Waves

Yet another play of two actors: inertia and elasticity

References.

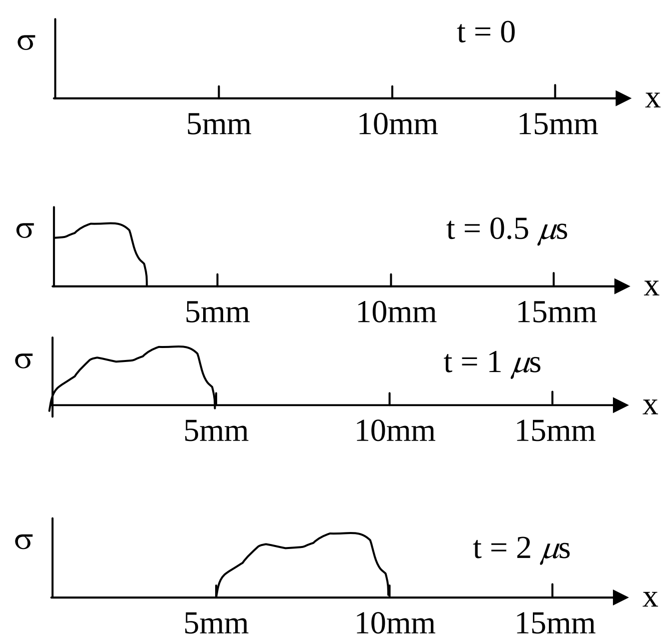
H. Kolsky, *Stress Waves in Solids*, Dover Publications, New York.
K.F. Graff, *Wave Motion in Elastic Solids*, Dover Publications, New York.
J.D. Achenbach, Wave propagation in elastic solids. North-Holland, Amsterdam. Also see Achenbach's speech upon receiving the Timoshenko Medal (http://imechanica.org/node/185).

Light and sound. A comparison between light and sound is instructive.

- *What's vibrating?* Vibrating electric field and magnetic field transport light. Vibrating stress field and displacement field transport sound.
- *Media.* Light can propagate in vacuum, as well as in certain materials. Sound wave must propagate in elastic media. Spring of air, liquid, solid. When a noisy watch is suspended in a glass jar with a thread. In the beginning, you can see the watch and hear its noise. When the air is sucked out of the jar, you can still see the watch, but cannot hear it.
- *Speeds.* The light wave speed is 3×10^8 m/s in vacuum. The sound speed is about 340 m/s in air, 1500 m/s in water, and 5000 m/s in steel.
- *Frequencies and wavelengths.* $f = c/\lambda$. Visible electromagnetic waves. Red: $\lambda = 0.7\mu$m, $f = 4.3\times 10^{14}$ Hz. Violet: $\lambda = 0.4\mu$m, $f = 7.5\times 10^{14}$ Hz. Audible sound waves. 20 Hz to 20 kHz. In air, they correspond to 17 m and 17 mm. In water, they correspond to 75 m and 7.5 cm.

Ultrasound. Ultrasound has frequencies too high to be detected by the human ear. Ultrasound can be generated using piezoelectric materials, which convert an electric field to a stress field. High frequencies correspond to short wavelengths.

- Ultrasound imaging: see baby inside mother.
- Non-destructive evaluation (NDE): detect flaws inside materials.
- Surface Acoustic Wave (SAW) devices for wireless applications.



Longitudinal wave in a rod. The speed of sound in steel is ~ 5 km/s. A frame-by-frame "movie" shows the stress profiles at several times.

- Before time zero, these is no stress in a steel rod.
- At t = 0, a hammer starts to hit the end of the rod. The hit will last for 1 μs.
- At t = 0.5 μs, 2.5 mm of rod is under compression, but the rest of the rod is stress-free. The delay is caused by the inertia of the matter.
- At t = 1 μs, 5 mm of rod is under compression, but the rest of the rod is stress-free.
- At t = 2 μs, the same 5 mm wave packet travels for 5 mm. The rod behind and ahead of the packet is stress-free.

The D'Alambert solution to the equation of motion. So far we have appealed to our daily experience about waves. What do the equations say? The equation of motion of the rod is

$$E\frac{\partial^2 u(x,t)}{\partial x^2} = \rho\frac{\partial^2 u(x,t)}{\partial t^2},$$

where x is the coordinate of a material particle when the rod is in the reference state, and $u(x,t)$ is the displacement of the material particle x at time t. We can determine the field $u(x,t)$ by solving this PDE, in conjunction with initial conditions and boundary conditions. For the time being, let us leave aside the initial and boundary conditions, and just look at the equation of motion by itself.

In the above, we have speculated that a wave can travel in the rod at a constant speed and an invariant waveform. This speculation can be written into a mathematical form. Let c be the wave speed. Define a composite variable involving both the material coordinate and time:

$$\xi = x - ct .$$

Let $f(\xi)$ be a function of the composite variable ξ. Consider a time-dependent field of displacement:

$$u(x,t) = f(\xi)$$

At t = 0, the displacement field is $u(x,0) = f(x)$. At time t, the displacement field is $u(x,t) = f(x - ct)$, which has the same shape as that at time t = 0, but moves to the right by a distance ct. That is, the function $u(x,t) = f(\xi)$ represents a fixed waveform traveling at speed c to the right.

What is the waveform $f(\xi)$ and the wave speed c? To find out, let us insert our guess

$$u(x,t) = f(\xi), \qquad \xi = x - ct$$

into the equation of motion. Using the chain rule in differential calculus, we obtain that

$$\frac{\partial u(x,t)}{\partial x} = \frac{df(\xi)}{d\xi}\frac{\partial \xi(x,t)}{\partial x} = \frac{df(\xi)}{d\xi}$$

and

$$\frac{\partial u(x,t)}{\partial t} = \frac{df(\xi)}{d\xi}\frac{\partial \xi(x,t)}{\partial t} = -c\frac{df(\xi)}{d\xi}$$

Insert into the equation of motion, and we have

$$E\frac{d^2 f}{d\xi^2} = \rho c^2 \frac{d^2 f}{d\xi^2}$$

Consequently, the equation of motion is satisfied by *any* function $f(\xi)$ provided that the wave speed is given by

$$c = (E/\rho)^{1/2} .$$

Similarly, $g(x+ct)$ represents a fixed profile of displacement traveling at speed c to the left. Let $\xi = x - ct$ and $\eta = x + ct$. Any functions $f(\xi)$ and $g(\eta)$ satisfy the equation of motion. Because the PDE is linear, it is also satisfied by the linear combination:

$$u(x,t) = f(x-ct) + g(x+ct)$$

Velocity of the material particle x at time t is

$$v = \frac{\partial u(x,t)}{\partial t}.$$

The general expression for the particle velocity is

$$v(x,t) = c[-F(x-ct) + G(x+ct)],$$

where

$$F(\xi) = df(\xi)/d\xi, \quad G(\eta) = dg(\eta)/d\eta.$$

The stress field is given by

$$\sigma = E\frac{\partial u(x,t)}{\partial x}.$$

The general expression for the stress field is

$$\sigma(x,t) = E[F(x-ct) + G(x+ct)].$$

To recap, the general solution to the equation of motion is

$$u(x,t) = f(x-ct) + g(x+ct),$$

where $c = (E/\rho)^{1/2}$. The two waveforms f and g are arbitrary functions, not determined by the equation of motion. The particle velocity and stress in the rod are

$$v(x,t) = c[-F(x-ct) + G(x+ct)]$$

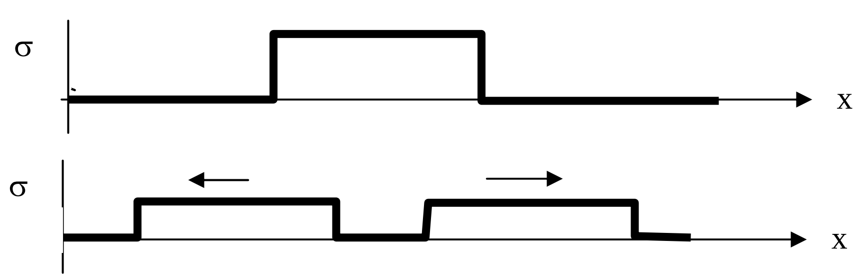
$$\sigma(x,t) = E[F(x-ct) + G(x+ct)]$$

where $F(\xi) = df(\xi)/d\xi$ and $G(\eta) = dg(\eta)/d\eta$.

The field $u(x,t)$ is governed by

- A PDE (i.e., the equation of motion)
- Initial conditions (i.e., the displacement field and velocity field in the rod at time zero.)
- Boundary conditions (i.e., the displacement and the stress at the two ends of the rod)

In reaching the D'Alambert solution, we have used the PDE, but not the initial conditions and the boundary conditions. We next illustrate how the initial conditions and boundary conditions come into play.



An initial-value problem. Pull a long steel rod with two grips. Hold the forces constant before time zero. Release the grips at time zero. We'd like to find out the waves generated in the rod afterward.

Before the grips are released. The rod has a static stress field, $s(x)$. When the grips are released, at time zero, the stress field is still the same as the static field:

$$E[F(x)+G(x)]=s(x).$$

At time zero, there is no velocity, so that

$$c[-F(x)+G(x)]=0.$$

A combination of the two equations gives that

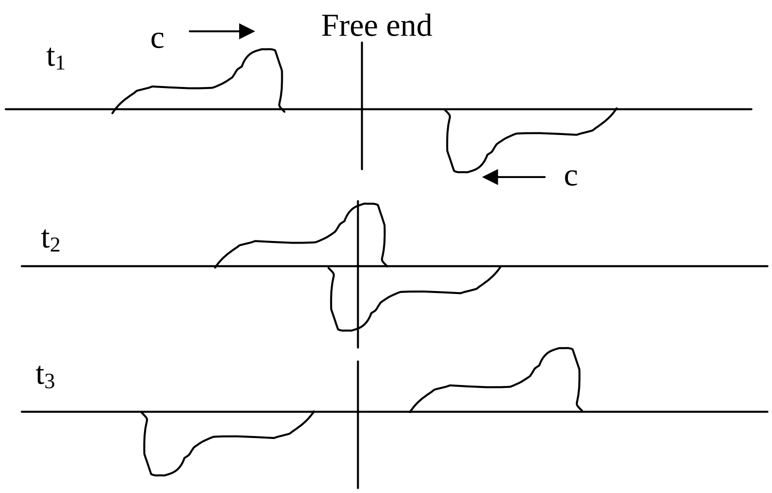
$$F(x)=G(x)=\frac{1}{2E}s(x).$$

The stress field afterward is

$$\sigma(x,t)=E[F(x-ct)+G(x+ct)]$$
$$=\frac{1}{2}[s(x-ct)+s(x+ct)]$$

After the grips are released, the static stress profile splits into two waves, one traveling to the right, and the other to the left. This solution is correct before either wave hits the end of the rod.

Reflection from a free end. We next illustrate how boundary conditions come into play. An incident wave in the rod hits the free end of the rod, and reflects back into the rod. Let's say the rod lies on the axis, $x < 0$, and the free end is at $x=0$. The incident wave travels from the left to the right. The boundary condition: Stress vanishes at all time at $x=0$. An incident compressive wave, after reflection, becomes a tensile wave. Dynamic fracture.



How do we obtain the solution by doing algebra? We know the shape of the incident wave,

$$\sigma^I=s(x-ct).$$

That is, we know the function $s(\xi)$. The reflected wave travels from the right to the left. It must be of the form

$$\sigma^R=h(x+ct).$$

Its shape $h(\eta)$ is to be determined. The stress in the rod is the sum of the incident and the reflected wave:

$$\sigma(x,t)=s(x-ct)+h(x+ct).$$

How to determine the function $h(\eta)$? The boundary condition: at the free end $x = 0$ the stress is zero at all time, namely, $\sigma(0,t) = 0$. Put this boundary condition to the general expression for the stress, and we have

$$0 = s(-ct) + h(ct).$$

Denote ct by Q. We have

$$h(Q) = -s(-Q).$$

That is, we have expressed the function h in terms of the known function s. The independent variable can be anything. The stress in the rod is given by

$$\sigma(x,t) = s(x - ct) - s(-x - ct).$$

The first term is the incident wave, running in the positive x-direction. The second term is the reflected wave, running in the negative x-direction. At the free end, $x = 0$, the stress indeed vanishes at all time. If an incident wave induces a compressive stress in the rod, the reflected wave induces a tensile stress in the rod.

Standing waves. A normal mode is a *standing wave*. For example, consider a rod fixed at one end and free to move on the other end. From the two boundary conditions, the standing wave of the longest wavelength has $\lambda = 4L$. Only a quarter of this wavelength is inside the rod. The frequency of the fundamental mode is given by $f = c/\lambda$, which recovers the result we obtained before:

$$\omega_1 = \frac{\pi}{2}\sqrt{\frac{E}{\rho}}.$$

We can understand other normal modes in the similar way.

Acoustic impedance. An alternative form of the D'Alambert solution is

$$u(x,t) = f(\alpha) + g(\beta)$$

where

$$\alpha = \frac{x}{c} - t, \quad \beta = -\frac{x}{c} - t.$$

Putting the above equations together, we write the D'Alambert solution as

$$u(x,t) = f\left(\frac{x}{c} - t\right) + g\left(-\frac{x}{c} - t\right).$$

The velocity of the material particle x at time t is $v = \partial u(x,t)/\partial t$, namely,

$$v(x,t) = -F\left(\frac{x}{c} - t\right) - G\left(-\frac{x}{c} - t\right),$$

where $F(\alpha) = df(\alpha)/d\alpha$ and $G(\beta) = dg(\beta)/d\beta$.

The axial force in the rod is $P = AE\partial u(x,t)/\partial x$, namely,

$$P(x,t) = RF\left(\frac{x}{c} - t\right) - RG\left(-\frac{x}{c} - t\right),$$

where

$$R = A\sqrt{E\rho}.$$

Note that the axial force varies from one material particle to another, and is time-dependent. Also note that

$$P = -Rv$$

for a wave propagates in the positive x direction, and

$$P = +Rv$$

for a wave propagates in the negative x direction. These relations may remind you of Hooke's law, except that force is proportional to the velocity, rather than the elongation. The quantity R is called the *acoustic impedance*, and has the unit of force per unit velocity. In words:

$$\text{Force} = \text{Impedence} \times \text{Velocity} .$$

For a given particle velocity, the axial force is large if the cross-sectional area is large, or the material is stiff, or the material is dense.

Reflection and transmission at a joint of two materials. Two rods are joined at $x = 0$. The rod on the left-hand side has properties

$$A_1, E_1, \rho_1, c_1 = (E_1 / \rho_1)^{1/2}, R_1 = A_1 \sqrt{E_1 \rho_1} .$$

The rod on the right-hand side has properties

$$A_2, E_2, \rho_2, c_2 = (E_2 / \rho_2)^{1/2}, R_2 = A_2 \sqrt{E_2 \rho_2} .$$

An incident wave comes from rod 1 towards the joint. Upon hitting the joint, the wave is partly reflected back to rod 1, and partly transmitted into rod 2.

Let the incident wave be

$$u^I(x,t) = f\left(\frac{x}{c_1} - t\right).$$

The function f is the incident waveform, and is known. We need to solve for the wave reflected into rod 1, and the wave transmitted into rod 2.

We expect that the reflected wave takes the form

$$u^R(x,t) = af\left(-\frac{x}{c_1} - t\right).$$

We interpret various pieces of this guess as follows:

- Both the displacement and the force are continuous at all time across the joint $x = 0$, so that the reflected wave in rod 1 should take a waveform similar to the waveform of the incident wave, $f(\cdot)$.
- The amplitude of the reflected wave can be different from that of the incident wave, so we multiply a dimensionless number a, which is the ratio of the amplitude of the reflected wave over the amplitude of the incident wave.
- The reflected wave runs in the negative x-direction, so we place the negative sign in front of x.
- The reflected wave runs in rod 1 at the wave speed c_1.

Everything about the reflected wave is known, except for the dimensionless number a.

Following a similar line of reasons, we expect that the transmitted wave takes the form

$$u^T(x,t) = bf\left(\frac{x}{c_2} - t\right).$$

Everything about the transmitted wave is known, except for the dimensionless number b, which is the ratio of the amplitude of the transmitted wave over the amplitude of the incident wave.

In rod 1, the net field is the superposition of the incident wave and the reflected wave:

$$u_1(x,t) = f\left(\frac{x}{c_1} - t\right) + af\left(-\frac{x}{c_1} - t\right).$$

The net axial force in rod 1 is

$$P_1(x,t) = R_1 F\left(\frac{x}{c_1} - t\right) - R_1 aF\left(-\frac{x}{c_1} - t\right),$$

where $F(\alpha) = df(\alpha)/d\alpha$.

In rod 2, the net field is the transmitted wave:

$$u_2(x,t)=bf\left(\frac{x}{c_2}-t\right).$$

The axial force in rod 2 is
.

$$P_2(x,t)=R_2bF\left(\frac{x}{c_2}-t\right)$$

To determine the two numbers, *a* and *b*, we invoke the boundary conditions: at the joint *x* = 0 and for all time, the displacement in rod 1 equals that in rod 2, and the axial force in rod 1 equals that in rod 2. Thus,

$$1+a=b$$

$$R_1(1-a)=R_2b$$

This set of linear algebraic equations is solved to give the two dimensionless numbers:

$$a=\frac{R_1-R_2}{R_1+R_2},\quad b=\frac{2R_1}{R_1+R_2}.$$

Note several special cases:

- When the two rods have the same impedance, $R_1=R_2$, the two numbers become $a=0$ and $b=1$. When the incident wave from rod 1 hits the joint, the wave does not reflect, but fully transmits into rod 2. The two rods are said to have matched impedance.
- When rod 2 has much lower impedance than rod 1, $R_2/R_1<<1$, the two numbers become $a=1$ and $b=2$. The force transmitted to rod 2 is vanishingly small, so that all the energy of the incident wave in rod 1 will be reflected back into rod 1. The force of the reflected wave has the sign opposite from that of the incident wave.
- When rod 2 has much higher impedance than rod 1, $R_2/R_1>>1$, the two numbers become $a=-1$ and $b=0$. The displacement transmitted to rod 2 is vanishingly small, so that all the energy of the incident wave in rod 1 will be reflected back into rod 1. The reflected wave keeps the sign of the force as that of the incident wave.

Dynamics of bending. We next consider bending of a beam. Let *x* be the coordinate of a cross section when the beam is in the reference configuration, i.e., when the beam is not bent. In the current state, the deflection of the beam is $w(x,t)$, the slope is

$$\theta=\frac{\partial w(x,t)}{\partial x},$$

and the curvature is

$$K=\frac{\partial \theta(x,t)}{\partial x}.$$

The deflection, slope, and curvature specify the deformation geometry.

The material model of the beam is specified by a relation between the curvature and the bending moment *M*, namely,

$$M=EIK,$$

where *E* is Young's modulus, and *I* is the second moment of the cross section. The quantity EI is the bending stiffness.

Next consider an element of the beam between *x* and $x+dx$. The balance of moment of the element dictates that

$$S=-\frac{\partial M(x,t)}{\partial x},$$

where $S(x,t)$ is the shear force acting on the cross section *x* at time *t*. Applying Newton's second law in the direction of deflection, we obtain that

$$\frac{\partial S(x,t)}{\partial x} = \rho A \frac{\partial^2 w(x,t)}{\partial t^2},$$

where ρ is the mass density, and *A* the area of the cross section.

A combination of the above equations gives

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} = -\rho A \frac{\partial^2 w(x,t)}{\partial t^2}.$$

This is the equation of motion for the deflection $w(x,t)$. This PDE, together with initial and boundary conditions, determines $w(x,t)$. Once $w(x,t)$ is determined, we can calculate other quantities such as the bending moment and shear force. For example, we have studied the vibration of the beam in a homework problem.

Bending wave. We now study a special kind of dynamic behavior of a beam: a wave traveling in the beam with a constant speed and an invariant waveform:

$$w(x,t) = f(\xi), \quad \xi = x - ct .$$

Both the wave speed *c* and the waveform *f* are unknown.

Inserting this guess into the PDE, we obtain that

$$EI\frac{d^4 f(\xi)}{d\xi^4} = -\rho A c^2 \frac{d^2 f(\xi)}{d\xi^2}.$$

This ODE has the general solution

$$f(\xi) = a_1 \sin\left(\sqrt{\frac{\rho A}{EI}} c\xi\right) + a_2 \cos\left(\sqrt{\frac{\rho A}{EI}} c\xi\right) + a_3\xi + a_4 ,$$

where a_1, a_2, a_3, a_4 are constants.

We will drop the rigid body motion $a_3\xi + a_4$, and focus one of the sinusoidal motion:

$$f(\xi) = a \sin\left(\sqrt{\frac{\rho A}{EI}} c\xi\right).$$

To interpret this solution, we write

$$w(x,t) = a\sin(kx - \omega t),$$

where *k* is the wave number, and ω the frequency. They relate to the wave speed *c* by

$$k = c\sqrt{\frac{\rho A}{EI}}, \quad \omega = c^2\sqrt{\frac{\rho A}{EI}} .$$

Thus, for a bending wave to travel at a constant speed and invariant waveform, the speed *c* can be arbitrary, and the waveform must be sinusoidal. Once a speed *c* is given, so are the wave number and the frequency.

Observe that the wave speed increases with the wave number, and becomes infinite for very short wavelengths. This is clearly an artifact of our model. The equation of motion is based on the classical theory of beams. The theory breaks down when the wavelength approaches the dimension of the cross section.

Dispersive wave. For a non-sinusoidal wave to travel in the beam, the wave may be a sum of many sinusoidal waves. Each sinusoidal wave has its own wave number and frequency, and travels at its own velocity. Consequently, as the non-sinusoidal wave travels, the waveform will change. A wave whose speed varies with its wave number is known as a dispersive wave. By contrast, the longitudinal wave in a rod is nondispersive, so that an arbitrary waveform can propagate in the rod without change.

Let us look at the sinusoidal wave again:

$$w(x,t) = a\sin(kx - \omega t).$$

The wave speed is given by $c = \omega / k$. For a nondispersive wave, the wave speed is constant, independent of the wave number, so that the frequency ω must be linear in the wave number k. For a dispersive wave, however, the wave speed varies with the wave number, so that the frequency is a *nonlinear function* of the wave number, $\omega(k)$. This function is known as the **dispersion relation**. For the bending wave, for example, the dispersion relation can be obtained by eliminating c from the $k-c$ and $\omega - c$ relations, so that

$$\omega(k) = k^2\sqrt{\frac{EI}{\rho A}}.$$

This is the dispersion relation for the bending wave.

Phase velocity. For a dispersive wave, it is useful to distinguish two kinds of velocities: phase velocity and group velocity. The speed of the sinusoidal wave

$$c_p(k) = \frac{\omega(k)}{k}$$

is known as the phase velocity. For example, the phase velocity of the pure sinusoidal wave in the beam is

$$c_p(k) = \sqrt{\frac{EI}{\rho A}}k.$$

Group velocity. The group velocity is defined as

$$c_g(k) = \frac{d\omega(k)}{dk}.$$

For the wave in the beam, the group velocity is

$$c_g(k) = 2\sqrt{\frac{EI}{\rho A}}k.$$

The definition of the group velocity is motivated as follows. Consider two sinusoidal waves with the same amplitude but slightly different wave numbers:

$$a\sin(k_1 x - \omega_1 t), \quad a\sin(k_2 x - \omega_2 t).$$

The overall response is the superposition of the two waves:

$$w(x,t) = a\sin(k_1 x - \omega_1 t) + a\sin(k_2 x - \omega_2 t)$$

$$= 2a\cos\left(\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right)\sin\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)$$

The combined wave is a wave of the average wave number $(k_1 + k_2)/2$, modulated by a wave of a smaller wave number $(k_1 - k_2)/2$ (i.e., a longer wave length). The former wave is called the carrier, and the latter the group. The group propagates at the velocity

$$\frac{\omega_1 - \omega_2}{k_1 - k_2} \approx \frac{d\omega(k)}{dk}.$$

Sketch a carrier wave modulated by a group wave.

Longitudinal wave vs. bending wave. (This section might be too subtle to be worthwhile.) We have seen two types of waves in a long rod: the longitudinal wave and the bending wave. The longitudinal wave is nondispersive, and the bending wave is dispersive. This difference arises from the difference in their governing equations.

The longitudinal wave is governed by

$$E\frac{\partial^2 u(x,t)}{\partial x^2} = \rho \frac{\partial^2 u(x,t)}{\partial t^2} .$$

Let L be an arbitrary length. We can then define dimensionless variables:

$$\hat{x} = \frac{x}{L}, \quad \hat{t} = \frac{t}{L}\sqrt{\frac{\rho}{E}} .$$

Using them as independent variables, we write

$$\phi(\hat{x},\hat{t}) = u(x,t).$$

The equation of motion for the longitudinal wave becomes

$$\frac{\partial^2 \phi(\hat{x},\hat{t})}{\partial \hat{x}^2} = \frac{\partial^2 \phi(\hat{x},\hat{t})}{\partial \hat{t}^2} .$$

This PDE is parameter-free, and so must be its solution. For the solution $u(x,t)$, all parameters appear in the scaling of x and t. Observe that both x and t are linear L. Consequently, the wave speed is independent of L.

The bending wave is governed by

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} = -\rho A \frac{\partial^2 w(x,t)}{\partial t^2} .$$

Let L be an arbitrary length. We can then define dimensionless variables:

$$\hat{x} = \frac{x}{L}, \quad \hat{t} = \frac{t}{L^2}\sqrt{\frac{\rho A}{EI}} .$$

Using them as independent variables, we write

$$\psi(\hat{x},\hat{t}) = w(x,t).$$

The equation of motion for the bending wave becomes

$$\frac{\partial^4 \psi(\hat{x},\hat{t})}{\partial \hat{x}^4} = -\frac{\partial^2 \psi(\hat{x},\hat{t})}{\partial \hat{t}^2} .$$

This PDE is parameter-free, and so must be its solution. For the solution $w(x,t)$, all parameters appear in the scaling of x and t. Observe that x is linear in L, but t is quadratic in L. Consequently, the wave speed must be inversely proportional to the wavelength.

Plane waves in a 3D homogeneous, elastic solid. Consider a wave traveling in a body. At a given time, the wave is localized in a region in the body. When size of the region affected by the wave is small compared to size of the body in all three directions, we may as well regard the body as an infinite body. A material particle in the body is unaffected before the wave arrives.

A *plane wave* is characterized by two directions:

1. **The direction of propagation**. The wave propagates in a fixed direction at all time.
2. **The direction of displacement**. The field of the displacement vector in the entire body is fixed in one direction at all time.

At a given time, the amplitude of the displacement is invariant for all material particles in any plane normal to the direction of propagation.

Without any calculation, we may make the following remarks:

- Because the governing equations have no length scale, a plane wave in an infinite body is nondispersive.

- Because the governing equations are linear, we can superimpose plane waves running in different directions to obtain any complex waves.

Longitudinal wave in an isotropic, homogeneous, elastic solid. An isotropic elastic solid supports plane waves of two types, longitudinal and transverse. For a longitudinal wave, the direction of displacement coincides with the direction of propagation. For a transverse wave, the direction of displacement is normal to the direction of propagation. We consider the longitudinal wave in this lecture, and leave the transverse wave as a homework problem.

Deformation geometry. Because all directions are equivalent in an isotropic material, we only need to consider a wave propagating in one direction, denoted by x. For a longitudinal wave, the only nonzero displacement component is $u(x,t)$. Consequently, the only nonzero strain component is

$$\varepsilon_x = \frac{\partial u(x,t)}{\partial x} .$$

That is, the solid is in a state of *uniaxial strain*.

Material model. Because of Poisson's effect, this strain induces normal stresses in all three directions. Symmetry dictates that $\sigma_y = \sigma_z$. Writing Hooke's law in the y-direction, we obtain that

$$\varepsilon_y = 0 = \frac{1}{E}\left(\sigma_y - \nu\sigma_z - \nu\sigma_x\right),$$

so that

$$\sigma_y = \sigma_z = \frac{\nu}{1-\nu}\sigma_x .$$

Writing Hooke's law in the x-direction, we obtain that

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y - \nu\sigma_z\right),$$

or

$$\sigma_x = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_x .$$

The coefficient is the stiffness under the uniaxial strain conditions.

Newton's second law reduces to

$$\frac{\partial \sigma_x}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} .$$

Putting the three ingredients together, we obtain the equation of motion:

$$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\frac{\partial^2 u}{\partial x^2} = \rho\frac{\partial^2 u}{\partial t^2} .$$

Except for the coefficient on the left-hand side, this equation of motion is identical to that for the rod. Thus, the same solution procedure applies. The longitudinal wave speed is

$$c_l = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} .$$

Take a representative value of Poisson's ratio, $\nu = 0.3$, and we obtain that

$$c_l = 1.2\sqrt{E/\rho} .$$

Recall that the wave speed in a rod is $\sqrt{E/\rho}$. Let us try to understand when the longitudinal wave in a rod is slower than the longitudinal wave in a 3D solid. For the longitudinal wave in a rod, we assume that the rod is in a state of uniaxial stress. This assumption is reasonable when the length scale of the wave is large compared to the cross section of the rod. Under this condition, the rod can deform freely in the transverse direction. The stress-strain relation is

$$\sigma_x = E\varepsilon_x$$

For the longitudinal wave in the 3D solid, the transverse deformation is prohibited. This constraint effectively stiffen the stress-strain relation, which becomes:

$$\sigma_x = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_x .$$

Consequently, the longitudinal wave in a rod is slower than the longitudinal wave in a three-dimensional solid.

Transverse wave in an isotropic elastic solid. For a transverse wave, the only nonzero displacement is in the direction normal to the direction of propagation. We may write $v(x,t)$. Following the same procedure, we find that the transverse wave speed is

$$c_t = \sqrt{\frac{E}{2\rho(1+\nu)}} .$$

Take a representative value of Poisson's ratio, $\nu = 0.3$, and we obtain that

$$c_t = 0.6\sqrt{E/\rho} .$$

Thus, the longitudinal wave is about twice as fast as the transverse wave.

The general form of a plane wave in an isotropic elastic solid. For a longitudinal wave, the time-dependent field of displacement is

$$\mathbf{u}(\mathbf{x},t) = \mathbf{s}f(\xi),$$

where

$$\xi = \frac{\mathbf{s}\cdot\mathbf{x}}{c_l} - t ,$$

and

$$c_l = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} .$$

We interpret various aspects of this form as follows:

- **s** is the unit vector pointing in the direction of propagation.
- In the three dimensional space, the distance measured along the direction of propagation **s** is given by the inner product $\mathbf{s}\cdot\mathbf{x}$. Thus, the variable $\xi = \frac{\mathbf{s}\cdot\mathbf{x}}{c_l} - t$ for the plane wave in three dimensions generalizes $\xi = \frac{x}{c} - t$ for the wave in one dimension.
- The plane wave is nondispersive, so that the waveform can be an arbitrary function $f(\cdot)$. This waveform is a scalar, representing the magnitude of the field of the displacement vector. At a fixed time, the magnitude of the displacement field varies for particles in the direction **s**, but is constant for particles in any plane normal to **s**.
- For the longitudinal wave, the direction of the field of displacement vector in the entire body coincides with the direction of propagation, so we multiply $\mathbf{s}$ in the front.
- For an isotropic material, the speed of longitudinal wave, c_l, is the same for any direction of propagation.

For a transverse wave, the field of displacement is

$$\mathbf{u}(\mathbf{x},t)=\mathbf{a}g\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c_t}-t\right),\quad c_t=\sqrt{\frac{E}{2\rho(1+\nu)}}$$

where **s** is the unit vector pointing in the direction of propagation, **a** is any unit vector normal to **s**, c_t is the speed of the transverse wave, and g is the waveform of displacement.

The general form of a plane wave propagates in direction **s** in an infinite isotropic material is the superposition of three waves:

$$\mathbf{u}(\mathbf{x},t)=\mathbf{s}f\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c_l}-t\right)+\mathbf{a}g\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c_t}-t\right)+\mathbf{b}h\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c_t}-t\right),$$

where **a** and **b** are unit vectors normal to **s**. Vectors **a** and **b** are also normal to each other. In time, the field of displacement will split into two lumps: the longitudinal wave runs faster than the transverse wave.

To obtain non-planar waves in an infinite body, we can superimpose plane waves in different directions. For example, we can think of a spherical wave as a sum of many longitudinal plane waves.

Exercise. A transverse wave propagates in an isotropic material. At any given material particle, it is observed that the magnitude of the displacement is time-independent, but the direction of the displacement rotates at frequency ω. Write a possible displacement field.

Plane waves in 3D, anisotropic, homogeneous, elastic solid. A combination of the momentum balance equation

$$\frac{\partial\sigma_{ij}}{\partial x_j}=\rho\frac{\partial^2 u_i}{\partial t^2}$$

and the stress-strain relation

$$\sigma_{ij}=C_{ijkl}\frac{\partial u_k}{\partial x_l}$$

leads to the equation of motion

$$C_{ijkl}\frac{\partial^2 u_k}{\partial x_l\partial x_j}=\rho\frac{\partial^2 u_i}{\partial t^2}.$$

This is a set of PDEs for the displacement field $\mathbf{u}(\mathbf{x},t)$.

For a plane wave the displacement field takes the form

$$\mathbf{u}(\mathbf{x},t)=\mathbf{a}f(\xi),\qquad \xi=\frac{\mathbf{x}\cdot\mathbf{s}}{c}-t\ ,$$

where **s** is the unit vector in the direction of propagation, **a** the unit vector in the direction of the displacement, f the waveform, and c the wave speed. In an anisotropic material, for a given direction of propagation **s**, the direction of displacement **a** may neither coincide with the direction of propagation nor be perpendicular to the direction of propagation. Indeed, the direction of displacement **a** can be determined as a part of the solution to the equation of motion.

Insert the expression of the plane wave into the equation of motion, and we obtain that

$$C_{ijkl}s_l s_j a_k=\rho c^2 a_i .$$

This is an eigenvalue problem of the matrix $C_{ijkl}s_l s_j$, known as the acoustic tensor, with ρc^2 being the eigenvalue, and **a** the eigenvector. Recall that this matrix is symmetric and positive-definite. Consequently, the matrix has three real and positive eigenvalues. Associated with each eigenvalue is an eigenvector, $\mathbf{a}$. Each eigenvector is determined up to a scalar; without loss of generality, we will set each eigenvector to be a unit vector. The three eigenvectors associated with the three eigenvalues are orthogonal to one another.

Thus, once a direction of propagation $\mathbf{s}$ is prescribed, there exist plane waves of three types. Their speeds, c', c'', c''', and their directions of displacement, $\mathbf{a}', \mathbf{a}'', \mathbf{a}'''$, are determined by the eigenvalue problem of the acoustic tensor $C_{ijkl}s_l s_j$. By liner superposition, the displacement field of a plane wave propagates in direction $\mathbf{s}$ is

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a}' f\left(\frac{\mathbf{x}\cdot\mathbf{s}}{c'} - t\right) + \mathbf{a}'' g\left(\frac{\mathbf{x}\cdot\mathbf{s}}{c''} - t\right) + \mathbf{a}''' h\left(\frac{\mathbf{x}\cdot\mathbf{s}}{c'''} - t\right).$$

The waveforms, f, g, h, are undetermined by the equation of motion.

Exercise. Determine the acoustic tensor of an isotropic material. Solve the eigenvalue problem of this acoustic tensor. Relate your results to the longitudinal and transverse waves.

Exercise. Given a single crystal of cubic symmetry, and an arbitrary direction of propagation, Determine the acoustic tensor. Use the elastic constants of single crystal copper to generate some plots of wave speeds in various directions.

Impedance tensor. Consider a plane wave in a body:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{a} f(\xi), \quad \xi = \frac{\mathbf{x}\cdot\mathbf{s}}{c} - t .$$

where $\mathbf{a}$ is the unit vector in the direction of displacement, $\mathbf{s}$ the unit vector in the direction of propagation, c the wave speed, and $f(\xi)$ the waveform.

The velocity of each material particle is given by

$$\mathbf{v} = \frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t}.$$

A direct calculation gives that

$$\mathbf{v}(\mathbf{x},t) = -\mathbf{a}\frac{df(\xi)}{d\xi}.$$

The stress field is given by

$$\sigma_{ij} = C_{ijpq}\frac{\partial u_p(\mathbf{x},t)}{\partial x_q}.$$

Let $\mathbf{n}$ be a unit vector. The traction on the plane normal to $\mathbf{n}$ is given by

$$t_i = \sigma_{ij} n_j .$$

A direct calculation gives that

$$t_i = \frac{C_{ijpq} n_j a_p s_q}{c}\frac{df(\xi)}{d\xi}.$$

A comparison between the velocity and the traction gives that

$$\mathbf{t} = -\mathbf{R}\mathbf{v},$$

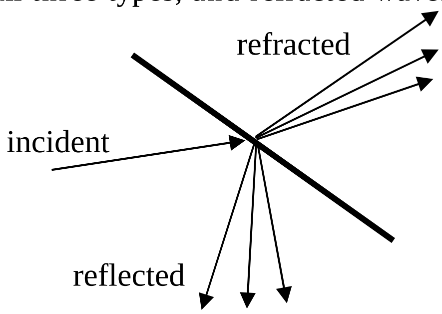
where

$$R_{ip} = \frac{C_{ijpq} n_j s_q}{c}.$$

We may call this object the impedance tensor. The impedance tensor depends on stiffness tensor, the two unit vectors $\mathbf{s}$ and $\mathbf{n}$, as well as on the wave speed. Note that the impedance tensor defined here has a different unit from that defined for the longitudinal wave in a rod.

Exercise. Determine the impedance tensors for the longitudinal and transverse waves in an isotropic solid.

Directions of propagation of the reflected and refracted waves. Two materials are bonded on a planar interface. To suspend any discussion involving material symmetry, we will consider generally anisotropic solids. Let $\mathbf{n}$ be the unit vector normal to the interface. An incident plane wave propagates in direction $\mathbf{s}$ and speed c. We would like to determine the directions of reflected and refracted waves. When the direction of the incident wave is normal to the interface, the incident wave may induce reflected waves all three types, and refracted waves of all three types. All the six waves propagate in the direction normal to the interface, $\mathbf{n}$.



We next consider the general case when the direction of the incident wave, $\mathbf{s}$, differs from the normal direction of the interface, $\mathbf{n}$. Once again, the incident wave may cause reflected waves of three types, and refracted waves of three types. Let one of the six waves propagates in direction $\mathbf{s}'$ at speed c'. The field of the incident wave is a function of the argument

$$\frac{\mathbf{s}\cdot\mathbf{x}}{c} - t ,$$

while the field of the other wave is a function of the argument

$$\frac{\mathbf{s}'\cdot\mathbf{x}}{c'} - t .$$

We place the origin of $\mathbf{x}$ on the interface. The displacement and the traction are continuous across the interface at all time. Consequently, for all $\mathbf{x}$ on the interface, the two arguments must be equal:

$$\frac{\mathbf{s}'\cdot\mathbf{x}}{c'} = \frac{\mathbf{s}\cdot\mathbf{x}}{c} ,$$

i.e.,

$$\left(\frac{\mathbf{s}'}{c'} - \frac{\mathbf{s}}{c}\right)\cdot\mathbf{x} = 0 .$$

Thus, the vector $\frac{\mathbf{s}'}{c'} - \frac{\mathbf{s}}{c}$ is perpendicular to the interface.

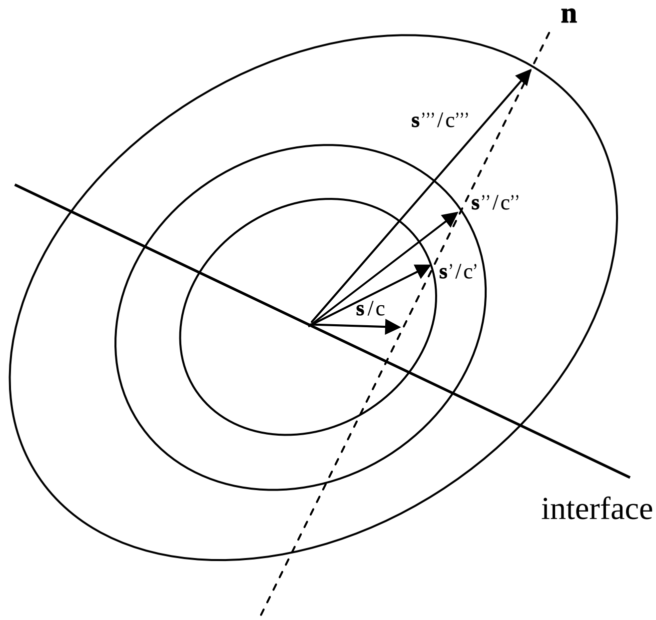
The directions of propagation of the reflected and refracted waves may be determined by using a geometric construction. Start at an origin, we plot a vector $\mathbf{s}$ in every direction, setting the length to be $1/c$, where c is the speed of a plane wave propagating in direction $\mathbf{s}$. The locus of the end points is a three-sheeted surface. For example, for an isotropic solid, this locus consists of two spheres, of radii $1/c_l$ and $1/c_t$. The latter is degenerated from two spheres with the same radii. For an anisotropic solid, the locus need not be spheres.

The figure illustrates the procedure to determine the directions of propagation of the three refracted waves, using the following information:

- the three-sheeted surface associated with the crystal where the refracted waves propagate,
- the normal direction of the interface, $\mathbf{n}$, and
- the vector associated with an incident wave, $\mathbf{s}/c$.

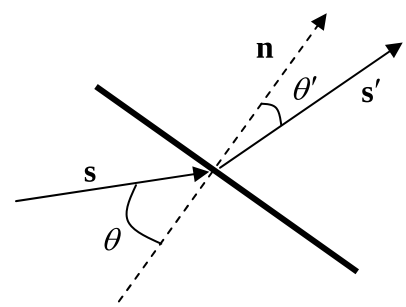
This procedure determines the vectors associated with the three refracted waves, $\mathbf{s}'/c', \mathbf{s}''/c'', \mathbf{s}'''/c'''$.

Total reflection. From this geometric construction, it is evident that when the speed c of the incident wave is small, i.e., $1/c$ is large, there may be fewer or no refracted waves. The condition of no refracted wave is known as total reflection. Also evident from the geometric construction that the condition of the total reflection also depends on the direction of propagation of the incident wave.



Snell's law. Because the vector $\frac{\mathbf{s}'}{c'}-\frac{\mathbf{s}}{c}$ is normal to the interface, the vector $\mathbf{s}'$ must lie in the plane spanned by $\mathbf{s}$ and $\mathbf{n}$, known as the **plane of incidence**. Also because the vector $\frac{\mathbf{s}'}{c'}-\frac{\mathbf{s}}{c}$ is normal to the interface, the projection of this vector on the interface must vanish, namely,

$$\frac{\sin\theta'}{c'}=\frac{\sin\theta}{c}.$$



where θ and θ' are the angles from the normal vector of the interface to $\mathbf{s}$ and $\mathbf{s}'$. This equation is known as Snell's law, which determines the directions of the reflected and refracted waves.

The arguments we have used are quite general, so that Snell's law is applicable to many waves, e.g., elastic waves and electromagnetic waves. I learned the argument from the book *Principles of Optics* by Born and Wolf.

Amplitudes of reflected and refracted waves. A solid fills a half space, with no traction acting on its surface. A known wave

$$\mathbf{u}(\mathbf{x},t)=\mathbf{a}f\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c}-t\right).$$

is incident upon the surface of the half space. Here, $\mathbf{a}$ is the unit vector in the direction of displacement, $\mathbf{s}$ is the unit vector in the direction of propagation, c is the wave speed, and $f(\cdot)$ is the waveform.

Waves of three types reflect back. For one of the reflected waves, let c' be the speed, $\mathbf{s}'$ be the direction of propagation, and $\mathbf{a}'$ the direction of displacement. The waveform of the reflected must be similar to that of the incident wave, $f(\cdot)$. Thus, the displacement field associated with this reflected wave is

$$A'\mathbf{a}'f\left(\frac{\mathbf{s}'\cdot\mathbf{x}}{c'}-t\right),$$

where A' is the ratio of the amplitude of the reflected wave over the amplitude of the incident wave. Similarly, let $(c'',\mathbf{s}'',\mathbf{a}'',A'')$ and $(c''',\mathbf{s}''',\mathbf{a}''',A''')$ be the corresponding quantities for the other two reflected waves. The net displacement field in the body is the linear superposition of the four waves:

$$\mathbf{u}(\mathbf{x},t)=\mathbf{a}f\left(\frac{\mathbf{s}\cdot\mathbf{x}}{c}-t\right)+A'\mathbf{a}'f\left(\frac{\mathbf{s}'\cdot\mathbf{x}}{c'}-t\right)+A''\mathbf{a}''f\left(\frac{\mathbf{s}''\cdot\mathbf{x}}{c''}-t\right)+A'''\mathbf{a}'''f\left(\frac{\mathbf{s}'''\cdot\mathbf{x}}{c'''}-t\right)$$

Everything else about the reflected waves has been determined except for A', A'', A'''.

To determine A', A'', A''', we use the traction free boundary conditions. In particular, when $\mathbf{x}$ is on the surface of the half space, all the arguments of the function are equal:

$$\frac{\mathbf{s}\cdot\mathbf{x}}{c}-t=\frac{\mathbf{s}'\cdot\mathbf{x}}{c'}-t=\frac{\mathbf{s}''\cdot\mathbf{x}}{c''}-t=\frac{\mathbf{s}'''\cdot\mathbf{x}}{c'''}-t=\xi$$

Let $\mathbf{n}$ be the unit vector normal to the surface of the half space. The traction vector on the surface is

$$t_i=n_jC_{ijpq}\frac{\partial u_p}{\partial x_q}=n_jC_{ijpq}\left(\frac{a_ps_q}{c}+\frac{A'a'_ps'_q}{c'}+\frac{A''a''_ps''_q}{c''}+\frac{A'''a'''_ps'''_q}{c'''}\right)\frac{df(\xi)}{d\xi}.$$

The traction free condition $t_i=0$ gives 3 linear algebraic equations:

$$n_jC_{ijpq}\left(\frac{a_ps_q}{c}+\frac{A'a'_ps'_q}{c'}+\frac{A''a''_ps''_q}{c''}+\frac{A'''a'''_ps'''_q}{c'''}\right)=0.$$

This set of equations can be rewritten in terms of the impedance tensors of the four waves:

$$\mathbf{Ra}+A'\mathbf{R}'\mathbf{a}'+A''\mathbf{R}''\mathbf{a}''+A'''\mathbf{R}'''\mathbf{a}'''=0.$$

This set of equations determines A', A'', A'''.

Exercise. A wave incident upon a bimaterial interface generates three reflected waves and three refracted waves. The amplitudes of the six waves are determined by the continuity of the displacement vector and the traction vector. Establish a set of six equations that determine the six amplitudes. Use the impedance tensors in the final result.

Reflection from a free surface of an isotropic material.

- If the incident wave is a transverse wave with the displacement vector perpendicular to the plane of incidence, the wave is entirely reflected as a wave of the same kind, and the amplitude is 1.
- If the incident wave is a transverse wave with the displacement vector in the plane of incidence, the reflected consists of a transverse wave of the same kind and a longitudinal wave. The amplitudes of the two reflected waves are

$$A_t=\frac{\sin 2\theta_l\sin 2\theta-(c_l/c_t)^2\cos^2 2\theta}{\sin 2\theta_l\tan 2\theta+(c_l/c_t)^2\cos^2 2\theta},\quad A_l=\frac{2(c_l/c_t)\sin 2\theta\cos 2\theta}{\sin 2\theta_l\sin 2\theta+(c_l/c_t)^2\cos^2 2\theta}.$$

- If the incident wave is a longitudinal wave, the reflected wave consists of a longitudinal wave and a transverse wave with the displacement vector in the plane of incidence. The amplitudes of the two reflected waves are

$$A_l=\frac{\sin 2\theta_t\sin 2\theta-(c_l/c_t)^2\cos^2 2\theta_t}{\sin 2\theta_t\sin 2\theta+(c_l/c_t)^2\cos^2 2\theta_t},\quad A_t=-\frac{2(c_l/c_t)\sin 2\theta\cos 2\theta_t}{\sin 2\theta_t\sin 2\theta+(c_l/c_t)^2\cos^2 2\theta_t}.$$

Stroh representation. (Stroh, A.N., 1962. Steady state problems in anisotropic elasticity. *Math. Phys.* 41, 77-103.) A body is in a state of stress independent of the coordinate x_3, and a source of stress is moving through the body at a constant speed v in the x_1 direction. Let (x,y) be a coordinate system moving at the source speed, relating to the fixed coordinates as

$$x=x_1-vt,\qquad y=x_2.$$

To an observer moving at speed v, the steady state field is time-independent. That is, the displacements are functions of (x,y).

Consider a specific steady-state displacement field,

$$\mathbf{u}(\mathbf{x},t)=\mathbf{A}f(x+py),$$

where $\mathbf{A}$ is a vector, p a number, and $f(\cdot)$ a function of one variable. The displacement field will satisfy the equation of motion provided

$$\left(C_{i1k1}-\rho v^2\delta_{ik}+pC_{i1k2}+pC_{i2k1}+p^2C_{i2k2}\right)A_k=0.$$

This is a set of linear algebraic equations for $\mathbf{A}$. To represent a nontrivial displacement field, $\mathbf{A}$ cannot be zero. Consequently, the determinant of the above equations must vanish, which leads to a polynomial equation of degree six in p. Denote the six roots by p_α, labeling $\alpha=\pm1,\pm2,\pm3$ so that, if complex roots occur, $p_{+\alpha},p_{-\alpha}$ are complex conjugates, and giving the positive α to the root with positive imaginary part. The labeling for real roots will be specified later. Following Stroh, we will only consider the case that the p_α are all distinct; equal roots may be regarded as the limiting case of distinct roots. For each p_α, we can determine a column $\mathbf{A}_\alpha$ up to a scaling factor. Make $\mathbf{A}_\alpha$ real when p_α is real, and $\mathbf{A}_\alpha$, $\mathbf{A}_{-\alpha}$ complex conjugates when p_α is complex.

For any six arbitrary functions $f_\alpha(\cdot)$, the linear combination

$$\mathbf{u}(\mathbf{x},t)=\sum_{\alpha=\pm1}^{\pm3}\mathbf{A}_\alpha f_\alpha(x+p_\alpha y)$$

satisfies the equation of motion. Summation over a Greek suffix will always be indicated explicitly. Any steady state solution can be represented in this form; the six Stroh functions f_α are to be determined by boundary conditions. Make f_α real when p_α is real, and f_α, $f_{-\alpha}$ complex conjugate when p_α is complex, so that the displacements will always be real-valued.

For later user, we write the above in terms of components:

$$u_i(\mathbf{x},t)=\sum_{\alpha=\pm1}^{\pm3}A_{i\alpha}f_\alpha(z_\alpha),$$

where $A_{i\alpha}$ are the components of the vector $\mathbf{A}_\alpha$, and $z_\alpha=x+p_\alpha y$. One can express the stresses in terms of the Stroh functions:

$$\sigma_{i2}=\sum_{\alpha=\pm1}^{\pm3}L_{i\alpha}f'_\alpha(z_\alpha),$$

$$\sigma_{i1}=-\sum_{\alpha=\pm1}^{\pm3}\left(L_{i\alpha}p_\alpha-\rho v^2A_{i\alpha}\right)f'_\alpha(z_\alpha),$$

with

$$L_{i\alpha}=\left(C_{i2k1}+p_\alpha C_{i2k2}\right)A_{k\alpha}.$$

We use $(\,'\,)$ to indicate the differentiation of any one-variable function.

The above equations hold for any speed v whether greater or less than the sonic speeds of the solid. When $v=0$, all the p_α are complex. When v is sufficiently large, all the p_α are real. There are three critical speeds, $V_3\le V_2\le V_1$. When v passes V_α, a pair of roots $p_{\pm\alpha}$ change from complex to real. If then $v<V_\alpha$, the roots $p_{\pm\alpha}$ are complex, and the functions $f_{\pm\alpha}$ are complex analytic functions. If $v>V_\alpha$, the two roots $p_{\pm\alpha}$ are real, and the equations,

$x + p_{\pm\alpha} y = \text{constant}$, are the characteristic lines, along which the real functions $f_{\pm\alpha}$ have constant values.

Rayleigh wave. A wave can propagate undiminished along the surface of a half space. However, the wave is localized near the surface, and the amplitude decays beneath the surface. This wave is known as the Rayleigh wave. There is no intrinsic length scale in this problem, so that the Rayleigh wave is nondispersive.

Say we seek a surface wave with the velocity v below the transverse wave, so that all the eigenvalues are complex. Let the complex potential be $f_1(z_1), f_2(z_2), f_3(z_3)$. The displacement field is

$$u_i = \sum_{\alpha=1}^{3} A_{i\alpha} f_\alpha(z_\alpha) + \sum_{\alpha=1}^{3} \bar{A}_{i\alpha} \bar{f}_\alpha(\bar{z}_\alpha).$$

The traction vector is

$$\sigma_{i2} = \sum_{\alpha=1}^{3} L_{i\alpha} f'_\alpha(z_\alpha) + \sum_{\alpha=1}^{3} \bar{L}_{i\alpha} \bar{f}'_\alpha(\bar{z}_\alpha).$$

On the surface of the half space, $y = 0$, the traction vanishes, so that

$$\mathbf{Lf}(x) + \overline{\mathbf{L}}\overline{\mathbf{f}}(x) = 0 .$$

The first term is analytic in the lower half plane, and the second term is analytic in the upper half plane. Consequently,

$$\mathbf{Lf}(z) = 0$$

for any z. This is an eigenvalue problem. To have a nonvanishing field, we must require that

$$\det \mathbf{L} = 0 .$$

This equation determines the velocity of the surface wave. The surface wave is also nondispersive.

The complex potentials are given by

$$\mathbf{f}(z) = \mathbf{h} g(z),$$

where $\mathbf{h}$ is an eigenvector of $\mathbf{L}$ associated with the Rayleigh wave, and g is an arbitrary function.

Rayleigh wave on the surface of an isotropic solid. For a half space, the anti-plane and the in-plane deformation decouple. We will only consider the in-plane deformation. We have

$$p_{\pm 1}^2 = v^2 / c_l^2 - 1 , \qquad p_{\pm 2}^2 = v^2 / c_t^2 - 1 .$$

The longitudinal and shear wave speeds are given by

$$c_l = \left[\frac{2(1-\nu)\mu}{(1-2\nu)\rho} \right]^{1/2}, \ c_t = \left(\frac{\mu}{\rho} \right)^{1/2},$$

It is evident that c_l and c_s are also the critical speeds. When the crack speed v surpasses c_l or c_s, a pair of roots change from complex to real.

The related matrices are all 2 by 2, as given by

$$\mathbf{A} = \begin{bmatrix} 1 & -p_2 \\ p_1 & 1 \end{bmatrix}, \quad \mathbf{L} = \mu \begin{bmatrix} 2p_1 & 1 - p_2^2 \\ -(1 - p_2^2) & 2p_2 \end{bmatrix},$$

For a subsonic crack, $v < c_t$, all roots are imaginary numbers:

$$p_1 = i\sqrt{1 - v^2 / c_l^2} , \qquad p_2 = i\sqrt{1 - v^2 / c_t^2} .$$

The speed of the surface wave is determined by $\det \mathbf{L} = 0$, namely,

$$4\sqrt{\left(1-v^2/c_l^2\right)\left(1-v^2/c_t^2\right)}=\left(2-v^2/c_t^2\right)^2.$$

This is an algebraic equation for v, and must be solved numerically. When $\nu = 1/3$, $v_R / c_t = 0.932$.

Waves in laminates. Let us now look at waves propagating in a direction parallel to the interfaces of a laminate of different materials. An example is the Love wave, propagating in a laminate of a layer bonded to a half space. The thickness of each layer provides a length scale, so that the waves are dispersive. Each material has its own set of complex functions. We look for solutions of the form

$$f_\alpha(z_\alpha)=a_\alpha \exp(ikz_\alpha)+b_\alpha \exp(-ikz_\alpha).$$

Such a function ensures that the x dependence is sinusoidal, with the wave number k. The wave velocity v and the constants a_α and b_α are determined by requiring continuity of traction and displacements, which leads to an eigenvalue problem. The algebra is messy but straightforward.