6. The basic equations and fundamental problem types of the linear theory of elasticity

Introduction. Here we assemble the eqs. governing linear elastodynamics and elastostatics, formulate the corresponding fundamental problems and deduce certain general results pertaining to these probs

Notation. In what follows R is a regular region (not necessarily bounded or simply connected) with closure $\bar{R}$ and boundary ∂R; $\underset{\sim}{n}$ is the unit outer normal vector of ∂R (at regular points of ∂R). $\mathcal{T}=[t_0,t_1]$ or $\mathcal{T}=[t_0,\infty)$, $\mathring{\mathcal{T}}=(t_0,t_1)$ or $\mathring{\mathcal{T}}=(t_0,\infty)$.

$\rho, \underset{\sim}{C}, \underset{\sim}{\varkappa}, \lambda, \mu, \eta, \nu, \kappa$ have their usual meaning and are taken to be defined on $\bar{R}$; they are constants for a homogeneous medium.

$\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{S}, \underset{\sim}{f}$ have their usual meaning and are taken to be defined on $\bar{R}\times\mathcal{T}$ (elastodynamics) or on $\bar{R}$ (elastostatics).

The fundamental system of field equations

The kinematic, kinetic, and constitutive ingredient of the linear theory, established in the previous chapters, furnish the following field equations, which must hold on $R \times \mathring{J}$ (elastodynamics) or on R (elastostatics).

Strain-displacement relations

$$\boldsymbol{\gamma} = \operatorname{sym} \nabla \mathbf{u} \quad \text{or} \quad \gamma_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad (6.1)$$

Stress-strain relations

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\gamma} \ \text{or} \ \sigma_{ij} = C_{ijkl}\gamma_{kl} \ , \quad \boldsymbol{\gamma} = \boldsymbol{\varkappa}\boldsymbol{\sigma} \ \text{or} \ \gamma_{ij} = \varkappa_{ijkl}\sigma_{kl}, \ (6.2$$

where $\mathbf{C}$ is the elasticity & $\boldsymbol{\varkappa}$ the elastic compliance tensor field.
In particular, in case of isotropy,

$$\boldsymbol{\sigma} = \lambda \mathbf{1}\operatorname{tr}\boldsymbol{\gamma} + 2\mu\boldsymbol{\gamma} \ \text{or} \ \sigma_{ij} = \lambda\delta_{ij}\gamma_{kk} + 2\mu\gamma_{ij} = 2\mu\left(\frac{\nu}{1-2\nu}\delta_{ij}\gamma_{kk} + \gamma_{ij}\right)$$

or

$$\boldsymbol{\gamma} = \frac{1}{\eta}[(1+\nu)\boldsymbol{\sigma} - \nu\mathbf{1}\operatorname{tr}\boldsymbol{\sigma}] \ \text{or} \ \gamma_{ij} = \frac{1}{\eta}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}] \qquad (6$$

Stress equations of motion or equilibrium

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = \rho\ddot{\mathbf{u}} \quad \text{or} \quad \sigma_{ij,j} + f_i = \rho\ddot{u}_i \qquad (6.3)$$

In particular, for the equilibrium case

$$\nabla \cdot \underset{\sim}{\sigma} + \underset{\sim}{f} = \underset{\sim}{0} \quad \text{or} \quad \sigma_{ij,j} + f_i = 0 \qquad (6.3')$$

Remarks. We omitted $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$ because it is implied by (6.2). Explain omission of compatibility equations $\nabla \wedge \nabla \wedge \underset{\sim}{\gamma} = \underset{\sim}{0}$ (implied by (6.1) if $\underset{\sim}{u}$ smooth enoug

(A) The fundamental boundary – initial-value problems of elastodynamics

Given: $R, \mathcal{T}$; ρ and $\underset{\sim}{c}$ on $\bar{R}$, $\underset{\sim}{f}$ on $\bar{R}\times\mathcal{T}$ (R bounded)

Find: $\underset{\sim}{u}, \underset{\sim}{\gamma}, \underset{\sim}{\sigma}$ suitably smooth on $\bar{R}\times\mathcal{T} \ni$ (6.1), (6.2), (6.3) hold on $R\times\mathring{\mathcal{T}}$, subject to the following initial and boundary conditions.

Initial conditions

$$\underset{\sim}{u}(\underset{\sim}{x}, t_0) = \underset{\sim}{\mathring{u}}(\underset{\sim}{x}), \quad \dot{\underset{\sim}{u}}(\underset{\sim}{x}, t_0) = \underset{\sim}{\mathring{v}}(\underset{\sim}{x}) \;\; \forall \; \underset{\sim}{x} \in R \qquad (6.4)$$

where $\underset{\sim}{\mathring{u}}$ and $\underset{\sim}{\mathring{v}}$ are functions pre-assigned on R

Boundary conditions

Problem AI ("displacement pb.")

$$\underset{\sim}{u} = \underset{\sim}{\overset{*}{u}} \quad \text{on } \partial R \times \mathcal{T}, \qquad (6.5)$$

where $\overset{*}{\underset{\sim}{u}}$ is pre-assigned on $\partial R \times \mathcal{T}$.

<u>Problem AII</u> ("traction pb.")

$$\underset{\sim}{s} = \underset{\sim}{\sigma}\, \underset{\sim}{n} = \overset{*}{\underset{\sim}{s}} \quad \text{on } \partial R \times \mathcal{T}, \tag{6.6}$$

where $\overset{*}{\underset{\sim}{s}}$ is pre-assigned on $\partial R \times \mathcal{T}$.

<u>Problem AIII</u> ("mixed pb.")

$$\underset{\sim}{u} = \overset{*}{\underset{\sim}{u}} \text{ on } \partial_1 R \times \mathcal{T}, \quad \underset{\sim}{s} = \overset{*}{\underset{\sim}{s}} \text{ on } \partial_2 R \times \mathcal{T}, \tag{6.7}$$

where $\partial_1 R, \partial_2 R$ are complementary subsets of ∂R, i.e. $\partial_1 R \cup \partial_2 R = \partial R$, $\partial_1 R \cap \partial_2 R = \emptyset$, and $\overset{*}{\underset{\sim}{u}}, \overset{*}{\underset{\sim}{s}}$ are given on $\partial_1 R \times \mathcal{T}$, $\partial_2 R \times \mathcal{T}$, respectively.

<u>Remarks</u>. Probs. AI, AII are special cases of AIII.
Mention terminology used by Muskhelishvili.
Mention "mixed-mixed" boundary conditions.
Mention time-dependent $\partial_1 R, \partial_2 R$ (eg moving punch

(B) <u>The fundamental boundary-value problems of elastostatics</u>

<u>Given</u>: Given R; $\underset{\sim}{c}$ and $\underset{\sim}{f}$ on $\bar{R}$. (R bounded)

Elastodynamic and elastostatic states

Let $\mathcal{R}$ be an arbitrary region in E (open, closed, or neither), $\mathring{\mathcal{R}}$ the interior of $\mathcal{R}$, $\mathcal{T} = [t_o, t_1]$ a time-interval, and $\mathring{\mathcal{T}} = (t_o, t_1)$.

Suppose $\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}]$ is an ordered array of a vector field $\underset{\sim}{u}$ and second-order tensor fields $\underset{\sim}{\chi}$, $\underset{\sim}{\sigma}$, all defined on $\mathcal{R} \times \mathcal{T}$. Suppose ρ and $\underset{\sim}{c}$ denote a scalar field and a fourth-order tensor field on $\mathcal{R}$, while $\underset{\sim}{f}$ is a vector field on $\mathcal{R} \times \mathcal{T}$.

We write

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}] \in \mathcal{E}(\rho, \underset{\sim}{c}, \underset{\sim}{f}; \mathcal{R} \times \mathcal{T}) \quad (*)$$

and say that $\mathcal{S}$ is an elastodynamic state on $\mathcal{R} \times \mathcal{T}$, with the displacement field $\underset{\sim}{u}$, the strain field $\underset{\sim}{\chi}$, and the stress field $\underset{\sim}{\sigma}$ — corresponding to the mass density ρ, the elasticity field $\underset{\sim}{c}$, and the body-force density $\underset{\sim}{f}$, provided:

(a) $\underset{\sim}{u} \in C^2(\mathring{\mathcal{R}} \times \mathring{\mathcal{T}}) \cap C^1(\mathcal{R} \times \mathcal{T})$, $\ddot{\underset{\sim}{u}} \in C(\mathcal{R} \times \mathcal{T})$,

$\rho \in C(\mathcal{R})$, $\rho > 0$ on $\mathcal{R}$, $\underset{\sim}{f} \in C(\mathcal{R} \times \mathcal{T})$,

$\underset{\sim}{c} \in C^1(\mathcal{R})$, $\underset{\sim}{c}$ is invertible on $\mathcal{R}$, and $\underset{\sim}{c}$ obeys the symmetry relations

$$c_{ijk\ell} = c_{jik\ell} = c_{k\ell ij} \quad \text{on} \quad \mathcal{R} \ ;$$

(b) The fields $\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}, \rho, \underset{\sim}{c}$, and $\underset{\sim}{f}$ satisfy the fundamental field equations of linear elastodynamics on $\mathring{\mathcal{R}} \times \mathring{\mathcal{T}}$.

If, in particular, $\underset{\sim}{c}$ is isotropic on $\mathcal{R}$ and corresponds to the shear modulus μ and Poisson's ratio ν, we write

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}] \in \mathcal{E}(\rho, \mu, \nu, \underset{\sim}{f}; \mathcal{R} \times \mathcal{T}). \quad (**)$$

Further, if (*) holds with $\underset{\sim}{f} = \underset{\sim}{0}$ on $\mathcal{R} \times \mathcal{T}$, i.e. in the absence of body forces, we write

$$\mathscr{S} = [\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}] \in \mathcal{E}(\rho, \underset{\sim}{c}; \mathcal{R} \times \mathcal{J}) \quad , (\S)$$

while

$$\mathscr{S} = [\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}] \in \mathcal{E}(\rho, \mu, \nu; \mathcal{R} \times \mathcal{J}) \quad (\S\S)$$

signifies that (**) holds in the same circumstances.

When (*) holds, but $\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}$, and $\underset{\sim}{f}$ are <u>time-independent</u>, we write

$$\mathscr{S} = [\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}] \in \mathcal{E}(\underset{\sim}{c}, \underset{\sim}{f}; \mathcal{R}) \quad (*)'$$

and call $\mathscr{S}$ an <u>elastostatic state</u> on $\mathcal{R}$, corresponding to the data $\underset{\sim}{c}$ and $\underset{\sim}{f}$. Similarly, if (**), (§), or (§§) hold for time-independent $\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}$, and $\underset{\sim}{f}$, we write

$$\mathscr{S} = [\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu, \underset{\sim}{f}; \mathcal{R}) \quad , \quad (**)'$$

$$\mathscr{S} = [\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}] \in \mathcal{E}(\underset{\sim}{c}; \mathcal{R}) \quad , \quad (\S)'$$

$$\mathscr{S} = [\underset{\sim}{u}, \underset{\sim}{\chi}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu; \mathcal{R}) \quad , (\S\S)'$$

respectively.

<u>Remarks</u>

(i) Note that $\mathcal{E}(\rho, \underset{\sim}{c}, \underset{\sim}{f}; \mathcal{R} \times \mathcal{J})$ is the class of all suitably regular solutions on $\mathcal{R} \times \mathcal{J}$ of the fundamental elastodynamic field equations, corresponding to sufficiently regular data.

(ii) Observe that $\mathcal{R} = \mathring{\mathcal{R}}$ if $\mathcal{R}$ is an open region (domain), while $\mathcal{R} = \overline{\mathcal{R}}$, if $\mathcal{R}$ is a closed region.

(iii) The hypotheses (a) and (b) imply

$$\underset{\sim}{\chi} \in C^1(\mathring{\mathcal{R}} \times \mathring{\mathcal{J}}) \cap C(\mathcal{R} \times \mathcal{J}) \quad , \quad \underset{\sim}{\sigma} \in C^1(\mathring{\mathcal{R}} \times \mathring{\mathcal{J}}) \cap C(\mathcal{R} \times \mathcal{J}) \quad .$$

The smoothness of $\underset{\sim}{f}$ assumed in (a) is, in fact, implied by the remaining smoothness assumptions in (a) together with (b).

Find: $\underset{\sim}{u}, \underset{\sim}{\gamma}, \underset{\sim}{\sigma}$ suitably smooth on $\bar{R} \ni$ (6.1), (6.2), (6.3) hold on R, subject to the following boundary conditions:

Problem BI. $$\underset{\sim}{u} = \overset{*}{\underset{\sim}{u}} \text{ on } \partial R \qquad (6.8)$$

Problem BII $$\underset{\sim}{s} \equiv \underset{\sim}{\sigma}\underset{\sim}{n} = \overset{*}{\underset{\sim}{s}} \text{ on } \partial R \qquad (6.9)$$

Problem BIII $$\underset{\sim}{u} = \overset{*}{\underset{\sim}{u}} \text{ on } \partial_1 R, \quad \underset{\sim}{s} = \overset{*}{\underset{\sim}{s}} \text{ on } \partial_2 R \qquad (6.10)$$

Discussion. Raise existence and uniqueness questions. We shall deal with uniqueness, but avoid existence issue (refer to literature). Note that (6.1), (6.2), and (6.3) or (6.3′) are $6+6+3=15$ equations in $3+6+6=15$ unknowns u_i, $\gamma_{ij}=\gamma_{ji}$, $\sigma_{ij}=\sigma_{ji}$.

In preparation for an economical treatment of various basic theorems pertaining to linear elastodynamics & elastostatics we introduce the following notions of an elastodynamic and elastostatic state.

Definition (Elastodynamic state)

Let $\mathcal{R}$ be a region in E (open, closed, or neither) and let

$\mathcal{J} = [t_0, t_1]$ be a time-interval. We say that the ordered array $[\underset{\sim}{u}, \underset{\sim}{\mathcal{E}}, \underset{\sim}{\sigma}]$ is a (regular) <u>elastodynamic state</u> on $\mathcal{R}\times\mathcal{J}$, corresponding to the mass density ρ, the elasticity field $\underset{\sim}{c}$, and the body-force density $\underset{\sim}{f}$, and write

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\mathcal{E}}, \underset{\sim}{\sigma}] \in \mathcal{E}(\rho, \underset{\sim}{c}, \underset{\sim}{f}; \mathcal{R}\times\mathcal{J}), \tag{6.11}$$

if:

(a) $\rho \in \mathcal{C}(\mathcal{R})$, $\rho > 0$ on $\mathcal{R}$, $\underset{\sim}{c} \in \mathcal{C}^1(\mathcal{R})$, $\underset{\sim}{c}$ is invertible and symmetric on $\mathcal{R}$ ($c_{ijkl} = c_{jikl} = c_{klij}$), $\underset{\sim}{f} \in \mathcal{C}(\mathcal{R}\times\mathcal{J})$; $\underset{\sim}{u} \in \mathcal{C}^2(\mathring{\mathcal{R}}\times\mathring{\mathcal{J}}) \cap \mathcal{C}^1(\mathcal{R}\times\mathcal{J})$, $\ddot{\underset{\sim}{u}} \in \mathcal{C}(\mathcal{R}\times\mathcal{J})$, $\underset{\sim}{\sigma} \in \mathcal{C}^1(\mathring{\mathcal{R}}\times\mathring{\mathcal{J}}) \cap \mathcal{C}(\mathcal{R}\times\mathcal{J})$

(b) Equations (6.1), (6.2), (6.3) hold on $\mathring{\mathcal{R}}\times\mathring{\mathcal{J}}$.

If, in particular, (6.2') hold, so that the medium is <u>isotropic</u> (with the shear mod. μ and Poisson ratio ν) we write

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\mathcal{E}}, \underset{\sim}{\sigma}] \in \mathcal{E}(\rho, \mu, \nu, \underset{\sim}{f}; \mathcal{R}\times\mathcal{J}) \tag{6.12}$$

<u>Discussion</u>. Thus $\mathcal{E}(\rho, \underset{\sim}{c}, \underset{\sim}{f}; \mathcal{R}\times\mathcal{J})$ is the class of all suitably smooth solutions of the fundamental field eqs. of linear elastodynamics on $\mathring{\mathcal{R}}\times\mathring{\mathcal{J}}$, corresponding to suitably smooth field data $\rho, \underset{\sim}{c}, \underset{\sim}{f}$. Note that (a), (b) $\Rightarrow$ $\underset{\sim}{\mathcal{E}} \in \mathcal{C}^1(\mathring{\mathcal{R}}\times\mathring{\mathcal{J}}) \cap \mathcal{C}(\mathcal{R}\times\mathcal{J})$
Mention simplification of (a) in case $\mathcal{R}$ is open ($\mathring{\mathcal{R}} = \mathcal{R}$).

<u>Definition</u> (Elastostatic state)

Let $\mathcal{R}$ be a region in E (open, closed, or neither)

We say that the ordered array $[\underset{\sim}{u}, \underset{\sim}{T}, \underset{\sim}{S}]$ is (regular) <u>elastostatic state</u> on $\mathcal{R}$, corresponding to the data $\underset{\sim}{C}, \underset{\sim}{f}$, and write

$$s = [\underset{\sim}{u}, \underset{\sim}{T}, \underset{\sim}{S}] \in \mathcal{E}(\underset{\sim}{C}, \underset{\sim}{f}; \mathcal{R}), \tag{6.13}$$

if

(a) $\underset{\sim}{C} \in \mathcal{C}^1(\mathcal{R})$, $\underset{\sim}{C}$ is invertible & symmetric on $\mathcal{R}$, $\underset{\sim}{f} \in \mathcal{C}(\mathcal{R})$, $\underset{\sim}{u} \in \mathcal{C}^2(\mathring{\mathcal{R}}) \cap \mathcal{C}^1(\mathcal{R})$, $\underset{\sim}{S} \in \mathcal{C}^1(\mathring{\mathcal{R}}) \cap \mathcal{C}(\mathcal{R})$

(b) Equations (6.1), (6.2), (6.3′) hold on $\mathring{\mathcal{R}}$

If, in particular, (6.2′) hold, we write

$$s = [\underset{\sim}{u}, \underset{\sim}{T}, \underset{\sim}{S}] \in \mathcal{E}(\mu, \nu, \underset{\sim}{f}; \mathcal{R}) \tag{6.14}$$

<u>Remarks.</u> Clearly, (6.11) ⇒ (6.13) and (6.12) ⇒ (6.14) if $\underset{\sim}{u}, \underset{\sim}{T}, \underset{\sim}{S}$ are time-independent.

<u>Thm</u> 6.1 (Principle of superposition). Let $\mathcal{R}$ be an arbitrary region, $\mathcal{T} = [t_0, t_1]$, and let α, β be real numbers. Suppose

$$s' = [\underset{\sim}{u}', \underset{\sim}{T}', \underset{\sim}{S}'] \in \mathcal{E}(\rho, \underset{\sim}{C}, \underset{\sim}{f}'; \mathcal{R} \times \mathcal{T})$$
$$s'' = [\underset{\sim}{u}'', \underset{\sim}{T}'', \underset{\sim}{S}''] \in \mathcal{E}(\rho, \underset{\sim}{C}, \underset{\sim}{f}''; \mathcal{R} \times \mathcal{T})$$

Define

$$s = \alpha s' + \beta s'' \equiv [\alpha \underset{\sim}{u}' + \beta \underset{\sim}{u}'', \alpha \underset{\sim}{T}' + \beta \underset{\sim}{T}'', \alpha \underset{\sim}{S}' + \beta \underset{\sim}{S}''] \text{ on } \mathcal{R} \times \mathcal{T}$$

Then,

$$s = [\underline{u}, \underline{E}, \underline{S}] \in \mathcal{E}(\rho, \underline{C}, \alpha\underline{f}' + \beta\underline{f}''; R \times T)$$

Proof. Immediate from definition of elastodynamic state and linearity of fundamental field eqs. of elastodynamics. Explain.

Remark. If, in particular, s' and s'' are time-independent Thm. 6.1 reduces to the principle of superposition for linear elastostatics.

Thm. 6.2 (Uniqueness theorem for Problem AIII, Neuman) Let R be a bounded (open) regular region and let $T = [t_0, t_1]$. Let $\partial_1 R \cup \partial_2 R = \partial R$, $\partial_1 R \cap \partial_2 R = \emptyset$. Suppose:

(a) $s' = [\underline{u}', \underline{E}', \underline{S}'] \in \mathcal{E}(\rho, \underline{C}, \underline{f}; \bar{R} \times T)$, $s'' = [\underline{u}'', \underline{E}'', \underline{S}''] \in \mathcal{E}(\rho, \underline{C}, \underline{f}; \bar{R} \times T)$

(b) $\underline{u}'(\underline{x}, t_0) = \underline{u}''(\underline{x}, t_0)$; $\dot{\underline{u}}'(\underline{x}, t_0) = \dot{\underline{u}}''(\underline{x}, t_0)$ $\forall\ \underline{x} \in R$

(c) $\underline{u}' = \underline{u}''$ on $\partial_1 R \times T$, $\underline{s}' = \underline{S}'\underline{n} = \underline{s}'' = \underline{S}''\underline{n}$ on $\partial_2 R \times T$

(d) $\underline{C}$ is positive semi-definite on R

Then,

$$s' = s'' \text{ on } \bar{R} \times T$$

Remark: The above asserts that Problem AIII for a bounded regular region R admits at most one sol. that is a regular elastodynamic state on $\bar{R}$.

Proof. Define

$$s = [\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{S}] = s' - s'' \text{ on } \bar{R}\times\mathcal{T} \qquad (1)$$

(1), Thm. 6.1, (a), (b), (c) $\Rightarrow$

(α) $s = [\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{S}] \in \mathcal{E}(\rho, \underset{\sim}{C}; \underset{\sim}{0}, \bar{R}\times\mathcal{T})$

(β) $\underset{\sim}{u}(\underset{\sim}{x}, t_0) = \underset{\sim}{0}, \quad \dot{\underset{\sim}{u}}(\underset{\sim}{x}, t_0) = \underset{\sim}{0} \quad \forall\, \underset{\sim}{x} \in R$

(γ) $\underset{\sim}{u} = \underset{\sim}{0}$ on $\partial_1 R\times\mathcal{T}$, $\underset{\sim}{s} \equiv \underset{\sim}{S}\underset{\sim}{n} = \underset{\sim}{0}$ on $\partial_2 R\times\mathcal{T}$

Now apply Thm. 5.3[1] to s (which meets required hyp.).

$$\int_{\partial R} \underset{\sim}{s}\cdot\dot{\underset{\sim}{u}}\, dA + \int_R \underset{\sim}{f}\cdot\dot{\underset{\sim}{u}}\, dV = \dot{K} + \dot{U} \text{ on } \mathcal{T}, \quad \underset{\sim}{f} = 0 \text{ on } R\times\mathcal{T} \quad (2)$$

$$K(t) = \frac{1}{2}\int_R \rho(\underset{\sim}{x})\, \dot{\underset{\sim}{u}}^2(\underset{\sim}{x}, t)\, dV, \quad U(t) = \int_R \Gamma(\underset{\sim}{E}(\underset{\sim}{x}, t), \underset{\sim}{x})\, dV \quad (3)$$

(2), (α), (γ) $\Rightarrow$

$$\dot{K} + \dot{U} = 0 \text{ on } \mathcal{T}, \quad K(t) + U(t) = K(t_0) + U(t_0) \ \forall\, t \in \mathcal{T}$$

But (3), (α), (β) $\Rightarrow$ $K(t_0) = 0$, $U(t_0) = 0$, so that

---

1 Power identity for linear elastic solids.

$$K(t) + U(t) = 0 \ \forall \ t \in \mathcal{T} \tag{4}$$

(3), (d) $\Rightarrow K \geq 0, U \geq 0$ on $\mathcal{T}$, whence

$$K = 0, \ U = 0 \ \text{on} \ \mathcal{T} \tag{5}$$

From the first of (5), the first of (3), and the continuity of $\dot{\underset{\sim}{u}}$ on $\bar{R} \times \mathcal{T}$ one has

$$\dot{\underset{\sim}{u}} = \underset{\sim}{0} \ \text{on} \ \bar{R} \times \mathcal{T} \tag{6}$$

(6), (β) and $\underset{\sim}{u} \in \mathcal{T}(\bar{R} \times \mathcal{T}) \Rightarrow$

$$\underset{\sim}{u}(\underset{\sim}{x}, t) = \underset{\sim}{u}(\underset{\sim}{x}, t_0) = \underset{\sim}{0} \ \ \forall \ (\underset{\sim}{x}, t) \in \bar{R} \times \mathcal{T} \tag{7}$$

(7), (α) $\Rightarrow \underset{\sim}{u} = \underset{\sim}{0}, \ \underset{\sim}{\varepsilon} = \underset{\sim}{0}, \ \underset{\sim}{\sigma} = \underset{\sim}{0}$ on $\bar{R} \times \mathcal{T}$, whence from (1),

$$\mathcal{S}' = \mathcal{S}'' \ \text{on} \ \bar{R} \times \mathcal{T}$$

qed.

Discussion. Emphasize generality of Thm. 6.2 (covers nonhomog., anisotropic media). Mention generalization to mixed-mixed prob. (see next Exercise), extension to unbounded R {Wheeler & Co., ARMA, 31, 1968; Wheeler, Quart. Appl. Math. 28, 1970}. Mention removal of hyp. (d) {Brun, Comptes Rendus, Acad. Sci, Paris, 261, 1965}

Remark on stringency of underlying regularity assumpt.

...: exclusion of discont. surface data and of non-matching initial & body conds. (illustrate).

Thm. 6.3 (Reciprocal thm. of elastostatics, Betti). Let R be a (open) bounded regular region. Suppose,

$$\mathcal{S}'=[\underset{\sim}{u}',\underset{\sim}{\varepsilon}',\underset{\sim}{\sigma}']\in\mathcal{E}(\underset{\sim}{c},\underset{\sim}{f}';\bar{R})\ ,\ \mathcal{S}''=[\underset{\sim}{u}'',\underset{\sim}{\varepsilon}'',\underset{\sim}{\sigma}'']\in\mathcal{E}(\underset{\sim}{c},\underset{\sim}{f}'';\bar{R})$$

Then,

$$\int_{\partial R}\underset{\sim}{s}'\cdot\underset{\sim}{u}''\,dA+\int_{R}\underset{\sim}{f}'\cdot\underset{\sim}{u}''\,dV=\int_{\partial R}\underset{\sim}{s}''\cdot\underset{\sim}{u}'\,dA+\int_{R}\underset{\sim}{f}''\cdot\underset{\sim}{u}'\,dV$$

$$=\int_{R}\underset{\sim}{\sigma}'\cdot\underset{\sim}{\varepsilon}''\,dV=\int_{R}\underset{\sim}{\sigma}''\cdot\underset{\sim}{\varepsilon}'\,dV \qquad (6.15)$$

Proof:

$$\int_{\partial R}\underset{\sim}{s}'\cdot\underset{\sim}{u}''\,dA=\int_{\partial R}s_i'u_i''\,dA=\int_{\partial R}\sigma_{ij}'u_i''n_j\,dA=\int_{\partial R}w_jn_j\,dA\,,\ \ldots \qquad (1)$$

provided

$$w_j=\sigma_{ij}'u_i'' \text{ on } \bar{R} \qquad (2)$$

By (2) and hyp.,

$$w_j\in C^1(R)\cap C(\bar{R}) \qquad (3)$$

$$\nabla\cdot\underset{\sim}{w}=w_{j,j}=\sigma_{ij,j}'u_i''+\sigma_{ij}'u_{(i,j)}''=-f_i'u_i''+\sigma_{ij}'\varepsilon_{ij}'' \text{ on } R$$

Hence, by hyp.,

$$w_{j,j}\in C(\bar{R}) \qquad (4)$$

(1), (3), (4), and Thm. 1.15 (divergence thm.) $\Rightarrow$

$$\int_{\partial R} \underset{\sim}{s}' \cdot \underset{\sim}{u}'' \, dA + \int_R \underset{\sim}{f}' \cdot \underset{\sim}{u}'' \, dV = \int_R \underset{\sim}{\sigma}' \cdot \underset{\sim}{\varepsilon}'' \, dV \qquad (5)$$

By symmetry,

$$\int_{\partial R} \underset{\sim}{s}'' \cdot \underset{\sim}{u}' \, dA + \int_R \underset{\sim}{f}'' \cdot \underset{\sim}{u}' \, dV = \int_R \underset{\sim}{\sigma}'' \cdot \underset{\sim}{\varepsilon}' \, dV \qquad (6)$$

Also, by hyp. and Thm. 5.5,

$$\underset{\sim}{\sigma}' \cdot \underset{\sim}{\varepsilon}'' = \underset{\sim}{\sigma}'' \cdot \underset{\sim}{\varepsilon}' \text{ on } \overline{R} \qquad (7)$$

Finally, (5), (6), (7) $\Rightarrow$ (6.15). qed.

<u>Remarks</u>. Mention extension to unbounded domains. Above thm. plays pivotal role in theory of Green's functions. See application to computation of average deformations in Exercise 24. Mention Graffi's reciprocal thm. for elastodynamics.

<u>Corollary</u>. Let R be a bounded regular region and

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma}] \in \mathcal{E}(\underset{\sim}{c}, \underset{\sim}{f}; \overline{R}), \; \underset{\sim}{s} = \underset{\sim}{\sigma}\underset{\sim}{n} \text{ on } \partial R.$$

Then,

$$\int_{\partial R} \underset{\sim}{s} \cdot \underset{\sim}{u} \, dA + \int_R \underset{\sim}{f} \cdot \underset{\sim}{u} \, dV = \int_R \underset{\sim}{\sigma} \cdot \underset{\sim}{\varepsilon} \, dV = 2\int_R W \, dV = 2U \qquad (6.16)$$

(Energy identity of elastostatics)

Proof. Take $\mathcal{S}'=\mathcal{S}''=\mathcal{S}$ in Thm. 6.3 and use (5.8), (5.4).

Thm. 6.4 (Uniqueness theorem of elastostatics, Kirchhoff).
Let R be a regular region (not necessarily bounded). Suppose:

(a) $\mathcal{S}'=[\underset{\sim}{u}',\underset{\sim}{E}',\underset{\sim}{S}']\in\mathcal{E}(\underset{\sim}{C},\underset{\sim}{f};\bar{R})$, $\mathcal{S}''=[\underset{\sim}{u}'',\underset{\sim}{E}'',\underset{\sim}{S}'']\in\mathcal{E}(\underset{\sim}{C},\underset{\sim}{f};\bar{R})$

(b) $\underset{\sim}{u}'=\overset{*}{\underset{\sim}{u}}$ on $\partial_1 R$, $\underset{\sim}{s}'=\overset{*}{\underset{\sim}{s}}$ on $\partial_2 R$, $\underset{\sim}{u}''=\overset{*}{\underset{\sim}{u}}$ on $\partial_1 R$, $\underset{\sim}{s}''=\overset{*}{\underset{\sim}{s}}$ on $\partial_2 R$

(c) in case R is unbounded,

$$\underset{\sim}{u}'(\underset{\sim}{x})=\overset{\infty}{\underset{\sim}{u}}(\underset{\sim}{x})+O(r^{-1}),\quad \underset{\sim}{S}'(\underset{\sim}{x})=\overset{\infty}{\underset{\sim}{S}}(\underset{\sim}{x})+O(r^{-2}) \text{ as } r=|\underset{\sim}{x}|\to\infty$$

$$\underset{\sim}{u}''(\underset{\sim}{x})=\overset{\infty}{\underset{\sim}{u}}(\underset{\sim}{x})+O(r^{-1}),\quad \underset{\sim}{S}''(\underset{\sim}{x})=\overset{\infty}{\underset{\sim}{S}}(\underset{\sim}{x})+O(r^{-2}) \text{ as } r\to\infty$$

where $\overset{\infty}{\underset{\sim}{u}}$ and $\overset{\infty}{\underset{\sim}{S}}$ are a vector field and a symmetric two-tensor field defined in the intersection of R with a neighborhood of infinity.

(d) $\underset{\sim}{C}$ is positive definite on R

Then,

$$\underset{\sim}{u}'=\underset{\sim}{u}''+\mathring{\underset{\sim}{u}},\quad \underset{\sim}{E}'=\underset{\sim}{E}'',\quad \underset{\sim}{S}'=\underset{\sim}{S}'' \text{ on } \bar{R}, \qquad (*)$$

where $\mathring{\underset{\sim}{u}}$ is an infinitesimally rigid displacement field[1].
Further, $\mathring{\underset{\sim}{u}}=\underset{\sim}{0}$ on $\bar{R}$ if $\partial_1 R$ contains at least three non-collinear points or if R is unbounded.

---

1 Remark on different connotation of $\mathring{\underset{\sim}{u}}$ in elastodynamics (initial displt.)

Proof. Define

$$s = [\mathbf{u}, \mathbf{E}, \mathbf{S}] = s' - s'' \text{ on } \overline{R} \tag{1}$$

(1), Thm. 6.1 $\Rightarrow$

(α) $s \in \mathcal{E}(\mathbf{C}; \overset{\circ}{\mathbf{f}}, \overline{R})$ (β) $\mathbf{u} = \mathbf{0}$ on $\partial_1 R$, $\mathbf{s} = \mathbf{0}$ on $\partial_2 R$

(γ) if R is unbounded,

$$\mathbf{u}(\mathbf{x}) = O(x^{-1}), \quad \mathbf{S}(\mathbf{x}) = O(x^{-2}) \text{ as } x = |\mathbf{x}| \to \infty$$

Assume first R is <u>bounded</u>. Apply (6.16) to s. Thus, by (α), (β), (5.8),

$$\int_{\partial R} \mathbf{s} \cdot \mathbf{u} \, dA + \int_R \mathbf{f} \cdot \mathbf{u} \, dV = \int_R \underbrace{C_{ijkl} \, \varepsilon_{ij} \, \varepsilon_{kl}}_{(\mathbf{C}\mathbf{E}) \cdot \mathbf{E}} dV = 0, \quad \mathbf{f} = \mathbf{0} \text{ on } \overline{R} \tag{2}$$

Hence, by (α) and (d),

$$\mathbf{E} = 0, \quad \mathbf{S} = \mathbf{0} \text{ on } \overline{R} \tag{3}$$

(3), Thm. 2.12 $\Rightarrow$ $\mathbf{u}(\mathbf{x}) = \overset{\circ}{\mathbf{u}}(\mathbf{x}) = \overset{\circ}{\mathbf{w}} \wedge \mathbf{x} + \overset{\circ}{\mathbf{b}}$ $\forall$ $\mathbf{x} \in \overline{R}$, where $\overset{\circ}{\mathbf{w}}$ and $\overset{\circ}{\mathbf{b}}$ are constant vectors. Also, one shows easily that $\overset{\circ}{\mathbf{u}}(\mathbf{x}^{(k)}) = \mathbf{0}$ for three non-collinear points[§] $\mathbf{x}^{(k)}$ (k=1, $\Rightarrow$ $\overset{\circ}{\mathbf{w}} = \overset{\circ}{\mathbf{b}} = \mathbf{0}$ and hence $\mathbf{u}(\mathbf{x}) = 0$ $\forall$ $\mathbf{x} \in \overline{R}$.

Next, suppose R is an <u>unbounded</u> regular region. According to the definition of such a region and by (γ)

§ $[\mathbf{x}^{(1)} - \mathbf{x}^{(2)}] \wedge [\mathbf{x}^{(1)} - \mathbf{x}^{(3)}] \neq \mathbf{0}$

$\exists\ \rho_0 > 0$ and $M > 0 \ni$

$$\left.\begin{array}{l} R_\rho = B_\rho \cap R \quad (\rho_0 < \rho < \infty) \text{ is a bounded regular reg.} \\ |\underset{\sim}{u}(\underset{\sim}{x})| < \dfrac{M}{\rho},\ |\underset{\sim}{\sigma}(\underset{\sim}{x})| < \dfrac{M}{\rho^2}\ \forall\ \underset{\sim}{x} \in S_\rho \cap R \quad (\rho_0 < \rho < \infty) \end{array}\right\} (4)$$

where $B_\rho = \{\underset{\sim}{x} \mid |\underset{\sim}{x}| < \rho\}$ and $S_\rho = \partial B_\rho$.

Apply (6.16) to the restriction of $\mathcal{S}$ to R_ρ and use $(\alpha), (\beta)$.

Thus,

$$\int_{\partial R_\rho} \underset{\sim}{s} \cdot \underset{\sim}{u}\, dA = \int_{R \cap S_\rho} \underset{\sim}{s} \cdot \underset{\sim}{u}\, dA = \int_{R_\rho} c_{ijkl}\, \gamma_{ij}\, \gamma_{kl}\, dV \quad (\rho_0 < \rho < \infty) \qquad (5)$$

(4) $\Rightarrow$

$$\left| \int_{R \cap S_\rho} \underset{\sim}{s} \cdot \underset{\sim}{u}\, dA \right| \le \int_{R \cap S_\rho} |\underset{\sim}{s} \cdot \underset{\sim}{u}|\, dA \le \int_{R \cap S_\rho} |\underset{\sim}{\sigma}\underset{\sim}{n}|\, |\underset{\sim}{u}|\, dA \le \int_{R \cap S_\rho} |\underset{\sim}{\sigma}|\, |\underset{\sim}{u}|\, dA$$

$$\le \frac{M^2}{\rho^3} \int_{S_\rho} dA = \frac{4\pi M^2}{\rho} = o(1) \text{ as } \rho \to \infty$$

Hence letting $\rho \to \infty$ in (5) one arrives at

$$\int_R c_{ijkl}\, \gamma_{ij}\, \gamma_{kl}\, dV = 0$$

$\Rightarrow \underset{\sim}{\gamma} = \underset{\sim}{\sigma} = \underset{\sim}{0}$ on $\overline{R} \Rightarrow \underset{\sim}{u}(\underset{\sim}{x}) = \mathring{\underset{\sim}{w}} \wedge \underset{\sim}{x} + \mathring{\underset{\sim}{c}}\ \forall\ \underset{\sim}{x} \in \overline{R}$. But $(\gamma) \Rightarrow$

$\mathring{\underset{\sim}{w}} = \mathring{\underset{\sim}{c}} = \underset{\sim}{0} \Rightarrow \underset{\sim}{u} = 0$ on $\overline{R}$. qed.

Remarks

(1) Note and explain indeterminacy of $\underset{\sim}{u}$ if R is bounded and $\partial_2 R = \partial R$ (Problem BII)

(2) Hypothesis (d) ($\underset{\sim}{c}$ pos. def. on R) is not removable but can be weakened in certain circumstances; e.g. if R is bounded, $\partial_1 R = \partial R$ (Problem BI), and $\underset{\sim}{c}$ is constant on R and isotropic, one can easily prove uniqueness if merely $\mu \neq 0$, $-\infty < \nu < 1/2$, $1 < \nu < \infty$. Yet $\Gamma(\underset{\sim}{x})$ here is pos. def. only if $\mu > 0$; $-1 < \nu < 1/2$. Mention Ericksen's non-uniqueness proof. Bramble-Payne results for Prob. BII.

(3) Discuss inadequacy of hyp. (c) for unbounded domains. For exterior regular regions and homog. isotropic elastic media one can proof the following satisfactory result.

Thm. 6.5. Let R be an exterior regular region and suppose hypotheses (a), (b), (d) of Thm. 6.4 hold with $\underset{\sim}{c}$ constant and isotropic, $\partial_2 R = \partial R$. Then (*) holds if hyp. (c) is replaced by

either

$$\underset{\sim}{u}'(\underset{\sim}{x}) = \overset{\infty}{\underset{\sim}{u}}(\underset{\sim}{x}) + o(1)\,,\quad \underset{\sim}{u}''(\underset{\sim}{x}) = \overset{\infty}{\underset{\sim}{u}}(\underset{\sim}{x}) + o(1) \quad \text{as } x \to \infty$$

or

$$\underset{\sim}{\sigma}'(\underset{\sim}{x}) = \overset{\infty}{\underset{\sim}{\sigma}}(\underset{\sim}{x}) + o(1)\,,\quad \underset{\sim}{\sigma}''(\underset{\sim}{x}) = \overset{\infty}{\underset{\sim}{\sigma}}(\underset{\sim}{x}) + o(1) \quad \text{as } x \to \infty$$

For a proof of Thm. 6.5 and related results for Pb. BIII in case R is exterior, see Gurtin & Co. A.R.M.A., 8, 1961. A much stronger uniqueness thm. holds true for the half-space (see Turteltaub & Co., A.R.M.A., 24, 1967). But counter-examples show that Thm. 6.5 is false if R is an arbitrary unbounded regular region. A satisfactory uniqueness theorem for such regions appea to be lacking.

<u>Thm.</u> 6.6. Let R be a <u>bounded</u> regular region. Suppose: $\underset{\sim}{\sigma} \in \mathcal{C}^1(R) \cap \mathcal{C}(\bar{R})$, $\underset{\sim}{f} \in \mathcal{C}(\bar{R})$, $\nabla \cdot \underset{\sim}{\sigma} + \underset{\sim}{f} = \underset{\sim}{0}$, $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$ on R, $\underset{\sim}{\sigma}\underset{\sim}{n} = \underset{\sim}{s}$ on ∂

Then,

$$\int_{\partial R} \underset{\sim}{s}\, dA + \int_R \underset{\sim}{f}\, dV = \underset{\sim}{0}\,,\quad \int_{\partial R} \underset{\sim}{x} \wedge \underset{\sim}{s}\, dA + \int_R \underset{\sim}{x} \wedge \underset{\sim}{f}\, dV = \underset{\sim}{0}\,. \quad (\S)$$

Proof. Use the divergence theorem, reversing the argument spelled out in the derivation of the Eulerian eqs. of motion. (3.18) in Thm. 3.3.

Discussion. Thm. 6.6 supplies a necessary condition for the existence of a sufficiently well behaved sol. to Prob. BII′ for a bounded regular region: the data $\underset{\sim}{f}$ and $\overset{*}{\underset{\sim}{s}}$ must conform to the over-all equil. requirement (§). Explain why this is not nec. if R unbounded. Eqs. (§) are needed in Exercise 19.

Field equations implied by the fundamental system

Here we deduce certain implications of the fundamental system of field eqs. in linear elasticity theory for homogeneous media with a view toward alternative formulations of the basic problems.

Thm. 6.7. Let $\mathcal{R}$ be an arbitrary open region and $\mathcal{T} = [t_0, t_1]$. Suppose

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{E}, \underset{\sim}{S}] \in \mathcal{E}(\rho, \underset{\sim}{C}, \underset{\sim}{f}; \mathcal{R} \times \mathcal{T}), \quad \underset{\sim}{C} = \text{const. on } \mathcal{R}. \quad (*)$$

Then the following equations hold on $R \times \mathring{J} = \mathring{R} \times \mathring{J}$

<u>Stress-displacement relations</u>

$$\underset{\sim}{\sigma} = \underset{\sim}{c}\, \nabla \underset{\sim}{u} \quad \text{or} \quad \sigma_{ij} = c_{ijkl}\, u_{k,l} \tag{6.17}$$

In particular if $\underset{\sim}{c}$ is isotropic,

$$\underset{\sim}{\sigma} = \lambda \underset{\sim}{1}\, \nabla\cdot\underset{\sim}{u} + 2\mu\, \mathrm{sym}\nabla\underset{\sim}{u} \quad \text{or} \quad \sigma_{ij} = \lambda \delta_{ij} u_{k,k} + \mu(u_{i,j} + u_{j,i}) \tag{6.17'}$$

[Recall $\lambda = 2\mu\nu/(1-2\nu)$]

<u>Displacement eqs. of motion (equilibrium)</u>

$$\nabla\cdot(\underset{\sim}{c}\,\nabla\underset{\sim}{u}) + \underset{\sim}{f} = \rho\ddot{\underset{\sim}{u}} \quad \text{or} \quad c_{ijkl}\, u_{k,lj} + f_i = \rho \ddot{u}_i \tag{6.18}$$

In particular, if $\underset{\sim}{c}$ is isotropic,

$$\mu \nabla^2 \underset{\sim}{u} + (\lambda+\mu)\nabla\nabla\cdot\underset{\sim}{u} + \underset{\sim}{f} = \rho\ddot{\underset{\sim}{u}}$$

$$\Leftrightarrow \quad \mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + f_i = \rho \ddot{u}_i$$

$$\Leftrightarrow \quad \nabla^2\underset{\sim}{u} + \frac{1}{1-2\nu}\nabla\nabla\cdot\underset{\sim}{u} + \frac{\underset{\sim}{f}}{\mu} = \frac{\rho}{\mu}\ddot{\underset{\sim}{u}}$$

(6.18′) Cauchy-Navier

(6.18′) are equivalent to

$$(\lambda+2\mu)\nabla\vartheta - 2\mu\nabla\wedge\underset{\sim}{w} + \underset{\sim}{f} = \rho\ddot{\underset{\sim}{u}}, \quad \vartheta \doteq \nabla\cdot\underset{\sim}{u}, \quad \underset{\sim}{w} \doteq \tfrac{1}{2}\nabla\wedge\underset{\sim}{u} \tag{6.18''}$$

Conversely, if $\underset{\sim}{u} \in C^2(R\times\mathring{J})$, $\rho \in C(R)$, $\underset{\sim}{c} = \text{const}$ on R,

$c_{ijkl} = c_{jikl} = c_{klij}$, then (6.17), (6.18), (6.1) $\Rightarrow$ (*)

<u>Proof.</u> (*) $\Rightarrow$ (6.1), (6.2), (6.3) hold on $\mathcal{R}\times\mathring{\mathcal{T}}$. Eliminate first $\underset{\sim}{\tau}$, then $\underset{\sim}{\sigma}$, to deduce (6.17), (6.18). Proceed analogously for the isotropic case, using (6.2'). Alternatively specialize (6.17), (6.18) by means of (4.9). To see the equivalence of (6.18') and (6.18"), recall from (1.44) that

$$\nabla\wedge\nabla\wedge\underset{\sim}{u} = \nabla\nabla\cdot\underset{\sim}{u} - \nabla^2\underset{\sim}{u} \quad \text{on } \mathcal{R}\times\mathring{\mathcal{T}}$$

To establish "converse" reverse argument.

<u>Remark</u>. Thm. 6.7 applies in particular to the equilibrium case.

Displacement-formulation of Problems AIII, BIII for a homogeneous medium

Thm. 6.7 enables one to reduce Problems AIII {BIII} to a boundary-initial value problem {boundary-value problem} with u_i as the only unknowns, if $\underset{\sim}{c}$ is constant.

Problem AIII. Given a regular region R and $\mathcal{T}=[t_0,t_1]$, ρ, $\underset{\sim}{c}$ constant on $\bar{R}$, $\underset{\sim}{f}$ on $\bar{R}\times\mathcal{T}$, find u_i (suitably smooth on $\bar{R}\times\mathcal{T}$) $\ni$ (6.18) or (6.18′) hold on $R\times\mathring{\mathcal{T}}$, subject to

Initial conds.: $u_i(\underset{\sim}{x},t_0)=\mathring{u}_i(\underset{\sim}{x})$, $\dot{u}_i(\underset{\sim}{x},t_0)=\mathring{v}_i(\underset{\sim}{x})$ $\forall\, \underset{\sim}{x}\in R$

Boundary conds:

$$u_i=\overset{*}{u}_i \text{ on } \partial_1 R\times\mathcal{T}, \quad c_{ijkl}u_{k,l}n_j=\overset{*}{s}_i \text{ on } \partial_2 R\times\mathcal{T} \qquad (6.19)$$

If $\underset{\sim}{c}$ is isotropic, (6.19) give way to

$$\left.\begin{aligned} &u_i=\overset{*}{u}_i \text{ on } \partial_2 R\times\mathcal{T},\\ &\mu\left[\frac{2\nu}{1-2\nu}\delta_{ij}u_{k,k}+u_{i,j}+u_{j,i}\right]n_j=\overset{*}{s}_i \text{ on } \partial_2 R\times\mathcal{T}\end{aligned}\right\} \qquad (6.19')$$

Problem BIII. Given a regular region R, $\underset{\sim}{c}$ constant on $\bar{R}$, $\underset{\sim}{f}$ on $\bar{R}$, find u_i (suitably smooth on $\bar{R}$) $\ni$

$$\nabla\cdot(\underset{\sim}{c}\nabla\underset{\sim}{u})+\underset{\sim}{f}=\underset{\sim}{0} \text{ or } c_{ijkl}u_{k,lj}+f_i=0 \text{ on } R \qquad (6.20)$$

or, if $\underset{\sim}{c}$ is isotropic,

$$\nabla^2\underset{\sim}{u}+\frac{1}{1-2\nu}\nabla\nabla\cdot\underset{\sim}{u}+\frac{\underset{\sim}{f}}{\mu}=0 \text{ or } u_{i,jj}+\frac{1}{1-2\nu}u_{j,ji}+\frac{f_i}{\mu}=0 \text{ on } R \qquad (6.20$$

Boundary Conditions

$$u_i = \overset{*}{u}_i \text{ on } \partial_1 R \,, \quad c_{ijkl} u_{k,l} n_j = \overset{*}{s}_i \text{ on } \partial_2 R \qquad (6.21)$$

If $\underset{\sim}{c}$ is isotropic (6.21) become

$$u_i = \overset{*}{u}_i \text{ on } \partial_1 R \,, \quad \mu\left[\frac{2\nu}{1-2\nu}\delta_{ij} u_{k,k} + u_{i,j} + u_{j,i}\right] n_j = \overset{*}{s}_i \text{ on } \partial_2 R \qquad (6.21')$$

Discussion

(a) Once $\underset{\sim}{u}$ has been found, $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\sigma}$ are computable from (6.1), (6.2) {(6.2')}.

(b) In comparing above with original formulation of Problems AIII {BIII} note reduction of number of unknowns from 15 to 3 at expense of more complicated boundary conditions unless $\partial_1 R = \partial R$. We shall see later that for a homog., isotropic solid one may further reduce the field eqs. in Problem AIII {BIII} to the wave equation {Laplace's equation} at the expense of additional complications of the boundary conditions. (See future introduction of displacement potentials).

(c) Consider Problem BII for a homog. & isotropic medium. Assume $\underset{\sim}{f}$ and $\overset{*}{\underset{\sim}{s}}$ are independent of elastic constants μ and ν. The displacement formulation of this problem reveals that then $\underset{\sim}{\sigma}$ is independent of μ. To see this, set $\bar{\underset{\sim}{u}} = \mu \underset{\sim}{u}$ and refer to (6.20′), (6.21′), noting that $\bar{\underset{\sim}{u}}$ is independent of μ. Then appeal to (6.17′).

Thm. 6.8: Let R be an <u>open</u> region and let

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\gamma}, \underset{\sim}{\sigma}] \in \mathcal{E}(\underset{\sim}{c}, \underset{\sim}{f}; R). \quad (*)$$

Suppose $\underset{\sim}{u} \in \mathcal{C}^3(R)$. Then $\underset{\sim}{\gamma}$ satisfies the <u>strain equations of compatibility</u> on R:

$$\left.\begin{aligned} &\nabla \wedge \nabla \wedge \underset{\sim}{\gamma} = \underset{\sim}{0} \text{ or } \varepsilon_{ipm} \varepsilon_{jqn} \gamma_{mn,pq} = 0 \\ \Leftrightarrow \quad &\gamma_{ij,kl} + \gamma_{kl,ij} - \gamma_{il,jk} - \gamma_{jk,il} = 0 \\ \Leftrightarrow \quad &\gamma_{ij,kk} + \gamma_{kk,ij} - \gamma_{ik,jk} - \gamma_{jk,ik} = 0 \end{aligned}\right\} \quad (6.22)$$

Conversely, let R be <u>simply connected</u>, let $\underset{\sim}{\gamma} \in \mathcal{C}^2(R)$, $\underset{\sim}{\gamma} = \underset{\sim}{\gamma}^T$ on R, and let (6.22) hold on R. Suppose $\underset{\sim}{c} \in \mathcal{C}^1(R)$, $c_{ijkl} = c_{jikl} = c_{klij}$ on R, $\underset{\sim}{\sigma} \in \mathcal{C}^1(R)$ and (6.2), (6.3′) hold on R

↑ $\underset{\sim}{c}$ is invertible on R, There $\exists$ $\underset{\sim}{u}$ on $R \ni$ (*) hold. Further, such a field $\underset{\sim}{u}$ is supplied by (2.63), (2.64).

Proof.. Immediate from Thm. 2.13, Exercise 13, and Thm. 2.16.

Thm. 6.9. Let R be an open region. Let

$$\delta = [\underset{\sim}{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu, \underset{\sim}{f}; R), \quad (*)$$

where μ, ν are constant, $\mu \neq 0$, $\nu \neq -1, 1/2, 1$. Suppose $\underset{\sim}{u} \in C^3(R)$. Then $\underset{\sim}{\varepsilon} \in C^2(R)$, $\underset{\sim}{\sigma} \in C^2(R)$, $\underset{\sim}{f} \in C^1(R)$ and $\underset{\sim}{\sigma}$ ~~(on R)~~ satisfies the "stress equations of compatibility" (Beltrami-Michell eqs.):

$$\left.\begin{aligned} &\nabla^2 \underset{\sim}{\sigma} + \frac{1}{1+\nu} \nabla\nabla \operatorname{tr} \underset{\sim}{\sigma} = -\frac{\nu}{1-\nu} \underset{\sim}{1}\, \nabla \cdot \underset{\sim}{f} - 2 \operatorname{sym} \nabla \underset{\sim}{f} \\ \text{or} \quad &\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} \sigma_{kk,ij} = -\frac{\nu}{1-\nu} \delta_{ij} f_{k,k} - f_{i,j} - f_{j,i} \end{aligned}\right\} \quad (6.23)$$

Further, (6.23) $\Rightarrow$

$$\nabla^2 \operatorname{tr} \underset{\sim}{\sigma} = -\frac{1+\nu}{1-\nu} \nabla \cdot \underset{\sim}{f} \quad \text{or} \quad \nabla^2 \sigma_{kk} = -\frac{1+\nu}{1-\nu} f_{k,k} \qquad (6.24)$$

Conversely, let R be <u>simply connected</u>, let μ, ν be as above, let $\underset{\sim}{f} \in C^1(R)$, $\underset{\sim}{\sigma} \in C^2(R)$, $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$ on R, and assume (6.3′), (6.23) hold on R. Define $\underset{\sim}{\gamma}$ on R through (6.2′). Then $\exists$ $\underset{\sim}{u}$ on R $\ni$ (*) holds. Further, such a field $\underset{\sim}{u}$ is supplied by (2.63), (2.64).

<u>Proof.</u> The asserted smoothness of $\underset{\sim}{\gamma}, \underset{\sim}{\sigma}, \underset{\sim}{f}$ is immediate from (*) and $\underset{\sim}{u} \in C^3(R)$ (explain).

By (*) and the hyp. on μ, ν, one has

$$\gamma_{ij} = \frac{1}{\eta}[(1+\nu)\sigma_{ij} - \delta_{ij}\nu\Theta], \quad \Theta = \sigma_{kk} \text{ on } R \qquad (1)$$

$$\sigma_{ik,k} + f_i = 0, \quad \sigma_{ik} = \sigma_{ki} \text{ on } R \qquad (2)$$

Also, by hyp. and Thm. 6.8 (See Exercise 11),

$$\gamma_{ij,kk} + \gamma_{kk,ij} - \gamma_{ik,jk} - \gamma_{jk,ik} = 0 \text{ on } R \qquad (3)$$

Substitution from (1) into (3), after simplifications, yields

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\Theta_{,ij} = \frac{\nu}{1+\nu}\delta_{ij}\nabla^2\Theta + \sigma_{ik,jk} + \sigma_{jk,ik} \qquad (4)$$

(2) $\Rightarrow$

$$\sigma_{ik,jk} = -f_{i,j}, \quad \sigma_{jk,ik} = -f_{j,i} \qquad (5)$$

Contract (4) and use (5) to obtain

$$\nabla^2\Theta = \frac{1+\nu}{1-\nu}\sigma_{ik,ik} = -\frac{1+\nu}{1-\nu}f_{k,k} \qquad (6)$$

But (4),(5),(6) $\Rightarrow$ (6.23). Trivially, (6.23) $\Rightarrow$ (6.24).

To establish the converse claim, reverse argument and appeal to Thm. 6.8.

Remark on misleading term "stress eqs. of compat."

~~Exercise 21. Carry out the proof of the "converse" part of Thm. 6.9.~~

Stress-formulation of Problem BII for homog., isotropic solids

Thm. 6.9 enables one to reduce this problem to a boundary value problem for σ if R is simply connected.

Reduced Problem BII. Given a regular simply connected region R, given a constant Poisson's ratio ν $(-1<\nu<1/2)$ and $f \in C^1(R) \cap C(\bar{R})$, find $\sigma \in C^2(R) \cap C(\bar{R}) \ni$

$$\sigma_{ij,j} + f_i = 0, \quad \sigma_{ij} = \sigma_{ji} \quad \text{on } R, \tag{6.3'}$$

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu}\sigma_{kk,ij} = -\frac{\nu}{1-\nu}\delta_{ij} f_{k,k} - f_{i,j} - f_{j,i} \text{ on } R, \tag{6.23}$$

subject to the boundary conditions

$$\sigma_{ij} n_j = \overset{*}{s}_i \text{ on } \partial R$$

Once $\underset{\sim}{\sigma}$ has been found, $\underset{\sim}{\tau}$ follows from (6.2′) for given $\mu > 0$ and $\underset{\sim}{u}$ may be found from (6.1). Such a $\underset{\sim}{u}$ is in fact supplied by (2.63), (2.64).

Remarks

(a) Emphasize incompleteness of above characterization of $\underset{\sim}{\sigma}$ if R fails to be simply connected.

(b) Observe that (6.3′), (6.23) constitute $3+6=\underline{9}$ equations for the $\underline{6}$ unknowns $\sigma_{ij} = \sigma_{ji}$. Apparent overdeterminacy is spurious in view of available existence thm. for Problem BII.

(c) Note that above formulation confirms that $\underset{\sim}{\sigma}$ is independent of μ if R simply connected. We already know this to be true regardless of the connectivity of R.

Thm. 6.7 (Some properties of elastostatic fields). Let R be an open region and suppose

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\tau}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu, \underset{\sim}{f}; R), \ \vartheta = \nabla \cdot \underset{\sim}{u}, \ \underset{\sim}{w} = \frac{1}{2} \nabla \wedge \underset{\sim}{u}, \ \Theta = \operatorname{tr} \underset{\sim}{\sigma}, \quad (*)$$

with μ, ν constant. Assume $\underset{\sim}{u} \in \mathcal{C}^5(R)$. Then $\underset{\sim}{\tau} \in \mathcal{C}^4(R)$,

$\underset{\sim}{\gamma} \in C^4(R)$, $\underset{\sim}{f} \in C^3(R)$ and

(a) $\nabla \cdot \underset{\sim}{f} = 0$ on R $\overset{\not\Leftarrow}{\Rightarrow}$ $\nabla^2 \vartheta = 0$, $\nabla^2 \Theta = 0$ on R

(b) $\nabla \wedge \underset{\sim}{f} = \underset{\sim}{0}$ on R $\overset{\not\Leftarrow}{\Rightarrow}$ $\nabla^2 \underset{\sim}{w} = \underset{\sim}{0}$ on R

(c) $\nabla \cdot \underset{\sim}{f} = 0$, $\nabla \wedge \underset{\sim}{f} = \underset{\sim}{0}$ on R $\overset{\not\Leftarrow}{\Rightarrow}$ $\nabla^4 \underset{\sim}{u} = \underset{\sim}{0}$, $\nabla^4 \underset{\sim}{\gamma} = \underset{\sim}{0}$, $\nabla^4 \underset{\sim}{\sigma} = \underset{\sim}{0}$ on R

<u>Proof</u>. The asserted smoothness of $\underset{\sim}{\gamma}, \underset{\sim}{\sigma}, \underset{\sim}{f}$ is clear from (*) and the hyp. $\underset{\sim}{u} \in C^5(R)$.

<u>Re (a)</u>. From (*) and Thm. 6.7

$$\nabla^2 \underset{\sim}{u} + \frac{1}{1-2\nu} \nabla \nabla \cdot \underset{\sim}{u} + \frac{\underset{\sim}{f}}{\mu} = \underset{\sim}{0} \quad \text{on } R \qquad (**)$$

Operate on (**) with $\nabla \cdot$ to see that $\nabla^2 \vartheta = 0$ on R.
Then note from (6.2′) that

$$\Theta = \operatorname{tr} \underset{\sim}{\sigma} = (3\lambda + 2\mu) \operatorname{tr} \underset{\sim}{\gamma} = (3\lambda + 2\mu)\vartheta$$

<u>Re (b)</u> Operate on (**) with $\nabla \wedge$

<u>Re (c)</u>. Here from (a), (b),

$$\nabla^2 \nabla \cdot \underset{\sim}{u} = 0, \quad \nabla^2 \nabla \wedge \underset{\sim}{u} = \underset{\sim}{0} \quad \text{on } R \qquad (1)$$

But from (1.44),

$$\nabla^2 \underset{\sim}{u} = \nabla \nabla \cdot \underset{\sim}{u} - \nabla \wedge \nabla \wedge \underset{\sim}{u} \qquad (2)$$

(1), (2) $\Rightarrow$

$$\nabla^4 \underset{\sim}{u} = \nabla^2(\nabla^2 \underset{\sim}{u}) = \nabla^2 \nabla \nabla\cdot\underset{\sim}{u} - \nabla^2 \nabla\wedge\nabla\wedge\underset{\sim}{u}$$

$$= \nabla \nabla^2(\nabla\cdot\underset{\sim}{u}) - \nabla\wedge\nabla^2(\nabla\wedge\underset{\sim}{u}) = \underset{\sim}{0} \quad \text{on } R \qquad (3)$$

$(3), (6.1), (6.2') \Rightarrow \nabla^4 \underset{\sim}{\tau} = \nabla^4 \underset{\sim}{\sigma} = \underset{\sim}{0}$ on R

qed.

Remarks

(i) The hyp. under (a), (b), (c) hold in particular if $\underset{\sim}{f} = \underset{\sim}{0}$ on R. Also, the hyp. under (c) holds clearly if $\underset{\sim}{f} = \nabla\varphi$, $\nabla^2\varphi = 0$ on R.

(ii) Let $\underset{\sim}{T}$ be a tensor-valued function defined on an open region R.

$\underset{\sim}{T} \in \mathcal{C}^2(R)$, $\nabla^2 \underset{\sim}{T} = \underset{\sim}{0} \Leftrightarrow \underset{\sim}{T}$ is harmonic on R

$\underset{\sim}{T} \in \mathcal{C}^4(R)$, $\nabla^4 \underset{\sim}{T} = \underset{\sim}{0} \Leftrightarrow \underset{\sim}{T}$ is biharmonic on R

Trivially, $\underset{\sim}{T}$ harmonic on $R \Rightarrow \underset{\sim}{T}$ biharmonic on R

On can show that:

$\underset{\sim}{T}$ harmonic or biharmonic on $R \Rightarrow \underset{\sim}{T} \in \mathcal{C}^\infty(R)$

Thus ϑ, Θ in (a), $\underset{\sim}{w}$ in (b), and $\underset{\sim}{u}, \underset{\sim}{\tau}, \underset{\sim}{\sigma}$ in (c) are all $\mathcal{C}^\infty(R)$.

The smoothness assumption $\underset{\sim}{u} \in \mathcal{C}^5(R)$ in Thm. 6.10 is artificial (explain). We now cite a theorem according to which this hypothesis is removable if $\underset{\sim}{f}$ is sufficiently smooth.

Thm. 6.11. Let R be an open region and suppose

$$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{T}, \underset{\sim}{E}] \in \mathcal{E}(\mu, \nu, \underset{\sim}{f}; R), \quad (\mu, \nu \text{ consts.})$$

Assume $\underset{\sim}{f} \in \mathcal{C}^\infty(R)$. Then,

$$[\underset{\sim}{u}, \underset{\sim}{T}, \underset{\sim}{E}] \in \mathcal{C}^\infty(R).$$

For a proof, see Duffin, J.R.M.A., 5 (1956). See also Gurtin (Encyclopedia of Physics). Thm. 6.11 reflects a general property of elliptic p.d.e.

Discussion.

(i) Thus conclusions in Thm. 6.10 remain valid if hyp. $\underset{\sim}{u} \in \mathcal{C}^5(R)$ is removed, provided $\underset{\sim}{f} \in \mathcal{C}^\infty(R)$.

(ii) Recall maximum principle of harmonic functions if φ is harm. on bounded open R and $\varphi \in \mathcal{C}(\bar{R})$, then the absolute maximum & minimum of φ on $\bar{R}$ occur on ∂R. Apply to $\vartheta, \Theta, \underset{\sim}{w}$ in Thm. 6.9.

Maximum principle does not hold for biharmonic functions. Thus even if $\nabla \cdot \underset{\sim}{f} = 0$, $\nabla \wedge \underset{\sim}{f} = \underset{\sim}{0}$ on R, maxima and minima of σ_{ij} may occur in R (in interior of body). Explain significance for experimental stress analysis.

<u>Polya's result</u>: Suppose R is bounded, open. Let

$\mathcal{S} = [\underset{\sim}{u}, \underset{\sim}{\gamma}, \underset{\sim}{\sigma}] \in \mathcal{E}(\mu, \nu, \underset{\sim}{f}; \bar{R})$, μ & ν const., $\nabla^2 \underset{\sim}{u} = 0$ on R

Then $\nabla^2 \underset{\sim}{\gamma} = \nabla^2 \underset{\sim}{\sigma} = 0$ on R and the fields

$$u_i, \gamma_{ij}, \sigma_{ij}, |\underset{\sim}{u}|, \gamma_i, \sigma_i, W, W', W''$$

assume their maxima & minima on ∂R.