$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} .$$

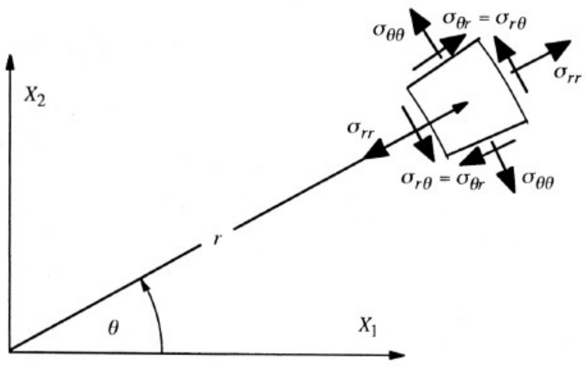

Figure 13. Stress components in polar coordinates.

Holes and stress concentrations. Consider a circular hole of radius a in a plate whose dimensions are much larger than a and can, for present purposes be taken as infinite (Figure 14). The plate is under remotely uniform stress σ^∞ in the 2 direction and the boundary of the hole is free of loading. The same conditions, interpreted as a plane strain problem, describe a circular tunnel in a large solid. Thus we wish to solve $\nabla^2(\nabla^2 U) = 0$ subject to the requirements that the stresses associated with U satisfy $\sigma_{22} \to \sigma^\infty$, $\sigma_{11} \to 0$ and $\sigma_{12} \to 0$ as $r \to \infty$, and that $\sigma_{rr} = \sigma_{r\theta} = 0$ on $r = a$. To make the solution to this type of problem unique, we must also specify the value of the integral of $\partial \mathbf{u}/\partial s$ with respect to arc length s around the hole, which is zero in the present case for which we require a *single-valued* displacement field, but which would be non-zero if the hole was to represent the core of a dislocation.

The proper stress state as $r \to \infty$ is given by writing $U = \sigma^\infty X_1^2 / 2 = \sigma^\infty r^2 (1 + \cos 2\theta) / 4$ and, while this does not meet the conditions on the boundary of the hole, it does encourage one to seek solutions to $\nabla^2(\nabla^2 U) = 0$, which is a *partial* differential equation, in the form $U = g(r) + h(r)\cos 2\theta$. The functions g(r) and h(r) must satisfy *ordinary* differential equations, which are easier to solve. After solving for the most general forms of g(r) and h(r) which, for example, for h is $h(r) = A\, r^4 + B\, r^2 + C + D\, r^{-2}$ where A, B, C and D are constants, and choosing all constants

to meet the given conditions on stress at $r = a$ and as $r \to \infty$, and to give single-valued displacements, one solves for U and from it finds the stresses:

$$\sigma_{rr} = (\sigma^\infty / 2)[(1 - a^2/r^2) - (1 - 4a^2/r^2 + 3a^4/r^4)\cos 2\theta]$$
$$\sigma_{r\theta} = -(\sigma^\infty / 2)(1 + 2a^2/r^2 - 3a^4/r^4)\sin 2\theta \qquad .$$
$$\sigma_{\theta\theta} = (\sigma^\infty / 2)[(1 + a^2/r^2) + (1 + 3a^4/r^4)\cos 2\theta]$$

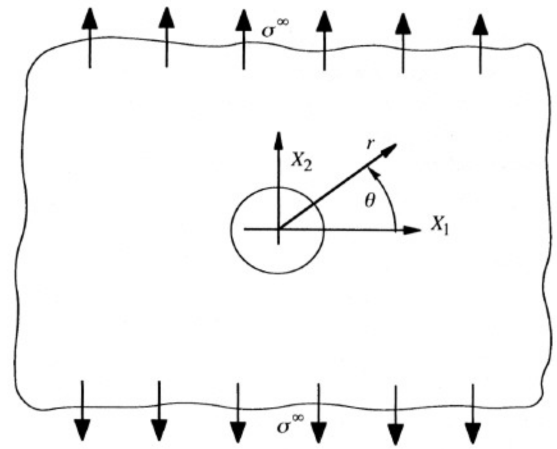

Figure 14. Circular hole on a large plate (or circular tunnel in a solid) under remote tensile stress σ^∞.

Thus, setting $r = a$, $\sigma_{\theta\theta} = \sigma^\infty(1 + 2\cos 2\theta)$ is the stress created around the boundary of the hole. This amounts at $\theta = 0$ and π, that is, at boundary points intersected by the X_1 axis, to a concentration of stress $\sigma_{\theta\theta} = \sigma_{22} = 3\,\sigma^\infty$. At $\theta = \pi/2$ and $-\pi/2$, points intersected by the X_2 axis, there is an oppositely signed stress $\sigma_{\theta\theta} = \sigma_{11} = -\sigma^\infty$. Thus, when we consider a circular hole in a brittle material which can support very little tensile stress, we expect failure to begin at $\theta = 0$ or π under remote tensile loading, but at $\theta = \pi/2$ or $-\pi/2$, and at three times the load level, under remote compressive loading.

As another problem showing an important aspect of stress concentration, consider the Kolosov-Inglis problem of an elliptical hole (Figure 15, at left) in large plate under remotely uniform stress

σ^∞ in the 2 direction as above. This also describes the tunnel cavity of elliptical cross section under plane strain. Let a denote the semi-axis of the ellipse along the 1 direction and b denote that along the 2 direction; the equation of the ellipse is $X_1^2 / a^2 + X_2^2 / b^2 = 1$. It is then found that the concentration of stress at points of the hole boundary intersected by the X_1 axis is $\sigma_{22} =$ $(1 + 2b/a)\,\sigma^\infty$, which can be rewritten as $\sigma_{22} = (1 + 2\sqrt{a / \rho_{tip}})\sigma^\infty$. In the latter form, $\rho_{tip} = b^2/a$ is the radius of curvature of the hole boundary, at the tip of the hole at $X_1 = \pm a$. This illustrates a result of general validity for notches with relatively small root radii compared to length: The elevation of stress over the value (σ^∞) in absence of the notch is, very approximately, given by $2\sqrt{a / \rho_{tip}}\sigma^\infty$ in all cases, where a is the half length of an internal notch like the elliptical hole just discussed, and is the length of a notch that has been cut in from the free surface of a solid. Thus good engineering design is always sensitive to the stress concentrating effect of holes and, especially, notches or other cut-outs of small root radius, avoiding them where possible. This is a lesson reinforced by the bitter experience of many structural failures beginning at unrecognized locations of stress concentration. In the particular case of the elliptical hole, the stress induced along the hole boundary where it is intersected by the X_2 axis is $\sigma_{11} = -\sigma^\infty$, independent of the b/a ratio.

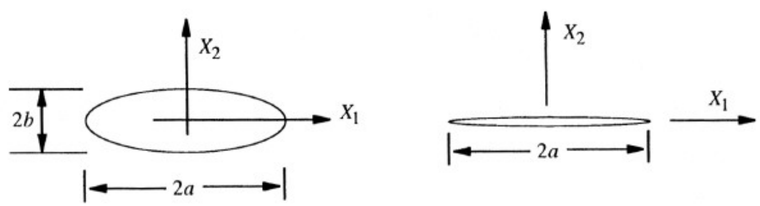

Figure 15. Elliptical tunnel hole, and limit as a flat crack.

Inclusions. Points around the boundary of the elliptical hole just discussed are found to displace according to the equations

$$u_1 = -(\sigma^\infty / E')X_1\,, \quad \text{and} \quad u_2 = (1 + 2a / b)(\sigma^\infty / E')X_2$$

where X_1 and X_2 are coordinates of points on the boundary, and satisfy $X_1^2 / a^2 + X_2^2 / b^2 = 1$. Also, $E' = E$ for the plane stress model but $E' = E / (1 - \nu^2)$ for plane strain. These results show that if we had considered not a solid with an elliptical hole but rather a solid with a uniform elliptical

inclusion of another material, in this case a material with vanishing small elastic modulus compared to that of the surrounding solid (so that the situation in the surrounding solid is indistinguishable from that for the case of a hole), then that inclusion would undergo uniform strain, the strains $\varepsilon_{ij}^{\text{incl}}$ within it being given by rewriting the above equations as $u_1 = \varepsilon_{11}^{\text{incl}} X_1$ and $u_2 = \varepsilon_{22}^{\text{incl}} X_2$ and noting that $\varepsilon_{12}^{\text{incl}} = 0$. A little reflection on this result will convince one that if an inclusion of arbitrary but uniform and isotropic material properties (the stress-strain relation for the inclusion material need not even be linear) were placed in the hole, then the inclusion would undergo a uniform stress and strain that could be calculated from its material properties and the information given so far here.

This discussion generalizes to an important three-dimensional result discovered by J. D. Eshelby and which is this: Let a uniform, possibly anisotropic, linear elastic solid of infinite extent be loaded by a remotely uniform stress tensor and let it contain an ellipsoidal inclusion of a material of uniform but different mechanical properties. The inclusion material can even be such that, in its stress-free state, it takes an ellipsoidal shape which differs finitely from the stress-free shape of the ellipsoidal hole into which it is to be inserted. Eshelby's result is that, regardless of all these factors, the inclusion undergoes a spatially uniform stress and strain state. To develop that result, Eshelby first solved the *transformation* problem of a misfitting linear elastic inclusion of identical properties as those of the surrounding material. That involves taking an ellipsoidal region of a uniform solid and, at least conceptually, transforming its stress-free state by a homogeneous infinitesimal strain, without changing its elastic properties. A uniform stress field within the transformed zone then suffices to deform it back to its original shape, and that stress field can be maintained *in-situ*, without disturbance of the region outside the ellipsoid, by application of a suitable layer of surface force. Since the actual transformation problem to be solved has no agent to supply that layer of force, the strains everywhere can be calculated as those due to removing such a layer (i.e., applying a force layer of opposite sign) within an elastically uniform full space. One calculates then that the stress and strain state induced within the inclusion is uniform, and the rest of what is needed for arbitrary inclusions can be developed from there.

Crack as limit of elliptical hole. Consider again the two-dimensional problem of the elliptical hole under remotely uniform tension, and let the semi-axis b go to zero (Figure 15, at right) so as to define a flat Griffith crack lying along the X_1 axis on $-a < X_1 < +a$. In this case the stress concentration at the hole becomes unbounded so that it can fairly be objected that the discussion lies outside the proper realm of linear elasticity. However, as will be seen, it proves quite useful to continue with the linear elastic model and to learn about its stress singularities. Letting $\Delta u_2 = u_2^+ - u_2^-$ denote the crack opening gap (superscripts + and – denote the upper and lower