neither the energy dissipation nor the path dependence in its formulation. However, the dissipation and path dependence are important phenomena, especially in processes where the unloading occurs. In order to ensure that the unloading does not lead to the healing of the failed material, the failed finite elements have to be removed from the mesh.[3]

3. New failure potential

It is evident from Fig. 1 and other examples reviewed in Volokh (2008) that formula $\psi = \Phi\{1 - \exp(-W/\Phi)\}$ is useful for a description of smooth failure with a flat limit point on the stress–strain curve which corresponds to a gradual process of the bond rupture. In the case of more abrupt bond ruptures, however, a much sharper transition to the material instability occurs. To describe such sharp transition to failure we propose the following universal formula for the strain energy density

$$\psi = \frac{\Phi}{m}\left\{\Gamma\left(\frac{1}{m},0\right) - \Gamma\left(\frac{1}{m},\frac{W^m}{\Phi^m}\right)\right\}. \tag{1}$$

Here the upper incomplete gamma function is used.

$$\Gamma(s,x) = \int_x^\infty t^{s-1}\exp(-t)dt. \tag{2}$$

Differentiating the modified strain energy with respect to the right Cauchy–Green tensor we get the following constitutive equation for the Cauchy stress

$$\boldsymbol{\sigma} = 2(\det\mathbf{F})^{-1}\mathbf{F}\frac{\partial\psi}{\partial\mathbf{C}}\mathbf{F}^T, \tag{3}$$

or

$$\boldsymbol{\sigma} = 2(\det\mathbf{F})^{-1}\mathbf{F}\frac{\partial W}{\partial\mathbf{C}}\mathbf{F}^T\exp\left(-\frac{W^m}{\Phi^m}\right). \tag{4}$$

Deriving (4) we used the chain rule and the following result

$$\frac{\partial\psi}{\partial W} = \exp\left(-\frac{W^m}{\Phi^m}\right), \tag{5}$$

[2] The gradual failure is characteristic of some soft biological tissues, for example.

[3] It is interesting to note that the models of continuum damage mechanics, which include the energy dissipation in their theoretical formulation still need the deletion of finite elements during simulation.

which comes from the derivative of the upper incomplete gamma function.

$$\frac{\partial \Gamma(s,x)}{\partial x} = -x^{s-1}\exp(-x). \tag{6}$$

The new material parameter m controls the sharpness of the transition to material instability on the stress strain curve. Increasing m it is possible to simulate the catastrophic rupture of the internal bonds. Finally, we should mention that the limiting value of the failure energy is $\Phi\Gamma(1/m,0)/m$ and it reduces to Φ for $m=1$.

4. Examples

To reveal the features of the proposed generalized strain energy we examine a few plane stress problems in this section.

We consider biaxial deformations in plane (x_1,x_2) of a thin rubber sheet by using principal stretches and stresses for an isotropic incompressible hyperelastic material

$$\sigma_i = \lambda_i \frac{\partial \psi}{\partial \lambda_i} - p, \quad \text{(no sum over } i\text{)}. \tag{7}$$

Ignoring the normal stress in the out-of-plane direction it is possible to find the Lagrange multiplier, p, and final expressions for the principal stresses in the plane of the sheet

$$\sigma_1 = \lambda_1 \frac{\partial \psi}{\partial \lambda_1} - \lambda_3 \frac{\partial \psi}{\partial \lambda_3}, \quad \sigma_2 = \lambda_2 \frac{\partial \psi}{\partial \lambda_2} - \lambda_3 \frac{\partial \psi}{\partial \lambda_3}, \tag{8}$$

where

$$\lambda_3 = \frac{1}{\lambda_1 \lambda_2}. \tag{9}$$

It is convenient to introduce a biaxiality ratio, n, as follows $\lambda_1=\lambda$, $\lambda_2=\lambda^n$, $\lambda_3=\lambda^{-(n+1)}$. In this case, we have uniaxial tension for $n=-0.5$; equal biaxial tension for $n=1.0$; pure shear for $n=0.0$.

Two materials are examined: a filled natural rubber (NR) vulcanizate (which is crystallizable) and a filled styrene–butadiene rubber (SBR) vulcanizate.

Hamdi et al. (2006) tested both materials and calibrated the Yeoh (1990) model for NR

$$W_{\mathrm{NR}} = \sum_{k=1}^{3} C_{k0}(\lambda_1^2+\lambda_2^2+\lambda_3^2-3)^k, \tag{10}$$

where $C_{10}=0.298$ MPa, $C_{20}=0.014$ MPa, $C_{30}=0.00016$ MPa, and the Ogden (1972) model for SBR

$$W_{\mathrm{SBR}} = \sum_{k=1}^{2} \frac{\mu_k}{\alpha_k}(\lambda_1^{\alpha_k}+\lambda_2^{\alpha_k}+\lambda_3^{\alpha_k}-3), \tag{11}$$

where $\mu_1=0.638$ MPa, $\alpha_1=3.03$, $\mu_2=-0.025$ MPa, $\alpha_2=2.35$.

Besides, Hamdi et al. (2006) found the following values of the critical stretches corresponding to the sample failure in uniaxial tension: $\lambda_{\mathrm{NR}}=7.12$, $\lambda_{\mathrm{SBR}}=6.88$. Based on this rupture data we fit material failure parameters Φ and m for various possibilities of the transition to failure as shown in Figs. 2 and 3. Particularly, we introduce the following modified Yeoh models (Fig. 2) with varying parameter $m=1, 10, 50$

$$\psi_{\mathrm{NR}m} = \frac{\Phi_{\mathrm{NR}m}}{m}\left\{\Gamma\left(\frac{1}{m},0\right) - \Gamma\left(\frac{1}{m},\frac{W_{\mathrm{NR}}^m}{\Phi_{\mathrm{NR}m}^m}\right)\right\}, \tag{12}$$

where $\Phi_{\mathrm{NR1}}=74.4$ MPa, $\Phi_{\mathrm{NR10}}=82.0$ MPa, and $\Phi_{\mathrm{NR50}}=69.63$ MPa.

Analogously, we introduce the modified Ogden models (Fig. 3) with varying $m=1, 10, 50$

$$\psi_{\mathrm{SBR}m} = \frac{\Phi_{\mathrm{SBR}m}}{m}\left\{\Gamma\left(\frac{1}{m},0\right) - \Gamma\left(\frac{1}{m},\frac{W_{\mathrm{SBR}}^m}{\Phi_{\mathrm{SBR}m}^m}\right)\right\}, \tag{13}$$

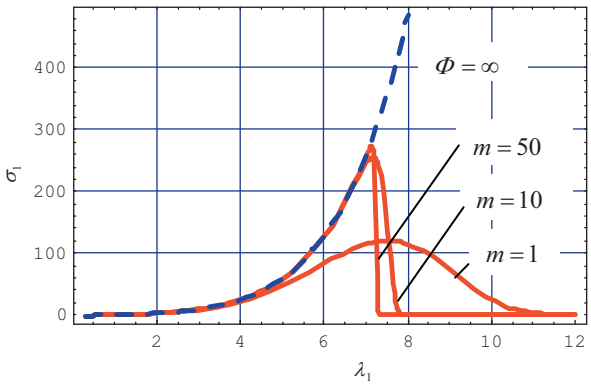

Fig. 2. Cauchy stress (MPa) versus stretch in uniaxial tension. Dashed line designates intact Yeoh model. Solid lines designate Yeoh model with energy limiters for varying m.

where $\Phi_{\mathrm{SBR1}}=108.65$ MPa, $\Phi_{\mathrm{SBR10}}=94.71$ MPa, and $\Phi_{\mathrm{SBR50}}=78.71$ MPa.

It is remarkable that models with large parameter m depart from the stress–strain curve of the intact material much later than models with small m.

We emphasize that though failure parameters Φ and m control the onset of failure and the decay of the stress–strain curve correspondingly they are not completely independent and they should be fitted simultaneously like any material parameters.

We analyze failure of thin sheets of NR and SBR in plane stress states with varying biaxiality, n, for the introduced material models with failure (12) and (13). Critical stretches corresponding to the onset of static instability – failure – are presented in Fig. 4.

The results obtained for various magnitudes of m coincide. In the case of stresses shown in Fig. 5 the situation is different. Stresses are significantly smaller for $m=1$ as compared to the cases of $m=10$ and $m=50$ which are close to each other. Remarkably, the magnitudes of the critical stretches and stresses decrease when the stretch state drifts from the uniaxial tension to the equal biaxial tension. The latter means that failure criteria of maximum principal stretch or stress are debatable from the presented theory standpoint: the failure indicator should essentially depend on the spatial stress–strain state.

Finally, we compare the critical stretches obtained by using the elasticity with energy limiters with the results obtained by using a multiscale model of rubber failure proposed by Dal and Kaliske (2009) and experiments performed by Hamdi et al. (2006) – Fig. 6.

There is a good correspondence between theoretical and experimental results for NR while for SBR there is some deviation of

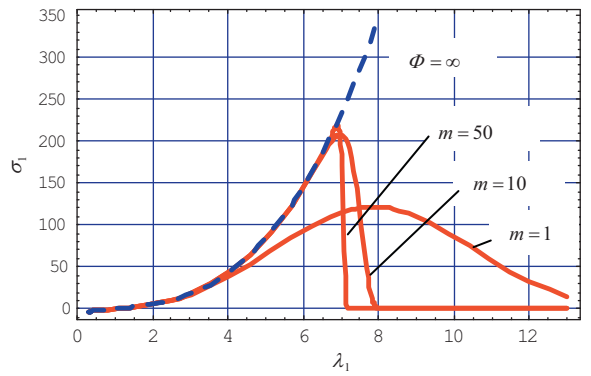

Fig. 3. Cauchy stress (MPa) versus stretch in uniaxial tension. Dashed line designates intact Ogden model. Solid lines designate Ogden model with energy limiters for varying m.

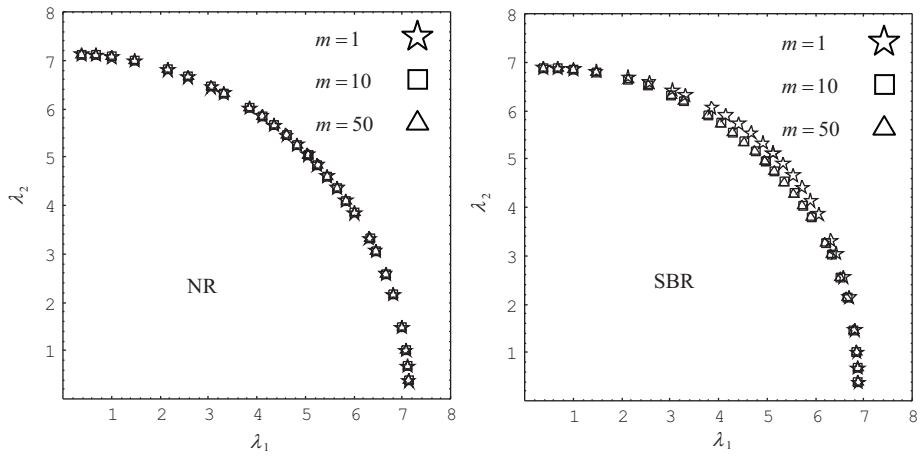

Fig. 4. Critical failure stretches for NR (left) and SBR (right) materials based on the hyperelasticity with energy limiters.

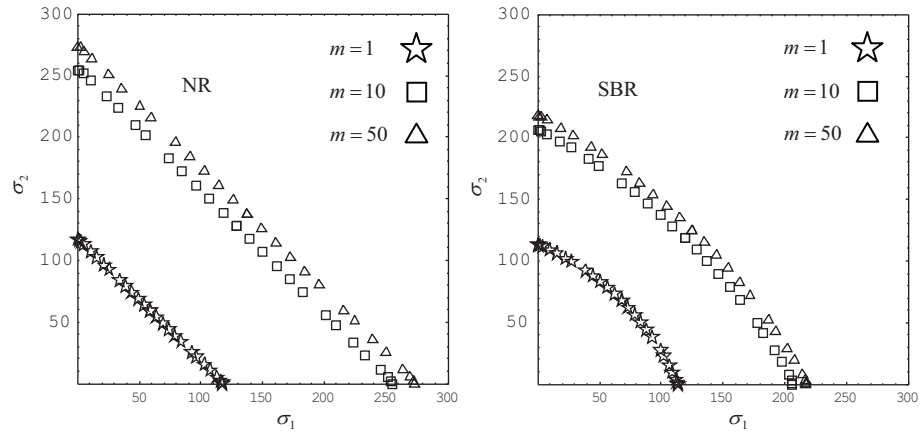

Fig. 5. Critical failure stresses (MPa) for NR (left) and SBR (right) materials based on the hyperelasticity with energy limiters.

theoretical prediction from experiments when the stress state tends to the equal biaxial tension. This deviation might be reasonable since the equal biaxial state might be sensitive to imperfections in samples and loads. The latter is the reason why it is practically impossible to execute equal biaxial tests of failure with the displacement control. It should not be missed, of course, that the energy limiters were fitted to the uniaxial tension experiments and they can be improved, in principle, by fitting to the more general biaxial data presented in Fig. 6. We also mention that the multiscale theory of Dal and Kaliske (2009) provides a better approximation of experimental data in equal biaxial tension of SBR than the present theory. However, the elasticity with energy limiters is dramatically simpler than any multiscale or CDM theory.

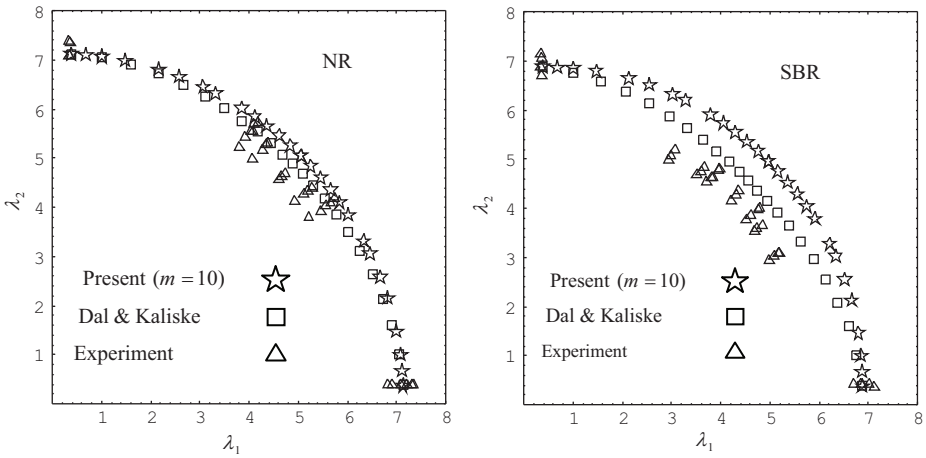

Fig. 6. Critical failure stretches for NR (left) and SBR (right) materials based on the hyperelasticity with energy limiters.

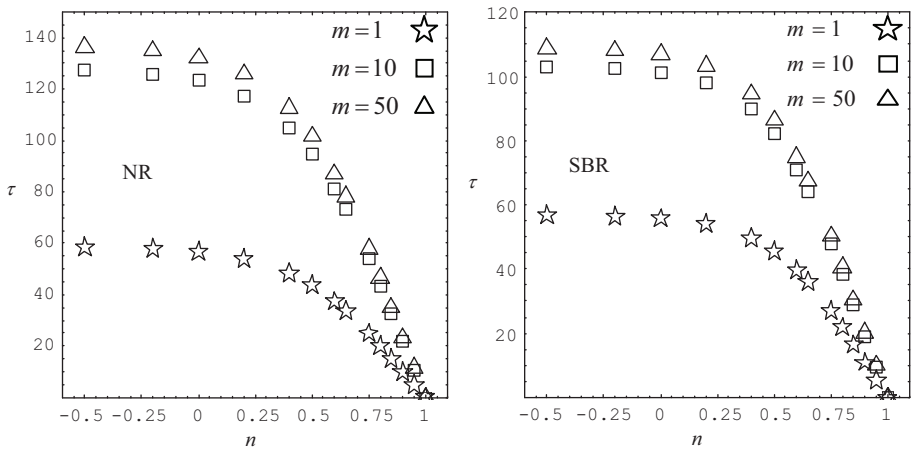

Fig. 7. Critical shear stress, $\tau=(\sigma_1-\sigma_2)/2$, (MPa) for NR (left) and SBR (right) materials based on the hyperelasticity with energy limiters.

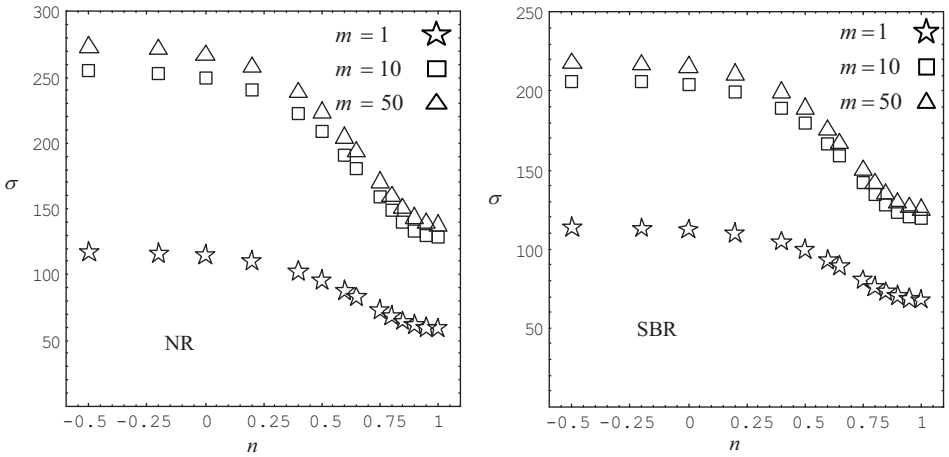

Fig. 8. Critical von Mises stress (MPa) for NR (left) and SBR (right) materials based on the hyperelasticity with energy limiters.

5. Concluding remarks

We proposed a universal formula (1) which allows enriching constitutive models of intact materials with a failure description. This new formula includes the energy limiter, Φ, and a new constant, m. Φ makes the crucial contribution to the separation energy that can be accumulated in an infinitesimal material volume prior to its failure while m is related to the abruptness of the bond rupture. The larger is m the faster is transition to failure. Small magnitudes of m are required in the cases of the gradual bond rupture.

We examined the proposed formula in the problems of plane stress for NR and SBR by using the modified Yeoh and Ogden models of conventional hyperelasticity. The comparison of the results obtained by using the proposed approach and experiments and other theories is favorable, especially, taking into account the simplicity of the new formulation.

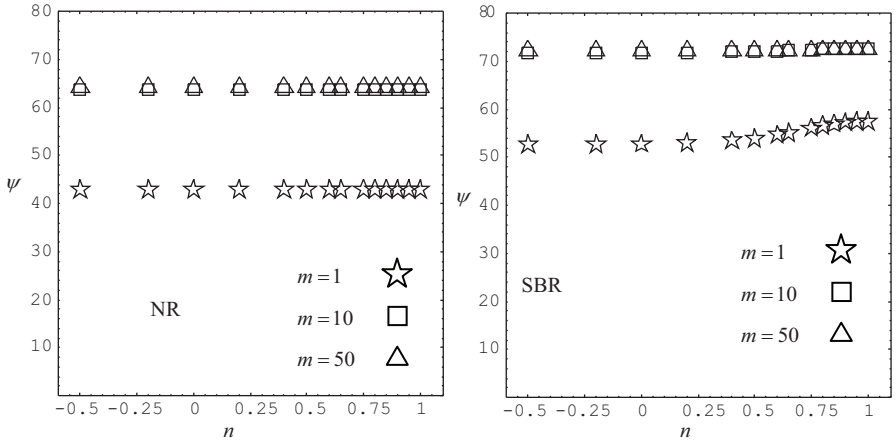

Fig. 9. Critical strain energy (MPa) for NR (left) and SBR (right) materials based on the hyperelasticity with energy limiters.

Table 1
Full separation energy versus critical failure energy.

	NR			SBR		
	$m=1$	$m=10$	$m=50$	$m=1$	$m=10$	$m=50$
Full separation energy (MPa)	74.4	78.0	68.9	108.7	90.1	77.8
Critical energy (MPa)	42.9	63.1	64.2	52.8	71.8	72.2

Encouraged by the experimental verification we can use the proposed models with energy limiters to examine various local (pointwise) failure criteria.

It is evident from Figs. 4 and 5 that the criteria of the maximum principal stretch or stress derived from the uniaxial tension experiments are questionable.

Another criterion is the maximum shear stress. Its variation in the critical cases of the plane stress considered in the previous section is depicted in Fig. 7. Evidently, the maximum shear stress is not a good choice for the failure indicator.

Von Mises stress, $\sigma=\sqrt{3(\boldsymbol{\sigma}:\boldsymbol{\sigma}-(tr\boldsymbol{\sigma})^2/3)/2}$, is often used as a failure indicator in order to account for the spatial variations of the stress–strain state. The critical failure magnitudes of the von Mises stress are shown in Fig. 8 for the cases analyzed in the present work. These results do not favor the von Mises stress as the failure criterion.

Finally, we checked the magnitudes of the accumulated strain energy in all considered cases of failure – Fig. 9.

The results definitely favor a constant magnitude of the strain energy as a failure criterion. It should not be missed that the critical failure energies that appear in Fig. 9 do not coincide with the energy of full separation, $\Phi\Gamma(1/m,0)/m$, – see Table 1. The failure occurs before the limit of full separation is reached and, consequently, the critical values of energy for various cases of loading are not coincident in advance!

Acknowledgement

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