A nonlinear field theory of deformable dielectrics

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Work with
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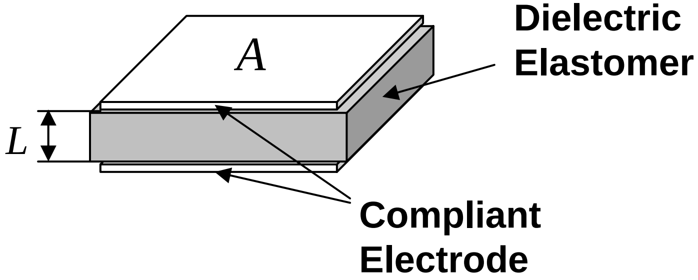
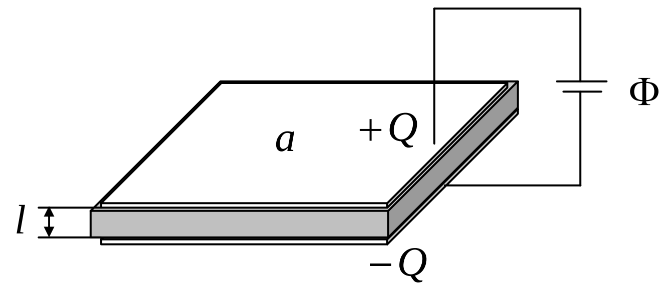
Talk 1 in Session 21-2-2, 10:00 am - 12:00 pm, Wednesday, 6 June 2007, McMat 2007, Austin, Texas
Slides are available at http://imechanica.org/node/635

Dielectric elastomer actuators

- Electromechanical coupling
- Large deformation, light weight, low cost…
- Soft robots, artificial muscles…





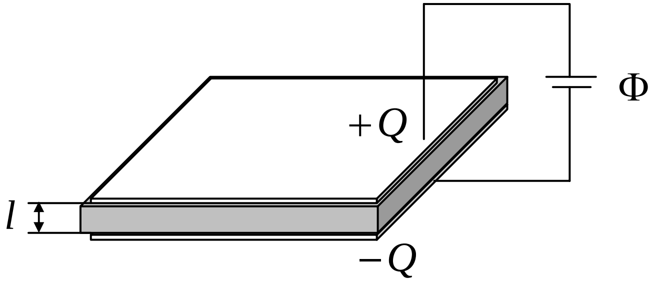
Reference State

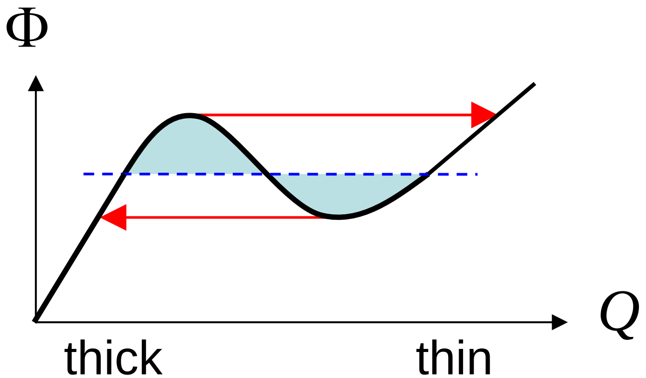
Current State

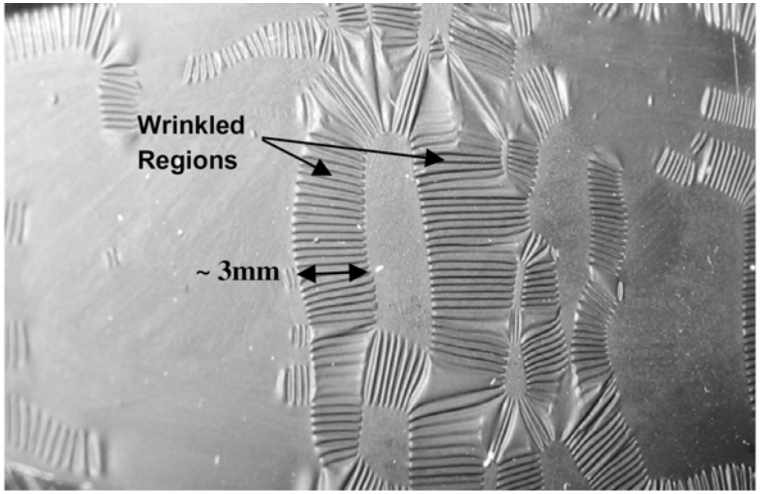
Pelrine, Kornbluh, Pei, Joseph
High-speed electrically actuated elastomers with strain greater than 100%.
Science 287, 836 (2000).

Electromechanical instability

Zhao, X, Hong, W., Suo, Z., 2007. http://imechanica.org/node/1283.



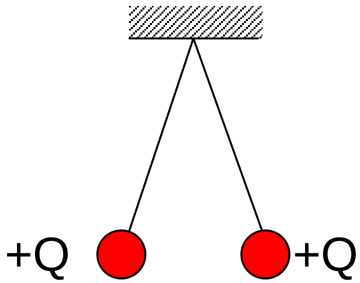




Coexistent states: flat and wrinkled

Plante, Dubowsky,
Int. J. Solids and Structures **43**, 7727 (2006).

Trouble with electric force



In a vacuum,
force between charges can be measured
Define electric field by E = F/Q

$$E_i = -\Phi_{,i} \qquad D_{i,i} = q$$

$$D_i = \varepsilon E_i$$

$$F_i = qE_i$$

In a solid,
force between charges is
NOT an operational concept

Historical work
- Toupin (1956)
- Eringen (1963)
- Tiersten (1971)
……

Recent work
- Dorfmann, Ogden (2005)
- Landis, McMeeking (2005)
- **Suo, Zhao, Greene (2007)**
……

Maxwell stress in vacuum (1864?)

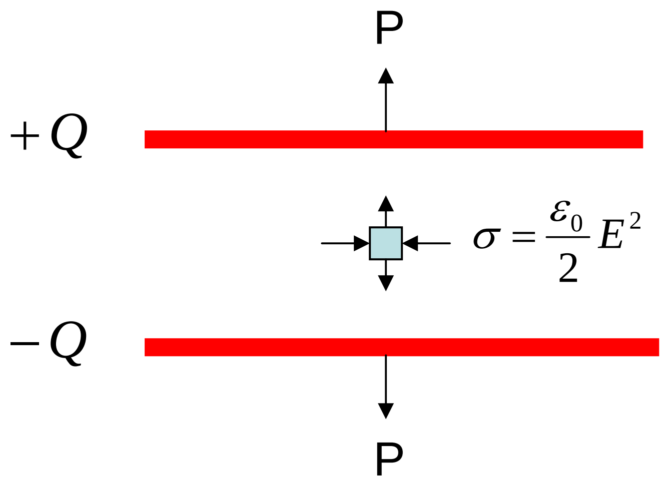
$$E_i = -\Phi_{,i} \qquad D_{i,i} = q \qquad D_i = \varepsilon_0 E_i$$

$$F_i = qE_i = D_{j,j}E_i = \left(D_j E_i\right)_{,j} - D_j E_{i,j}$$

$$D_j E_{i,j} = \varepsilon_0 E_j E_{j,i} = \frac{\varepsilon_0}{2}\left(E_k E_k\right)_{,i} \qquad E_{i,j} = -\Phi_{,ij} = E_{j,i}$$

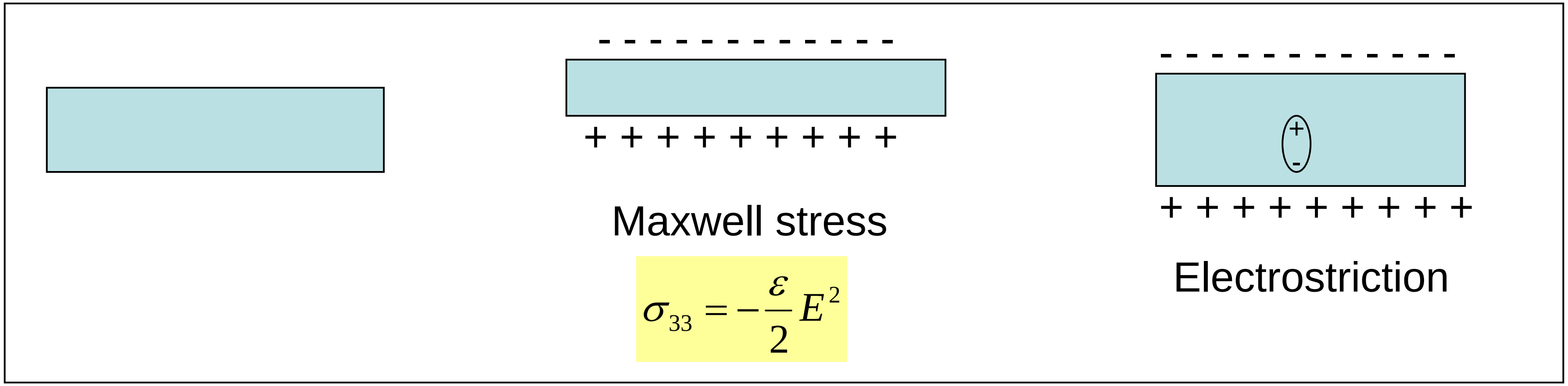
$$F_i = \left(\varepsilon_0 E_j E_i - \frac{\varepsilon_0}{2} E_k E_k \delta_{ij}\right)_{,j}$$

$$\sigma_{ij} = \varepsilon_0 E_j E_i - \frac{\varepsilon_0}{2} E_k E_k \delta_{ij}$$



Trouble with Maxwell stress in dielectrics

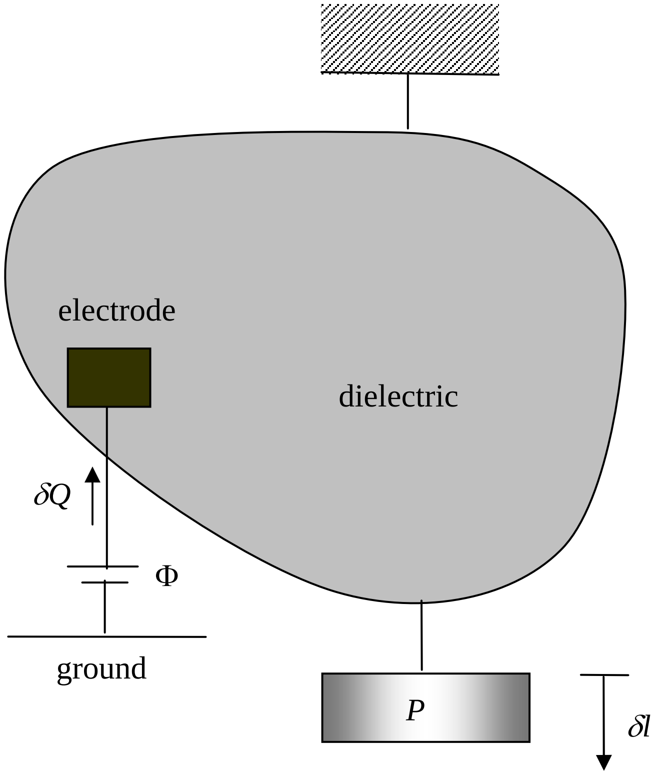
Maxwell said that he didn't make progress with dielectrics, but his qualms have not prevented people from using his name anyway…



- In dielectrics, electric force is not an operational concept.
- ε varies with deformation in general.
- Why E^2 dependence?
- How about the sign of the Maxwell stress?

In solid, Maxwell stress has NO theoretical basis

All troubles are gone if we focus on measurable quantities



A transducer

Work done by the weight $P\delta l$

Work done by the battery $\Phi\delta Q$

A system of elastic conductors and dielectrics is specified by a free-energy function $U(l,Q)$

$$\delta U = P\delta l + \Phi\delta Q$$

$$P(l,Q) = \frac{\partial U(l,Q)}{\partial l}, \qquad \Phi(l,Q) = \frac{\partial U(l,Q)}{\partial Q}$$

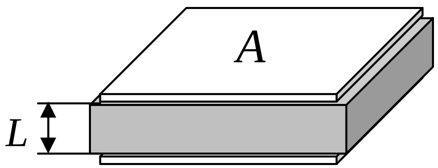
$U(l,Q)$ is measurable

Suo, Zhao, Greene

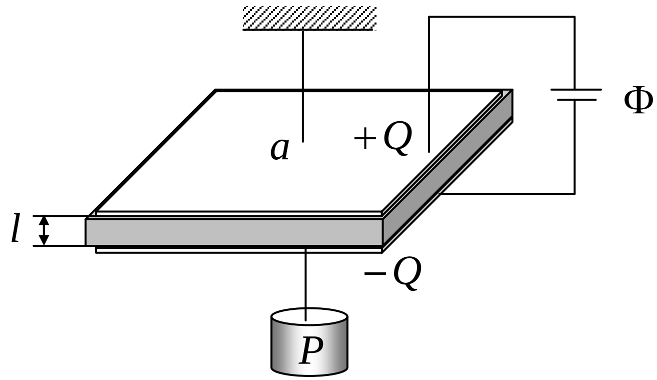
JMPS, in press. http://imechanica.org/node/635

Intensive quantities

Reference State



Current State



Intensive Quantities

$\lambda = l / L$

$s = P / A$

$\tilde{E} = \Phi / L \qquad (E = \Phi / l)$

$\tilde{D} = Q / A \qquad (D = Q / a)$

Work done by the weight $P\delta l = (sA)\delta(\lambda L) = (AL)s\delta\lambda$

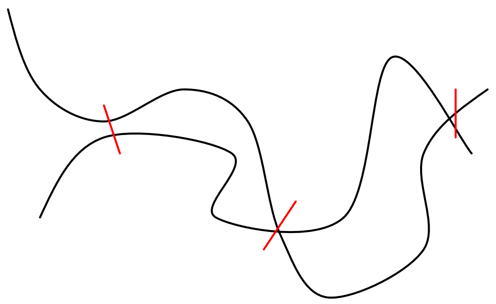
Work done by the battery $\Phi\delta Q = (\tilde{E}L)\delta(\tilde{D}A) = (AL)\tilde{E}\delta\tilde{D}$

Free energy per unit volume $W(\lambda, \tilde{D})$

$$\delta U = P\delta l + \Phi\delta Q \Longrightarrow \delta W = s\delta\lambda + \tilde{E}\delta\tilde{D} \Longrightarrow \begin{cases} s = \partial W / \partial\lambda \\ \tilde{E} = \partial W / \partial\tilde{D} \end{cases}$$

Ideal dielectric elastomer

Decouple elastic and dielectric energy



$$W(\lambda, \tilde{D}) = W_0(\lambda) + W_1(D)$$

Linear dielectric liquid $W_1(D) = \frac{D^2}{2\varepsilon}$

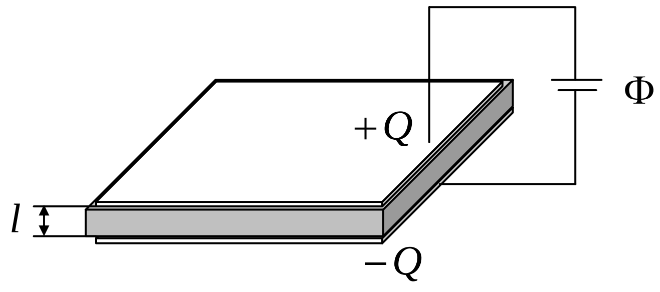
$$D = \frac{Q}{a} = \frac{Q}{\lambda_1 \lambda_2 A} = \lambda \tilde{D}$$

Arruda-Boyce elastomer: $W_0 = \mu \left[\frac{1}{2}(I-3) + \frac{1}{20N}(I^2 - 9) + \frac{11}{1050N^2}(I^3 - 27) + \ldots \right]$

μ: small-strain shear modulus
N: number of rigid units between neighboring crosslinks

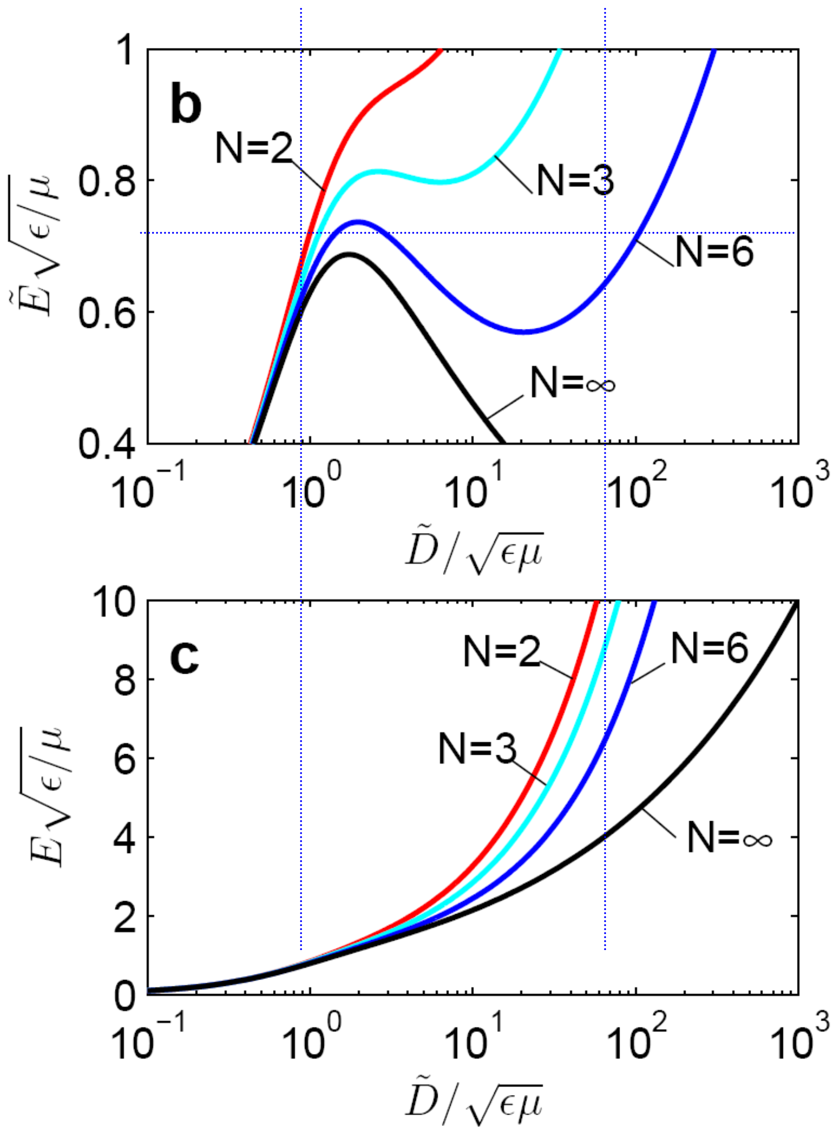
$I = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$

Hysteresis and coexistent states



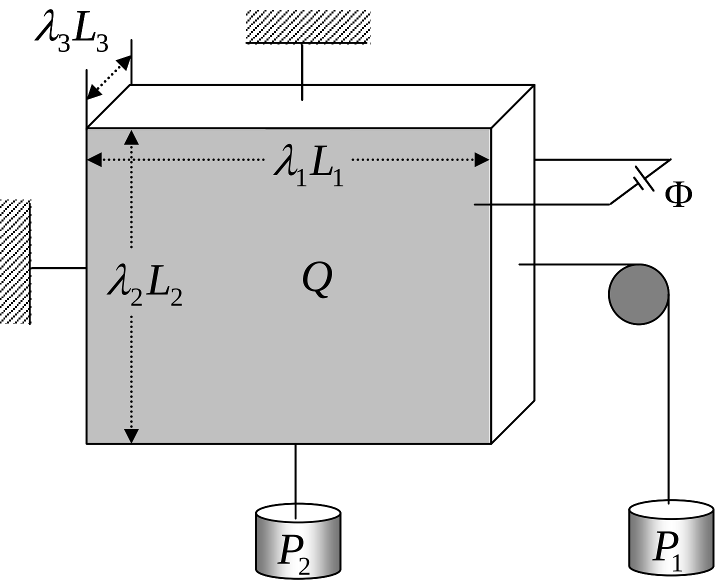
$$\tilde{E} = \Phi / L \qquad (E = \Phi / l)$$

$$\tilde{D} = Q / A \qquad (D = Q / a)$$



Zhao, X, Hong, W., Suo, Z., 2007. http://imechanica.org/node/1283.

Equilibrium & Stability



Free energy of the system

$$G = L_1 L_2 L_3 W\left(\lambda_1, \lambda_2, \tilde{D}\right) - P_1 \lambda_1 L_1 - P_2 \lambda_2 L_2 - \Phi Q$$

Elastomer weights battery

$$\frac{\delta G}{L_1 L_2 L_3} = \left(\frac{\partial W}{\partial \lambda_1} - s_1\right)\delta\lambda_1 + \left(\frac{\partial W}{\partial \lambda_2} - s_2\right)\delta\lambda_2 + \left(\frac{\partial W}{\partial \tilde{D}} - \tilde{E}\right)\delta\tilde{D}$$

$$+ \frac{1}{2}\frac{\partial^2 W}{\partial \lambda_1^2}\delta\lambda_1^2 + \frac{1}{2}\frac{\partial^2 W}{\partial \lambda_2^2}\delta\lambda_2^2 + \frac{1}{2}\frac{\partial^2 W}{\partial \tilde{D}^2}\delta\tilde{D}^2$$

$$+ \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}\delta\lambda_1\delta\lambda_2 + \frac{\partial^2 W}{\partial \lambda_1 \partial \tilde{D}}\delta\lambda_1\delta\tilde{D} + \frac{\partial^2 W}{\partial \lambda_2 \partial \tilde{D}}\delta\lambda_2\delta\tilde{D}$$

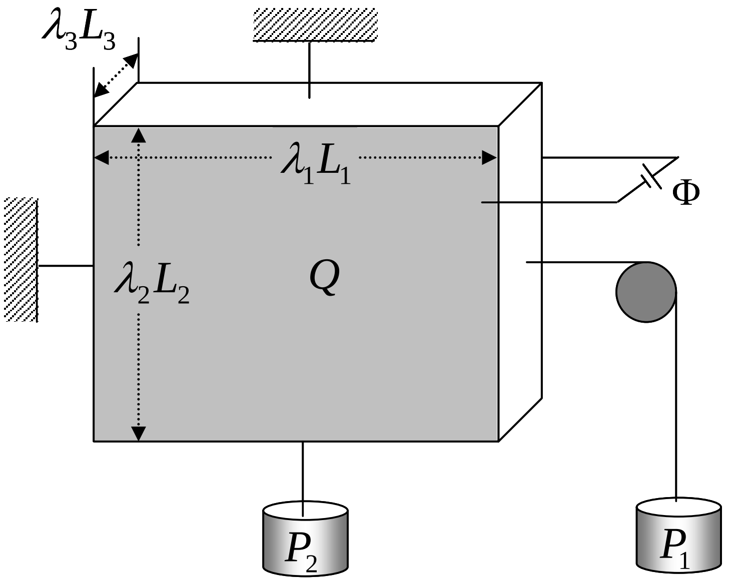
Equilibrium state

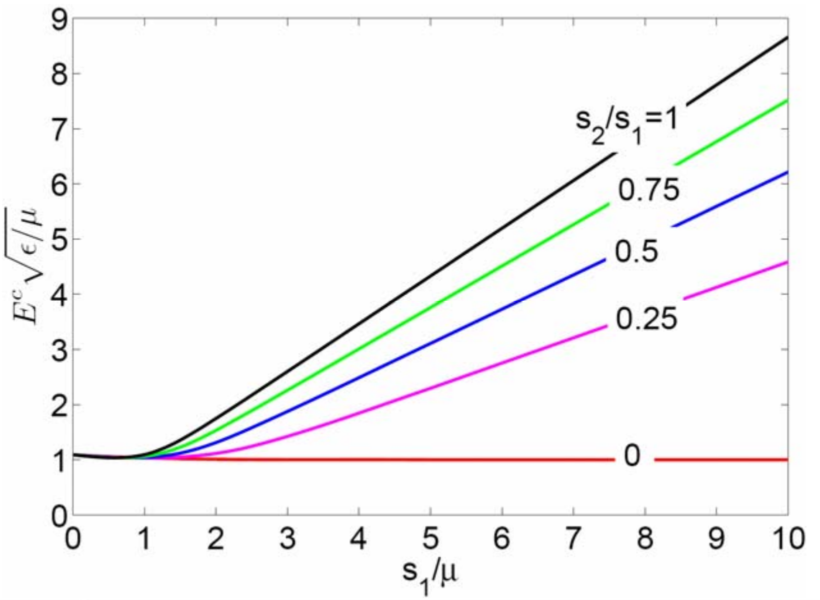
$$s_1 = \frac{\partial W}{\partial \lambda_1} \qquad s_2 = \frac{\partial W}{\partial \lambda_2} \qquad \tilde{E} = \frac{\partial W}{\partial \tilde{D}}$$

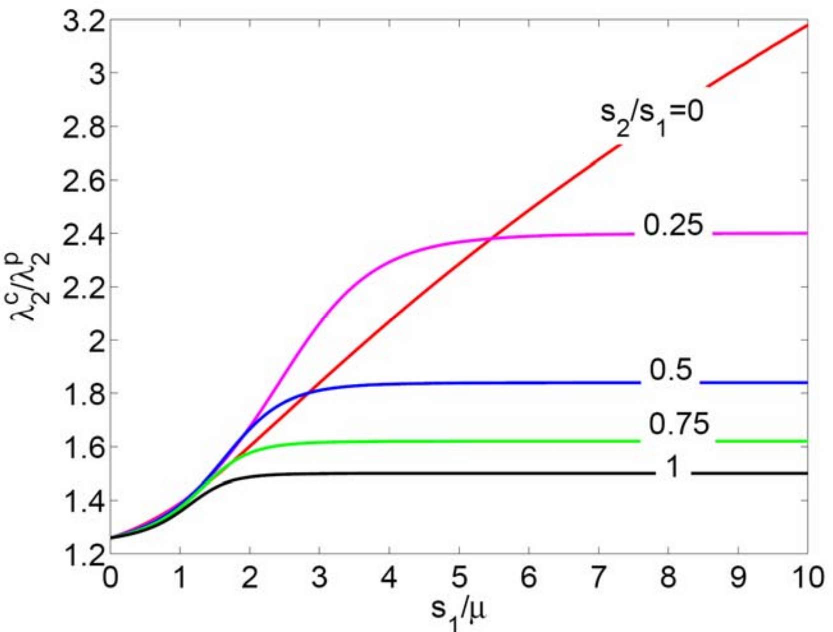
Equilibrium state becomes unstable when the Hessian ceases to be positive-definite

$$\det(\mathbf{H}) = 0$$

Pre-stresses enhance actuation







Experiment: Pelrine, Kornbluh, Pei, Joseph Science 287, 836 (2000).

Theory: Zhao, Suo
http://imechanica.org/node/1456

Inhomogeneous field

A field of weights

$$F_{iK}(\mathbf{X},t) = \frac{\partial x_i(\mathbf{X},t)}{\partial X_K} \qquad \int s_{iK} \frac{\partial \xi_i}{\partial X_K} dV = \int \left(\tilde{b}_i - \tilde{\rho} \frac{\partial^2 x_i}{\partial t^2} \right) \xi_i dV + \int \tilde{t}_i \xi_i dA$$

A field of batteries

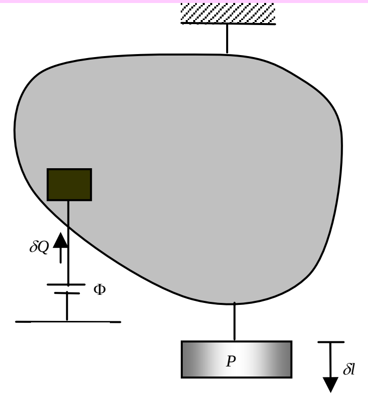
$$\tilde{E}_K(\mathbf{X},t) = -\frac{\partial \Phi(\mathbf{X},t)}{\partial X_K} \qquad \int \left(-\frac{\partial \eta}{\partial X_K} \right) \tilde{D}_K dV = \int \eta \tilde{q} dV + \int \eta \tilde{\omega} dA$$

Material laws

$$\delta W = s_{iK} \delta F_{iK} + \tilde{D}_K \delta \tilde{E}_K$$

$$s_{iK}(\mathbf{F}, \tilde{\mathbf{D}}) = \frac{\partial W(\mathbf{F}, \tilde{\mathbf{D}})}{\partial F_{iK}} \qquad \tilde{E}_K(\mathbf{F}, \tilde{\mathbf{D}}) = \frac{\partial W(\mathbf{F}, \tilde{\mathbf{D}})}{\partial \tilde{D}_K}$$

- Linear PDEs
- Nonlinear material laws



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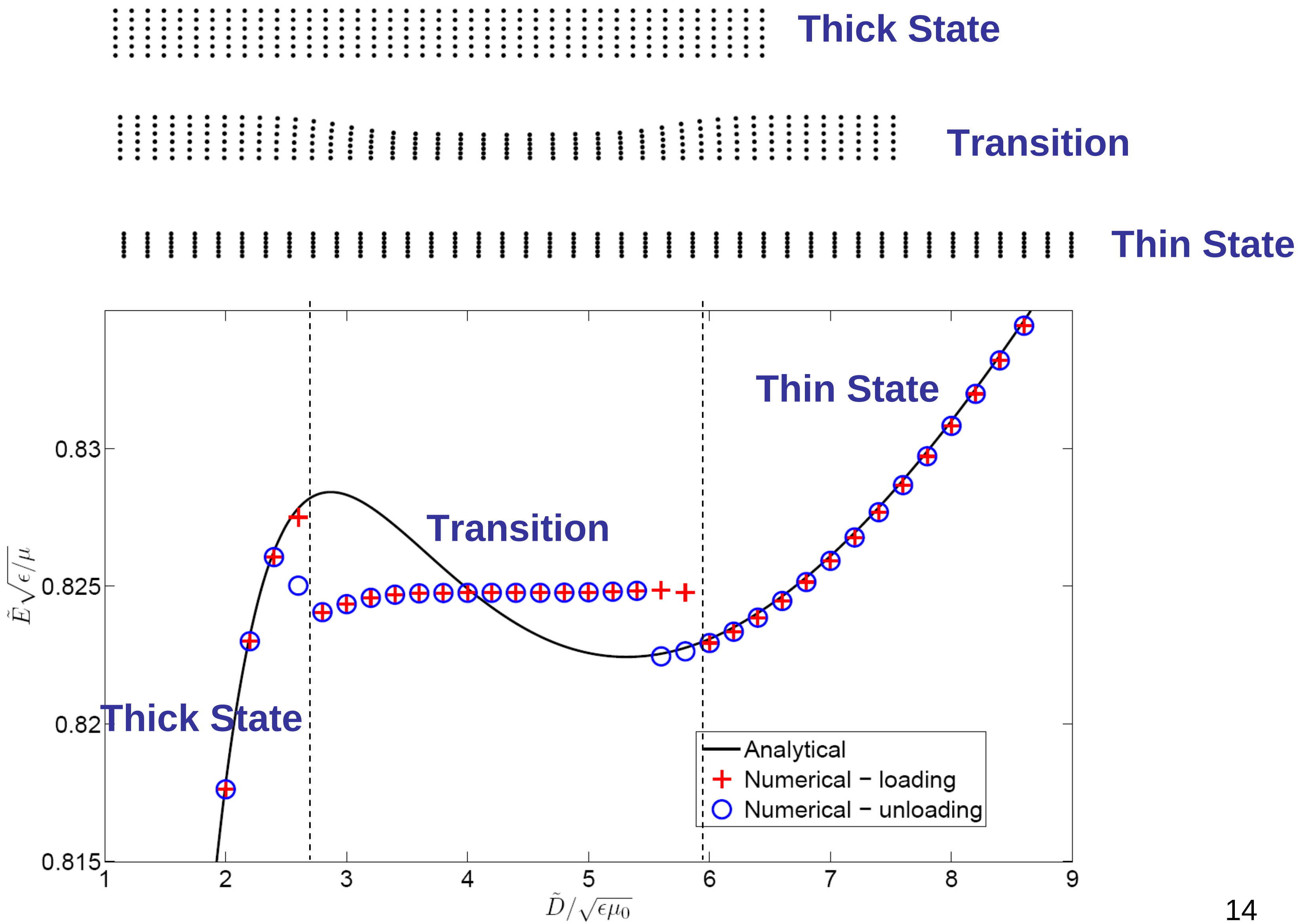
Finite element method



Zhou, Hong, Zhao, Zhang, Suo http://imechanica.org/node/1447

Summary

- A nonlinear field theory. No Maxwell stress. No electric body force. No polarization vector.
- Electromechanical instability.
- Hysteresis and coexistent states.
- Finite element method.

These slides are available at http://imechanica.org/node/635.
iMechanica get together. Wednesday, 5.45pm-9:00pm, Room 2.120. Beer, snacks…