

A nonlinear field theory of deformable dielectrics

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Abstract

Two difficulties have long troubled the field theory of dielectric solids. First, when two electric charges are placed inside a dielectric solid, the force between them is not a measurable quantity. Second, when a dielectric solid deforms, the true electric field and true electric displacement are not work conjugates. These difficulties are circumvented in a new formulation of the theory in this paper. Imagine that each material particle in a dielectric is attached with a weight and a battery, and prescribe a field of virtual displacement and a field of virtual voltage. Associated with the virtual work done by the weights and inertia, define the nominal stress as the conjugate to the gradient of the virtual displacement. Associated with the virtual work done by the batteries, define the nominal electric displacement as the conjugate to the gradient of virtual voltage. The approach does not start with Newton's laws of mechanics and Maxwell-Faraday theory of electrostatics, but produces them as consequences. The definitions lead to familiar and decoupled field equations. Electromechanical coupling enters the theory through material laws. In the limiting case of a fluid dielectric, the theory recovers the Maxwell stress. The approach is developed for finite deformation, and is applicable to both conservative and dissipative dielectrics. As applications of the theory, we discuss material laws for conservative dielectrics, and study infinitesimal fields superimposed upon a given field, including phenomena such as vibration, wave propagation, and bifurcation.

Keywords: dielectric, elastomer, large deformation, electrostriction, piezoelectricity

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1. Introduction

All materials contain electrons and protons. In a dielectric, these charged particles form bonds, and move relative to one another by short distances in response to a voltage or a force. That is, all dielectrics are deformable. The notion of a rigid dielectric is as fictitious as that of a rigid body: they are idealizations useful for some purposes, but misleading for others.

Deformable dielectrics are central in diverse technologies (Newhham, 2005; Uchino, 1997; Sessler, 1987; Campbell, 1998). Our own interest is renewed by recent innovations in materials, principally organics capable of large deformation, including electrostrictive polymers (Zhang, 1998; Chu et al., 2006), cellular electrets (Graz, et al., 2006), liquid crystal elastomers (Warner and Terentjev, 2003), and elastomers capable of large deformation under electric field (e.g., Pelrine, et al, 2000). Also emerging are technologies to produce patterns of electric charge with small features (Jacobs and Whitesides, 2001; McCarty et al., 2006). Potential applications of these materials and technologies include transducers in large-area, flexible electronics (e.g., displays, artificial muscles, and sensitive skins), as well as in devices at small length scales. Phenomena of electric-field induced motion and instability have also been actively studied (e.g., Li and Aluru, 2002; Gao and Suo, 2003; Suo and Hong, 2004; Huang, 2005; Lu and Salac, 2006; Zhu et al., 2006).

Although the atomic origin of dielectric deformation has long been understood, how to formulate a field theory remains controversial. Many theories have been formulated (e.g., Becker, 1982; Landau and Lifshitz, 1984; Toupin, 1956; Eringen, 1963; Pao, 1978; Eringen and Maugin, 1989; Maugin, et al., 1992; Kuang, 2002), invoking different approximations and postulates. On these theories, Pao (1978) remarked, “That there are so many coexisting theories and results for a subject so fundamental in nature may sound very surprising to experimentalists, for theories can usually be sorted out, or proven to be fallacious by carefully designed experiments. The difficulty here is that the electromagnetic fields inside matter are expressed in terms of field variables which cannot be directly measured in laboratories.” Recent critiques of these theories may be found in Rinaldi and Brenner (2002), and in McMeeking and Landis (2005).

Pao’s remarks were directed to general theories of electromagnetism in matter, but we find that his remarks apt for theories of deformable dielectrics. To give some ideas of the controversies involved, we mention two difficulties.

One difficulty has to do with the notion of electric force. Consider, for example, a parallel-plate capacitor, consisting of an insulating medium and two electrodes, with a battery maintaining a positive charge on one electrode, and a negative charge on the other (Fig. 1). If the insulating medium is a vacuum or a fluid, we must apply a force (e.g., by using a weight) to maintain equilibrium. In this case, there is no ambiguity as to what the electric force is: the force between the two electrodes can be measured by the weight. Maxwell (1891) converted this force into a state of stress in the medium. When the insulating medium is a solid dielectric, however, the electric force cannot be measured. Indeed, for many common solid dielectrics subject to a voltage, the two electrodes appear to repel, rather than attract, each other (Newhham, 2005). The atomic origin of this phenomenon is clear. Influenced by the voltage between the electrodes, charges inside the dielectric tend to displace relative to one another, often accompanied by an elongation of the material in the direction of the electric field.

On the force between electric charges in a solid, Feynman (1964) remarked, “This is a very difficult problem which has not been solved, because it is, in a sense, indeterminate. If you put charges inside a dielectric solid, there are many kinds of pressures and strains. You cannot deal with virtual work without including also the mechanical energy required to compress the

solid, and it is a difficult matter, generally speaking, to make a unique distinction between the electrical forces and mechanical forces due to solid material itself. Fortunately, no one ever really needs to know the answer to the question proposed. He may sometimes want to know how much strain there is going to be in a solid, and that can be worked out. But it is much more complicated than the simple result we got for liquids.”

A second difficulty has to do with work conjugates. It is a simple matter to show that, when a dielectric solid deforms, the true electric field and the true electric displacement are *not* work conjugates. Although this fact does not preclude them from being used to formulate the field theory of deformable dielectrics, the non-conjugates do lead to complications, and their almost exclusive, and sometimes erroneous, use in the literature contributes to the controversies.

While the first difficulty was noted by nearly all authors on the subject, we have not found any explicit discussion on the second. The dubious status of electric force is unsettling as most textbooks start by defining electric field by the force acting on a test charge divided by the amount of charge. In a solid dielectric, the electric force is not a measurable quantity, so that this definition is not operational. A common approach is to forgo this definition, and regard the field equations of electrostatics as the starting point. But to discuss deformation one has to link the electric field to a force, and this connection is usually made by invoking work. In this connection, some authors (e.g., Landau and Lifshitz, 1984; Line and Glass, 1977) assumed that the true electric field and the true electric displacement are work conjugates. While this assumption does not lead to serious errors for infinitesimal deformation, it does for finite deformation.

In this paper, in the spirit of Feynman’s remark, instead of dwelling on the meaningless notion of electric force, we ask questions concerning measurable quantities. In effect, we ask, given an applied voltage and applied force, how much does one electrode move relative to the other, and how much charge flows from one electrode to the other? Instead of leaving various fields undefined, we define them by operational procedures.

As a mental aid in formulating the theory, imagine that each material particle in a dielectric is attached with a weight and a battery, and then prescribe a field of virtual displacement and a field of virtual voltage. We will use virtual work to define fields inside media, an approach well established in mechanics, but perhaps less so in electrostatics. Associated with the work done by the weights and inertia, we define the stress inside the dielectric as the conjugate to the gradient of displacement. Associated with the work done by the batteries, we define the electric displacement inside the dielectric as the conjugate to the gradient of electric potential. The approach requires no additional postulate beyond what we mean by work, displacement, charge and inertia. The approach does not start with field equations, but produces them as consequences. The theory is applicable to finite deformation, and to both conservative and dissipative dielectrics.

We write the body of the paper with minimal digression, hoping that a reader with basic knowledge of electrostatics and mechanics can appreciate the theory. Section 2 reviews elementary facts of work, energy and electromechanical coupling, using a generic transducer. Section 3 uses a homogenous field to illustrate a procedure to define quantities per unit length, area, and volume, a procedure that we generalize in Section 4 to inhomogeneous fields in three dimensions. Section 5 sketches the material laws for conservative media. Section 6 applies the theory to fluid dielectrics, and recovers the Maxwell stress. Section 7 discusses solid dielectrics. Section 8 applies the theory to infinitesimal fields superimposed upon a given field.

Our approach allows us to choose, among many alternatives, measures of stress, strain, electric field, and electric displacement. The body of the paper will focus on nominal quantities using material coordinates. Various Appendices describe alternative formulations and link to the

existing literature. We show that our theory recovers the results of McMeeking and Landis (2005), who formulated a theory of deformable dielectrics by using spatial coordinates and the true electric field and true electric displacement. These authors started with an electric force, but concluded that this force cannot be measured in solid dielectrics. A parallel reading of that paper and the present one should provide a fuller understanding of both approaches.

2. Work, electromechanical coupling, and energy

Work. Figure 2 illustrates a system of insulators and conductors, loaded by a field of weights and batteries, of which only one of each is drawn. All batteries are connected to a common ground. We can measure the displacement δl of the weight, and the amount of charge δQ pumped by the battery from the ground to the electrode. There might be other weights dropping or rising and other batteries pumping charge from or to the ground, but the work done by this particular weight is $P\delta l$, and the work done by this particular battery is $\Phi\delta Q$. If we regard work, displacement and charge as primitive, measurable quantities, the above statements of work define the force P supplied by the weight, and the voltage Φ supplied by the battery. The force is said to be the work conjugate to the displacement, and the voltage the work conjugate to the charge. We will use the word weight as shorthand for all mechanisms (including inertia) that do work through displacements, and the word battery as shorthand for all mechanisms that do work through flows of charge. We will neglect the effects of magnetism and electromagnetic radiation.

Electromechanical coupling. Now imagine that the weight and battery are adjustable, so that the force P and the voltage Φ can vary. When the displacement is held constant, a change in the charge may cause the force to change. When the charge is held constant, a change in the displacement may cause the voltage to change. These electromechanical coupling effects are universal to all dielectrics, because all dielectrics have electrons and protons, and the charged particles can move relative to one another.

Conservative system. The two electromechanical coupling effects are linked for conservative systems. A conservative system is one for which, under isothermal conditions, the work done by the weight and the battery is fully stored as the Helmholtz free energy of the system. That is, associated with small changes δl and δQ , the free energy of the system, U , changes by

$$\delta U = P\delta l + \Phi\delta Q. \quad (2.1)$$

To this equation we should add the work done by all other weights and batteries. For simplicity, however, here we assume that only one weight and one battery do work. This may be achieved by removing all other weights and batteries, and making sure that every other part in the system other than the particular electrode is either grounded or charge neutral.

These idealizations ensure that the free energy of the system is a function of two variables, $U(l, Q)$. We only need to measure the difference in U , l and Q between the current state and a reference state. For the conservative system, the force and the voltage are partial derivatives:

$$P(l, Q) = \frac{\partial U(l, Q)}{\partial l}, \quad \Phi(l, Q) = \frac{\partial U(l, Q)}{\partial Q}. \quad (2.2)$$

Associated with small changes δl and δQ , the force and the voltage change by

$$\delta P = \frac{\partial^2 U(l, Q)}{\partial l^2} \delta l + \frac{\partial^2 U(l, Q)}{\partial l \partial Q} \delta Q, \quad (2.3)$$

$$\delta\Phi = \frac{\partial^2 U(l, Q)}{\partial l \partial Q} \delta l + \frac{\partial^2 U(l, Q)}{\partial Q^2} \delta Q. \quad (2.4)$$

We may call $\partial^2 U(l, Q)/\partial l^2$ the mechanical tangent stiffness of the system, and $\partial^2 U(l, Q)/\partial Q^2$ the electrical tangent stiffness of the system. The two electromechanical coupling effects are both characterized by the same cross derivative, namely,

$$\frac{\partial P(l, Q)}{\partial Q} = \frac{\partial^2 U(l, Q)}{\partial l \partial Q} = \frac{\partial \Phi(l, Q)}{\partial l}. \quad (2.5)$$

Consequently, for a conservative system, the two electromechanical coupling effects reciprocate.

Vacuum. As an illustration, consider the parallel-plate capacitor, loaded by both the voltage and the weight. The two electrodes are separated by a vacuum. The separation between the two electrodes l may vary, but the area of either electrode remains to be A . Recall the elementary fact:

$$U(l, Q) = \frac{lQ^2}{2\varepsilon_0 A}, \quad (2.6)$$

where ε_0 is the permittivity of vacuum, so that

$$P = \frac{\partial U(l, Q)}{\partial l} = \frac{Q^2}{2A\varepsilon_0}. \quad (2.7)$$

This force is due to the attraction of the opposite charges on the two electrodes, and is balanced by the weight. The force divided by the area of the electrode is a special case of the Maxwell stress. See Section 6 for how the Maxwell stress comes out from our formulation for inhomogeneous fields in fluids in three dimensions.

3. A homogeneous field in a parallel-plate capacitor

To exhibit the essentials of our approach in a simple setting, we first analyze a parallel-plate capacitor (Fig. 1). We assume that the capacitor is made such that the field in the capacitor is homogenous, an assumption that enables us to readily define intensive quantities (i.e., quantities per unit length, area, and volume). We wish to endow these quantities no more significance than merely being intensive, work-conjugating quantities.

Take any state of the dielectric as the reference state, in which the dielectric has cross-sectional area A and thickness L . In the current state, the dielectric deforms to cross-sectional area a and thickness l . Define the stretch λ as the thickness of the dielectric in the current state divided by the thickness of the dielectric in the reference state:

$$\lambda = \frac{l}{L}. \quad (3.1)$$

Define the nominal stress s as the force supplied by the weight in the current state divided by the area in the reference state:

$$s = \frac{P}{A}. \quad (3.2)$$

When the thickness changes by δl , the weight does work $P\delta l = ALs\delta\lambda$. Note that $V = AL$ is the volume of the dielectric in the reference state. Thus, $s\delta\lambda$ is the work done by the weight in the current state divided by the volume of the dielectric in the reference state, and the nominal stress is work-conjugate to the stretch.

To generalize the above definitions to inhomogeneous deformation in three dimensions, we will use virtual displacement and virtual work to define the stress. Here let us see what these

ideas mean in parallel-plate capacitor. Let Δ be a virtual displacement of the weight. Replace δl by Δ , and call Δ/L the virtual stretch of the capacitor, and $P\Delta$ the virtual work done by the weight. Define the nominal stress s as the work conjugate to the stretch. That is, we insist the equation

$$Vs \frac{\Delta}{L} = P\Delta \quad (3.3)$$

holds true for arbitrary virtual displacement Δ . This definition recovers $s = P/A$. We intend to use $s\delta\lambda$ to represent the mechanical work (i.e., work done by external mechanisms through displacements). Consequently, only the work done by the weight need be included, and there is no need to consider any electric effect in defining the stress.

The virtual displacement Δ has no relation with the actual displacement. In fact, it need not have any physical interpretation; in the literature on Galerkin method, for example, the virtual displacement is simply called a test function. We deviate from the convention in many textbooks on continuum mechanics and drop the symbol δ in front of the virtual displacement for two reasons: (a) The virtual displacement need not be small and can have arbitrary magnitude and unit, and (b) We will reserve the symbol δ to indicate an actual, small change of a physical quantity.

Define the nominal electric field \tilde{E} as the voltage supplied by the battery in the current state divided by the thickness in the reference state:

$$\tilde{E} = \frac{\Phi}{L}. \quad (3.4)$$

Define the nominal electric displacement \tilde{D} by the charge Q on an electrode in the current state divided by the area in the reference state:

$$\tilde{D} = \frac{Q}{A}. \quad (3.5)$$

When a small amount of electric charge δQ flows from the negative electrode to the positive electrode, the work done by the battery is $\Phi\delta Q = AL\tilde{E}\delta\tilde{D}$. The work done by the battery in the current state divided by the volume of the dielectric in the reference state is $\tilde{E}\delta\tilde{D}$. Consequently, the nominal electric field is work-conjugate to the nominal electric displacement.

Alternatively, we can regard the nominal electric displacement as defined by the work conjugate to the nominal electric field. To this end, let ζ be the virtual voltage supplied by the battery. Replace Φ by ζ , and call ζ/L the virtual electric field, and $\zeta\delta Q$ the virtual work done by the battery. Define the nominal electric displacement \tilde{D} such that the equation

$$V \frac{\zeta}{L} \delta\tilde{D} = \zeta\delta Q \quad (3.6)$$

holds true for arbitrary virtual voltage ζ . This definition recovers $\tilde{D} = Q/A$ once we integrate the charge from zero to its current value. We intend to use $\tilde{E}\delta\tilde{D}$ to represent the electrical work (i.e., work done by the flow of charge). Consequently, only the work done by the battery need be included, and there is no need to consider any mechanical effect in defining the electric displacement.

Incidentally, when a solid dielectric deforms, the true electric field and electric displacement are not work-conjugate to each other. This can be seen readily as follows. Define

the true electric field by the voltage supplied by the battery in the current state divided by the thickness in the current state:

$$E = \frac{\Phi}{l}, \quad (3.7)$$

Define the true electric displacement D by the charge Q in the current state divided by the area in the current state:

$$D = \frac{Q}{a}. \quad (3.8)$$

In terms of the true electric field and the true electric displacement, the work done by the battery is

$$\Phi \delta Q = (lE)\delta(aD) = lED\delta a + laE\delta D. \quad (3.9)$$

For a solid dielectric, $\delta a \neq 0$, so that the true electric displacement is *not* work-conjugate to the true electric field.

There is considerable flexibility as to how we define measures of stress, strain, electric field, and electric displacement. So long as a theory relates measurable quantities, all alternative definitions are equally valid. To argue for the superiority of one set of definitions to another amounts to glorifying prejudices. In particular, true electric field is no more realistic than nominal electric field. Certain definitions, however, may save mental efforts for people brought up in certain ways. To avoid confusion while respect diversity, we will develop one set of definitions (the nominal quantities) in the body of the paper and discuss alternatives in Appendix A.

4. Inhomogeneous fields in three dimensions

Consider a continuous body of material particles. As a mental aid, imagine that each material particle is connected to a battery, which maintains the electric potential of the material particle with respect to the ground. The material itself is an insulator, but the battery may pump charge from the ground to the material particle. Similarly, we imagine that each material particle is connected to a weight.

Any state of the body may serve as the reference state. Following the usual practice in continuum mechanics (Truesdell and Noll, 2003), we use the coordinates \mathbf{X} of each material particle in the reference state to name the material particle. Let $dV(\mathbf{X})$ be a material element of volume in the reference state. The body extends in the entire space, but may contain interfaces between dissimilar media. Let $N_K(\mathbf{X})dA(\mathbf{X})$ be a material element of an interface, where $dA(\mathbf{X})$ is the area of the element, and $N_K(\mathbf{X})$ is the unit vector normal to the interface between media labeled as $-$ and $+$, pointing toward medium $+$.

Virtual work associated with virtual displacement. Nominal stress. At time t , the material particle \mathbf{X} moves to a place with coordinate \mathbf{x} . The function $\mathbf{x}(\mathbf{X}, t)$ describes the motion of the body. In the current state (at time t), the material particle \mathbf{X} has velocity $\partial \mathbf{x}(\mathbf{X}, t) / \partial t$ and acceleration $\partial^2 \mathbf{x}(\mathbf{X}, t) / \partial t^2$. Let $\tilde{\mathbf{b}}(\mathbf{X}, t)dV(\mathbf{X})$ be the force due to the weights on a material element of volume, $\tilde{\mathbf{t}}(\mathbf{X}, t)dA(\mathbf{X})$ be the force due to the weights on a material element of interface, and $\tilde{\rho}(\mathbf{X})dV(\mathbf{X})$ be the mass of the material element of volume. We do not connect each material particle with a ‘‘pump of mass’’, so that $\tilde{\rho}(\mathbf{X})$ is time-independent, and is taken to be known.

Let $\Delta_i(\mathbf{X})$ be a field of virtual displacement. We have dropped the time dependence in

the virtual displacement because we have no use of it. The virtual displacement has no relation to the actual motion, and has no physical significance. All we require is that $\Delta_i(\mathbf{X})$ be a continuous field of vector. We replace $\delta\lambda$ by the gradient in virtual displacement, $\partial\Delta_i/\partial X_K$, and define the tensor of nominal stress, $s_{iK}(\mathbf{X}, t)$ as the work conjugate to the gradient of the virtual displacement. That is, the following equation holds true for any virtual displacement $\Delta_i(\mathbf{X})$:

$$\int s_{iK} \frac{\partial\Delta_i}{\partial X_K} dV = \int \left(\tilde{b}_i - \tilde{\rho} \frac{\partial^2 x_i}{\partial t^2} \right) \Delta_i dV + \int \tilde{t}_i \Delta_i dA. \quad (4.1)$$

The right-hand side is the virtual work associated with the virtual displacement, done by the weights and inertia. The above expression is known as the Principle of Virtual Work. However, this expression as used here introduces no principle of any kind; we merely use this expression to define the tensor of nominal stress. As such, there is no need to invoke the electrical force. For the parallel-plate capacitor, the procedure reduces to defining the nominal stress as the force supplied by the weight in the current state divided by the area in the reference state. In that simpler setting, it would be comical to call such a procedure a principle.

Having defined the tensor of nominal stress, we can find out what equation this tensor satisfies. Applying the divergence theorem, one obtains that

$$\int s_{iK} \frac{\partial\Delta_i}{\partial X_K} dV = \int \left[\frac{\partial(s_{iK}\Delta_i)}{\partial X_K} - \frac{\partial s_{iK}}{\partial X_K} \Delta_i \right] dV = \int (s_{iK}^- - s_{iK}^+) N_K \Delta_i dA - \int \frac{\partial s_{iK}}{\partial X_K} \Delta_i dV. \quad (4.2)$$

The surface integral extends over the area of all interfaces. Across the interface, the virtual displacement $\Delta_i(\mathbf{X})$ is assumed to be continuous, but the stress need not be continuous. Insisting that the defining equation for the nominal stress holds true for any field of virtual displacement $\Delta_i(\mathbf{X})$, we find that the nominal stress obeys that

$$\frac{\partial s_{iK}(\mathbf{X}, t)}{\partial X_K} + \tilde{b}_i(\mathbf{X}, t) = \tilde{\rho}(\mathbf{X}) \frac{\partial^2 x_i(\mathbf{X}, t)}{\partial t^2} \quad (4.3)$$

in the volume of the body, and

$$(s_{iK}^-(\mathbf{X}, t) - s_{iK}^+(\mathbf{X}, t)) N_K(\mathbf{X}, t) = \tilde{t}_i(\mathbf{X}, t), \quad (4.4)$$

on the surface of the body. These equations express momentum balance in every current state in terms of the nominal fields, and is well known in continuum mechanics. Because we define the stress as a conjugate to the gradient of displacement to calculate the work done by the weights and inertia, we do not invoke the nebulous notion of forces due to charges in our theory.

Virtual work associated with virtual voltage. Nominal electric displacement. As an aid to formulate the theory, we attach each material particle a battery, and connect all the batteries to a common ground. The medium is insulating, but the batteries can vary the charge on every material particle. Let the charge on the element of volume vary by $\delta\tilde{q}(\mathbf{X}, t)dV(\mathbf{X})$, and the charge on the element of an interface vary by $\delta\tilde{\omega}(\mathbf{X}, t)dA(\mathbf{X})$.

Let $\zeta(\mathbf{X})$ be a field of virtual voltage. The virtual voltage has no relation with the actual potential, and is simply used as a test function. Define the vector of nominal electric displacement, $\tilde{D}_K(\mathbf{X}, t)$, as the work-conjugate to the gradient of the virtual voltage. That is, the following equation

$$\int \left(-\frac{\partial \zeta}{\partial X_K} \right) \delta \tilde{D}_K dV = \int \zeta \delta \tilde{q} dV + \int \zeta \delta \tilde{\omega} dA \quad (4.5)$$

holds true for any field of virtual voltage $\zeta(\mathbf{X})$. The right-hand side is the virtual work associated with the virtual voltage.

We apply the divergence theorem to the left-hand side, and obtain that

$$\int \frac{\partial \zeta}{\partial X_K} \delta \tilde{D}_K dV = \int \left[\frac{\partial (\zeta \delta \tilde{D}_K)}{\partial X_K} - \zeta \frac{\partial (\delta \tilde{D}_K)}{\partial X_K} \right] dV = \int \zeta (\delta \tilde{D}_K^- - \delta \tilde{D}_K^+) N_K dA - \int \zeta \frac{\partial (\delta \tilde{D}_K)}{\partial X_K} dV. \quad (4.6)$$

The virtual voltage $\zeta(\mathbf{X})$ is assumed to be continuous across the interface, but the electric displacement need not be continuous across the interface. Insisting that the defining equation for the nominal electric displacement holds true for any field of virtual voltage $\zeta(\mathbf{X})$, we find that the nominal electric displacement obeys that

$$\frac{\partial \tilde{D}_K(\mathbf{X}, t)}{\partial X_K} = \tilde{q}(\mathbf{X}, t) \quad (4.7)$$

in the volume of the body, and

$$(\tilde{D}_K^+(\mathbf{X}, t) - \tilde{D}_K^-(\mathbf{X}, t)) N_K(\mathbf{X}, t) = \tilde{\omega}(\mathbf{X}, t). \quad (4.8)$$

on the surface of the body. These equations express Gauss's law in every current state in terms of the nominal fields. We have dropped the sign of increment, so that $\tilde{q}(\mathbf{X}, t) dV(\mathbf{X})$ is the difference in electric charge on a material element of volume between the current state and the reference state, and $\tilde{\omega}(\mathbf{X}, t) dA(\mathbf{X})$ is the difference in electric charge on a material element of an interface between the current state and the reference state.

From the work done by weights and inertia through displacement, we have defined the stress, which satisfies the usual field equations of mechanics. From the work done by batteries through flow of charge, we have defined the electric displacement, which satisfies the usual field equations of electrostatics. We have used the nominal quantities and material coordinates. A benefit of using material coordinates is that the domain of the problem remains fixed as the system evolves in time. Appendix B will develop this approach using spatial coordinates.

The field equations formulated in this section are applicable to both conservative and dissipative media. In the following section we will consider material laws for conservative media.

5. Material laws for conservative media

One bothers to define stress and electric displacement because one conjectures that each material particle will behave like a miniaturized parallel-plate capacitor, i.e., a material in homogenous field. Whether this conjecture is true can be examined by comparing its consequences with experimental observations. If the conjecture turns out to be untrue, one can define higher-order stress-like quantities, or simply abandon this approach and just focus on the system as a whole, as we did in Section 2. In this paper, we assume that the conjecture holds true, and examine its consequences.

At time t , the material particle \mathbf{X} occupies the place $\mathbf{x}(\mathbf{X}, t)$. In Section 3, we have defined a measure of deformation, the stretch, by the thickness in the current state divided by thickness in the reference state. For inhomogeneous fields in three dimensions, the stretch is generalized to the deformation gradient

$$F_{iK}(\mathbf{X}, t) = \frac{\partial x_i(\mathbf{X}, t)}{\partial X_K}. \quad (5.1)$$

At time t , the material particle \mathbf{X} has electric potential $\Phi(\mathbf{X}, t)$. In Section 3, we have defined the nominal electric field by the voltage in the current state divided by the thickness of the dielectric in the reference state. For inhomogeneous fields in three dimensions, we define the nominal electric field as the gradient of the electric potential:

$$\tilde{E}_K(\mathbf{X}, t) = -\frac{\partial \Phi(\mathbf{X}, t)}{\partial X_K}. \quad (5.2)$$

Associated with small, actual change in the deformation gradient and in the electric displacement, δF_{iK} and $\delta \tilde{D}_{Ki}$, the field of weights and batteries do actual work. According to the definition of the nominal stress and the nominal electric displacement, (4.1) and (4.5), the actual work done by the weights is $\int s_{iK} \delta F_{iK} dV$, and actual work done by the batteries is $\int \tilde{E}_K \delta \tilde{D}_K dV$.

The above statements are true for both dissipative and conservative dielectrics. For conservative dielectrics, the actual work is fully stored as free energy in the body. The conjecture that each material particle behaves like a miniaturized parallel-capacitor is interpreted as follows. Let $W dV(\mathbf{X})$ be the free energy of a material element of volume. According to the above definitions of the nominal fields, associated with the actual changes, δF_{iK} and $\delta \tilde{D}_{Ki}$, the free energy of the material element of volume changes by

$$\delta W = s_{iK} \delta F_{iK} + \tilde{E}_K \delta \tilde{D}_K. \quad (5.3)$$

Thus, the energy density is a function of the deformation gradient and the nominal electric displacement, $W(\mathbf{F}, \tilde{\mathbf{D}})$. The four fields, the nominal stress, the nominal electric field, the deformation gradient, and the nominal electric displacement are related as

$$s_{iK}(\mathbf{F}, \tilde{\mathbf{D}}) = \frac{\partial W(\mathbf{F}, \tilde{\mathbf{D}})}{\partial F_{iK}}, \quad \tilde{E}_K(\mathbf{F}, \tilde{\mathbf{D}}) = \frac{\partial W(\mathbf{F}, \tilde{\mathbf{D}})}{\partial \tilde{D}_K}. \quad (5.4)$$

A benefit of using work conjugates is that the material laws take these familiar forms.

Invariance under rigid-body rotation. When the entire system in the current state undergoes a rigid-body rotation, the nominal electric displacement $\tilde{\mathbf{D}}$ is invariant, but the deformation gradient, F_{iK} , varies. To ensure that W is invariant under rigid-body rotation, we invoke the Lagrangian strain

$$L_{KM} = \frac{1}{2}(F_{iK} F_{iM} - \delta_{KM}),$$

which is invariant when the entire system in the current state undergoes a rigid-body rotation. Consequently, a conservative dielectric is characterized by the energy function $W(\mathbf{L}, \tilde{\mathbf{D}})$. The nominal stress and the nominal electric field are obtained from partial derivatives:

$$s_{iK}(\mathbf{L}, \tilde{\mathbf{D}}) = F_{iM} \frac{\partial W(\mathbf{L}, \tilde{\mathbf{D}})}{\partial L_{KM}}, \quad \tilde{E}_K(\mathbf{L}, \tilde{\mathbf{D}}) = \frac{\partial W(\mathbf{L}, \tilde{\mathbf{D}})}{\partial \tilde{D}_K}. \quad (5.5)$$

Material laws expressed in true fields. Appendix C lists the well known relations between the true fields and the nominal fields. For example, the true stress σ_{ij} relates to the nominal stress by

$$\sigma_{ij} = \frac{F_{jK}}{\det(\mathbf{F})} s_{iK}. \quad (5.6)$$

The true electric displacement relates to the nominal electric displacement as

$$D_i = \frac{F_{iK}}{\det(\mathbf{F})} \tilde{D}_K. \quad (5.7)$$

The true electric field relates to the nominal electric field as

$$E_i = H_{iK} \tilde{E}_K(\mathbf{X}, t), \quad (5.8)$$

where H_{iK} is the inverse of the deformation gradient, namely, $H_{iK} F_{iL} = \delta_{KL}$ and $H_{iK} F_{jK} = \delta_{ij}$.

The material laws can be expressed using the true stress and the true electric field:

$$\sigma_{ij}(\mathbf{L}, \tilde{\mathbf{D}}) = \frac{F_{iM} F_{jM}}{\det(\mathbf{F})} \frac{\partial W(\mathbf{L}, \tilde{\mathbf{D}})}{\partial L_{KM}}, \quad E_i(\mathbf{L}, \tilde{\mathbf{D}}) = H_{iK} \frac{\partial W(\mathbf{L}, \tilde{\mathbf{D}})}{\partial \tilde{D}_K}. \quad (5.9)$$

While in general the nominal stress is not a symmetric tensor, the true stress is.

Nonpolar material. For a nonpolar material, a reference state exists such that

$$W(\mathbf{L}, \tilde{\mathbf{D}}) = W(\mathbf{L}, -\tilde{\mathbf{D}}) \quad (5.10)$$

for all \mathbf{L} and $\tilde{\mathbf{D}}$. For example, vacuum and fluid dielectrics are nonpolar; so are many solid dielectrics. For a nonpolar dielectric, in the absence of a voltage, a change in applied force will not cause any electric charge to flow to or from the electrode.

Isotropic material. For an isotropic material, a reference state exists such that the energy density is a function of the invariants formed by the tensor \mathbf{L} and the vector $\tilde{\mathbf{D}}$:

$$L_{KK}, L_{KN} L_{KN}, L_{KN} L_{NM} L_{MK}, \tilde{D}_K \tilde{D}_K, L_{AB} \tilde{D}_A \tilde{D}_B, L_{KA} \tilde{D}_A L_{KB} \tilde{D}_B. \quad (5.11)$$

An isotropic material is nonpolar.

Electrical Gibbs Function. Note that nominal electric field obeys similar field equations as the deformation gradient, and the nominal electric displacement obeys similar field equations as the nominal stress. Consequently, it is often convenient to replace the nominal electric displacement with the nominal electric field as an independent variable. Define the electrical Gibbs free energy by

$$\hat{W} = W - \tilde{E}_K \tilde{D}_K. \quad (5.12)$$

Thus, associated with small changes in the deformation gradient and nominal electric field, the Gibbs free energy changes by

$$\delta \hat{W} = s_{iK} \delta F_{iK} - \tilde{D}_K \delta \tilde{E}_K. \quad (5.13)$$

The Gibbs free energy is a function of the deformation gradient and the nominal electric field, $\hat{W}(\mathbf{F}, \tilde{\mathbf{E}})$. The stress and the electric displacement are given by

$$s_{iK} = \frac{\partial \hat{W}(\mathbf{F}, \tilde{\mathbf{E}})}{\partial F_{iK}}, \quad \tilde{D}_K = -\frac{\partial \hat{W}(\mathbf{F}, \tilde{\mathbf{E}})}{\partial \tilde{E}_K}. \quad (5.14)$$

6. Fluid dielectrics

We will treat a common fluid that has no memory of its reference state. Consequently, the fluid is characterized by a free energy per unit volume in the current state and, under isothermal conditions, is a function of two variables: the density in the current state, ρ , and the invariant of the true electric displacement, $\Lambda = D_i D_i$. Let $f(\rho, \Lambda)$ be the free energy of the fluid divided by its volume in the current state.

To use the apparatus developed in the previous sections, we still imagine a reference state, and let \mathbf{F} be the deformation gradient of the current state. The free energy per unit volume in the reference state is

$$W(\mathbf{F}, \tilde{\mathbf{D}}) = \det(\mathbf{F}) f(\rho, \Lambda), \quad (6.1)$$

where, in terms of the nominal quantities,

$$\rho = \frac{\tilde{\rho}}{\det \mathbf{F}}, \quad \Lambda = \frac{F_{jM} F_{jN}}{(\det \mathbf{F})^2} \tilde{D}_M \tilde{D}_N. \quad (6.2)$$

The nominal electric field is

$$\tilde{E}_K = \frac{\partial W(\mathbf{F}, \tilde{\mathbf{D}})}{\partial \tilde{D}_K} = \det(\mathbf{F}) \frac{\partial f(\rho, \Lambda)}{\partial \Lambda} \frac{\partial}{\partial \tilde{D}_K} \left(\frac{F_{jM} F_{jN}}{(\det \mathbf{F})^2} \tilde{D}_M \tilde{D}_N \right) = 2 \frac{\partial f(\rho, \Lambda)}{\partial \Lambda} F_{jK} D_j. \quad (6.3)$$

Recall that $\tilde{E}_K = F_{iK} E_i$, and we obtain the true electric field in terms of the true electric displacement:

$$E_i = 2 \frac{\partial f(\rho, \Lambda)}{\partial \Lambda} D_i. \quad (6.4)$$

The nominal stress is

$$s_{iK} = \frac{\partial W(\mathbf{F}, \tilde{\mathbf{D}})}{\partial F_{iK}} = \frac{\partial \det(\mathbf{F})}{\partial F_{iK}} f(\rho, \Lambda) + \det(\mathbf{F}) \frac{\partial f(\rho, \Lambda)}{\partial \rho} \frac{\partial \rho}{\partial F_{iK}} + \det(\mathbf{F}) \frac{\partial f(\rho, \Lambda)}{\partial \Lambda} \frac{\partial \Lambda}{\partial F_{iK}}. \quad (6.5)$$

Recall that $\partial \det(\mathbf{F}) / \partial F_{iK} = H_{iK} \det(\mathbf{F})$. A direct calculation gives that

$$s_{iK} = H_{iK} \left(\det(\mathbf{F}) f - \tilde{\rho} \frac{\partial f}{\partial \rho} \right) + \frac{\tilde{D}_M \tilde{D}_N}{\det(\mathbf{F})} (2F_{iN} \delta_{MK} - 2F_{jM} F_{jN} H_{iK}) \frac{\partial f}{\partial \Lambda}. \quad (6.6)$$

Recall that the true stress relates to the nominal stress by $\sigma_{ij} = s_{iK} F_{jK} / \det \mathbf{F}$, so that

$$\sigma_{ij} = \left(f - \rho \frac{\partial f}{\partial \rho} - 2 \frac{\partial f}{\partial \Lambda} D_k D_k \right) \delta_{ij} + 2 \frac{\partial f}{\partial \Lambda} D_i D_j. \quad (6.7)$$

This result agrees with that in Becker (1982) and Landau and Lifshitz (1984). The agreement is expected because these authors derived this result by adhering to the statements of work, even though their steps are quite different from ours.

In a limiting case that the fluid is incompressible and linearly dielectric, the free energy density is

$$f = \frac{1}{2\varepsilon} D_i D_i, \quad (6.8)$$

where ε is the permittivity of the fluid. The material laws are

$$E_i = \frac{1}{\varepsilon} D_i \quad (6.9)$$

and

$$\sigma_{ij} = \frac{1}{\varepsilon} \left(D_i D_j - \frac{1}{2} D_m D_m \delta_{ij} \right). \quad (6.10)$$

We regard a vacuum as another limiting case, when the fluid has vanishingly mechanical stiffness. The free-energy density is

$$f = \frac{1}{2\varepsilon_0} D_i D_i, \quad (6.11)$$

so that

$$E_i = \frac{1}{\varepsilon_0} D_i, \quad (6.12)$$

$$\sigma_{ij} = \frac{1}{\varepsilon_0} \left(D_i D_j - \frac{1}{2} D_m D_m \delta_{ij} \right). \quad (6.13)$$

Equations (6.10) and (6.13) recover the Maxwell stress (Maxwell, 1891). See Pao (1978) for a review of literature on Maxwell stress. Within our approach, the Maxwell stress requires no special treatment: the Maxwell stress comes out as a material law once an incompressible fluid or a vacuum is viewed as a limiting form of material. In both limits, both mechanical work and electrical work are stored as electrostatic energy. Consequently, the electric force can be measured with no ambiguity.

By contrast, for compressible fluids, and for solid dielectrics discussed below, the stress depends on both deformation and electric displacement. Any attempt to separate them and call part of the stress the Maxwell stress must be arbitrary. The practice may provide temporary mental comfort, but on close examination is without merit.

The material laws (6.5) and (6.7), in conjunction with the field equations in Section 4, determine fields in compressible fluids in equilibrium. Note that the material laws couple the electrical and mechanical fields in both ways, so that in general the two sets of fields must be solved simultaneously. For an incompressible fluid and for a vacuum, however, the material laws only couple the fields in one way, so that the electrical field is determined without the knowledge of the mechanical field, except that the mechanical field may deform the shape of the fluid (i.e. the shape of its container), and the deformed shape will enter as a boundary condition in solving the electrical field.

7. Solid dielectrics

A solid dielectric remembers its reference state. We will formulate the material laws using the nominal fields.

Small-strain, small-electric-displacement approximation. When the magnitudes of various fields are small, a commonly used approach is to expand the energy function into the Taylor series, and retain the leading terms. For nonpolar materials, the most general form up to quadratic terms of the strain and electric displacement is

$$W(\mathbf{L}, \tilde{\mathbf{D}}) = \frac{1}{2} C_{ABKM} L_{AB} L_{KM} + \frac{1}{2} \gamma_{AB} \tilde{D}_A \tilde{D}_B + R_{ABKM} \tilde{D}_A \tilde{D}_B L_{KM}, \quad (7.1)$$

where the tensors C , γ and R are independent of strain and electric displacement. The condition of being nonpolar requires that each term be an even function in the electric displacement. The material laws are

$$s_{iK} = F_{iM} \left(C_{ABKM} L_{AB} + R_{ABKM} \tilde{D}_A \tilde{D}_B \right), \quad (7.2)$$

$$\tilde{E}_K = (R_{KMAB} L_{AB} + \gamma_{KM}) \tilde{D}_M. \quad (7.3)$$

To further reduce to commonly used laws for electrostrictive materials (e.g., Newnham, 2005), one has to retain just the terms linear in the gradient of displacement. This is yet another distinct approximation, and has to be justified on a case by case basis, as illustrated by the von Karman plate theory.

Landau and Lifshitz (1984) assumed that the electrical Gibbs free energy is a function of infinitesimal strain and true electric field. This assumption raises several concerns. When the body is subject to a rigid-body rotation in the current state, the true electric field is not invariant,

so that the true electric field by itself is an inappropriate variable to formulate material laws. Also, the true electric displacement is not work-conjugate to the electric displacement. Given these concerns, it is unclear to us if the derivation in Landau and Lifshitz (1984) should lead to any measurable difference from the Taylor expansion.

For a polar material, one can expand the energy function to a quadratic form:

$$W(\mathbf{L}, \tilde{\mathbf{D}}) = \frac{1}{2} C_{ABKM} L_{AB} L_{KM} + \frac{1}{2} \gamma_{AB} \tilde{D}_A \tilde{D}_B + g_{AKM} \tilde{D}_A L_{KM} + R_{ABKM} \tilde{D}_A \tilde{D}_B L_{KM}. \quad (7.4)$$

Since W need not be an even function of the electric displacement, the leading term that couples the electric displacement and the strain is linear in the electric displacement. The coupling term quadratic in the electric displacement can often be neglected. When the infinitesimal strain is used, the above form is commonly used to describe linearly piezoelectric material.

Mechanically compliant but electrically stiff material. The large deformation of elastomers and the large electric displacement of water are both due to entropy. Common elastomers seem to exhibit large deformation but modest electric displacement. For such materials, we will only retain the Taylor series of $W(\mathbf{L}, \tilde{\mathbf{D}})$ up to quadratic terms in $\tilde{\mathbf{D}}$. We assume that the material is mechanically compliant, and allow arbitrary value of stretch. Thus, the free-energy function takes the form

$$W(\mathbf{L}, \tilde{\mathbf{D}}) = W_0(\mathbf{L}) + b_A(\mathbf{L}) \tilde{D}_A + \frac{1}{2} \beta_{AB}(\mathbf{L}) \tilde{D}_A \tilde{D}_B, \quad (7.5)$$

where $W_0(\mathbf{L})$, $b_A(\mathbf{L})$ and $\beta_{AB}(\mathbf{L})$ are functions of strain tensor, to be fitted to experimental data.

If the material is nonpolar, $b_A(\mathbf{L}) = 0$. Furthermore, if the material is isotropic, $W_0(\mathbf{L})$ is a function of the three invariants of the strain tensor, as reviewed by, e.g., Holzapfel (2000). For an isotropic material, the electrical stiffness $\beta_{AB}(\mathbf{L})$ takes the form

$$\beta_{AB}(\mathbf{L}) = \beta_0 \delta_{AB} + \beta_1 L_{AB} + \beta_2 L_{KA} L_{KB}, \quad (7.6)$$

where β_0 , β_1 and β_2 are each a function of the three invariants of the strain tensor. For a layer of dielectric between two electrodes, empirically, it is known that, subject to a voltage and under a constant force, some materials contract, while others elongate. The former is like vacuum; and the latter, unlike. This difference can be represented by a suitable choice of the function $\beta_{AB}(\mathbf{L})$.

One can also formulate a theory for electrically compliant and mechanically stiff materials by using a free energy function

$$W(\mathbf{L}, \tilde{\mathbf{D}}) = W_1(\tilde{\mathbf{D}}) + d_{AB}(\tilde{\mathbf{D}}) L_{AB} + \frac{1}{2} C_{ABKM}(\tilde{\mathbf{D}}) L_{AB} L_{KM}. \quad (7.7)$$

Such materials are treated extensively in the literature on ferroelectric materials (e.g., Lines and Glass, 1977), and will not be discussed here.

8. Infinitesimal fields superimposed upon a given field

Let us now summarize the basic equations. On each material element of volume, we prescribe mass $\tilde{\rho}(\mathbf{X})dV$, electric charge $\tilde{q}(\mathbf{X}, t)dV$ and mechanical force $\tilde{\mathbf{b}}(\mathbf{X}, t)dV$. On each material element of interface, we prescribe electric charge $\tilde{\omega}(\mathbf{X}, t)dA$ and mechanical force $\tilde{\mathbf{t}}(\mathbf{X}, t)dA$. We describe an evolving system by the motion $\mathbf{x}(\mathbf{X}, t)$ and the potential $\Phi(\mathbf{X}, t)$. The corresponding deformation gradient and electric field are

$$F_{iK} = \frac{\partial x_i(\mathbf{X}, t)}{\partial X_K}, \quad \tilde{E}_K = -\frac{\partial \Phi(\mathbf{X}, t)}{\partial X_K}. \quad (8.1)$$

The material is conservative, characterized by the electrical Gibbs free energy $\hat{W}(\mathbf{F}, \tilde{\mathbf{E}})$, so that the stress and the electric displacement relate to the deformation gradient and electric field as

$$s_{iK} = \frac{\partial \hat{W}(\mathbf{F}, \tilde{\mathbf{E}})}{\partial F_{iK}}, \quad \tilde{D}_K = -\frac{\partial \hat{W}(\mathbf{F}, \tilde{\mathbf{E}})}{\partial \tilde{E}_K}. \quad (8.2)$$

The stress and electric displacement satisfy

$$\frac{\partial s_{iK}(\mathbf{X}, t)}{\partial X_K} + \tilde{b}_i(\mathbf{X}, t) = \tilde{\rho}(\mathbf{X}) \frac{\partial^2 x_i(\mathbf{X}, t)}{\partial t^2}, \quad \frac{\partial \tilde{D}_K(\mathbf{X}, t)}{\partial X_K} = \tilde{q}(\mathbf{X}, t) \quad (8.3)$$

in the volume of the body, and

$$(s_{iK}^- - s_{iK}^+) N_K = \tilde{t}_i, \quad (\tilde{D}_K^+ - \tilde{D}_K^-) N_K = \tilde{q}(\mathbf{X}, t) \quad (8.4)$$

on the surface of the body.

The structure of the theory immediately lends itself to theoretical studies in parallel with similar ones in finite elasticity; see, e.g., Truesdell and Noll (2003). As an example, we sketch a study of infinitesimal fields superimposed upon a given field.

Except for mass density, perturb every field by a small amount; for example, perturb the motion by $\delta \mathbf{x}(\mathbf{X}, t)$ and the potential by $\delta \Phi(\mathbf{X}, t)$. Associated with the perturbation, the deformation gradient and the nominal electric field change by

$$\delta F_{iK}(\mathbf{X}, t) = \frac{\partial \delta x_i(\mathbf{X}, t)}{\partial X_K}, \quad \delta \tilde{E}_K(\mathbf{X}, t) = -\frac{\partial \delta \Phi(\mathbf{X}, t)}{\partial X_K}. \quad (8.5)$$

The stress and the electric displacement change by

$$\delta s_{iK} = K_{iKjL} \delta F_{jL} - e_{iKL} \delta \tilde{E}_L, \quad (8.6)$$

$$\delta \tilde{D}_L = e_{iKL} \delta F_{iK} + \varepsilon_{LK} \delta \tilde{E}_K. \quad (8.7)$$

Various tangent moduli are calculated from the second derivatives of the Gibbs free energy:

$$K_{iKjL}(\mathbf{F}, \tilde{\mathbf{E}}) = \frac{\partial^2 \hat{W}(\mathbf{F}, \tilde{\mathbf{E}})}{\partial F_{jL} \partial F_{iK}}, \quad \varepsilon_{KL}(\mathbf{F}, \tilde{\mathbf{E}}) = -\frac{\partial^2 \hat{W}(\mathbf{F}, \tilde{\mathbf{E}})}{\partial \tilde{E}_K \partial \tilde{E}_L}, \quad e_{iKL}(\mathbf{F}, \tilde{\mathbf{E}}) = -\frac{\partial^2 \hat{W}(\mathbf{F}, \tilde{\mathbf{E}})}{\partial F_{iK} \partial \tilde{E}_L}. \quad (8.8)$$

As indicated, the derivatives are calculated at the given field \mathbf{F} and $\tilde{\mathbf{E}}$. So long as the perturbation around a given field is concerned, all dielectrics, including vacuum, act like a linear piezoelectric.

The perturbation satisfies the field equations:

$$\frac{\partial \delta s_{iK}}{\partial X_K} + \delta \tilde{b}_i = \tilde{\rho}(\mathbf{X}) \frac{\partial^2 \delta x_i}{\partial t^2}, \quad \frac{\partial \delta \tilde{D}_K}{\partial X_K} = \delta \tilde{q} \quad (8.9)$$

in volume, and

$$(\delta s_{iK}^- - \delta s_{iK}^+) N_K = \delta \tilde{t}_i, \quad (\delta \tilde{D}_K^+ - \delta \tilde{D}_K^-) N_K = \delta \tilde{q}(\mathbf{X}, t) \quad (8.10)$$

on interface. Consequently, the perturbation satisfies the statements:

$$\int \delta s_{iK} \delta F_{iK} dV = \int \left(\delta \tilde{b}_i - \tilde{\rho} \frac{\partial^2 \delta x_i}{\partial t^2} \right) \delta x_i dV + \int \delta \tilde{t}_i \delta x_i dA \quad (8.11)$$

$$\int \delta \tilde{E}_K \delta \tilde{D}_K dV = \int \delta \Psi \delta \tilde{q} dV + \int \delta \Psi \delta \tilde{q} dA \quad (8.12)$$

Adding the two equations, and inserting the material laws, we obtain that

$$\int (K_{iKjL} \delta F_{jL} \delta F_{iK} + \varepsilon_{LK} \delta \tilde{E}_L \delta \tilde{E}_K) dV = \int \left(\delta \tilde{b}_i \delta x_i - \tilde{\rho} \frac{\partial^2 \delta x_i}{\partial t^2} \delta x_i + \delta \Psi \delta \tilde{q} \right) dV + \int (\delta \tilde{t}_i \delta x_i + \delta \Psi \delta \tilde{q}) dA.$$

(8.13)

This relation was derived for linear piezoelectric undergoing infinitesimal deformation (Suo et al., 1992). This relation may be interpreted in several ways, depending on whether the tensor of elasticity and the tensor of permittivity are positive-definite.

First we assume that both tensors are positive-definite. Further assume that the external loads do not vary, i.e., $\delta\tilde{b}_i = 0$, $\delta\tilde{q} = 0$, $\delta\tilde{r}_i = 0$, $\delta\tilde{w} = 0$. The following conclusions are obvious.

(i) If the perturbation is static, the perturbation in the deformation gradient and nominal electric field must vanish.

(ii) If the perturbed is dynamic, it vibrates around the given field. Let

$$\delta x_i(\mathbf{X}, t) = f_i(\mathbf{X}) \sin \omega t, \quad \delta \Phi(\mathbf{X}, t) = f_4(\mathbf{X}) \sin \omega t. \quad (8.14)$$

The positive-definite tangent moduli ensure that the natural frequency is real.

(iii) If the given field is static and homogenous, then all the tangent moduli are constant. The equations of motion for perturbation become

$$K_{iKjL} \frac{\partial^2 \delta x_j}{\partial X_K \partial X_L} - e_{iKL} \frac{\partial^2 \delta \Phi}{\partial X_K \partial X_L} = \rho \frac{\partial^2 \delta x_i}{\partial t^2}. \quad (8.15)$$

$$e_{iKL} \frac{\partial^2 \delta x_i}{\partial X_K \partial X_L} + \varepsilon_{LK} \frac{\partial^2 \delta \Phi}{\partial X_K \partial X_L} = 0. \quad (8.16)$$

Consider an infinite medium. Assume a plane wave with unit normal vector \mathbf{N} and speed c . The disturbance takes the form

$$\delta x_i(\mathbf{X}, t) = \delta a_i f(\mathbf{N} \cdot \mathbf{X} - ct), \quad (8.17)$$

$$\delta \Phi(\mathbf{X}, t) = \delta a_4 f(\mathbf{N} \cdot \mathbf{X} - ct), \quad (8.18)$$

where $\delta \mathbf{a}$ is the direction of the displacement, and f is the profile of the wave.

The equations of motion become

$$K_{iKjL} N_K N_L \delta a_j - e_{iKL} N_L N_K \delta a_4 = \rho c^2 \delta a_i, \quad (8.19)$$

$$e_{iKL} N_L N_K \delta a_i + \varepsilon_{KL} N_L N_K \delta a_4 = \rho c^2 \delta a_4. \quad (8.20)$$

This is an eigenvalue problem. When the tangent modulus is positive-definite, we can find four distinct plane waves.

(iv) Under the same assumption as in (3), the equations of motion are the same as those for infinitesimal field in a linear piezoelectric. One can use the complex-variable method to solve boundary value problems and eigenvalue problems, such as cracks, surface waves and transonic waves; see Suo et al. (1992) and Yu and Suo (2000) for examples.

Now consider the bifurcation from a homogenous field. Let the given field be homogenous. The tensor of tangent modulus depends on the state of homogeneous field, and may no longer be positive-definite. A static, inhomogeneous field of perturbation may set in. Write the perturbation as

$$\delta x_i(\mathbf{X}) = \delta a_i f(\mathbf{N} \cdot \mathbf{X}), \quad (8.21)$$

$$\delta \Phi(\mathbf{X}) = \delta a_4 f(\mathbf{N} \cdot \mathbf{X}). \quad (8.22)$$

Substitute into the equation of motion, and we obtain that

$$K_{iKjL} N_K N_L \delta a_j - e_{iKL} N_L N_K \delta a_4 = 0, \quad (8.23)$$

$$e_{iKL} N_L N_K \delta a_i + \varepsilon_{KL} N_L N_K \delta a_4 = 0. \quad (8.24)$$

This is a set of homogenous algebraic equation for δa_1 , δa_2 , δa_3 and δa_4 . To have a nontrivial

solution, the determinant of the equation must vanish:

$$\det((\varepsilon_{AB} N_A N_B)(K_{ijkl} N_K N_L) + (e_{LmK} N_L N_K)(e_{AmB} N_A N_B) \delta_{ij}) = 0. \quad (8.25)$$

This Hadamard-type condition can be used to search for the direction \mathbf{N} and the state of the given field when inhomogeneous field sets in.

9. Concluding remarks

It appears that basic field equations can be derived by definitions consistent with what we mean by work. The troublesome notion of electrical force is not invoked: for deformable dielectric solids, such force can be neither defined nor measured. Also for deformable dielectric solids, the true electric field and true electric displacement are not work-conjugate. Instead, we use nominal electric field and nominal electric displacement in the body of the paper, and define various other work-conjugates in Appendices. Gauss's law takes the same form in the material description and spatial description. Our theory recovers the Maxwell stress for dielectric fluids, and provides a basis to characterize finite deformation in dielectric solids. As an application of the theory, we study infinitesimal fields superimposed on a given field, including phenomena such as vibration, wave propagation, and bifurcation. In the beginning of Section 8, we have summarized the basic equations. Alternatively, a combination of (4.1), (4.5) and material laws (5.5) or (5.14) provide a basis for a finite element for deformable dielectrics.

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Appendix A. Define work conjugates for parallel-plate capacitor normalized by the current thickness and area

In the body of the paper, we have elected to use the thickness and area in the reference state to normalize various quantities. Now we list definitions using the thickness and area in the current state. Deform the dielectric from the current thickness l by a small amount to $l + \delta l$. Define the increment in the natural strain $\delta\varepsilon$, as the increment in the thickness divided by the current thickness, namely,

$$\delta\varepsilon = \frac{\delta l}{l}. \quad (\text{A.1})$$

Define the true stress, σ , as the force supplied by the weight in the current state divided by the area in the current state, namely,

$$\sigma = \frac{P}{a}. \quad (\text{A.2})$$

In terms of the natural strain and the true stress, the work done by the weight is

$$P\delta l = al\sigma\delta\varepsilon. \quad (\text{A.3})$$

Note that al is the volume of the dielectric in the current state. Associated with the change in thickness, the work done by the weight divided by the volume of the dielectric in the current state is $\sigma\delta\varepsilon$. That is, the true stress is work-conjugate to the natural strain.

In Section 3, we have shown that the true electric field and the true electric displacement are not work conjugates. If we do not wish to use such a pair of non-conjugates, we may define a natural electric displacement such that when the battery pumps charge δQ from the ground to the electrode, the increment in the natural electric displacement field is

$$\delta\hat{D} = \frac{\delta Q}{a}. \quad (\text{A.4})$$

In terms of the natural electric displacement field and the true electric field, the work done by the battery is

$$\Phi\delta Q = alE\delta\hat{D}. \quad (\text{A.5})$$

Associated with the relocation of charge, the work done by the battery divided by the volume of the dielectric in the current state is $E\delta\hat{D}$. That is, the true electric field is work-conjugate to the natural electric displacement.

It may be convenient for computation to formulate the theory using the complementary work, $-Q\delta\Phi$. We may define the incremental natural electric field by the increment in the voltage in the current state divided by the thickness in the current state:

$$\delta\hat{E} = \frac{\delta\Phi}{l}. \quad (\text{A.6})$$

Because $-Q\delta\Phi = -alD\delta\hat{E}$, the true electric displacement is work-conjugate to the natural electric field.

Appendix B Spatial description

McMeeking and Landis (2005) formulated a theory of deformable dielectrics using spatial description, along with true electric displacement and true electric field. The two true fields are not work-conjugate, but these authors started with the field equations of electrostatics, and derived a correct expression of the rate of work due to electric charges. To show our approach reproduces their results, this Appendix develops our approach using spatial description.

Unlike their approach, however, we will start with statements of work, and show that it is unnecessary to invoke the notions of electric force and Maxwell stress.

Associated with the motion $\mathbf{x}(\mathbf{X}, t)$, a material element of volume $dV(\mathbf{X})$ in the reference state becomes $dv(\mathbf{x}, t)$ in the current state, and a material element of interface $N_K(\mathbf{X})dA(\mathbf{X})$ in the reference state becomes $n_i(\mathbf{x}, t)da(\mathbf{x}, t)$ in the current state, where n_i is the unit vector normal to the interface.

In terms of the spatial coordinates, the velocity and acceleration of material particle $\mathbf{X}(\mathbf{x}, t)$ takes the familiar expressions:

$$\mathbf{v}(\mathbf{x}, t) = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t}, \quad \mathbf{a}(\mathbf{x}, t) = \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} + \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial x_j} v_j(\mathbf{x}, t). \quad (\text{B.1})$$

In the current state, let $\mathbf{b}(\mathbf{x}, t)dv$ be the force due to the weights on a material element of volume, $\mathbf{t}(\mathbf{x}, t)da$ be the force due to the weights on a material element of interface, and $\rho(\mathbf{x}, t)dv$ be the mass of a material element of volume.

Let $\Delta_i(\mathbf{x})$ be the field of virtual displacement. Define the tensor of true stress σ_{ij} as the work conjugate to $\partial \Delta_i / \partial x_j$. That is, we insist that

$$\int \sigma_{ij} \frac{\partial \Delta_i}{\partial x_j} dv = \int (b_i - \rho \mathbf{a}) \Delta_i dv + \int t_i \Delta_i da, \quad (\text{B.2})$$

holds true for any fields of virtual displacement $\Delta_i(\mathbf{x})$. Using the divergence theorem, we find that the true stress satisfies the following equations:

$$\frac{\partial \sigma_{ij}(\mathbf{x}, t)}{\partial x_j} + b_i(\mathbf{x}, t) = \rho \mathbf{a} \quad (\text{B.3})$$

in the volume of the body, and

$$(\sigma_{ij}^- - \sigma_{ij}^+) n_j = t_i \quad (\text{B.4})$$

on the interfaces. Because we define the true stress as the conjugate to the gradient of displacement to calculate the work done by the weights and inertia, we do not involve electrical force in our theory.

Electrostatics using true electric field and natural electric displacement. The potential as a function of the spectral coordinates, $\phi(\mathbf{x}, t)$, relates to the potential as a function of material coordinates, $\Phi(\mathbf{X}, t)$, by a change of variable:

$$\phi(\mathbf{x}, t) = \Phi(\mathbf{X}, t). \quad (\text{B.5})$$

Define the true electric field by

$$\mathbf{E}(\mathbf{x}, t) = -\frac{\partial \phi(\mathbf{x}, t)}{\partial \mathbf{x}}. \quad (\text{B.6})$$

Let $\delta \hat{q}(\mathbf{x}, t)dv$ be the change in charge on a material element of volume, and $\delta \hat{\omega}(\mathbf{x}, t)da$ be the change in charge on a material element of interface. Let $\zeta(\mathbf{x})$ be a test function, which we may call the virtual voltage. Define the vector of natural electric displacement $\delta \hat{D}_i$ as the work-conjugate to $-\partial \zeta / \partial x_i$. That is, we insist that

$$\int \left(-\frac{\partial \zeta}{\partial x_i} \delta \hat{D}_i \right) dv = \int \zeta \delta \hat{q} dv + \int \zeta \delta \hat{\omega} da, \quad (\text{B.7})$$

holds true for any test function $\zeta(\mathbf{x})$. Using the divergence theorem, we find that natural electric displacement field satisfies the following equations:

$$\frac{\partial \delta \hat{D}_i(\mathbf{x}, t)}{\partial x_i} = \delta \hat{q}(\mathbf{x}, t) \quad (\text{B.8})$$

in the volume of the body, and

$$(\delta \hat{D}_i^+ - \delta \hat{D}_i^-) n_i = \delta \hat{w}(\mathbf{x}, t) \quad (\text{B.9})$$

on the interfaces. Because the place \mathbf{x} will be occupied by different material particles at different times, we cannot drop the sign of increment in the above equations involving natural electric displacement.

Electrostatics using natural electric field and true electric displacement. In computation, it might be convenient to use the increment of the natural electric field of a material particle. Let $\Phi(\mathbf{X}, t)$ be the electric potential of material particle \mathbf{X} at time t , and let $\delta \Phi(\mathbf{X}, t)$ be an increment. By a change of variable, we write that

$$\delta \phi(\mathbf{x}, t) = \delta \Phi(\mathbf{X}, t) \quad (\text{B.10})$$

Define the vector of the increment of natural electric field by

$$\delta \hat{E}_i(\mathbf{x}, t) = -\frac{\partial \delta \phi(\mathbf{x}, t)}{\partial x_i}. \quad (\text{B.11})$$

Let $\zeta(\mathbf{x})$ be a test function. Define the vector of true electric displacement $D_i(\mathbf{x}, t)$ as the work conjugate to the natural electric field. That is, we insist that the following hold true for any variation in the electric potential

$$\int D_i \frac{\partial \zeta}{\partial x_i} dv = -\int q \zeta dv - \int \omega \zeta da, \quad (\text{B.12})$$

where $q(\mathbf{x}, t)dv(\mathbf{x}, t)$ is the charge on a material element of volume in the current state, and $\omega(\mathbf{x}, t)da(\mathbf{x}, t)$ is the charge on a material element of an interface in the current state. This definition leads to the usual expression of Gauss's law in spatial description:

$$\frac{\partial D_i(\mathbf{x}, t)}{\partial x_i} = q(\mathbf{x}, t) \quad (\text{B.13})$$

in the volume of the body, and

$$(D_i^+ - D_i^-) n_i = \omega(\mathbf{x}, t). \quad (\text{B.14})$$

on the interfaces.

We can also state a relation similar to (B.12) using the material description:

$$\int \tilde{D}_\kappa \frac{\partial \zeta}{\partial X_\kappa} dV = -\int \tilde{q} \zeta dV - \int \tilde{\omega} \zeta dA. \quad (\text{B.15})$$

This statement once again leads to (4.7) and (4.8)

Appendix C Relating fields in material and spatial descriptions

The fields in material and spatial descriptions relate in usual ways (e.g., Truesdell and Noll, 2003; Kuang, 2002, Holzapfel, 2000; Huang, 2003). Here we list some of these relations for convenience. A few of the relations concerning electrical quantities are possibly new, but are derived using similar methods.

Some identities of kinematics. Associated with motion $\mathbf{x}(\mathbf{X}, t)$, the volume of a material element in the current state relates to that in the reference state by

$$dv = \det(\mathbf{F})dV, \quad (\text{C.1})$$

and the area of a material element in the current state relates to that in the reference state by

$$F_{iK}n_i da = \det(\mathbf{F})N_K dA, \quad (\text{C.2a})$$

and

$$da = \det(\mathbf{F})\sqrt{H_{iK}N_K H_{iL}N_L} dA, \quad (\text{C.2b})$$

where H_{iK} is the inverse of the deformation gradient, i.e., $H_{iK}F_{iL} = \delta_{KL}$ and $H_{iK}F_{jK} = \delta_{ij}$.

Note that

$$\frac{\partial H_{iK}}{\partial F_{mN}} = -H_{iN}H_{mK}, \quad (\text{C.3})$$

$$\frac{\partial \det(\mathbf{F})}{\partial F_{mN}} = \det(\mathbf{F})H_{mN}. \quad (\text{C.4})$$

In the time between t and $t + \delta t$, the material particle \mathbf{X} moves by small, actual displacement

$$\delta \mathbf{x}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t + \delta t) - \mathbf{x}(\mathbf{X}, t). \quad (\text{C.5})$$

By a change of variable, we define the actual displacement by

$$\delta \mathbf{u}(\mathbf{x}, t) = \mathbf{x}(\mathbf{X}, t + \delta t) - \mathbf{x}(\mathbf{X}, t). \quad (\text{C.6})$$

Associated with this small actual displacement, the material element of volume and the material element of interface changes by

$$\delta(dv) = \frac{\partial \delta u_k}{\partial x_k} dv, \quad \delta(da) = (\delta_{ij} - n_i n_j) \frac{\partial \delta u_i}{\partial x_j} da. \quad (\text{C.7})$$

Mechanical fields. By definition, the true and nominal mass density relate to each other as

$$\rho(\mathbf{x}, t)dv(\mathbf{x}, t) = \tilde{\rho}(\mathbf{X})dV(\mathbf{X}), \quad (\text{C.8})$$

so that

$$\rho(\mathbf{x}, t)\det(\mathbf{F}) = \tilde{\rho}(\mathbf{X}). \quad (\text{C.9})$$

Note that

$$\frac{\partial \Delta_i(\mathbf{X})}{\partial X_K} = \frac{\partial \Delta_i(\mathbf{x})}{\partial x_j} \frac{\partial x_j(\mathbf{X}, t)}{\partial X_K} = \frac{\partial \Delta_i(\mathbf{x})}{\partial x_j} F_{jK}(\mathbf{X}, t). \quad (\text{C.10})$$

By definition, the true stress relates to the nominal stress as

$$\sigma_{ij}(\mathbf{x}, t) \frac{\partial \Delta(\mathbf{x})_i}{\partial x_j} dv(\mathbf{x}, t) = s_{iK}(\mathbf{X}, t) \frac{\partial \Delta_i(\mathbf{X})}{\partial X_K} dV(\mathbf{X}), \quad (\text{C.11})$$

which reduces to

$$\sigma_{ij}(\mathbf{x}, t) = \frac{F_{jK}}{\det(\mathbf{F})} s_{iK}(\mathbf{X}, t). \quad (\text{C.12})$$

Electrical fields. Note that

$$\frac{\partial \zeta(\mathbf{X}, t)}{\partial X_K} = \frac{\partial \zeta(\mathbf{x}, t)}{\partial x_i} \frac{\partial x_i(\mathbf{X}, t)}{\partial X_K} = \frac{\partial \zeta(\mathbf{x}, t)}{\partial x_i} F_{iK}(\mathbf{X}, t), \quad (\text{C.13})$$

so that the nominal electric field relates to the true electric field as

$$\tilde{E}_K(\mathbf{X}, t) = F_{iK}(\mathbf{X}, t)E_i(\mathbf{x}, t), \quad (\text{C.14})$$

and the natural electric field relates to the nominal electric field as

$$\delta\tilde{E}_K(\mathbf{X}, t) = F_{iK}(\mathbf{X}, t)\delta\hat{E}_i(\mathbf{x}, t). \quad (\text{C.15})$$

By definition, the natural electric displacement field relates to the nominal electric field as

$$\frac{\partial\zeta}{\partial x_i}\delta\hat{D}_i dv = \frac{\partial\zeta}{\partial X_K}\delta\tilde{D}_K dV, \quad (\text{C.16})$$

which reduces to

$$\delta\hat{D}_i(\mathbf{x}, t) = \frac{F_{iK}}{\det(\mathbf{F})}\delta\tilde{D}_K(\mathbf{X}, t). \quad (\text{C.17})$$

By definition, the true electric displacement field relates to the nominal electric field as

$$D_i \frac{\partial\zeta}{\partial x_i} dv = \tilde{D}_K \frac{\partial\zeta}{\partial X_K} dV, \quad (\text{C.18})$$

which reduces to

$$D_i(\mathbf{x}, t) = \frac{F_{iK}}{\det(\mathbf{F})}\tilde{D}_K(\mathbf{X}, t). \quad (\text{C.19})$$

Using this relation and Gauss's law in the spatial description, one can derive Gauss's law in the material description (e.g., Kuang, 2002).

By definition,

$$\delta\hat{q}dv = \delta(qdv) = \delta\tilde{q}dV, \quad \delta\hat{\omega}da = \delta(\omega da) = \delta\tilde{\omega}dA, \quad (\text{C.20})$$

so that

$$\delta\hat{q} = \delta q + \frac{\partial\delta u_k}{\partial x_k}q, \quad \delta\hat{\omega} = \delta\omega + (\delta_{ij} - n_i n_j)\frac{\partial\delta u_i}{\partial x_j}\omega, \quad (\text{C.21})$$

and

$$q = \frac{\tilde{q}}{\det(\mathbf{F})}, \quad \omega = \frac{\tilde{\omega}}{\det(\mathbf{F})\sqrt{H_{iK}H_{iL}N_KN_L}}. \quad (\text{C.22})$$

Using Gauss's law and the divergence theorem, McMeeking and Landis (2005) show that the material rate of electrical work is

$$\int -\frac{\partial\phi}{\partial x_i}\left(\frac{\delta D_i}{\delta t} + \frac{\partial v_k}{\partial x_k}D_i - \frac{\partial v_i}{\partial x_j}D_j\right)dv. \quad (\text{C.23})$$

This expression once again shows that the true electric field and the true electric displacement are not work conjugates. Using the relations among various measures of electric displacement, however, one can confirm that (C.23) is equivalent to (B.7) and (4.5), each defining a set of work-conjugate measures.

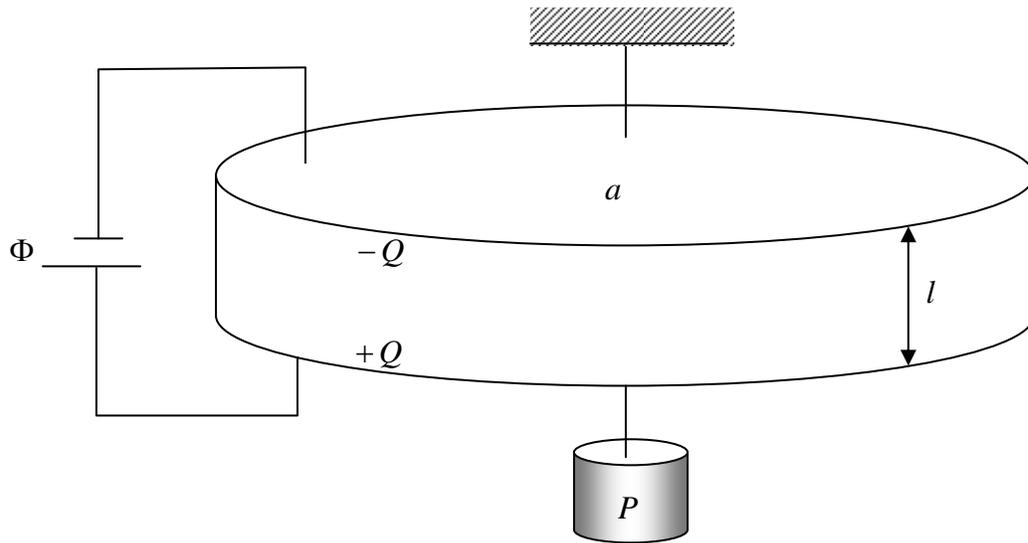


Figure 1 A parallel capacitor gives rise to a homogeneous field inside the dielectric.

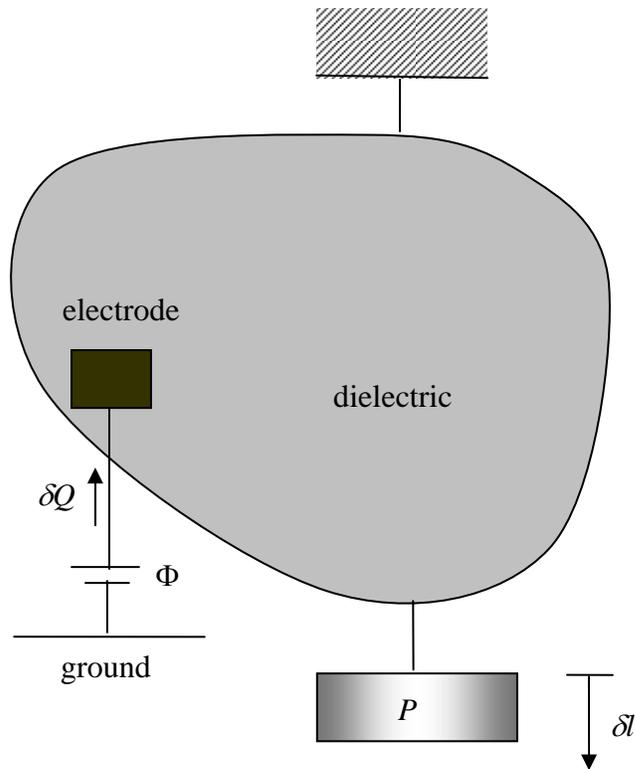


Figure 2 A dielectric is loaded by two mechanisms: a weight and a battery. When the weight drops by a displacement δl , the weight does work $P\delta l$. The dielectric is perfectly insulating, but battery can pump electric charges from the ground to the electrode. When the electrode gains charge δQ , the battery does work $\Phi\delta Q$.