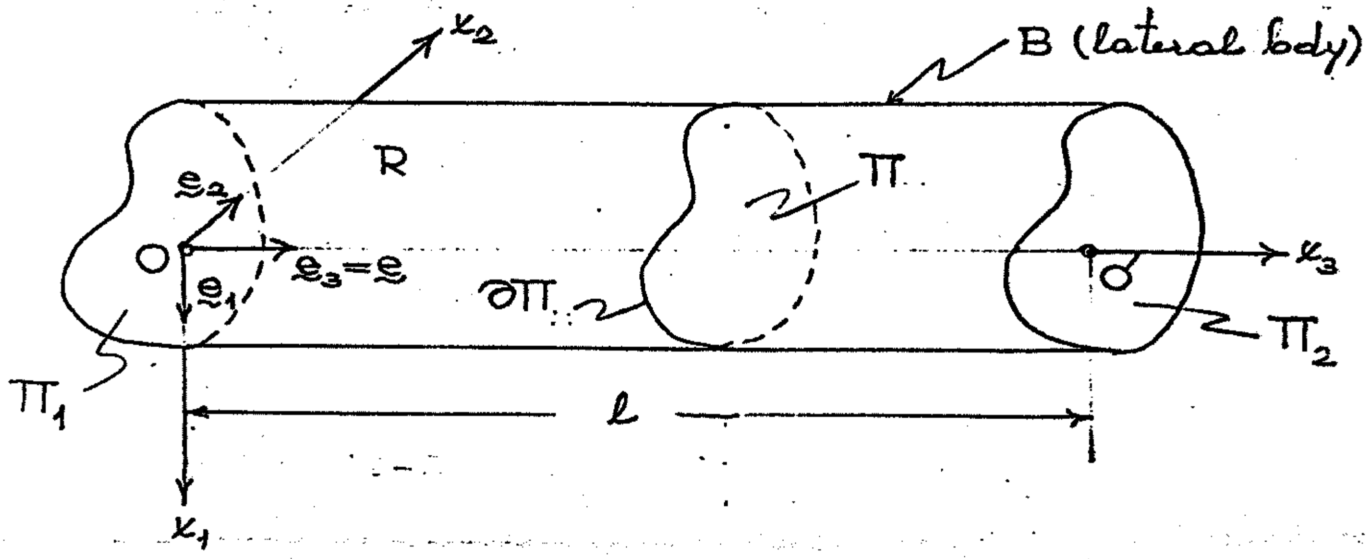
13. Saint-Venant's problem: extension, torsion, and bending of cylindrical or prismatic bodies.

References: Sokolnikoff, Timoshenko-Goodier

Formulation of original problem.

R ... open cylindrical or prismatic region of length l

Π ... open cross-sectional region of R



Π ... bounded, regular plane region: bounded open plane region whose boundary $\partial\Pi$ is the union of a finite number of nonintersecting simple closed, piecewise smooth curves.

Note: Π need not be simply connected

$$\left.\begin{aligned} &R=\{\underset{\sim}{x} \mid (x_1,x_2)\in\Pi,\ 0<x_3<\ell\} \\ &\Pi_1=\{\underset{\sim}{x} \mid (x_1,x_2)\in\Pi,\ x_3=0\},\ \Pi_2=\{\underset{\sim}{x} \mid (x_1,x_2)\in\Pi,\ x_3=\ell\} \\ &B=\{\underset{\sim}{x} \mid (x_1,x_2)\in\partial\Pi,\ 0\le x_3\le\ell\} \end{aligned}\right\} \quad (13$$

R is a bounded regular region in E, $\partial R=\Pi_1\cup\Pi_2\cup B$

We seek

$$\delta=[\underset{\sim}{u},\underset{\approx}{E},\underset{\approx}{S}]\in\mathcal{E}(\mu,\nu,\bar{R}),\quad \mu>0,-1<\nu<1/2 \qquad (13.$$

subject to the <u>boundary conds.</u>

$$\underset{\sim}{s}=\underset{\sim}{0} \text{ on } B,\ \underset{\sim}{s}=\overset{*}{\underset{\sim}{s}}{}^{(1)} \text{ on } \Pi_1,\ \underset{\sim}{s}=\overset{*}{\underset{\sim}{s}}{}^{(2)} \text{ on } \Pi_2, \qquad (13.$$

with

$$\int_{\Pi_1}\overset{*}{\underset{\sim}{s}}{}^{(1)}dA+\int_{\Pi_2}\overset{*}{\underset{\sim}{s}}{}^{(2)}dA=0,\quad \int_{\Pi_1}\underset{\sim}{x}\wedge\overset{*}{\underset{\sim}{s}}{}^{(1)}dA+\int_{\Pi_2}\underset{\sim}{x}\wedge\overset{*}{\underset{\sim}{s}}{}^{(2)}dA=\underset{\sim}{0} \qquad (13$$

Recall from Thm. 6.6 that (13.4) are necessary for the existence of sol. to (13.2), (13.3).

Remark on formidable complexity of above second boundary-value problem.

Relaxed boundary conds.

Here one assigns merely the resultant force $\underset{\sim}{L}$ and the resultant moment $\underset{\sim}{M}$ about O′ of the tractions on Π_2. Thus, replace (13.3) by

$$\underset{\sim}{s} = \underset{\sim}{0} \text{ on } B, \quad \int_{\Pi_2} \underset{\sim}{s}\, dA = \underset{\sim}{L}, \quad \int_{\Pi_2} (\underset{\sim}{x} - \underset{\sim}{e}_3 \ell) \wedge \underset{\sim}{s}\, dA = \underset{\sim}{M} \quad (13$$

Note that (13.2), (13.5) $\Rightarrow$ (13.4). ?

Describe original S.V. principle. Explain its practical and theoretical importance. Mention Boussinesq's generalization. Refer to Quart. Appl. Math. 11, 1954 p. 393 for a statement and proof of an amended version of unusual principle. See Toupin, ARMA 18, 2, 1965 for an attempt to prove original prin Mention Knowles & Co.

Conditions (13.5) are equivalent to

$$\sigma_{i\alpha} n_\alpha = 0 \quad \text{on } B \quad (\alpha = 1,2)^{\S}$$

$$\int_{\Pi_2} \sigma_{3i}\, dA = L_i$$

$$\int_{\Pi_2} \sigma_{33} x_2\, dA = M_1 , \quad \int_{\Pi_2} \sigma_{33} x_1\, dA = -M_2 , \quad \int_{\Pi_2} (\sigma_{32} x_1 - \sigma_{31} x_2)\, dA = M_3$$

From here on assume O is the centroid of Π_1, so that

$$\int_{\Pi} x_\alpha\, dA = 0 , \quad A = \int_{\Pi} dA \quad (\text{area of } \Pi) \qquad (13.7$$

Classification of the relaxed Saint-Venant problem

By virtue of the principle of superposition it suffices to deal separately with the following four loading cases:

I.) $\underset{\sim}{L} = L \underset{\sim}{e}_3 , \underset{\sim}{M} = \underset{\sim}{0}$ ... "pure extension"

II.) $\underset{\sim}{L} = \underset{\sim}{0}$, say $\underset{\sim}{M} = M \underset{\sim}{e}_2$ ... "pure bending"

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§ Greek indices are henceforth understood to have the range

III.) $\underset{\sim}{L} = \underset{\sim}{0}$, $\underset{\sim}{M} = M \underset{\sim}{e}_3$ ... "pure torsion"

IV.) Say, $\underset{\sim}{L} = L \underset{\sim}{e}_1$, $\underset{\sim}{M} = \underset{\sim}{0}$ ... "Saint-Venant bending"

Case I: Pure extension

$$L_\alpha = 0 \, , \, L_3 = L \, , \, M_i = 0 \tag{13.8}$$

A state $\mathcal{S}$ conforming to (13.2), (13.6), (13.8) is trivially given by ...

$$\mathcal{S}: \quad \sigma_{ij} = \delta_{3i} \delta_{3j} \frac{L}{A} \, , \quad u_\alpha = -\frac{\nu L x_\alpha}{\eta A} \, , \quad u_3 = \frac{L x_3}{\eta A} \tag{13.9}$$

Verify (13.9) directly. Emphasize lack of uniqueness, suppression of arb. additive infinites. rigid displt. Note that, but for the usual indeterminacy in $\underset{\sim}{u}$, $\mathcal{S}$ given by (13.9) is the unique sol. of the original S.V. problem corresponding to the data

$$\overset{*}{\underset{\sim}{s}}{}^{(1)} = -\frac{L}{A} \underset{\sim}{e}_3 \text{ on } \Pi_1 \, , \quad \overset{*}{\underset{\sim}{s}}{}^{(2)} = \frac{L}{A} \underset{\sim}{e}_3 \text{ on } \Pi_2 \, .$$

Mention extension of beam under its own weight (Sok. a

* Note weak dependence on shape of Π, σ indept. of elast

Case II : Pure bending

$$L_i = 0 \,, \; M_1 = M_3 = 0 \,, \; M_2 = M \tag{13.1}$$

Assume for convenience that the x_α-axes are principal axes of inertia of Π_1 at O. Thus,

$$\int_{\Pi} x_1 x_2 \, dA = 0 \;, \qquad \int_{\Pi} x_1^2 \, dA = I_2 \equiv I \tag{13.1}$$

Guided by the elementary beam theory (Euler-Bernou
assume that $\mathcal{S}$ has the stress field

$$\sigma_{ij} = -\delta_{3i}\delta_{3j}\frac{M x_1}{I} \tag{13.}$$

This field clearly satisfies the stress eqs. of equil. & compatibility (6.3′) and (6.23) [Beltrami-Michell]. Furthe the existence of single-valued $\underset{\sim}{u}$ is assured even if Π is multiply connected since $\underset{\sim}{\sigma}$ may be continued analytic throughout E. One easily verifies that $\underset{\sim}{\sigma}$ meets (13.6), (13
Carry out.

By means of (6.2′), Thm. 2.16 or by direct integration of (6.17′) [displacement-stress rels.] one finds

$$
\left.\begin{aligned}
u_1 &= \frac{M}{2\eta I}(x_3^2 + \nu x_1^2 - \nu x_2^2) + \underline{\mathring{a}_1 + \mathring{w}_2 x_3 - \mathring{w}_3 x_2} \\
u_2 &= \frac{M}{\eta I}\nu x_1 x_2 + \underline{\mathring{a}_2 + \mathring{w}_3 x_1 - \mathring{w}_1 x_3} \\
u_3 &= -\frac{M}{\eta I} x_1 x_3 + \underline{\mathring{a}_3 + \mathring{w}_1 x_2 - \mathring{w}_2 x_1} \\
& \qquad\qquad (\mathring{a}_i, \mathring{w}_i \ldots \text{ constants})
\end{aligned}\right\} (13.
$$

(13.13) may be verified by substitution into (6.17').

Discussion. Identify additive rigid displt. Note $\mathbf{s}$ is exact sol. of original S.V. problem governed by (13.2), (13.3) if and only if the loading is applied consistent with (13.12). Note weak dependence on shape of Π, $\underset{\sim}{\sigma}$ inde of elastic consts.

Consider a cantilever beam fixed at $x_3 = 0$: note singular mixed problem. We may meet relaxed fixity conds. by suitable disposition over $\underset{\sim}{\mathring{a}}, \underset{\sim}{\mathring{w}}$. Thus take

$$u_i(\underset{\sim}{0}) = 0\ , \quad u_{1,3}(\underset{\sim}{0}) = u_{2,3}(\underset{\sim}{0}) = u_{2,1}(\underset{\sim}{0}) = 0 \qquad (*)\ \S$$

which correspond to fixing O, an element of the x_1-axis, and an element of the plane $x_3 = 0$ at O. (*) are met i

§ Suppose $\underset{\sim}{0} \in \Pi_0$. Note $(*) \Rightarrow u_{i,j}(\underset{\sim}{0}) = 0$.

$$\overset{\circ}{\underset{\sim}{a}} = \underset{\sim}{0}, \quad \overset{\circ}{\underset{\sim}{w}} = \underset{\sim}{0}$$

and (13.13) now yield

$$u_1(\underset{\sim}{e}_3 z) = \frac{Mz^2}{2\eta I}, \quad u_2(\underset{\sim}{e}_3 z) = u_3(\underset{\sim}{e}_3 z) = 0 \qquad (0 \leq z \leq l)$$

in agreement with the elementary beam theory.

Case III: Pure torsion

$$L_i = 0, \quad M_\alpha = 0, \quad M_3 = M \tag{13.14}$$

(13.14), (13.6) yield the boundary conds.:

$$\sigma_{\alpha\beta} n_\beta = 0 \text{ on } B \text{ (a)} \qquad \sigma_{3\beta} n_\beta = 0 \text{ on } B \text{ (b)} \qquad (13.1$$

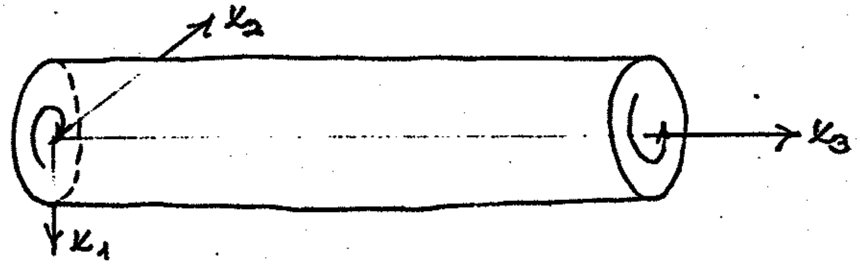
$$\int_{\Pi_2} \sigma_{3\alpha}\, dA = 0 \qquad \text{(a)}$$

$$\int_{\Pi_2} \sigma_{33}\, dA = 0, \quad \int_{\Pi_2} x_\alpha \sigma_{33}\, dA = 0 \qquad \text{(b)} \quad (1$$

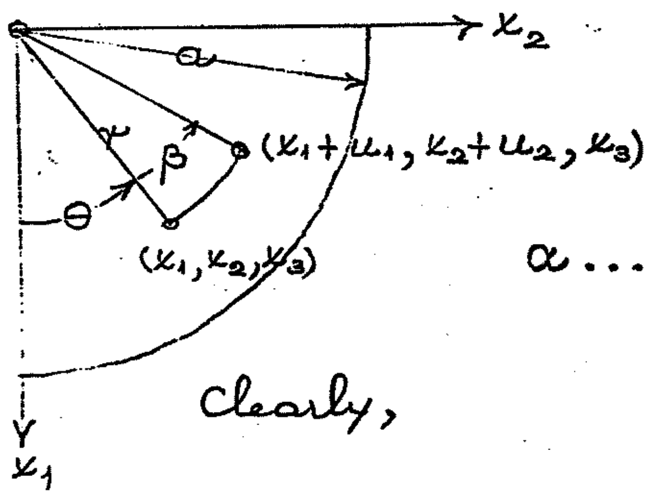
$$\int_{\Pi_2} (\sigma_{32} x_1 - \sigma_{31} x_2)\, dA = M \qquad \text{(c)}$$

The special case of the circular cylinder (Coulomb)

$$\Pi = \{ (x_1, x_2) \mid 0 \leq x_1^2 + x_2^2 < a^2 \}$$



Coulomb's assumption: each circular cross-section rotates rigidly about the x_3-axis relative to $x_3 = 0$ through an angle proportional to x_3. (No warping)



$$\beta = \alpha x_3$$

α ... angle of twist per unit length (specific angle of twist)

Clearly,

$$u_1 = r\cos(\theta+\beta) - r\cos\theta \doteq x_1(\cos\beta - 1) - x_2 \sin\beta$$

$$u_2 = r\sin(\theta+\beta) - r\sin\theta \doteq x_2(\cos\beta - 1) + x_1 \sin\beta$$

Expand in powers of β and linearize (infinites. defor to reach

$$u_1(\underline{x}) \doteq -\alpha x_2 x_3 \ , \quad u_2(\underline{x}) \doteq \alpha x_1 x_3 \ , \quad u_3(\underline{x}) \doteq 0 \quad \forall\, \underline{x} \in \overline{R} \qquad (13.$$

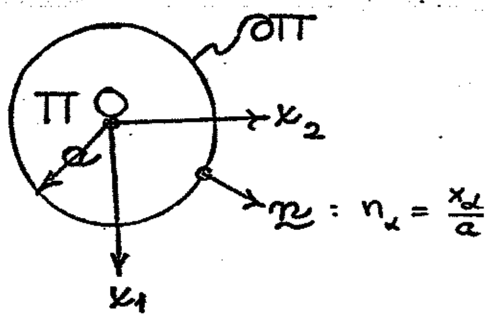
(13.17) $\Rightarrow \vartheta = \nabla \cdot \underline{u} = u_{k,k} = 0$ and thus (13.17), (6.17′) $\Rightarrow$

$$\sigma_{ii} = 0 \ (\text{no sum}) \, , \ \sigma_{12} \doteq 0 \, , \ \sigma_{31} \doteq -\mu\alpha x_2 \, , \ \sigma_{32} \doteq \mu\alpha x_1 \quad \text{on } \overline{R} \qquad (13$$

Evidently $\underline{\sigma}$ satisfies (6.3′), i.e. $\sigma_{ij,j} = 0$.

<u>Re boundary conds.</u>: (13.15 a) hold. (13.15b) here bec

$$\sigma_{31}\frac{x_1}{a} + \sigma_{32}\frac{x_2}{a} = 0 \ \text{on } \partial\Pi$$



which is met by $\underline{\sigma}$ of (13.18).

(13.16 a, b) hold. (13.16c) $\Leftrightarrow$

$$\mu\alpha \int_{\Pi} (x_1^2 + x_2^2)\, dA = M \quad \text{whence}$$

$$\alpha = \frac{M}{\mu I} \ , \quad I = \int_{\Pi} (x_1^2 + x_2^2)\, dA \qquad (13.$$

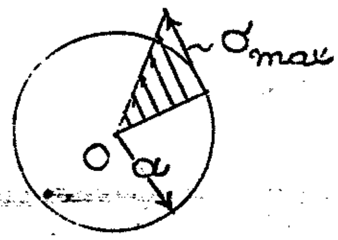
Consistent with (3.17), (13.18) one has

$$\underline{\sigma}_3(\underline{x}) = \underline{S}(\underline{x}, \underline{e}_3) = \underline{e}_1 \sigma_{31}(\underline{x}) + \underline{e}_2 \sigma_{32}(\underline{x}) \quad \text{or}$$

$$\boldsymbol{\sigma}_3(\mathbf{x}) = \mathbf{e}_1(-\mu\alpha x_2) + \mathbf{e}_2(\mu\alpha x_1)$$

Hence and by (13.19),

$$\left.\begin{aligned} &\boldsymbol{\sigma}_3(\mathbf{x})\cdot\mathbf{r} = 0\ , \quad \mathbf{r} = \mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 \\ &\sigma(\mathbf{x}) \equiv |\boldsymbol{\sigma}_3(\mathbf{x})| = \mu\,|\alpha|\,r = \frac{|M|\,r}{I}\ , \quad r = |\mathbf{r}| \end{aligned}\right\} \quad (13.$$



Coulomb's sol. is the exact sol. of original S.V. prob. for loads applied consistent with (13.18), (13.20). Mention hollow circular cylinder.

Remark on exact sol. for arbitrary axisymm. shear on Π_1, Π_2 (see Sok., art 50). [Special quantitative confirmation of S.V. principle]. Remark on pure torsion of solids of revolution: see, for example, Sok., art. 49. For a quant. treatment of S.V. princ. in this particular context, see Knowles & Co., ARMA, 22, 2, 1966.

Torsion of beams of arbitrary cross-section

Navier erroneously assumed that Coulomb's solution applies to arbitrary Π. We show first that this is impossible

Thus suppose (13.18) holds and invoke (13.15b). Hence

$$-x_2 n_1 + x_1 n_2 = 0 \text{ on } \partial\Pi \quad (1)$$

Assume each component of $\partial\Pi$ parametrized by

$$x_\alpha = x_\alpha(s) \quad (0 \le s \le s_1) \quad (2)$$

where s ... arc-length. Then from calculus, at each regular point of $\partial\Pi$,

$$n_1 = \frac{dx_2}{ds}, \quad n_2 = -\frac{dx_1}{ds} \qquad (13.21)$$

(1), (13.21) $\Rightarrow$

$$x_1 \frac{dx_1}{ds} + x_2 \frac{dx_2}{ds} = 0 \Rightarrow \frac{d}{ds}(x_1^2 + x_2^2) = 0 \Rightarrow x_1^2 + x_2^2 = a^2 \text{ on } \partial\Pi$$

i.e. $\partial\Pi$ is circular.

In the light of above, and adopting once again a "semi-inverse" approach, we modify (13.17) by setting

$$u_1(\underset{\sim}{x}) = -\alpha x_2 x_3, \quad u_2(\underset{\sim}{x}) = \alpha x_1 x_3, \quad u_3(\underset{\sim}{x}) = \alpha\varphi(x_1, x_2) \quad (13.2$$

$$\forall \underset{\sim}{x} \in \overline{R}$$

Interpret (13.22) geometrically: α ... specific angle of twist, φ ... warping function. Here again $\vartheta = u_{k,k} = 0$ on R and (13.22), (6.17') $\Rightarrow$

$$\left.\begin{aligned} &\sigma_{ii} = 0 \ (\text{no sum}),\ \sigma_{12} = 0, \\ &\sigma_{31} = \mu\alpha\left(\frac{\partial\varphi}{\partial x_1} - x_2\right),\ \sigma_{32} = \mu\alpha\left(\frac{\partial\varphi}{\partial x_2} + x_1\right) \end{aligned}\right\} \quad (13.$$

State of pure shear. Subst. from (13.23) into (6.3') to get

$$\nabla^2\varphi = 0 \text{ on } \Pi \qquad (13.$$

Turn to the <u>boundary conds</u> (13.15), (13.16):

(13.15a) ok. (13.15b), (13.23) $\Rightarrow$

$$\left(\frac{\partial\varphi}{\partial x_1} - x_2\right)n_1 + \left(\frac{\partial\varphi}{\partial x_2} + x_1\right)n_2 = 0 \text{ on } \partial\Pi \text{ or}$$

$$\frac{\partial\varphi}{\partial n} = x_2 n_1 - x_1 n_2 \text{ on } \partial\Pi \qquad (13.$$

which, in view of (13.21), may be written as

$$\frac{\partial\varphi}{\partial n} = \frac{1}{2}\frac{d}{ds}\left(x_1^2 + x_2^2\right) \text{ on } \partial\Pi \qquad (13.$$

Consider next (13.16a). (13.23), (13.24) $\Rightarrow$

$$\sigma_{3\beta,\beta} = 0 \quad \text{on } \Pi .$$

Hence, and from the two-dimensional divergence thm.,

$$\int_\Pi \sigma_{3\alpha}\, dA = \int_\Pi [(x_\alpha \sigma_{3\beta})_{,\beta} - x_\alpha \sigma_{3\beta,\beta}]\, dA = \int_\Pi (x_\alpha \sigma_{3\beta})_{,\beta}\, dA$$

$$= \oint_{\partial\Pi} x_\alpha \sigma_{3\beta} n_\beta\, ds = 0 \quad \text{because of (13.15b).}$$

Hence (13.16a) ok. Further (13.23) $\Rightarrow$ (13.16b).

Finally, consider (13.16c). In view of (13.23), one has

$$\int_\Pi (\sigma_{32} x_1 - \sigma_{31} x_2)\, dA = \mu\alpha \int_\Pi \left(x_1^2 + x_2^2 + x_1 \frac{\partial\varphi}{\partial x_2} - x_2 \frac{\partial\varphi}{\partial x_1}\right) dA = M$$

Thus (13.16c) holds provided,

$$\alpha = \frac{M}{K}, \quad K = \mu \int_\Pi \left(x_1^2 + x_2^2 + x_1 \frac{\partial\varphi}{\partial x_2} - x_2 \frac{\partial\varphi}{\partial x_1}\right) dA \quad (13.$$

K ... torsional rigidity of Π.

<u>Summary</u>: A solution s of current problem is given by (13.22), (13.23), provided φ is a solution of the two-dimensional Neumann problem (13.24), (13.25) and α is subsequently computed from (13.27).

<u>Discussion</u>. According to two-dim. divergence thm.

$$\int_{\Pi} \nabla^2\varphi \, dA = \int_{\Pi} \nabla\cdot\nabla\varphi \, dA = \oint_{\partial\Pi} \underset{\sim}{n}\cdot\nabla\varphi \, ds = \oint_{\partial\Pi} \frac{\partial\varphi}{\partial n} ds = 0 .$$

by (13.24). This nec. cond. for the existence of φ is satisfied because of (13.26). Note that φ is unique up to arb. additive constant, which affects $\underset{\sim}{u}$ merely within an additive translation according to (13.22)
Observe that above anal. holds if (13.7), (13.11) are not met
(13.22), (13.23), (13.24) $\Rightarrow \nabla^2 u_i = \nabla^2 \sigma_{ij} = \nabla^2 \gamma_{ij} = 0$ on R
Recall Polya's results for harmonic elastostatic states.

$s = [\underset{\sim}{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma}]$ is the "unique" sol. for the original S.V. problem governed by end-loads consistent with (13.2
Alternatively, s may be regarded as the unique sol of the mixed-mixed prob. characterized by (13.2), (13
and

$$\sigma_{33} = 0\ ,\quad u_1 = -\alpha x_2 x_3\ ,\quad u_2 = \alpha x_1 x_3 \text{ on } \Pi_1 \cup \Pi_2$$

Mention minimum energy characterization (ARMA, 21, 2, 1966)

The conjugate warping function, Prandtl's stress fun

Here we assume, for simplicity, that Π is simply connecte

See Sok. art. 47 for generalization to multiply connected

Objective: Reduction of Case III to a two-dimensional Dirichl problem.

Let $\varphi \in C^2(\Pi)$ satisfy (13.24) and define ψ on Π through

$$\psi(x_1, x_2) = \int_{(x_1^0, x_2^0)}^{(x_1, x_2)} [-\varphi_{,2}(\xi_1, \xi_2)\, d\xi_1 + \varphi_{,1}(\xi_1, \xi_2)\, d\xi_2]\ \forall\ (x_1, x_2) \in \Pi\ ($$

Then from two-dim. Stokes thm., $\psi \in C^1(\Pi)$ and

$$\frac{\partial\varphi}{\partial x_1} = \frac{\partial\psi}{\partial x_2}\ ,\quad \frac{\partial\varphi}{\partial x_2} = -\frac{\partial\psi}{\partial x_1} \text{ on } \Pi, \qquad (13.$$

which are the Cauchy-Riemann eqs. Thus,

$$f(z) = f(x_1 + i x_2) = \varphi(x_1, x_2) + i\psi(x_1, x_2) \text{ anal. on } \Pi \quad (13$$

$$\nabla^2\psi = 0 \text{ on } \Pi. \qquad (13$$

Note that ψ is uniquely determined by (13.29) except for an arb. additive constant.

φ and ψ are "conjugate harmonic fcs" on Π.

If $\varphi \in C^1(\bar{\Pi})$, $\psi \in C^1(\bar{\Pi})$ one has from (13.29), (13.21),

$$\frac{\partial \varphi}{\partial n} = \varphi_{,\alpha} n_\alpha = \frac{\partial \psi}{\partial x_2}\frac{dx_2}{ds} + \left(-\frac{\partial \psi}{\partial x_1}\right)\left(-\frac{dx_1}{ds}\right) = \psi_{,\alpha}\frac{dx_\alpha}{ds} \text{ on } \partial\Pi$$

Thus,

$$\frac{\partial \varphi}{\partial n} = \frac{d\psi}{ds} \text{ on } \partial\Pi \tag{13.32}$$

Observe that (13.32) hold for every pair of conjugate harmonic functions belonging to $C^1(\bar{\Pi})$.

(13.26), (13.32) $\Rightarrow$

$$\psi(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) \text{ on } \partial\Pi \qquad (13.3$$

if we suppress an inessential additive constant. (Explai

Accordingly ψ is the (unique) solution of the two-di Dirichlet problem (13.31), (13.33). Now (13.29), (13.23) =

$$\left.\begin{array}{l} \sigma_{ii} = 0 \text{ (no sum)},\ \sigma_{12} = 0, \\ \sigma_{31} = \mu\alpha\left(\frac{\partial \psi}{\partial x_2} - x_2\right),\ \sigma_{32} = \mu\alpha\left(-\frac{\partial \psi}{\partial x_1} + x_1\right) \end{array}\right\} (1$$

Introduce the Prandtl stress function ϕ on Π by setting

$$\phi(x_1, x_2) = \psi(x_1, x_2) - \frac{1}{2}(x_1^2 + x_2^2) \quad \forall (x_1, x_2) \in \bar{\Pi}. \quad (13$$

(13.35), (13.34) $\Rightarrow$

$$\sigma_{ii} = 0 \text{ (no sum)}, \; \sigma_{12} = 0, \; \sigma_{31} = \mu\alpha \frac{\partial\phi}{\partial x_2}, \; \sigma_{32} = -\mu\alpha \frac{\partial\phi}{\partial x_1} \quad (13.$$

Also, (13.31), (13.33), (13.35) $\Rightarrow$

$$\nabla^2\phi = -2 \text{ on } \Pi \quad (13.37) \qquad \phi = 0 \text{ on } \partial\Pi \quad (13.38)$$

Accordingly, when cast in terms of ϕ, the relaxed torsi problem reduces to a Dirichlet problem for the two-dim. Poisson eq. (13.37) with a homog. boundary con

To express K via ϕ, note from (13.27), (13.29), (13.35) th

$$K = \frac{M}{\alpha} = -\mu \int_\Pi x_\alpha \phi_{,\alpha} dA = -\mu \int_\Pi (x_\alpha \phi)_{,\alpha} dA + 2\mu \int_\Pi \phi d$$

But

$$\int_\Pi (x_\alpha \phi)_{,\alpha} dA = \oint_{\partial\Pi} \phi x_\alpha n_\alpha ds = 0 \text{ by (13.38), whenc}$$

$$K = \frac{M}{\alpha} = 2\mu \int_\Pi \phi \, dA. \quad (13.39$$

Consider the surface

$$\Sigma : \; z = \phi(x_1, x_2) \;\; \forall \; (x_1, x_2) \in \bar{\Pi} \quad (13.40$$

The contour lines (level curves) of Σ are given by $\phi(x_1, x_2) = k$ (constant). Hence the slope of the countour lines $\frac{dx_2}{dx_1}$ obeys

$$\frac{\partial\phi}{\partial x_1} + \frac{\partial\phi}{\partial x_2}\frac{dx_2}{dx_1} = 0 \quad \text{or, in view of (13.36);}$$

$$\frac{dx_2}{dx_1} = -\frac{\partial\phi}{\partial x_1}\Big/\frac{\partial\phi}{\partial x_2} = \frac{\sigma_{32}}{\sigma_{31}} \quad \text{along } \phi(x_1, x_2) = k \qquad (13.$$

Also, from (13.36),

$$\left.\begin{aligned} \underset{\sim}{\sigma}_3(\underset{\sim}{x}) &= \underset{\sim}{S}(\underset{\sim}{x}, \underset{\sim}{e}_3) = \underset{\sim}{e}_1\sigma_{31}(\underset{\sim}{x}) + \underset{\sim}{e}_2\sigma_{32}(\underset{\sim}{x}) \quad \forall\, \underset{\sim}{x} \in \overline{R} \\ \sigma &= |\underset{\sim}{\sigma}_3| = \sqrt{\sigma_{31}^2 + \sigma_{32}^2} = \mu|\alpha|\sqrt{\phi_{,1}^2 + \phi_{,2}^2} = \mu|\alpha||\nabla\phi| \end{aligned}\right\} (13$$

(13.40), (13.41), (13.42), (13.39) yield the conclusions:

a) The contour lines of Σ are the trajectories of the resultant (shear) stress on Π;

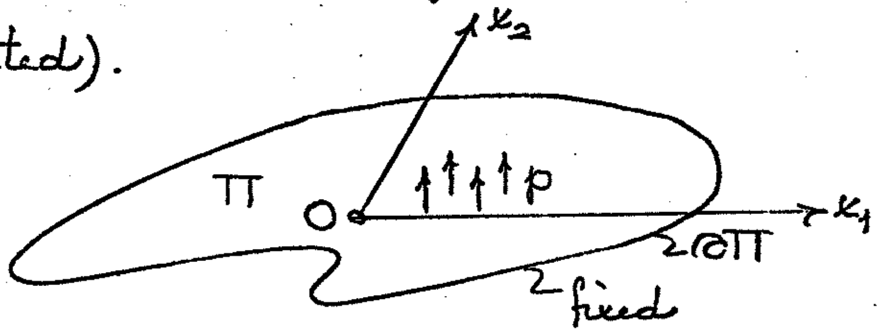
b) The mag. σ of the resultant stress on Π is proportio to the corresponding slope of line of steepest descent;

c) The torsional rigidity K is $2\mu \times$ volume of the region enclosed by $\Pi \cup \Sigma$.

Observe that a contour map of Σ associated with a given Π supplies all relevant information concerning the sol. of the relaxed torsion problem for a cylinder with the cross-section Π.

<u>Prandtl's membrane analogy</u>

Consider "small" deflections of an ideal membrane (zero shear and bending stiffness) spanned over Π. (simply connected).



p ... transverse pressure $[FL^{-2}]$. Assume $p = \text{const.}$ on Π

T ... constant membrane tension per unit length $[FL^{-1}]$

w ... transverse deflection $[L]$

Then w is the solution of the boundary-value problem

$$\nabla^2 w = -\frac{p}{T} \text{ on } \Pi, \quad w = 0 \text{ on } \partial\Pi \qquad (13.\cdot$$

Apply the change of variable

$$w = \beta\phi \text{ on } \Pi \,, \quad \beta = \frac{p}{2T} \tag{13.44}$$

Then (13.43) becomes

$$\nabla^2\phi = -2 \text{ on } \Pi \,, \quad \phi = 0 \text{ on } \partial\Pi \,, \tag{13.45}$$

which coincides with (13.37), (13.38). Hence, from (13.4

$$\Sigma : \; \zeta = \frac{w(x_1, x_2)}{\beta} \quad \forall \; (x_1, x_2) \in \bar{\Pi} \tag{13.4}$$

Thus a contour map of the deflection surface of the membrane furnishes a contour map for Σ if β is known. Explain soap-film apparatus, calibratio via known solution for circular Π, extension to m.c. Π. See Mindlin & Salvadori, Handbook of Exp. Stress Anal. Mention sand-heap analogy for plastic torsion (Nadai), mixed membrane-sand-hea anal. for elastic-plastic torsion. See Nadai, Theory of Flow and Fracture of Solids. Refer to hydrodynami and electrostatic analogies, photoelasticity, conformal mapping, numerical methods.

Application: Torsion of elliptic cylinders.

Recall (13.30) and describe S.V.'s inverse procedure.

$f(z) = \text{const.} \rightarrow$ circular Π

Consider,

$$f(z) = i c^2 z^2 + i k^2 \qquad (c, k \ldots \text{real consts.})$$

Since $z^2 = (x_1 + i x_2)^2 = x_1^2 - x_2^2 + 2 i x_1 x_2$ we have here

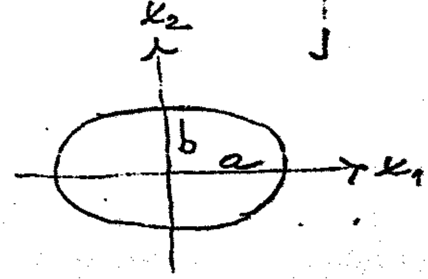
$$\varphi = -2c^2 x_1 x_2 \;, \quad \underset{(11)}{\psi} = c^2(x_1^2 - x_2^2) + k^2 \qquad (13.4\text{:}$$

Subst. into (13.33) yields

$$c^2(x_1^2 - x_2^2) + k^2 = \tfrac{1}{2}(x_1^2 + x_2^2) \quad \text{on } \partial\Pi \text{ or}$$

$$(\tfrac{1}{2} - c^2)\, x_1^2 + (\tfrac{1}{2} + c^2)\, x_2^2 = k^2 \text{ on } \partial\Pi \text{, whence, } \underline{\text{if}}\ 0 \le c^2 <$$

$$\left.\begin{aligned} &\partial\Pi: \ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \quad \text{ellipse} \\ &a = k/\sqrt{\tfrac{1}{2} - c^2}\,, \quad b = k/\sqrt{\tfrac{1}{2} + c^2} \\ &c^2 = \frac{1}{2}\,\frac{a^2 - b^2}{a^2 + b^2}\,, \quad k^2 = \frac{a^2 b^2}{a^2 + b^2} \qquad (a \ge b) \end{aligned}\right\} \ (13.48$$



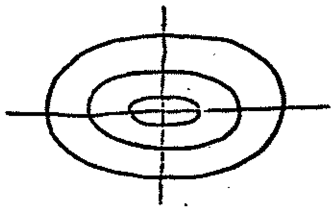
By (13.47), (13.48),

$$\varphi(x_1, x_2) = -\frac{(a^2-b^2)}{a^2+b^2}\, x_1 x_2 \tag{13.49}$$

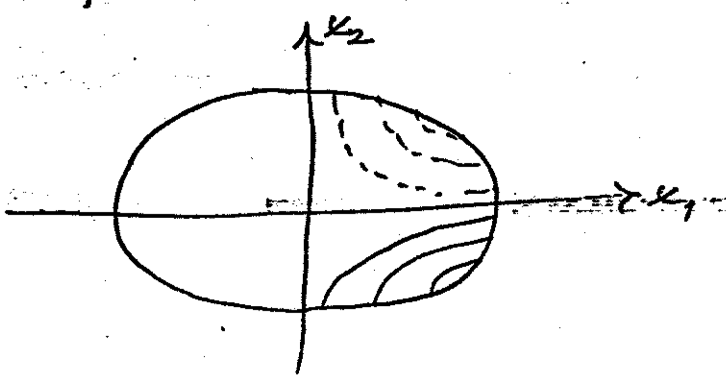
By (13.47), (13.48), (13.35), after elementary manipulation,

$$\phi(x_1, x_2) = \frac{-a^2b^2}{a^2+b^2}\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) \tag{13.50}$$

Thus the lines of shearing stress are the ellipses $\phi(x_1,x_2) =$ ~~which are homothetic with C.~~



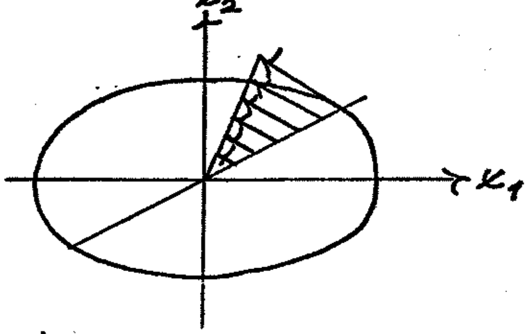
The contour lines of the warping surface are the rectang hyperbolas $\varphi(x_1, x_2) = \text{const.}$



From (13.50), (13.36) follows

$$\left.\begin{aligned} \sigma_{31} &= -\frac{2\mu\alpha a^2 x_2}{a^2+b^2}, \quad \sigma_{32} = \frac{2\mu\alpha b^2 x_1}{a^2+b^2} \\ \sigma &= |\underset{\sim}{\sigma}_3| = \sqrt{\sigma_{31}^2+\sigma_{32}^2} = \frac{2\mu|\alpha|}{a^2+b^2}\sqrt{a^4x_2^2+b^4x_1^2} \end{aligned}\right\} \quad (13.$$

From (13.51) and anal. geometry one concludes easily: $\underset{\sim}{\sigma}_3(\underset{\sim}{x})$ along each dia. of Π is par. to conjugate dia. and prop. in mag. to distance $r = \sqrt{x_1^2+x_2^2}$ from center of Π, (const. of proportionality depending on location of dia (see Sok.):



By (13.51), (13.48),

$$\text{On } \partial\Pi: \ \sigma = \frac{2\mu|\alpha| ab}{a^2+b^2}\sqrt{a^2-e^2x_1^2}, \quad e = \frac{1}{a}\sqrt{a^2-b^2}, \quad (1$$

so that

$$\sigma_{max} = \sigma\Big|_{(x_1=0)} = \frac{2\mu|\alpha| a^2 b}{a^2+b^2} \quad (!) \qquad (13.$$

By (13.50), (13.39),

$$K = M/\alpha = \frac{\pi\mu a^3 b^3}{a^2+b^2} \qquad (13.$$

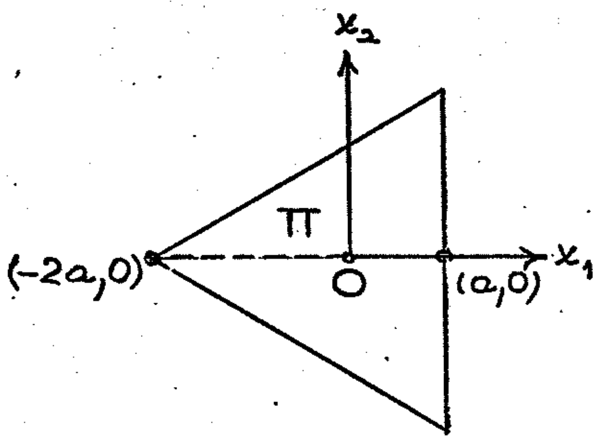
Note: for $a=b$ one recovers Coulomb's solution

AM 135. Exercise 29

(a) Show that the analytic function

$$f(z) = \varphi(x_1, x_2) + i\,\psi(x_1, x_2) = -\frac{i z^3}{6a} + \frac{2ia^2}{3}$$

generates the solution for the torsion of a beam whose cross-section Π is the equilateral triangle:

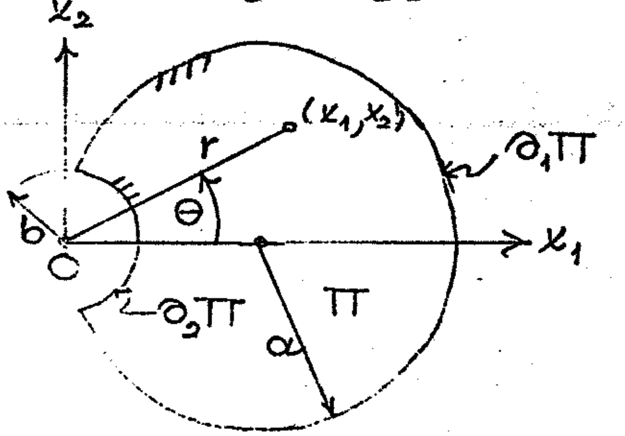


Obtain this solution and discuss it fully.

(b) Show that the analytic function

$$f(z) = \varphi(x_1, x_2) + i\,\psi(x_1, x_2) = iaz - \frac{iab^2}{z} + \frac{ib^2}{2}$$

generates the solution for the torsion of a beam whose cross-section Π is the notched circle:



Confirm that the magnitude of the resultant shear stress obeys

$$\sigma = \mu|\alpha|\left(a - \frac{b^2 \sec^2\theta}{4a}\right) \quad \text{on } \partial_1\Pi$$

$$\sigma = \mu|\alpha|(2a\cos\theta - b) \quad \text{on } \partial_2\Pi$$

Remarks

Mention additional exact solutions:
Series solutions (obtained by separation of variab
for Π rectangular or an isoscelese right
triangle {See Sokolnikoff, art 38}; circular
sector; circle with eccentric circular hole.
See also Weber-Günther, Torsionstheorie

Related topics

Solution of S.V. torsion problem via conform
mapping {See Sok., art 40-45}. Mention po
gonal Π via Schwarz-Christoffel transf.
Upper and lower bounds for K via variati
methods, isoperimetric problems.
Approximate theory of torsion of thin-walled
hollow Π {See Sok., art 47; Timo-Goodier, art 1
Torsion of solids of revolution {See Sok., art. 49
Timo-Goodier, art. 119}