THE FORMATION OF EQUILIBRIUM CRACKS DURING BRITTLE FRACTURE. GENERAL IDEAS AND HYPOTHESES. AXIALLY-SYMMETRIC CRACKS

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A large number of investigations has been devoted to the problem of the formation and the development of a crack during brittle fracture of solids. The first of these was the well-known work of Griffith [1] devoted to the determination of the critical length of a crack at a given load, i.e. the length of a crack at which it begins to widen catastrophically. Assuming an elliptical form of a crack forming in an infinite body subjected to an infinitely homogeneous tension, Griffith obtained an expression for the critical length of a crack as that corresponding to the total of the full increase in energy (equal to the sum of the surface energy plus the elastic energy released due to the formation of the crack).

In recent years, in connection with the numerous technical applications regarding the problem of cracks, the number of investigations has increased, among the first of which we ought to name the works of Orowan and Irwin, generalizing and refining Griffith's theory. A bibliography and a short résumé of these works can be found in the recent works of Orowan [2], Irwin [3], and Bueckner [4].

The development of cracks in brittle materials can be depicted in the following fashion. In the material there are a large number of micro-cracks. Upon an increase in load in a given spot of the body, a stress is reached sufficient for the development of the micro-crack existing at that spot to a certain size. The beginning of the development of the micro-crack is determined by some condition, because in view of the fact that usually the size of the micro-crack is small in comparison with the characteristic linear dimension of the stress change, the state of stress in the surrounding area of the micro-crack can be represented in accordance with Griffith's scheme in the form of a uniform infinite tension.

In the course of its development, the dimensions of the crack increase and finally become equal to the characteristic dimension of the stress change. Under determined conditions (for example, when the forces are not too large and are applied sufficiently far removed from the boundaries of the solid) the developing crack is stopped upon reaching its determined length, and the solid can remain in this state a long time provided there is no change in the loading.

In the following study equilibrium cracks are considered, i.e. cracks forming in a brittle solid subjected to a given system of forces, constant and not decreasing in time. Apparently the first ideas concerning equilibrium cracks are met in the works of Mott [5] and Frenkel' [6]. These ideas are similar but were arrived at independently. These works contain also a critical analysis of Griffith's theory. Both these authors, however, limited themselves to qualitative considerations, proceeding from the assumption of a crack of infinite length.

In our work the question concerning equilibrium cracks forming during brittle fracture of a material is presented as a problem in the classical theory of elasticity, based on certain very general hypotheses concerning the structure of a crack and the forces of interaction between its opposite sides, and also on the hypothesis of finite stresses at the ends of the crack, or, which amounts to the same thing, the smoothness of the joining of opposite sides of the crack at its ends. The latter hypothesis was first put forth by Khristianovich [7] in his consideration of certain problems in the formation of cracks in rocks. Using this hypothesis it seemed possible to solve a series of problems related to the development of cracks in rocks [7-11].

In the consideration of problems related to rocks, one may evidently neglect the effect of the cohesive of the material as compared with the effect of rock pressure, so that the neglect of the cohesive force of the material in reference [7-11] can be justified. However, in other problems of brittle fracture (for example, in problems of brittle fracture of metallic structures) factors of the type of rock pressure are absent, and the consideration of cohesive forces of the material becomes absolutely necessary. It seems that the intensity of the cohesive forces and their distribution can be characterized with sufficient accuracy by some new universal property of the material which we call the modulus of cohesion. Moreover, the dimensions of the cracks and their other properties are uniquely determined by the applied loads and the cohesive modulus. To determine the cohesive modulus of a material one can use comparatively simple tests.

1. Basic ideas and hypotheses. Statement of the problem. Let us consider equilibrium cracks in a brittle material, i.e. cracks maintaining constant dimensions under the influence of a given system of forces. Moreover, we shall be limited here to the consideration of the simplest case, cracks lying in one plane (so that points at the surface of the crack in the undeformed state are located in one plane, the plane of the crack). This case occurs when the applied stresses are symmetrical relative to the plane of the crack; the general case will be considered separately.

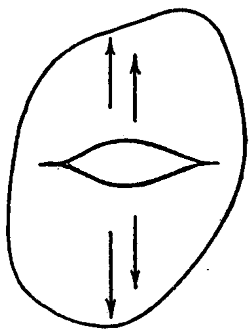

Fig. 1.

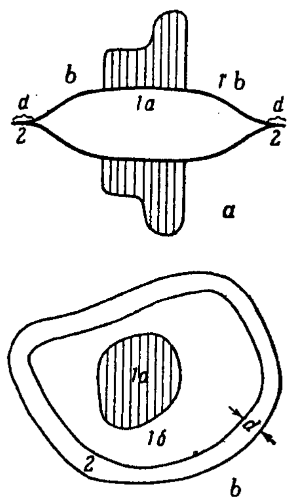

Fig. 2.

Thus, the following problem is considered (Fig. 1). A given breaking load symmetrical relative to the plane of the crack is applied to an isotropic, brittle, elastic solid whose dimensions we assume are large in comparison with the length of the crack. Moreover, it is assumed that the composite force applied from each side of the crack is limited. In particular the load can be applied on part of the very surface of the crack. The stresses in the solid away from the crack evidently converge to zero.

If the applied load is sufficiently large, there occurs a brittle fracture of the material which, if left to chance, must occur in the plane of symmetry of the applied load. Moreover, a crack is formed which widens and reaches certain dimensions. The problem is to find the crack dimensions corresponding to the given load and other crack parameters.

Let us turn first to the investigation of a more simple case when the load is applied to the edge of the crack surface. Thus, it is assumed that in the body there is a certain original dimension on which is imposed some rupturing load, which we assume is normal to the plane of the crack. Brittle fracture occurs and this fracture is widened (remaining flat because of symmetry of the load and the isotropy of the solid) to certain definite measurements.

The cross-section (a) and the plane (b) of such a crack are shown schematically in Fig. 2. The crack is divided into two regions: region 1 (the inner region) and region 2 (the terminal region). In the inner region the opposite sides of the cracks are at a significant spacing so that interaction between these sides does not occur. The inner region of the crack falls into two sub-regions 1a and 1b; in the first the applied load acts on the opposite sides of the crack, and in the second the opposite sides of the crack are free from stress.

In the terminal region the opposite sides of the crack come close to each other so that there are very large interaction forces attracting one side of the crack to the other. As is known, the intensity of the attractive force acting in the material strongly depends upon the distance, at first growing rapidly with an increase in the distance y between the attracting bodies, from a normal interatomic distance $y = b$ for which the intensity is equal to zero, up to some critical distance where it reaches a maximum value equal to the order of magnitude of Young's modulus, after which it rapidly falls with increasing distance (Fig. 3).

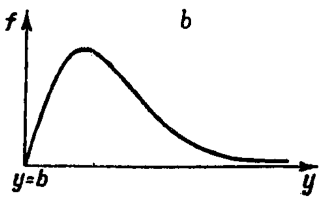

Fig. 3

The accurate determination of the system of cohesive forces acting in the terminal region is difficult. However, one can introduce certain hypotheses which permit limitation to one composite universal characteristic distribution of cohesive forces for a given material.

First hypothesis. The dimension d of the terminal region is small in comparison with the size of the whole crack.

Second hypothesis. The distribution of the displacement in the terminal region does not depend upon the acting load and for the given material under given conditions (temperature, composition and pressure of the surrounding atmosphere and so forth) is always the same.

In other words, according to this hypothesis, the ends in all cracks in a given material under given conditions are always the same. During the propagation of the crack the end region merely moves over to another place, but the distribution of the distortion in the sections of the terminal region with planes normal to the crack contour remains exactly the same. The cohesive forces attracting the opposite sides of the crack to each other depend only on the mutual distribution of the sides (i.e. on the distribution of the displacement); therefore, according to the formulated hypothesis, these stresses will be the same.

The fixed shape of the terminal region of the crack corresponds to the maximum possible resistance. We emphasize that in view of the irreversibility of the cracks occurring in the majority of materials, the second hypothesis is applied only to those equilibrium cracks which are formed during the primary rupture of the initially unbroken brittle solid and not to those cracks which are formed at an artificial notch without subsequent propagation or during a decrease in load resulting from previous cracking at some larger load. For the latter types of cracks the stress in the terminal region can be different (smaller); these types of cracks are excluded from this discussion.

Third hypothesis. The opposite sides of the crack are smoothly joined at the ends or, which amounts to the same thing, the stress at the end of a crack is finite.

(As we noted above, this hypothesis was first expressed by Khristianovich in relation to the formation of cracks in rocks.)

The indicated hypotheses make possible the solution of the problem under consideration. Let us mention the difference appearing in the case where the applied loads are not on the surface of the crack but inside the solid. In principle nothing is changed in this case. Actually, we shall represent the state of stress acting in the solid with a crack under the action of a certain load applied inside the solid, by the sum of two states of stress. One of these states corresponds to the state of the continuous solid without the crack, a state appearing under the action of a given loading system. The other state of stress corresponds to the state in a body with a load applied on the surface of the crack. The composite normal and shear stresses on the surface of the inner region of the crack must be equal to zero because the surface of the inner region of the crack is free of stress. Therefore, the load applied to the surface of the inner region of the crack represents a compressive normal stress, equal in magnitude and opposite in sign to the tensile stress appearing in the plane of symmetry of the applied load in the body without the crack. In the terminal region for the second state of stress the normal stresses are equal to the stress of the cohesive forces, by deduction corresponding to the stress of the first state of stress. In view of the fact that the plane of the crack is the plane of symmetry of the applied load, the shear stresses on the surface of the crack are absent. The normal displacements of the points of the surface of the crack are determined only by the second state of stress, in so far as they are equal to zero in the first state.

Thus, the change in comparison with the special case considered earlier consists in the fact that the inner region of the crack now does not fall into two sub-regions, and the loading of the surface of the crack occurs along the whole inner region.

The three hypotheses formulated above are applied also to this more general case. With reference to the first and the third hypothesis this does not require elucidation. As for the second hypothesis, the possibility of its application is explained by the fact that changes in the applied stresses in the terminal region under the influence of the applied load are of a very much larger order than the applied load. As was shown earlier, the stresses in the terminal region are of the order of Young's modulus, i.e. they significantly surpass the magnitude of the applied load. Therefore, in the general case too one can neglect changes in stresses applied to the surface of the crack in the terminal region under the influence of the applied load and consider only the distribution of stresses and displacements in the terminal region not dependent upon the load, i.e. one can apply the second hypothesis.

2. Axially symmetric cracks. (1) Let the crack have the form of a round slit of radius R. On both sides of the portion of the surface of the round slit (Fig. 4) at values of the running radius r, smaller than

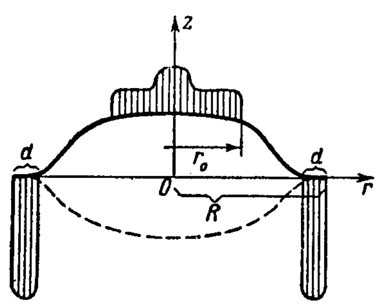

Fig. 4.

some radius r_0 (radius of the region of the load application), compressive normal forces $Z_z = -p(r)$ are active. The part of the surface of the slit corresponding to the intermediate region $r_0 < r < R - d$, is free of stress. In the terminal region $R - d < r < R$, tensile stresses $G(r)$ controlled by cohesive forces act.

As was shown by Sneddon [12], the distribution of normal displacements of points of the surface of the round crack in an infinite elastic solid, if a normal stress $- g(r)$ acts on this surface, and shear stresses are absent, have the form:

$$w = \frac{4(1-\nu^2)R}{\pi E}\int_\rho^1 \frac{\mu d\mu}{\sqrt{\mu^2-\rho^2}}\int_0^\mu \frac{xg(x\mu R)\,dx}{\sqrt{1-x^2}} \qquad \left(\rho = \frac{r}{R}\right) \tag{2.1}$$

Here E is Young's modulus and ν is Poisson's ratio. Partial integration produces this expression in the form

$$w = \frac{4(1-\nu^2)R}{\pi E}\left\{\sqrt{1-\rho^2}\int_0^1 \frac{xg(xR)\,dx}{\sqrt{1-x^2}} - \int_\rho^1 \sqrt{\mu^2-\rho^2}\left[\frac{d}{d\mu}\int_0^1 \frac{xg(x\mu R)\,dx}{\sqrt{1-x^2}}\right]d\mu\right\} \tag{2.2}$$

According to the condition of smooth joining of the opposite sides of the crack at its ends, this evidently leads to the relationship

$$\left(\frac{\partial w}{\partial r}\right)_{r=R} = 0, \quad \text{or} \quad \left(\frac{\partial w}{\partial \rho}\right)_{\rho=1} = 0 \tag{2.3}$$

Differentiating (2.2) we obtain

$$\frac{\partial w}{\partial \rho} = \frac{4(1-\nu^2)R}{\pi E}\left(-\frac{\rho}{\sqrt{1-\rho^2}}\int_0^1 \frac{xg(xR)\,dx}{\sqrt{1-x^2}} + \rho\int_\rho^1 \frac{d\mu}{\sqrt{\mu^2-\rho^2}}\left[\frac{d}{d\mu}\int_0^1 \frac{xg(x\mu R)\,dx}{\sqrt{1-x^2}}\right]\right)$$

The second member of the right-hand side of this relationship, under broad assumptions relative to the function g, converges to zero for $\rho \to 1$, from which it is evident that to fulfill condition (2.3) it is necessary to fulfill the relationship

$$\int_0^1 \frac{xg(xR)\,dx}{\sqrt{1-x^2}} = 0 \tag{2.4}$$

In this same form relationship (2.4) was obtained in reference [9] as the condition of finiteness of the stress at the edge of an axially-symmetric crack. In the case we are considering

$$g(r) = \begin{cases} p(r) & (0 \leqslant r \leqslant r_0) \\ 0 & (r_0 < r \leqslant R-d) \\ -G(r) & (R-d < r \leqslant R) \end{cases} \tag{2.5}$$

Substituting (2.5) into (2.4) we obtain

$$\int_0^{r_0/R} \frac{xp(xR)\,dx}{\sqrt{1-x^2}} - \int_{1-d/R}^1 \frac{xG(xR)\,dx}{\sqrt{1-x^2}} = 0 \tag{2.6}$$

Let us consider in more detail the second integral (2.6):

$$J = \int_{1-d/R}^1 \frac{xG(xR)\,dx}{\sqrt{1-x^2}} \tag{2.7}$$

On the strength of the first hypothesis, d is small in comparison with R, so that x in all regions of integration of $1 - d/R < x < 1$ differs little from unity and one way take

$$J = \int_{1-d/R}^{1} \frac{G(xR)\,dx}{\sqrt{1-x^2}} \qquad (2.8)$$

Let us pass over in this integral from the dimensionless variable x to the dimensional coordinate $s = R - Rx$, calculated from the edge of the crack and changing within the integration limits from zero to d. We have

$$dx = -\frac{1}{R}\,ds, \qquad \sqrt{1-x^2} = \frac{1}{R}\sqrt{(R+Rx)(R-Rx)} \approx \frac{1}{R}\sqrt{2Rs}$$

since Rx differs little from R within the limits of integration.

Substituting this into (2.8) we obtain

$$J = \frac{1}{\sqrt{2R}} \int_0^d \frac{F(s)\,ds}{\sqrt{s}}, \qquad F(s) = G(r) \qquad (2.9)$$

On the strength of the second hypothesis the distribution $F(s)$ does not depend upon the applied load, so that the integral

$$\int_0^d \frac{F(s)\,ds}{\sqrt{s}}$$

represents for a given material at given conditions a constant velue which we shall signify by K and shall call the cohesive modulus.

It is easy to show that the dimensions of the cohesive modulus are

$$[K] = FL^{-3/2}$$

where F is the dimension of force and L the dimension of length. The cohesive modulus represents a property of a material which has evidently has basic significance in the theory of brittle fracture; it enters into the basic relationships regardless of the character of loading and the geometrical shape of the crack. Thus we have

$$J = \frac{K}{\sqrt{2R}} \qquad (2.10)$$

Substituting this relationship into equation (2.6) we obtain, passing from the dimensionless variable x to the dimensional variable r:

$$\int_0^{r_0} \frac{rp(r)\,dr}{\sqrt{R^2 - r^2}} = K\sqrt{\frac{R}{2}} \qquad (2.11)$$

This equation determines the radius of crack R as a function of the applied load.

(2) Let us consider several examples. Let the applied load $p(r)$ be constant, $p(r) = P_0$. Then the equation (2.11) takes the form:

$$P_0 [R - \sqrt{R^2 - r_0^2}] = K\sqrt{\frac{R}{2}} \tag{2.12}$$

It is convenient to bring this equation to the dimensionless form

$$\frac{P_0 \sqrt{r_0}}{K} = \frac{1}{\sqrt{2}}\left(\frac{r_0}{R}\right)^{-1/2}\left(1 + \sqrt{1 - \frac{r_0^2}{R^2}}\right), \tag{2.13}$$

A graph of the relationship in (2.13) is given in Fig. 5. We see, that for $P_0\sqrt{r_0} < K/\sqrt{2}$, equation (2.13) does not have real solutions. This fact permits a very simple interpretation: there exists a certain minimum stress, the application of which on a circle of a given radius guarantees the possibility of opening the crack. At stresses less than this minimum the crack does not open. The magnitude of the minimum stress decreases in inverse proportion to the square root of the radius of the region of the load application. Each load surpassing the minimum corresponds to one single radius of the crack. With an increase in load the radius of the crack naturally grows.

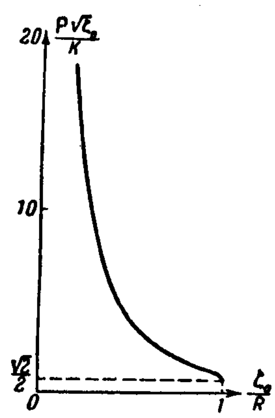

Fig. 5.

It is an especially curious particular case when a crack is formed by oppositely directed concentrated forces equal in magnitude and applied to opposite sides of the crack. This limiting case corresponds to $r_0/R \ll 1$ and $P_0\pi r_0^2 = T$, where T is the value of the concentrated force. We have from (2.12)

$$P_0\left[1 - \sqrt{1 - \frac{r_0^2}{R^2}}\right] \approx \frac{P_0}{2}\frac{r_0^2}{R^2} = \frac{T}{2\pi R^2} = \frac{K}{\sqrt{2R}} \tag{2.14}$$

Whence we obtain

$$R = \left(\frac{T}{\sqrt{2\pi}K}\right)^{2/3} \tag{2.15}$$

Using the *P*-theorem of the theory of similarity (see the book by Sedov [11]), the latter result can be obtained with an accuracy within the constant factor of one of the considered dimensions.

(3) With certain assumptions one can show the existence of a unique solution of equation (2.11) for an arbitrary load $p(r)$. Indeed, let us assume that

$$\int_0^{r_0} \frac{rp(r)\,dr}{\sqrt{r_0^2 - r^2}} > K\sqrt{\frac{r_0}{2}} \tag{2.16}$$

It is evident that with growth of R the function

$$\int_0^{r_0} \frac{rp(r)\,dr}{\sqrt{R^2 - r^2}} = \Phi_1(R)$$

decreases monotonically, and the function $\Phi_2(R) = K\sqrt{R}/\sqrt{2}$ increases monotonically. From (2.16) it follows that $\Phi_1(r_0) > \Phi_2(r_0)$ and therefore, there exists one unique value $R > r_0$, for which $\Phi_1(R) = \Phi_2(R)$, i.e. for which equation (2.11) is fulfilled. The inequality (2.16) is the condition under which the applied load will surpass the minimum load, and thus be able to open the crack.

3. Axially-symmetric cracks (continuation). (1) Let us consider normal displacements of points of the surface of the crack, determining its width. At a given load the radius of the crack R is determined by the relationship (2.11). Knowing the radius R one can determine the displacements mentioned using the formulas (2.1) and (2.5). We have

$$\psi = \frac{\pi E w}{4(1-\nu^2)R} = \psi_1 - \psi_2 \tag{3.1}$$

where for $\rho < r_0/R$

$$\psi_1 = \int_\rho^{r_0/R} \frac{\mu\,d\mu}{\sqrt{\mu^2 - \rho^2}} \int_0^1 \frac{xp(x\mu R)\,dx}{\sqrt{1-x^2}} + \int_{r_0/R}^1 \frac{\mu\,d\mu}{\sqrt{\mu^2 - \rho^2}} \int_0^{r_0/R\mu} \frac{xp(x\mu R)\,dx}{\sqrt{1-x^2}} \tag{3.2}$$

$$\psi_2 = \int_{1-d/R}^1 \frac{\mu\,d\mu}{\sqrt{\mu^2 - \rho^2}} \int_{(R-d)/R\mu}^1 \frac{xG(x\mu R)\,dx}{\sqrt{1-x^2}} \tag{3.3}$$

Let us consider the inner integral in the expression for ψ_2. Since μ is close to unity this integral is close to the integral

$$\int_{1-d/R}^{1} \frac{xG(xR)\,dx}{\sqrt{1-x^2}} \approx \frac{K}{\sqrt{2R}} \tag{3.4}$$

Whence also from (3.3) it follows that

$$\psi_2 \approx \int_{1-d/R}^{1} \frac{\mu d\mu}{\sqrt{\mu^2-\rho^2}} \frac{K}{\sqrt{2R}} = \frac{K}{\sqrt{2R}}\left[\sqrt{1-\rho^2} - \sqrt{\left(1-\frac{d}{R}\right)^2-\rho^2}\right] \approx$$
$$\approx \frac{K}{\sqrt{2R}}\sqrt{1-\rho^2}\left[1-\sqrt{1-\frac{2d/R}{1-\rho^2}}\right] \approx \frac{K}{\sqrt{2R}}\frac{(d/R)}{\sqrt{1-\rho^2}} \tag{3.5}$$

Since d/R is small, ψ_2 is small in comparison with ψ_1. Now let $\rho > r_0/R$, but $1-\rho \gg d/R$, i.e. the considered point of the surface of the crack remains at a significant distance from the ends in comparison with the dimension of the terminal region d. In this case

$$\psi_1 = \int_{\rho}^{1} \frac{\mu d\mu}{\sqrt{\mu^2-\rho^2}} \int_{0}^{r_0/R\mu} \frac{xp(x\mu R)\,dx}{\sqrt{1-x^2}} \tag{3.6}$$

and ψ_2 as before is determined by the relationship (3.3) and consequently, by the relationship (3.5). It is evident that also in this case $\psi_2 \ll \psi_1$. But the value of ψ_1 represents the relative displacement, determined *at a given fixed radius* R by the applied load in the absence of cohesive forces (and, of course, not satisfying the condition of smooth joining at $r = R$), and the value of ψ_2 - the component of relative displacement, dependent on the cohesive forces. Therefore, everywhere except directly near the edges of the crack, the distribution of the displacements almost corresponds to the distribution of the displacements occurring at a given fixed radius of crack R in the absence of cohesive forces. *This means that the cohesive forces essentially have an effect only on the value of the radius of the crack and on the distribution of the displacements close to the edges of the crack and not on the distribution of displacements in the main part of the crack.*

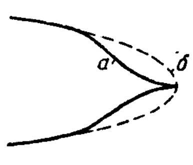

Fig. 6

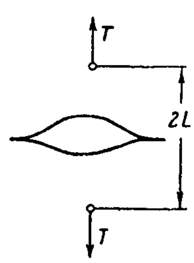

Fig. 7

In Fig. 6*a* is given the actual distribution of displacements; in Fig. 6*b* is given the distribution of displacement at the same radius of crack but found without considering the cohesive forces.

(2) Let us now consider on those changes which must be introduced to the preceding discussion if the breaking load is applied inside the solid and not on the surface of the crack. According to our assumption, since the load is symmetrical relative to the plane of the crack, shear stresses in this plane are absent. Let us assume, according to the foregoing, that the state of stress acting in the considered brittle solid with a crack is in the form of the sum of two states of stress; one of which represents the state of stress in a continuous body without a crack under the action of a given load, and the other a state of stress in a body with a crack in the presence of a breaking load at the surface of a crack. Let $p(r)$ represent the distribution of normal stress in the plane of symmetry under a given load in the absence of a crack. Then the compressive force at the surface of the crack determining the second state of stress is equal to $-p(r)$. Repeating the former reasoning, we obtain in place of equation (2.11) an equation determining the radius of the crack in the form

$$\int_0^R \frac{rp(r)\,dr}{\sqrt{R^2-r^2}} = K\sqrt{\frac{R}{2}} \tag{3.7}$$

which differs from equation (2.11) only by the fact that in place of radius r_0, we use the full radius of the crack R.

Since the normal displacement of the points of the surface of the crack, corresponding to the first state of stress; is equal to zero, all conclusions relative to the distribution of the displacement of the points of the surface of the crack, formulated in the foregoing section also hold true.

(3) Let us consider as an example the problem of the formation of a crack by two oppositely directed concentrated forces T, applied at points separated by a distance of $2L$ (Fig. 7). It is natural to suppose that the crack lies in the plane of symmetry of the load. Summarizing the well-known solutions of Boussinesq [14], it is easy to obtain that

$$p(r) = \frac{T}{4\pi(1-\nu)}\frac{L}{\sqrt{(r^2+L^2)^3}}\left\{3\frac{L^2}{r^2+L^2}+1-2\nu\right\} \tag{3.8}$$

Condition (3.7) can be rewritten in the form

$$\int_0^1 \frac{xp(xR)\,dx}{\sqrt{1-x^2}} = \frac{K}{\sqrt{2R}} \tag{3.9}$$

Substituting (3.8) in (3.9), we obtain

$$\frac{T}{4\pi(1-\nu)L^2}\int_0^1 \frac{1}{\sqrt{(1+R^2x^2/L^2)^3(1-x^2)}}\left[\frac{3}{1+R^2x^2/L^2}+1-2\nu\right]x\,dx = \frac{K}{\sqrt{2R}} \quad (3.10)$$

Carrying out the integration, we obtain

$$g\left(\frac{L}{R}\right)=\sqrt{2}\left(\frac{L}{R}\right)^{-3/2}\left(1+\frac{L^2}{R^2}\right)^2\left[1+\frac{2-\nu}{1-\nu}\frac{L^2}{R^2}\right]^{-1}=\frac{T}{\pi K L^{3/2}} \quad (3.11)$$

The graph of the function $g(L/R)$ has the form shown in Fig. 8. (On Fig. 8 is represented the case with $\nu = 0.5$). There is physical significance only to the left-hand part of the curve up to the minimum point,

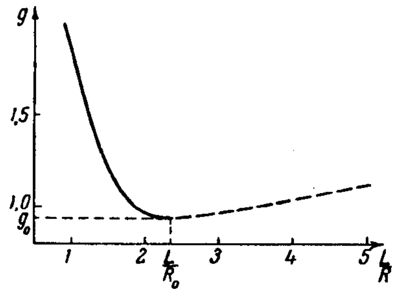

Fig. 8.

shown by the solid line. In fact, the right-hand part of the curve corresponds to the growth of the crack radius during decrease in load. As is evident from the graph of Fig. 8, at

$$T < \pi g_0 K L^{3/2} \quad (3.12)$$

equation (3.11) does not have a solution. In complete analogy to the foregoing, this means that for such small T the crack is not formed. At $T = \pi g_0 K L^{3/2}$ a crack of a determined final radius R_0 is formed at once with further increase in T the radius of the crack R is increased. The values g_0 and R_0 are found by elementary means: at $\nu = 0.5$ we have $g_0 = 0.945$, $L/R_0 = 2.35$, at $\nu = 0$ we obtain $g_0 = 1.39$, $L/R_0 = 2.05$.

Generalizing suitably the discussion of Section 3, part (2) to equation (3.7), one can show that such a condition is typical at an application of load inside the solid for a body of infinite dimensions. The following is true: if the load, applied inside the body, is proportional to some parameter, then at sufficiently small values of this parameter an equilibrium crack in general does not form. Upon reaching some critical value of the parameter, a crack of finite radius is immediately formed. Upon further increase of the parameter the crack grows continuously.

Let us also note the peculiar limiting case corresponding to the convergence of L to zero, i.e. the convergence of the points of application of forces. From the relationship (3.11) at $L/R \to 0$ we obtain again the relationship

$$R = \left(\frac{T}{\sqrt{2\pi}K}\right)^{2/3}$$

coincident with the relationship (2.15), as was to be expected.

The case of plane cracks will be considered in a separate paper.

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