5. Strain-energy, the linear elastic solid

Introduction. The notion of "elasticity" is based on energetic considerations that involve the concept of strain energy. The latter concept also plays a pivotal role in various fundamental theorems of linear elastostatic and elastodynamics (e.g. reciprocal thms. uniqueness thms, minimum energy principles and variational principles. These principles, in turn, serve as a basis for important approximative methods for the solution of boundary and boundary initial value problems in linear elast. theory (energy methods, direct variational methods. Finally, the strain-energy concept enters into certain failure hypotheses.

We turn first to a basic theorem concerning admissible motions of a continuous medium in the presence of infinitesimal deformations.

Thm. 5.1 (Power identity). Let R be bounded regular (open) region and $\mathcal{R} = \bar{R}$. Let $\hat{\underset{\sim}{y}}(\cdot,t) : \mathcal{R} \to \mathcal{R}_t$ $(t_0 \le t \le t_1)$ be an admissible motion of a body and suppose

$$\rho \in \mathcal{C}(\mathcal{R}),\ \underset{\sim}{f} \in \mathcal{C}(\mathcal{R}\times\mathcal{T})\ ;\ \underset{\sim}{\sigma} \in \mathcal{C}^1(\mathcal{R}\times\mathcal{T}),\ \mathcal{T} = [t_0, t_1] \qquad (a)$$

$$\underset{\sim}{\gamma} = \text{sym}\,\nabla\underset{\sim}{u}\ ,\ \nabla\cdot\underset{\sim}{\sigma} + \underset{\sim}{f} = \rho\ddot{\underset{\sim}{u}},\ \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T \text{ on } R\times\mathcal{T} \qquad (b)$$

$$\underset{\sim}{s}(\underset{\sim}{x},t) \equiv \underset{\sim}{s}(\underset{\sim}{x},\underset{\sim}{n}(\underset{\sim}{x}),t) = \underset{\sim}{\sigma}(\underset{\sim}{x},t)\,\underset{\sim}{n}(\underset{\sim}{x})\ \forall\ (\underset{\sim}{x},t) \in \partial R\times\mathcal{T}, \qquad (c)$$

where $\underset{\sim}{n}$ is the unit outer normal vector of ∂R. Then,

$$\int_{\partial R} \underset{\sim}{s}\cdot\dot{\underset{\sim}{u}}\,dA + \int_R \underset{\sim}{f}\cdot\dot{\underset{\sim}{u}}\,dV = \int_R \underset{\sim}{\sigma}\cdot\dot{\underset{\sim}{\gamma}}\,dV + \dot{K} \text{ on } \mathcal{T}, \qquad (5.1)$$

if K is the "kinetic energy" defined by

$$K(t) = \frac{1}{2}\int_R \rho(\underset{\sim}{x})\,\dot{\underset{\sim}{u}}^2(\underset{\sim}{x},t)\,dV \quad \forall\ t \in \mathcal{T}. \qquad (5.2)$$

Further, (5.1) remains valid if one assumes merely

$$\underset{\sim}{u} \in \mathcal{C}^2(R\times\mathcal{T}) \cap \mathcal{C}^1(\bar{R}\times\mathcal{T}),\ \ddot{\underset{\sim}{u}} \in \mathcal{C}(\bar{R}\times\mathcal{T}),\ \underset{\sim}{\sigma} \in \mathcal{C}^1(R\times\mathcal{T}) \cap \mathcal{C}(\bar{R}\times\mathcal{T}).$$

Remarks. By (a) & smoothness of motion, $\underset{\sim}{u} \in \mathcal{C}^2(\bar{R}\times\mathcal{T})$, $\underset{\sim}{\gamma} \in \mathcal{C}^1(\bar{R}\times\mathcal{T})$. One calls $\int_R \underset{\sim}{\sigma}\cdot\dot{\underset{\sim}{\gamma}}\,dV$ the stress-power. Interpret conclusion physically. Emphasize absence of constitutive assumptions.

Proof. (a), (c) $\Rightarrow$

$$\int_{\partial R} \underset{\sim}{s} \cdot \dot{\underset{\sim}{u}}\, dA + \int_R \underset{\sim}{f} \cdot \dot{\underset{\sim}{u}}\, dV = \int_{\partial R} \underbrace{\sigma_{ij} \dot{u}_i}_{w_j} n_j\, dA + \int_R f_i \dot{u}_i\, dA \qquad (1)$$

Set
$$w_j = \sigma_{ij} \dot{u}_i \text{ on } \bar{R} \times \mathcal{T} \qquad (2)$$

(2), (a), (b) $\Rightarrow$

$$\nabla \cdot \underset{\sim}{w} = w_{j,j} = \sigma_{ij,j} \dot{u}_i + \sigma_{ij} \dot{u}_{i,j} = \sigma_{ij,j} \dot{u}_i + \sigma_{ij} \dot{u}_{(i,j)}$$
$$= (\rho \ddot{u}_i - f_i) \dot{u}_i + \sigma_{ij} \dot{\gamma}_{ij} \text{ on } R \times \mathcal{T} \qquad (3)$$

(2), (3) imply even under relaxed smoothness assumpts:

$$\underset{\sim}{w} \in \mathcal{C}^1(R \times \mathcal{T}) \cap \mathcal{C}(\bar{R} \times \mathcal{T}),\ \nabla \cdot \underset{\sim}{w} \in \mathcal{C}(\bar{R} \times \mathcal{T}),$$

so that (1), (3), and the divergence theorem give

$$\int_{\partial R} \underset{\sim}{s} \cdot \dot{\underset{\sim}{u}}\, dA + \int_R \underset{\sim}{f} \cdot \dot{\underset{\sim}{u}}\, dV = \int_R \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\gamma}}\, dV + \int_R \rho \dot{\underset{\sim}{u}} \cdot \ddot{\underset{\sim}{u}}\, dV =$$

$$\int_R \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\gamma}}\, dV + \int_R \frac{1}{2} \rho \frac{\partial}{\partial t} \dot{\underset{\sim}{u}}^2\, dV = \int_R \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\gamma}}\, dV + \frac{d}{dt}\left\{\frac{1}{2} \int_R \rho \dot{\underset{\sim}{u}}^2\, dV\right\}.$$

qed.

Remark. Clearly, (5.1) continues to hold if R is replaced by $\mathcal{P} \subset R$, if $\mathcal{P}$ is a regular region.

Gradient of a scalar function of a two-tensor

Let $\mathcal{L}$ be the set of all two-tensors and φ a a scalar-valued function defined on $\mathcal{L}$. Thus,

$$\varphi(\underset{\sim}{T}) = f^{X}([\underset{\sim}{T}]^{X}) \quad \forall \underset{\sim}{T} \in \mathcal{L}, \forall X \in \mathcal{F}$$

We say $\varphi \in \mathcal{C}^{N}(\mathcal{L})$ $(N \geq 1)$ if f^{X} is N times continuously differentiable with respect to T_{ij}^{X} for some $X \in \mathcal{F}$ (and hence $\forall X \in \mathcal{F}$). We say that φ is <u>isotropic</u> if f^{X} is independent of X.

<u>Claim</u>. Let $\varphi \in \mathcal{C}^{N}(\mathcal{L})$ be scalar-valued and let

$$F_{ij}^{X}(\underset{\sim}{T}) = \partial f^{X}([\underset{\sim}{T}]^{X}) / \partial T_{ij}^{X} \quad \forall \underset{\sim}{T} \in \mathcal{L}, \forall X \in \mathcal{F}.$$

Then $F_{ij}^{X}(\underset{\sim}{T})$ are the compts. in X of a two-tensor $\underset{\sim}{F}(\underset{\sim}{T})$, called the gradient of φ at $\underset{\sim}{T}$. We write

$$\underset{\sim}{F}(\underset{\sim}{T}) = \nabla_{\underset{\sim}{T}} \varphi(\underset{\sim}{T}) \quad \text{or} \quad F_{ij} = \partial \varphi(\underset{\sim}{T}) / \partial T_{ij} \quad \forall \underset{\sim}{T} \in \mathcal{L}.$$

<u>Proof</u>. By hyp., $\forall [\underset{\sim}{A}] : X \to X'$ one has

$$f^{X}([\underset{\sim}{T}]^{X}) = f^{X'}([\underset{\sim}{T}]^{X'}), \quad T_{pq}^{X} = A_{mp} A_{nq} T_{mn}^{X'}$$

whence from the chain-rule.

$$F_{ij}^{X'} = \frac{\partial f^{X'}}{\partial T_{ij}^{X'}} = \frac{\partial f^{X}}{\partial T_{pq}^{X}} \frac{\partial T_{pq}^{X}}{\partial T_{ij}^{X'}} = F_{pq}^{X} A_{ip} A_{jq} \quad /$$

Recall that $\mathcal{B}$ is a <u>linear simple solid</u> if

$$\sigma_{ij}(\underset{\sim}{x},t) = c_{ijkl}(\underset{\sim}{x})\, \gamma_{kl}(\underset{\sim}{x},t) \;\; \forall\, (\underset{\sim}{x},t)\in\mathcal{R}\times\mathcal{T} \quad (*)$$

for every admissible motion $\hat{\underset{\sim}{y}}(\cdot,t):\mathcal{R}\to\mathcal{R}_t$ $(t_0\le t\le t_1)$, provided $\mathcal{R}$ is the region occupied by $\mathcal{B}$ in an undeformed configuration. Also,

$$c_{ijkl} = c_{jikl} = c_{ijlk} \text{ on } \mathcal{R} \quad (**)$$

We assume henceforth that $\underset{\sim}{c}\in\mathcal{C}^1(\mathcal{R})$.

<u>Definition</u>. A linear simple solid is <u>elastic</u> (linear elastic solid) if $\exists$ a scalar-valued function $\Gamma(\cdot,\underset{\sim}{x})$ (elastic potential) defined on $\mathcal{L}$ $\forall\, \underset{\sim}{x}\in\mathcal{R}$ with the following properties:

(a) $\Gamma(\cdot,\underset{\sim}{x})\in\mathcal{C}^2(\mathcal{L})$ $\forall\, \underset{\sim}{x}\in\mathcal{R}$, $\nabla_{\underset{\sim}{T}}\Gamma(\underset{\sim}{T},\underset{\sim}{x}) = \{\nabla_{T_{ji}}\Gamma(\underset{\sim}{T},\underset{\sim}{x})\}^T$ $\forall\, (\underset{\sim}{T},\underset{\sim}{x})\in\mathcal{L}\times$

(b) $\underset{\sim}{\sigma} = \nabla_{\underset{\sim}{T}}\Gamma(\underset{\sim}{\gamma},\underset{\sim}{x})$ or $\sigma_{ij} = \dfrac{\partial\Gamma(\underset{\sim}{\gamma},\underset{\sim}{x})}{\partial\gamma_{ij}} = \left.\dfrac{\partial\Gamma(\underset{\sim}{T},\underset{\sim}{x})}{\partial T_{ij}}\right|_{\underset{\sim}{T}=\underset{\sim}{\gamma}}$ on $\mathcal{R}\times\mathcal{T}$ (5

for every admissible motion $\hat{\underset{\sim}{y}}(\cdot,t):\mathcal{R}\to\mathcal{R}_t$ $(t_0\le t\le t_1$, $\mathcal{T}=[t_0,t_1]$.

One calls the function W defined by

$$W(\underset{\sim}{x},t) = \Gamma(\underset{\sim}{\gamma}(\underset{\sim}{x},t),\underset{\sim}{x}) \ \forall \ (\underset{\sim}{x},t) \in \mathcal{R}\times\mathcal{T} \qquad (5.4)$$

the <u>strain-energy density</u> of the linear elastic solid.

<u>Thm.</u> 5.2. A necessary and sufficient condition that a linear simple solid, with the response tensor field $\underset{\sim}{c}$, be elastic is that

$$c_{ijkl} = c_{klij} \ \text{ or } \ [\underset{\sim}{C}] = [\underset{\sim}{C}]^T \ \text{ on } \mathcal{R} \qquad (5.5)$$

In this instance,

$$\left.\begin{aligned} &= \Gamma(\underset{\sim}{\gamma},\underset{\sim}{x}) = \tfrac{1}{2} c_{ijkl}(\underset{\sim}{x})\gamma_{ij}\gamma_{kl} \ \forall \ (\underset{\sim}{\gamma},\underset{\sim}{x}) \in \mathcal{L}\times\mathcal{R} \\ &= W = \tfrac{1}{2}\underset{\sim}{\sigma}\cdot\underset{\sim}{\gamma} = \tfrac{1}{2}\sigma_{ij}\gamma_{ij} \ \text{ on } \mathcal{R}\times\mathcal{T}. \end{aligned}\right\} (5.6)$$

for every admissible motion

<u>Proof.</u>

<u>Re nec.</u> Let $\mathcal{B}$ be an <u>elastic</u> l.s.s. Then, by (*), (**) and def. $\exists\,\Gamma$ with appropriate regul. & symm. properties

$$\sigma_{ij} = \frac{\partial\Gamma(\underset{\sim}{\gamma},\underset{\sim}{x})}{\partial\gamma_{ij}} = c_{ijpq}(\underset{\sim}{x})\gamma_{pq} \ ; \ \frac{\partial^2\Gamma(\underset{\sim}{\gamma},\underset{\sim}{x})}{\partial\gamma_{kl}\partial\gamma_{ij}} = c_{ijkl}(\underset{\sim}{x}),$$

whence (5.5) follow from $\Gamma(\cdot,\underset{\sim}{x}) \in \mathcal{C}^2(\mathcal{L}) \ \forall \ \underset{\sim}{x}\in\mathcal{R}$.

Re suff. Suppose (*), (**) and (5.5) hold. Define $\Gamma(\underset{\sim}{\gamma},\underset{\sim}{x})$ through the first of (5.6). Clearly, Γ has required regularity. Also,

$$\frac{\partial\Gamma(\underset{\sim}{\gamma},\underset{\sim}{x})}{\partial\gamma_{ij}}=\frac{\partial}{\partial\gamma_{ij}}\left(\tfrac{1}{2}c_{pqkl}\gamma_{pq}\gamma_{kl}\right)$$

$$=\tfrac{1}{2}c_{pqkl}\,\delta_{pi}\delta_{qj}\gamma_{kl}+\tfrac{1}{2}c_{pqkl}\gamma_{pq}\,\delta_{ki}\delta_{lj}$$

$$=\tfrac{1}{2}c_{ijkl}\gamma_{kl}+\tfrac{1}{2}c_{pqij}\gamma_{pq}=c_{ijkl}\gamma_{kl}=\frac{\partial\Gamma(\underset{\sim}{\gamma},\underset{\sim}{x})}{\partial\gamma_{ji}}=\sigma_{ij},$$

so that the l.s.s. is elastic. The second of (5.6) is immediate from the first and (5.4), (*). /

Remarks. For a [illegible]

From (5.6) follows $\Gamma(\cdot,\underset{\sim}{x})\in C^{\infty}(\mathcal{L})\ \forall\,\underset{\sim}{x}\in\mathcal{R}$ and, since we assumed $\underset{\sim}{c}\in C^{1}(\mathcal{R})$, $\Gamma\in C^{1}(\mathcal{L}\times\mathcal{R})$.

Note that for a <u>homogeneous</u> linear elastic solid $\Gamma(\underset{\sim}{\gamma},\underset{\sim}{x})=\Gamma(\underset{\sim}{\gamma})$ (no explicit dependence on $\underset{\sim}{x}$).

<u>Remark</u>. Thus for a simple linear <u>elastic</u> solid $W(\underset{\sim}{x},t) \equiv \Gamma(\underset{\sim}{\gamma}(\underset{\sim}{x},t),\underset{\sim}{x})$ is a quadratic form in γ_{ij} and a bilinear form in the components of $\underset{\sim}{\sigma}$ and $\underset{\sim}{\gamma}$.

<u>Corollary</u>. If a linear simple solid is <u>isotropic</u> it is necessarily elastic. (See (4.12))

According to (4.1), (4.2), (5.5) a <u>linear elastic solid</u> obeys the stress-strain relations

$$\left.\begin{aligned} \sigma_{ij}(\underset{\sim}{x},t) &= c_{ijkl}(\underset{\sim}{x})\,\gamma_{kl}(\underset{\sim}{x},t) \quad \forall\ (\underset{\sim}{x},t)\in R\times T \\ c_{ijkl} &= c_{jikl} = c_{klij} \quad \text{on } R \end{aligned}\right\} \quad (5.7)$$

<u>Remarks</u>. Note that the symm. rels. in $\overset{(5.7)}{\cancel{(5.9)}} \Rightarrow c_{ijlk} = c_{ijkl}$ (explain). The symm. rels. (5.7) reduce the number of response functions c_{ijkl} from 36 to 21. From here on call $\underset{\sim}{c}$ the <u>elasticity tensor field</u>.

Inverting (5.7) one arrives at

$$\gamma_{ij} = \varkappa_{ijkl}\,\sigma_{kl}\ , \quad \varkappa_{ijkl} = \varkappa_{jikl} = \varkappa_{klij} \qquad (5.8)$$

$\underset{\sim}{\varkappa}$ ... <u>elastic compliance tensor field</u>.

Thm. 5.3 (Power identity for linear elastic solids). Let the hypotheses of Thm. 5.1 hold and assume $\mathcal{B}$ is a linear elastic solid. Then, if $\mathcal{P} \subset \mathcal{R}$ is any regular region,

$$\left.\begin{aligned} &\int_{\partial\mathcal{P}} \underset{\sim}{s} \cdot \dot{\underset{\sim}{u}}\, dA + \int_{\mathcal{P}} \underset{\sim}{f} \cdot \dot{\underset{\sim}{u}}\, dV = \dot{K} + \dot{U} \quad \text{on } \mathcal{T}, \\ &K(t) = \frac{1}{2} \int_{\mathcal{P}} \rho(\underset{\sim}{x})\, \dot{\underset{\sim}{u}}^2(\underset{\sim}{x},t)\, dV \quad \forall\, t \in \mathcal{T}; \end{aligned}\right\} (5.9)$$

and $U(t)$ is the total strain energy stored in $\mathcal{P}$ at time t, i.e

$$U(t) = \int_{\mathcal{P}} W(\underset{\sim}{x},t)\, dV = \int_{\mathcal{P}} \Gamma(\underset{\sim}{\gamma}(\underset{\sim}{x},t),\underset{\sim}{x})\, dV \quad (t_0 \le t \le t_1) \quad (5.10)$$

Proof. Appeal to (5.1), (5.2) with R replaced by $\mathcal{P}$ & note from def. of a l.e.s. that now

$$\int_{\mathcal{P}} \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{\gamma}}\, dV = \int_{\mathcal{P}} \sigma_{ij} \dot{\gamma}_{ij}\, dV = \int_{\mathcal{P}} \frac{\partial \Gamma}{\partial \gamma_{ij}} \dot{\gamma}_{ij}\, dV = \int_{\mathcal{P}} \dot{W}\, dV$$

$$= \frac{d}{dt} \int_{\mathcal{P}} W(\underset{\sim}{x},t)\, dV = \dot{U}(t) \qquad \text{q.e.d.}$$

Discussion (Significance of elasticity assumption)

Consider an admiss. motion $\hat{\underset{\sim}{y}}(\cdot,t) : \mathcal{R} \to \mathcal{R}_t \ (t_0 \le t \le t_1)$ and assume the motion is cyclic in the sense that

$$\underset{\sim}{u}(\underset{\sim}{x},t_0) = \underset{\sim}{u}(\underset{\sim}{x},t_1) \quad , \quad \dot{\underset{\sim}{u}}(\underset{\sim}{x},t_0) = \dot{\underset{\sim}{u}}(\underset{\sim}{x},t_1) \quad \forall\, \underset{\sim}{x} \in \mathcal{R}$$

Let

$$w_{\mathcal{P}} = \int_{t_0}^{t_1} \Big[\int_{\partial\mathcal{P}} \underset{\sim}{s} \cdot \dot{\underset{\sim}{u}}\, dA + \int_{\mathcal{P}} \underset{\sim}{f} \cdot \dot{\underset{\sim}{u}}\, dV \Big] dt,$$

so that $w_{\mathcal{P}}$ is the total external work done over $[t_0,t_1]$ by the forces acting on the part of the body occupying $\mathcal{P}$ in the reference configuration. Then by Thm. 5.3, $w_{\mathcal{P}} = 0$ (no energy dissipation). Mention converse of this result.

Thm. 5.4. Let $\underset{\sim}{\varkappa}$ be the elastic compliance tensor field of a linear elastic solid undergoing an admiss. motion. Let $\underset{\sim}{\gamma}, \underset{\sim}{\sigma}, W$ have their previous meanings. Define

$$\Sigma(\underset{\sim}{\sigma}, \underset{\sim}{x}) = \frac{1}{2} \varkappa_{ijkl}(\underset{\sim}{x}) \sigma_{ij} \sigma_{kl} \;\forall\; (\underset{\sim}{\sigma}, \underset{\sim}{x}) \in \mathcal{L} \times R. \quad (5.10)$$

Then,

$$W(\underset{\sim}{x}, t) = \Sigma(\underset{\sim}{\sigma}(\underset{\sim}{x}, t), \underset{\sim}{x}) = \frac{1}{2} \varkappa_{ijkl}(\underset{\sim}{x}) \sigma_{ij}(\underset{\sim}{x}, t) \sigma_{kl}(\underset{\sim}{x}, t) \;\forall\; (\underset{\sim}{x}, t) \in R \quad (5.11)$$

and

$$\underset{\sim}{\gamma} = \nabla_{\underset{\sim}{\tau}} \Sigma(\underset{\sim}{\sigma}, \underset{\sim}{x})\Big|_{\underset{\sim}{\tau}=\underset{\sim}{\sigma}} \text{ or } \gamma_{ij} = \frac{\partial \Sigma(\underset{\sim}{\sigma}, \underset{\sim}{x})}{\partial \sigma_{ij}} \equiv \frac{\partial \Sigma(\underset{\sim}{\tau}, \underset{\sim}{x})}{\partial \tau_{ij}}\Big|_{\underset{\sim}{\tau}=\underset{\sim}{\sigma}} \text{ on } R \times T \quad (5.12)$$

Proof. (5.6), (5.8), (5.10) $\Rightarrow$

$$W(\underset{\sim}{x}, t) = \frac{1}{2} \sigma_{ij}(\underset{\sim}{x}, t) \gamma_{ij}(\underset{\sim}{x}, t) = \frac{1}{2} \varkappa_{ijkl}(\underset{\sim}{x}) \sigma_{ij}(\underset{\sim}{x}, t) \sigma_{kl}(\underset{\sim}{x}, t)$$

$$= \Sigma(\underset{\sim}{\sigma}(\underset{\sim}{x}, t), \underset{\sim}{x})$$

Also, (5.12) follows easily from (5.10), (5.8).

Note complete duality between Σ and Γ (explain)

(5.[illegible]) [illegible] (5.10), (5.[illegible]), (5.12).

Thm. 5.5. Let $\underset{\sim}{c}$ be a four-tensor satisfying the symmetr[illegible] relations in (5.7). Let $(\underset{\sim}{\sigma}', \underset{\sim}{\gamma}')$ and $(\underset{\sim}{\sigma}'', \underset{\sim}{\gamma}'')$ be two pairs of symmetric two-tensors $\ni$

$$\sigma'_{ij} = c_{ijkl}\gamma'_{kl}\ , \quad \sigma''_{ij} = c_{ijkl}\gamma''_{kl}. \qquad (*)$$

Then one has the reciprocal relation

$$\boldsymbol{\sigma}'\cdot\boldsymbol{\gamma}'' = \boldsymbol{\sigma}''\cdot\boldsymbol{\gamma}' \qquad (5.13)$$

Further, if

$$\Gamma(\boldsymbol{\gamma}) = \tfrac{1}{2}c_{ijkl}\gamma_{ij}\gamma_{kl} \ \forall\ \boldsymbol{\gamma}\in\mathcal{L} \qquad (**)$$

one has

$$\Gamma(\boldsymbol{\gamma}'+\boldsymbol{\gamma}'') = \Gamma(\boldsymbol{\gamma}') + \Gamma(\boldsymbol{\gamma}'') + \boldsymbol{\sigma}'\cdot\boldsymbol{\gamma}'' \qquad (5.14)$$

<u>Proof</u>. By hyp.,

$$\boldsymbol{\sigma}'\cdot\boldsymbol{\gamma}'' = \sigma'_{ij}\gamma''_{ij} = c_{ijkl}\gamma'_{kl}\gamma''_{ij} = c_{klij}\gamma'_{ij}\gamma''_{kl} = c_{ijkl}\gamma'_{ij}\gamma''_{kl}$$
$$= \sigma''_{ij}\gamma'_{ij} = \boldsymbol{\sigma}''\cdot\boldsymbol{\gamma}'$$

whence (5.13) holds. Next, from (**), (*) and (5.13),

$$\Gamma(\boldsymbol{\gamma}'+\boldsymbol{\gamma}'') = \tfrac{1}{2}(\boldsymbol{\sigma}'+\boldsymbol{\sigma}'')\cdot(\boldsymbol{\gamma}'+\boldsymbol{\gamma}'') =$$

$$= \tfrac{1}{2}(\boldsymbol{\sigma}'\cdot\boldsymbol{\gamma}' + \boldsymbol{\sigma}''\cdot\boldsymbol{\gamma}'' + \boldsymbol{\sigma}'\cdot\boldsymbol{\gamma}'' + \boldsymbol{\sigma}''\cdot\boldsymbol{\gamma}') = \Gamma(\boldsymbol{\gamma}') + \Gamma(\boldsymbol{\gamma}'') + \boldsymbol{\sigma}'\cdot\boldsymbol{\gamma}'',$$

whence (5.14) holds.

Remarks. The reciprocal relation (5.13) will be used to prove Betti's reciprocal thm. of linear elastostatics. Emphasize non-addivity of strain-energy density (and hence of total strain energy) in the absence of local (or at least global orthogonality) of the fields $\underset{\sim}{\sigma}'$ and $\underset{\sim}{\gamma}''$.
↳ for isotropic: "orthog." $\Leftrightarrow (1+\nu)\,\underset{\sim}{\sigma}'\cdot\underset{\sim}{\sigma}'' = \nu\,(\mathrm{tr}\,\underset{\sim}{\sigma}')(\mathrm{tr}\,\underset{\sim}{\sigma}'')$
Mention applications in structural engineering.

Def. An elasticity tensor field $\underset{\sim}{c}$ on $\mathcal{R}$ is positive definite on $\mathcal{R}$ if $c_{ijkl}(\underset{\sim}{x})\gamma_{ij}\gamma_{kl} > 0 \;\forall\, \underset{\sim}{x}\in\mathcal{R}$ and $\forall$ symmetric two-tensor $\underset{\sim}{\gamma}$ other than the null-tensor. (Explain "pos. semi-definite $\underset{\sim}{c}$")

Thus $\underset{\sim}{c}$ is pos. def. on $\mathcal{R} \Leftrightarrow T(\underset{\sim}{\gamma},\underset{\sim}{x})$ is a pos. def. quadratic form in $\gamma_{ij}\;\forall\,\underset{\sim}{x}\in\mathcal{R}$.

Thm. 5.6. A necessary and sufficient condition that $\underset{\sim}{c}$ be pos. def. on $\mathcal{R}$ is that $\det[\underset{\sim}{C}^{X}]$ and all the principal minor determinants of the elasticity matrix $[\underset{\sim}{C}^{X}]$ be positive on $\mathcal{R}$ for a single $X\in\mathcal{F}$, i.e.

$$c^{X}_{1111}>0,\quad \det\begin{bmatrix} c^{X}_{1111} & c^{X}_{1122} \\ c^{X}_{2211} & c^{X}_{2222}\end{bmatrix}>0,\;\dots\;\det[\underset{\sim}{C}^{X}]>0 \text{ on } \mathcal{R} \tag{5.15}$$

For a proof see theorems on pos. def. quadratic forms in algebra books.

Strain-energy density for isotropic linear elastic solids

Here, for brevity, we shall write $\sigma, \gamma, \lambda, \mu, \nu, \Gamma(\gamma), \Sigma(\sigma)$ in place of $\sigma(x,t), \gamma(x,t), \lambda(x), \ldots \Gamma(\gamma, x), \Sigma(\sigma, x)$. Recall from (4.11), (4.18) that at present

$$\sigma_{ij} = \lambda \delta_{ij} \gamma_{kk} + 2\mu \gamma_{ij} \quad (i) \qquad \gamma_{ij} = \frac{1}{\eta}[(1+\nu)\sigma_{ij} - \nu \delta_{ij} \sigma_{kk}] \quad (ii)$$

One could obtain $\Gamma(\gamma)$ by specializing (5.6) in accordance with (4.9). Instead, note that (i) and (5.6) $\Rightarrow$

$$\Gamma(\gamma) = \frac{1}{2} \sigma_{ij} \gamma_{ij} = \frac{1}{2}(\lambda \delta_{ij} \gamma_{kk} + 2\mu \gamma_{ij}) \gamma_{ij} \text{ , whence}$$

$$\Gamma(\gamma) = \frac{1}{2} \lambda \gamma_{ii} \gamma_{jj} + \mu \gamma_{ij} \gamma_{ij} \qquad (5.16)$$

Recall from (1.26) that

$$I_1(\gamma) = \gamma_{ii} \; ; \; I_2(\gamma) = \frac{1}{2}(\gamma_{ii}\gamma_{jj} - \gamma_{ij}\gamma_{ij})$$

Hence (5.16) may be written as

$$\Gamma(\gamma) = \frac{\lambda + 2\mu}{2} I_1^2(\gamma) - 2\mu I_2(\gamma) \qquad (5.17)$$

Thus, in case of isotropy, involvement of $\underset{\sim}{\gamma}$ is via its invariants: here Γ is an <u>isotropic</u> scalar valued function on $\mathcal{L}$

Next, (ii) and (5.6) $\Rightarrow$

$$\Sigma(\underset{\sim}{\sigma}) = \frac{1}{2}\sigma_{ij}\gamma_{ij} = \frac{1}{2}\sigma_{ij}\frac{1}{\eta}[(1+\nu)\sigma_{ij} - \nu\,\delta_{ij}\,\sigma_{kk}],$$ whence

$$\Sigma(\underset{\sim}{\sigma}) = \frac{1}{2\eta}[(1+\nu)\sigma_{ij}\sigma_{ij} - \nu\,\sigma_{ii}\sigma_{jj}] \qquad (5.18)$$

or

$$\Sigma(\underset{\sim}{\sigma}) = \frac{1}{2\eta}[I_1^2(\underset{\sim}{\sigma}) - 2(1+\nu)I_2(\underset{\sim}{\sigma})] \qquad (5.19)$$

<u>Thm.</u> 5.7. A necessary and sufficient condition that $\underset{\sim}{c}$ be positive def. on $\mathcal{R}$ for an <u>isotropic</u> linear elastic solid, i.e. that $\Gamma(\underset{\sim}{\gamma},\underset{\sim}{x})$ and hence $\Sigma(\underset{\sim}{\sigma},\underset{\sim}{x})$ be a pos. def. quadratic form $\forall\, \underset{\sim}{x} \in \mathcal{R}$ is that (see 54)

$$\left.\begin{array}{l} \mu > 0,\ 2\mu + 3\lambda > 0 \ \text{ or } \ \mu > 0,\ -1 < \nu < \frac{1}{2} \\ \text{or } \ \eta > 0,\ -1 < \nu < \frac{1}{2} \ \text{ or } \ \mu > 0,\ \kappa > 0 \ \text{ on } \mathcal{R} \end{array}\right\} \qquad (5.20)$$

<u>Proof.</u> The equivalence of the four pairs of inequalities is easily seen by means of the table of relations among $\lambda, \mu, \eta, \nu, \kappa$. The above result could be obtained by specializing Thm. 5.6 in accordance with Thm. 4.2.

Instead we give the following independent proof. Referring $\underset{\sim}{\sigma}$ to its p. axes one may write (5.18) in terms of p. stresses as follows:

$$\Sigma(\underset{\sim}{\sigma}) = \frac{1}{2\eta}\left[(1+\nu)(\sigma_1^2+\sigma_2^2+\sigma_3^2) - \nu(\sigma_1+\sigma_2+\sigma_3)^2\right] \quad (*)$$

<u>Re necessity</u>. Assume $\Sigma(\underset{\sim}{\sigma})>0 \ \forall \ \underset{\sim}{\sigma} \neq \underset{\sim}{0}$ and consider the subsequent special choices of $\underset{\sim}{\sigma}$:

(a) $\sigma_1 = 1$, $\sigma_2 = \sigma_3 = 0$ ($\underset{\sim}{\sigma}$ uni-axial). Then $(*) \Rightarrow \eta > 0$

(b) $\sigma_1 = -(\sigma_2+\sigma_3) \neq 0$ ($\underset{\sim}{\sigma}$ deviatoric). Then $(*), \eta>0 \Rightarrow -1<\nu$

(c) $\sigma_i = 1$ ($\underset{\sim}{\sigma}$ isotropic). Then $(*), \eta>0 \Rightarrow \nu < \frac{1}{2}$

Hence $\eta>0, -1<\nu<\frac{1}{2}$ and thus (5.20) hold.

<u>Re sufficiency</u> Assume (5.20) hold true and note that $(*)$ is equivalent to:

$$\Sigma(\underset{\sim}{\sigma}) = \frac{1}{6\eta}\{(1-2\nu)(\sigma_1+\sigma_2+\sigma_3)^2 + (1+\nu)[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)$$

so that $\Sigma(\underset{\sim}{\sigma})>0 \ \forall \ \underset{\sim}{\sigma} \neq 0$. Hence $\Sigma(\underset{\sim}{\sigma})$ is positive definite.

qed

<u>Exercise 18</u>

(a) Let $\underset{\sim}{\sigma}$ be an instantaneous local stress tensor in a homogeneous, isotropic linear elastic solid and let

$\underset{\sim}{\sigma}'$ and $\underset{\sim}{\sigma}''$ be the isotropic and the deviatoric parts of $\underset{\sim}{\sigma}$ (see Thm. 1.12) so that

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}' + \underset{\sim}{\sigma}'', \quad \underset{\sim}{\sigma}' = \tfrac{1}{3}(\mathrm{tr}\,\underset{\sim}{\sigma})\underset{\sim}{1}, \quad \underset{\sim}{\sigma}'' = \underset{\sim}{\sigma} - \underset{\sim}{\sigma}'$$

Show that

$$\Sigma(\underset{\sim}{\sigma}) = \Sigma(\underset{\sim}{\sigma}') + \Sigma(\underset{\sim}{\sigma}'') \quad !$$

$$\left.\begin{aligned} \Sigma(\underset{\sim}{\sigma}') &= \frac{1-2\nu}{6\eta} I_1^2 = \frac{1-2\nu}{6\eta}(\sigma_1+\sigma_2+\sigma_3)^2 = \frac{3(1-2\nu)}{2\eta} p_0^2 \\ \Sigma(\underset{\sim}{\sigma}'') &= \frac{1}{6\mu}(I_1^2 - 3I_2) = \frac{1}{12\mu}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] = \frac{3s_0^2}{4\mu} \end{aligned}\right] \text{(5}$$

where $I_i = I_i(\underset{\sim}{\sigma})$, σ_i are the p. values of $\underset{\sim}{\sigma}$, while p_0 and s_0 are the octahydral normal and shear stress of $\underset{\sim}{\sigma}$. (see (3.35). Reconcile (5.21) with Thm. 5.5.

(b) Let $\underset{\sim}{\sigma}$ have the same meaning as in part (a). and assume $\Sigma(\underset{\sim}{\sigma})$ positive definite. Show that

$$\Sigma(\underset{\sim}{\sigma}) \geq \frac{g(\nu)}{4\mu}\underset{\sim}{\sigma}\cdot\underset{\sim}{\sigma} \;\forall \text{ symm. two-tensor } \underset{\sim}{\sigma}, \quad (*)$$

where

$$g(\nu) = 1 \quad (-1<\nu<0), \quad g(\nu) = \frac{1-2\nu}{1+\nu} \quad (0 \leq \nu < \tfrac{1}{2}).$$

Show further that the estimate (*) is optimal in the sense that $\exists\, \underset{\sim}{\sigma} \ni$ equality holds in (*).

Remark. One calls $\Sigma(\underset{\sim}{\sigma}')$ and $\Sigma(\underset{\sim}{\sigma}'')$ of (5.21) the strain-energy density of "volume-change" and of "distortion", respectively. Explain reason.