Electric potential

Electric charge. All materials are made of electrons, protons and neutrons. Every electron has the same electric charge of one type. Every proton has the same electric charge of the other type. Neutrons have no electric change. Electrical phenomena at a macroscopic scale can be analyzed by adding charges like integers. The charge on the electron is taken to be negative, and the charge on the proton is taken to be positive. The two types of charges have different signs but the same magnitude. The charge of each proton is called the elementary charge.

The net charge in a system is the elementary charge times the number of protons minus the number of electrons. A molecule of water, for example, contains 10 electrons and 10 protons, so that the molecule is neutral. A glass of wine has a large number of electrons and protons. Such a large system is usually nearly neutral: the difference in the number of electrons and the number of protons is much smaller than the number of electrons. Thus, to obtain the net charge of a large system, counting the two large numbers and then subtracting is usually a bad method. More effective methods exist to measure the net charge of a system, as described in textbooks of electrostatics.

The total amount of electric charge is conserved. If the amount of charge increases some place, it is because charge moves to this place from elsewhere.

Electric charge is commonly reported in two units. The first unit is just the number; for example, if a drop of oil picks up three more electrons than the number of protons, we say that the charge of the drop is -3. The second unit is called Coulomb. The two units of charge are converted by factor:

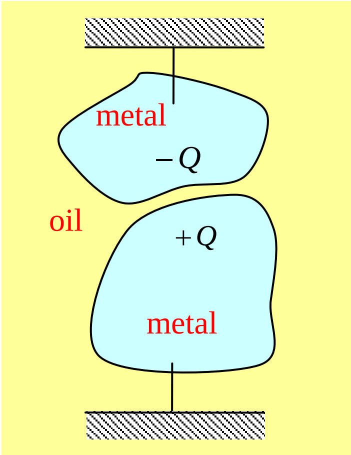
$$1 \text{ elementary charge} = 1.60\times10^{-19} \text{ Coulomb}.$$

Thus, the charge on the drop of oil with three extra electrons is -4.80×10^{-19} Coulomb.

Conductors and insulators. As an idealization, materials are divided into two kinds:

- *Conductors.* In a conductor, charged particles move over macroscopic distances. Examples include electrons moving in space, electrons moving in a metal line, ions moving in an electrolyte, and holes moving in a semiconductor.
- *Insulators.* A vacuum is an insulator. Air is an insulator before it breaks down. Many materials are insulators. Charged particles move relative to each other by a small distance, and is restrained by inter-particle forces. Examples include a small distortion of the electron cloud of an atom, a small change in distance between a positive ion and a negative ion in a dielectric.

Capacitor. Consider two pieces of a metal immersed in a jar of oil. The metal conducts electrons, and the two pieces are called the electrodes. The oil is an insulator and a liquid. We place on one electrode an amount of charge, $+Q$,



and place on the other electrode an opposite amount of charge, $-Q$. The positive and the negative charges cause the two electrodes to attractive each other. We fix the electrodes in space by using threads of an insulator.

Following Faraday, we visualize an electric field in the oil. The electric field is concentrated in the narrow gap between the two electrodes. So long as the oil is a good insulator, the device can separate the positive and the negative charges on the two electrodes for a long time. The device is called a capacitor.

For the time being we will place the same magnitude of charge on the two electrodes, so that from far away the capacitor is neutral. When we say "the charge Q of a capacitor", we mean $+Q$ on one electrode, and $-Q$ on the other.

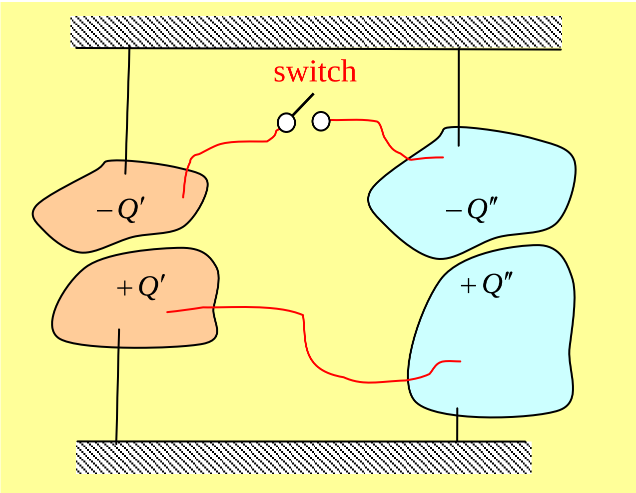
The capacitor is a system. We will fix its temperature by keeping the capacitor in thermal contact with a reservoir of energy. We will study isothermal processes, and will not consider the temperature explicitly. Consequently, the capacitor is a system of only one variable: the amount of charge on either electrode, Q. The thermodynamics of the capacitor is characterized by the Helmholtz free energy of the capacitor, F, ss a function of the charge, namely,

$$F = F(Q).$$

When the capacitor has a different amount of change, $Q + \delta Q$, the free energy changes by

$$\delta F = \frac{\partial F(Q)}{\partial Q}\delta Q .$$

We would like to find the experimental significance of the partial derivative.



Electric potential. Next consider two different capacitors, A' and A''. The two capacitors are characterized by different free-energy functions: $F'(Q')$ and $F''(Q'')$. Because the electric field is localized in the narrow gap between the two electrodes for either capacitor, when the two capacitors are far apart, the two capacitors independently retain the two functions

$F'(Q')$ and $F''(Q'')$.

The two capacitors are then connected by metal wires in parallel, with the positive electrode of one capacitor connected to the positive electrode of the other capacitor. The wire connecting the two negative electrodes has a switch. When the switch is off, the two capacitors do not exchange charge. When the switch is on, electric charge goes from one capacitor to the other. Which way will the charge go?

The net charge on the two positive electrodes remains fixed, so is the net charge on the two negative electrodes. That is,

$$Q' + Q'' = \text{constant}.$$

The composite of the two capacitors is a system with an internal variable: the partition of a certain amount of charge between the two capacitors. The free energy of the composite is the sum

$$F'(Q') + F''(Q'').$$

Thermodynamics dictates that, when the two capacitors equilibrate, the free energy of the composite should minimize. This minimization is subject to the conservation of charge. Thus, the condition of equilibrium is

$$\frac{\partial F'(Q')}{\partial Q'} = \frac{\partial F''(Q'')}{\partial Q''}.$$

When the switch is off, the two derivatives are in general unequal. When the switch is on, thermodynamics dictates that charge goes from the capacitor with a larger derivative to the capacitor with a smaller derivative.

We call the derivative the electric potential of the capacitor, and give the derivative a symbol Φ, namely,

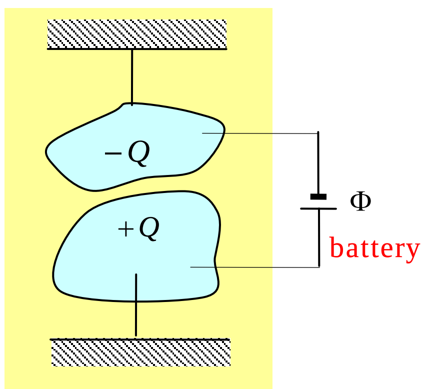
$$\Phi = \frac{\partial F(Q)}{\partial Q}.$$

That is, the electric potential of a capacitor is the change in the free energy of the capacitor associated with a unit change in the charge of the capacitor.

When the electric charge is counted as a number, the unit of the electric potential is the same as that of energy. When the electric charge is reported in units of Coulomb, the unit for the electric potential is called Volt. The two units of electric potential are converted by a factor:

$$1\text{Volt} = 1.60 \times 10^{-19}\ \text{Joule}.$$

If you find this factor revolting, perhaps you should argue with whoever assigned the elementary charge the number 1.60×10^{-19} Coulomb.



Battery as a reservoir of charge. A capacitor that stores a large amount of charge is called a reservoir of charge, or a battery for brevity. When the battery is connected to a smaller capacitor, the charge drawn from the battery is so small that the electric potential of the battery, $\Phi_{battery}$, remains unchanged as the charge is being drawn. Let $F(Q)$ be the free-energy function of the small capacitor. After being connected for some time, the small capacitor and the battery equilibrate, and the electric potential of the small capacitor equals that of the battery, namely,

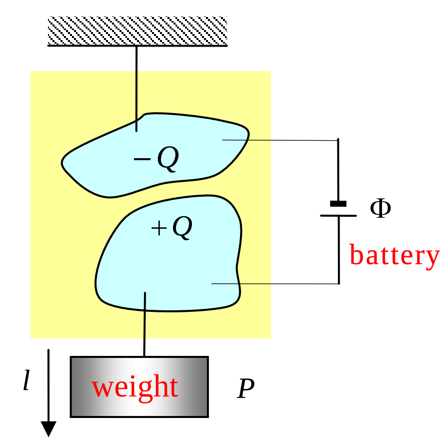
$$\frac{\partial F(Q)}{\partial Q} = \Phi_{battery}.$$

When $F(Q)$ and $\Phi_{battery}$ are given, the above condition of equilibrium is an algebraic equation that determines the charge in the small capacitor.

At the risk of causing confusion, we may drop the subscript and simply write Φ as the electric potential for the capacitor and the battery in equilibrium. Furthermore, we will use the word battery as shorthand for a device that can apply an electric potential between two electrodes of a capacitor.

Equilibrium between a capacitor, a weight, and a battery. Now we allow one electrode to move relative to the other. The charges of opposite signs, $+Q$ and $-Q$, cause the two electrodes to attract each other. To maintain equilibrium, we pull the two electrodes by a force. We use the word weight as shorthand for a device that applies this force. (In this particular case, a weight applying a constant force will cause instability. But we ignore this detail for the time being.)

We view the capacitor as a system of two variables: l and Q. The two variables can vary independently. On the plane with coordinates l and Q, a point represents a state of the capacitor, and a curve represents a process undergone by the capacitor. After charging the capacitor, we can disconnect the capacitor from the battery, but keep the charge Q in the capacitor fixed. When the electrodes move relative to each other, the weight does mechanical work. Alternatively, we can fix the separation of the electrodes, l, but add charges to the electrodes, so that the battery does electrical work.

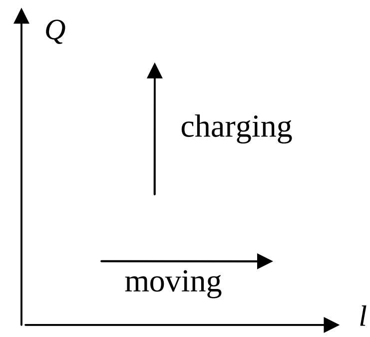


The free energy of the capacitor, F, is a function of the two independent variables, namely,

$$F = F(l, Q).$$

Associated with small changes in the independent variables, the free energy of the capacitor changes by

$$\delta F = \frac{\partial F(l,Q)}{\partial l}\delta l + \frac{\partial F(l,Q)}{\partial Q}\delta Q .$$



The weight applies a force *P*. Upon moving by a distance δl, the weight does work $P\delta l$. The battery applies an electric potential Φ. Upon giving the capacitor a small amount of charge, δQ, the battery does work $\Phi \delta Q$.

When the capacitor, the battery, and the weight equilibrate, the sum of the work done by the battery and by the weight equals the change in the energy stored in the capacitor:

$$\delta F = P\delta l + \Phi \delta Q .$$

Comparing this condition of equilibrium with the differential form of the function $F(l,Q)$, we relate the force and the voltage to the partial derivatives:

$$P = \frac{\partial F(l,Q)}{\partial l}, \quad \Phi = \frac{\partial F(l,Q)}{\partial Q} .$$

This pair of equations constitutes an alternative expression the condition of equilibrium between the capacitor, the weight, and the battery.

Actuator, sensor, and generator. Now imagine that the weight and battery are adjustable, so that the force *P* and the voltage Φ can vary. When the displacement is fixed, a change in the charge may cause the force to change. When the charge is fixed, a change in the displacement may cause the voltage to change. This electromechanical interaction converts electrical energy and mechanical energy, and is the basis for many designs of actuators, sensors, and generators.

Exercise. Find in the literature some designs of electrostatic actuator, sensor and generator.

Hessian of the function $F(l,Q)$. Associated with small changes δl and δQ, the force and the voltage change by

$$\delta P = \frac{\partial^2 F(l,Q)}{\partial l^2}\delta l + \frac{\partial^2 F(l,Q)}{\partial l \partial Q}\delta Q ,$$

$$\delta \Phi = \frac{\partial^2 F(l,Q)}{\partial l \partial Q}\delta l + \frac{\partial^2 F(l,Q)}{\partial Q^2}\delta Q .$$

Each element of the Hessian can be interpreted in physical terms. We may call $\partial^2 F(l,Q)/\partial l^2$ the mechanical tangent stiffness of the system, and $\partial^2 F(l,Q)/\partial Q^2$ the electrical tangent stiffness of the system. The two electromechanical coupling effects are both characterized by the same cross derivative, namely,

$$\frac{\partial P(l,Q)}{\partial Q} = \frac{\partial^2 F(l,Q)}{\partial l \partial Q} = \frac{\partial \Phi(l,Q)}{\partial l} .$$

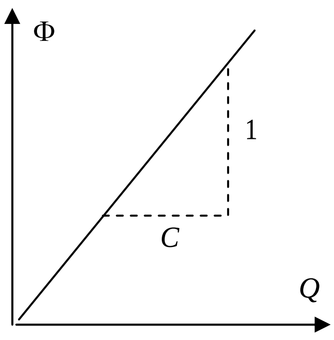
Consequently, for a conservative system, the two electromechanical coupling effects reciprocate.

Linear capacitor. When the charge of a capacitor is linear in the electric potential, we write

$$Q = C\Phi ,$$

where C is called the capacitance, and is independent of the electric potential. Integrating $\delta F = \Phi \delta Q$, we obtain that

$$F(Q) = \frac{Q^2}{2C} .$$



We have set the reference $F(0) = 0$.

Exercise. Find in the literature designs of capacitors. Find how the capacitance in each design depends on various geometric parameters.

Adjustable linear capacitor. For a linear capacitor,

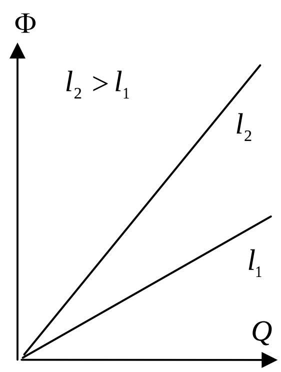
$$Q = C\Phi ,$$

the capacitance C usually varies with l, namely,

$$C = C(l).$$

Integrating $\Phi = \partial F(l,Q)/\partial Q$, we obtain that

$$F(l,Q) = \frac{Q^2}{2C(l)} ,$$



We have set $F(l,0) = 0$. That is, in the absence of the charge, we neglect any variation of the free energy with the displacement.

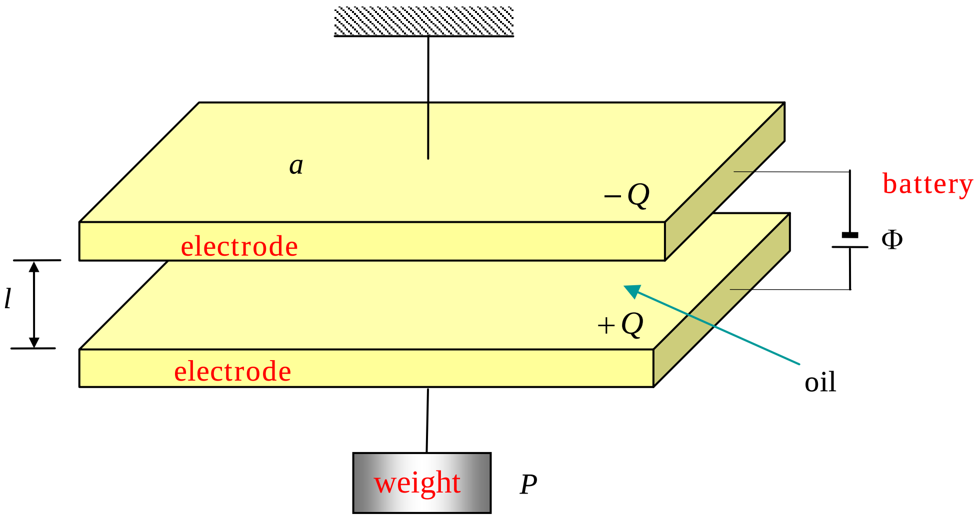
To maintain equilibrium the weight applies the force $P = \partial F(l,\Phi)/\partial l$ or

$$P = -\frac{Q^2}{2C^2} \frac{\partial C(l)}{\partial l} .$$

We neglect any force when $Q = 0$, so that $g(l)$ is a constant.

Exercise. Calculate the Hessian of the function $F(l,Q)$ of an adjustable linear capacitor. Interpret experimental significance of each element.

Parallel-plate capacitor. In a parallel-plate capacitor, a layer of oil separates two metal plates, known as electrodes. The area of either electrode is a, and the separation between the electrodes is l. The two electrodes are then connected through a metal wire to a battery, which pumps electrons from one electrode to the other. Let $+Q$ be the electric charge on one electrode, and $-Q$ be the electric charge on the other electrode. The metal plates and the metal wire are electrical conductors, and the oil is an electrical insulator.



The two charges, $+Q$ and $-Q$, cause the two electrodes to attract each other. To maintain equilibrium, the two electrodes are pulled by a pair of forces. We use the word weight as shorthand for any device that applies a force. Let us say that we have done the experiment, or a more practical variation of the experiment. We find that that the charge on either electrode is linear in the voltage:

$$\frac{Q}{a} = \varepsilon \frac{\Phi}{l}.$$

The constant ε is characteristic of the oil, and is known as the permittivity of the oil. The experiment determines the value of ε. The equation is valid when the oil is a linear dielectric, the electrodes are rigid, and the gap between the electrode is small compared to the lateral dimensions.

Recall that $\Phi = \partial F(l,Q)/\partial Q$. Integrating in Q gives

$$F(l,Q) = \frac{lQ^2}{2\varepsilon a}.$$

We have set $F(l,0) = 0$.

Recall that $P = \partial F(l,Q)/\partial l$. The weight needed to balance the electric attraction of the two electrodes is

$$P = \frac{Q^2}{2a\varepsilon}.$$

This force is independent of the separation between the electrodes, l.

Experimental determination of electric potential. Assuming that we can measure charge of the parallel capacitor, and the force needed to keep the two electrodes in equilibrium, equation

$$P = \frac{Q^2}{2a\varepsilon}$$

enables us to determine the permittivity form these measurements. Furthermore, equation

$$\frac{Q}{a} = \varepsilon \frac{\Phi}{l}$$

enables us to determine the electric potential of the parallel-plate capacitor.

In general, the experimental determination of the electric potential is analogous to that of temperature and chemical potential. Once we know how the electric potential of one capacitor varies with a measurable parameter, we can use this capacitor to determine electric potential of another capacitor by connecting the two capacitors in parallel. When the two capacitors equilibrate, a reading of the electric potential of one capacitor gives that of the other.

Once we know how to determine the electric potential by measurements, we also know how to determine the free energy of a capacitor as a function of change by integrating

$$\frac{\partial F(Q)}{\partial Q} = \Phi .$$

Electric field, electric displacement, and stress. Define electric field as

$$E = \frac{\Phi}{l},$$

And define electric displacement as

$$D = \frac{Q}{a}.$$

Thus, the linear relation between Φ and Q is expressed as.

$$D = \varepsilon E .$$

The free energy per unit volume is

$$\frac{F}{al} = \frac{D^2}{2\varepsilon}.$$

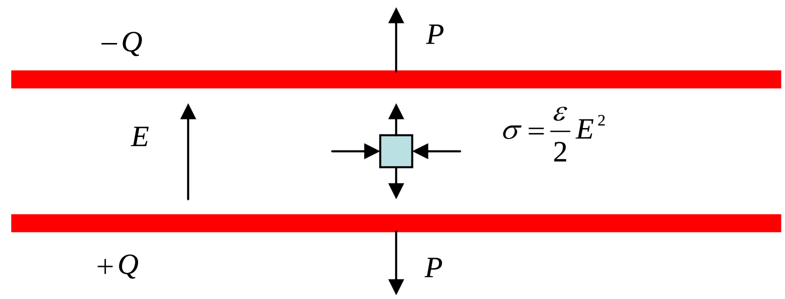
Furthermore, define the stress as

$$\sigma = \frac{P}{a}.$$

Consequently, to counteract the attraction due to the opposite charges on the two electrodes, we need to apply weight divided by the area:

$$\sigma = \frac{1}{2}\varepsilon E^2 .$$

This stress is known as the Maxwell stress. Let $E = 10^6$ V/m, for a representative value $\varepsilon = 10^{-11}$ F/m, the stress is $\sigma = 10$ N/m². Thus, the Maxwell stress is typically small compared to the strength of most materials.



The Maxwell stress is a tensor. In a parallel-plate capacitor, the electric field is a vector normal to the electrodes, but Maxwell showed that the stress is a tensor. The electrodes of the capacitor are plates normal to the *z* axis, so that the electric field in the capacitor, E, is along the *z* axis. The component stress in the *z* direction is

$$\sigma_z = \frac{\varepsilon}{2}E^2 .$$

This stress acts in the oil, balancing the stress due to applied force P. The same electric field also causes two components of stress in the directions normal to the *z* direction:

$$\sigma_x = \sigma_y = -\frac{\varepsilon}{2}E^2 .$$

We will describe a phenomenon related to these components shortly.

If the gap between the electrodes is a vacuum, the state if stress acts in the vacuum. Perhaps you can be conditioned to think about a state of stress in the vacuum. After all, you have been conditioned to think about electric field in the vacuum.

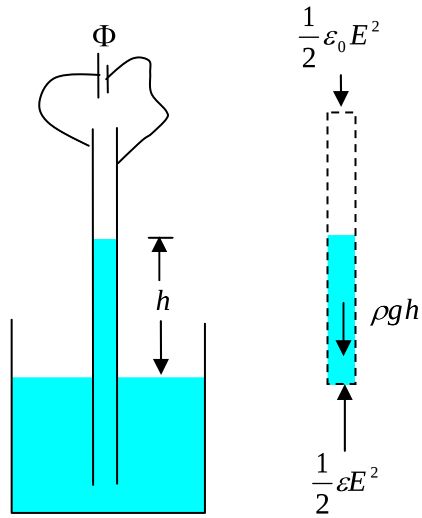
Rise of a dielectric liquid. We have already interpreted the component of the stress normal to the electrodes as the Coulomb attraction. We now look at the component of the stress parallel to the electrodes. Consider a capacitor illustrated in the figure. The stress parallel to the electrodes in the air is

$$\sigma_{air} = -\frac{1}{2}\varepsilon_0 E^2$$

The stress parallel to the electrodes in the liquid is

$$\sigma_{liquid} = -\frac{1}{2}\varepsilon E^2$$

The electric field near the air/water interface

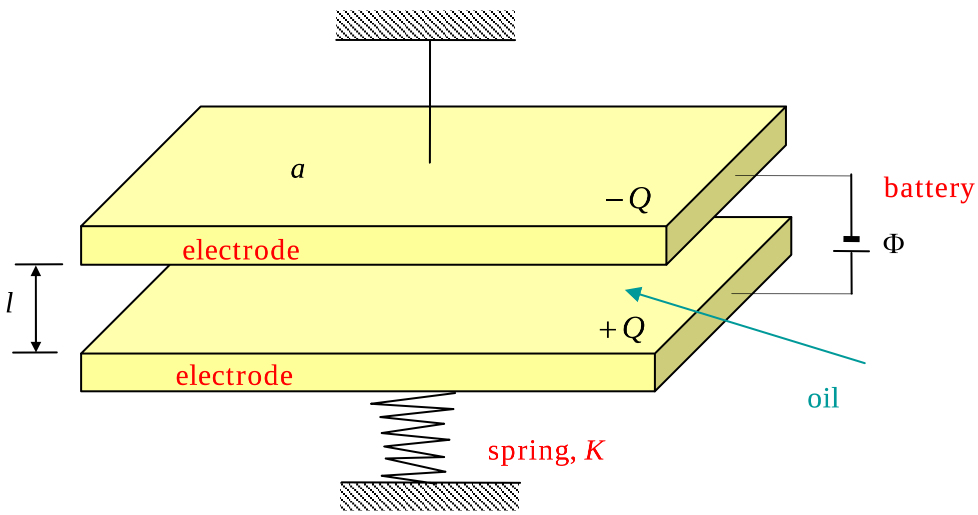


is distorted, so that the above two formulas are correct only at some distance away from the interface. Because $\varepsilon > \varepsilon_0$, the difference in the stresses in the two media will draw the liquid up against gravity. Examine the free-body diagram. The force balance requires that

$$\rho g h = \frac{1}{2}(\varepsilon - \varepsilon_0)E^2$$

Inserting representative numbers, the electric field may draw the liquid about to a height on the order of mm.

Exercise. Obtain the last result by using the method of energy.



Pull-in instability. In a parallel capacitor, one electrode is fixed in space, and the other electrode is connected to a spring of stiffness K. The capacitor is connected to a battery, which applies a constant electric potential Φ. The charges on the electrodes cause the electrodes to attract each other. When the electrodes approach each other, the charges increase. This positive feedback may cause a pull-in instability, unless the spring is stiff.

Before the capacitor is connected to the battery, the spring is unstretched, and the separation between the two electrode is l_0. After the capacitor is connected to the spring, the separation between the electrodes becomes l, so that the energy of the spring is

$$\frac{K}{2}(l - l_0)^2.$$

The free energy of the capacitor is

$$\frac{\varepsilon}{2}\left(\frac{\Phi}{l}\right)^2 al.$$

After the battery gives charge $Q = \varepsilon a \Phi / l$ to the capacitor, the free energy of the battery changes by

$$-\Phi Q .$$

The free energy of the composite is the sum of the free energy of the capacitor, the battery, and the spring, giving

$$G(l,\Phi) = -\frac{\varepsilon a \Phi^2}{2l} + \frac{K}{2}(l - l_0)^2 .$$

At a constant Φ, the separation l changes to minimize the function $G(l,\Phi)$.

In a state of equilibrium, we obtain the condition

$$\frac{\partial G(l,\Phi)}{\partial l} = 0 .$$

Thus,

$$\frac{\varepsilon a}{2}\left(\frac{\Phi}{l}\right)^2 + K(l - l_0) = 0 .$$

The first term is the attractive force between the two electrodes of the capacitor. The second term is the force in the spring. The equation balances these two forces, and solves the separation of the electrodes in the state of equilibrium, l_{eq}.

The state of equilibrium is stable if the free energy minimizes. Consequently, the critical condition is

$$\frac{\partial^2 G(l,\Phi)}{\partial l^2} = 0 .$$

This critical condition for stability is

$$-\varepsilon a \frac{\Phi^2}{l^3} + K = 0 .$$

A combination of the condition of equilibrium and the critical condition determines the critical values

$$l_c = \frac{2}{3} l_0 ,$$

$$\Phi_c = \left(\frac{K}{\varepsilon a}\right)^{1/2} \left(\frac{2}{3} l_0\right)^{3/2} .$$

Exercise. Sketch the free energy of the composite as a function of l for several values of Φ. Use the diagram to interpret the instability.

Electric potential as an independent variable. Define a new quantity:

$$\hat{F} = F(l,Q) - \Phi Q .$$

In equilibrium,

$$\delta F = P \delta l + \Phi \delta Q .$$

The differential form of the new quantity is

$$d\hat{F} = Pdl - Qd\Phi ,$$

so that $\hat{F}$ is a function of (l,Φ), and

$$\hat{F} = \hat{F}(l,\Phi)$$

$$P = \frac{\partial \hat{F}(l,\Phi)}{\partial l}, \quad Q = -\frac{\partial \hat{F}(l,\Phi)}{\partial \Phi} .$$

Exercise. Interpret the Hessian of the function $\hat{F}(l,\Phi)$.

Exercise. Interpret $F - \Phi Q$ as the Helmholtz free energy of a composite.

Exercise. Interpret another quantity $F - \Phi Q - Pl$.