Stress intensity factor

We have modeled a body by using the linear elastic theory. We have modeled a crack in the body by a flat plane, and the front of the crack by a straight line. Within this idealized model, the field around the front of the crack is singular. The singular field is clearly an artifact of the idealized model, but Irwin and others made the singular field a centerpiece of fracture mechanics.

The mathematics of this singular field had been known long before Irwin entered the field. We will focus on the mathematics in this lecture, and will describe Irwin's way of using the singular field in the following lecture.

In previous lectures, we have described the work of Griffith (http://imechanica.org/node/7470), and the reinterpretation of Irwin and Orowan (http://imechanica.org/node/7507). The descriptions centered on two quantities: energy release rate and fracture energy. This energy-based approach leads us into applications of fracture mechanics (http://imechanica.org/node/7531). Before showing you more applications, I'd like to tell you about a basic concept, possibly also due to Irwin, concerned with the modes of fracture.

The concept of the modes appeals to our intuition, but this concept is absent in the energy-based approach. Energy release rate by itself does not differentiate the modes of fracture. To describe the modes of a crack will require us to talk about the field around the front.

Modes of fracture. Depending on the symmetry of the field around the tip of a crack, fracture may be classified into three modes.

- Mode I: tensile mode, or opening mode.
- Mode II: in-plane shear mode, or sliding mode.
- Mode III: anti-plane shear mode, or tearing mode.

The modes describe the local condition around a point on the front. For example, consider a penny-shaped crack in an infinite body. The front of the crack is a circle. When a load pulls the body in the direction perpendicular to the plane of the crack, every point along the front of the crack is under the mode I condition. When a load shears the body in the direction parallel to the plane of the crack, every point along the front of the crack is under a mixed mode II and mode III condition. Only a few special points on the front are under either a pure mode II condition, or a pure mode III condition.

Governing equations of linear elasticity. The state of a body is characterized by three fields: displacement $u_i(x_1,x_2,x_3)$, strain $\varepsilon_{ij}(x_1,x_2,x_3)$, and stress $\sigma_{ij}(x_1,x_2,x_3)$. In the body, the three fields satisfy the governing equations:

- Strain-displacement relations $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$
- Balance of forces $\sigma_{ij,j} = 0$
- Stress-strain relations $\sigma_{ij} = C_{ijpq}\varepsilon_{pq}$

At every point on the surface of the body, in each direction, we prescribe either the displacement, or the traction $\sigma_{ij}n_j$. If these equations are vague to you, please review the notes on the elements of linear elasticity (http://imechanica.org/node/205).

The singular field around the front of a crack. While a body is in a three-dimensional space, the singular field around the front of a crack is nearly two-dimensional. Let us clarify this reduction in dimension. In the three-dimensional space, we model a body by a volume, a crack by a smooth surface, and the front of the crack by a smooth curve. For any point on the front, we use the point as the origin to set up local coordinates, with the axis x pointing in the direction of propagation of the front, y normal to the plane of the crack, and z tangent to the front.

Because the front is assumed to be a smooth curve, the field around the front is singular in x and y, but smooth in z. That is, for any component of the field, $f(x,y,z)$, the derivative $\partial f / \partial z$ is small compared to the derivatives $\partial f / \partial x$ and $\partial f / \partial y$. Consequently, we may drop all partial derivatives with respect to z in the governing equations. Consequently, the singular field around the front is locally characterized by a field of the form

$$u(x,y), v(x,y), w(x,y).$$

We further assume that the elastic behavior of the material is isotropic. The field decouples into two types:

- Plane-strain deformation: $u(x,y) \neq 0, v(x,y) \neq 0$, but $w(x,y) = 0$.
- Anti-plane deformation: $u(x,y) = 0, v(x,y) = 0$, but $w(x,y) \neq 0$.

The plane-strain deformation describes mode I and mode II cracks, and the anti-plane deformation describes mode III cracks. In class we will mostly talk about mode I cracks.

Plane-strain conditions. To analyze this locally plane-strain field, we focus on an actual plane-strain problem:

- The body is under the plane-strain conditions.
- The crack is a flat plane.
- The front of the crack is a straight line.
- The field is elastic all the way to the front.

The fields reduced to

- Displacement field: $u(x,y)$ and $v(x,y)$. Note that $w = 0$.
- Strain field: $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$ are functions of x and y. Other strain components vanish.
- Stress field: $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{zz}$ are functions of x and y. Other stress components vanish.

Strain-displacement relations:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

Balance of forces:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0.$$

Hooke's law:

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\right),$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}\right),$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}\right),$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy},$$

Under the plane strain conditions, $\varepsilon_{zz} = 0$, so that

$$\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right).$$

Substituting this relation to Hooke's law, we obtain that

$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\left(\sigma_{xx} - \frac{\nu}{1-\nu}\sigma_{yy}\right),$$

$$\varepsilon_{yy} = \frac{1-\nu^2}{E}\left(\sigma_{yy} - \frac{\nu}{1-\nu}\sigma_{xx}\right),$$

$$\varepsilon_{xy} = \frac{(1+\nu)}{E}\sigma_{xy}.$$

The quantity

$$\bar{E} = \frac{E}{1-\nu^2},$$

is known as the plane-strain modulus.

Under the plane strain conditions, the elastic field is represented by 2 displacements, 3 strains, and 4 stresses. The 9 functions are governed by 9 field equations (3 strain-displacement relations, 3 stress-strain relations, 2 equations to balance forces, and 1 relation between σ_z, σ_x, σ_y. If you have not studied plane-strain problems before, please review the notes http://imechanica.org/node/319.

Airy's function. For a very interesting historical account involving Airy, Stokes and Maxwell, see V.V. Meleshko, Selected topics in the history of the two-dimensional biharmonic problem, Applied Mechanics Review 56, 33-85 (2003). We now have 9 equations for 9 functions. Many methods have been devised to solve these equations. Here we will follow a particular method initiated by Airy (1863). The system of equations can be reduced to one equation for one function as follows. Recall a theorem in calculus. If functions $f(x,y)$ and $g(x,y)$ satisfy the following relation

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y},$$

then a function $\alpha(x,y)$ exists, such that

$$f = \frac{\partial \alpha}{\partial y}, \quad g = \frac{\partial \alpha}{\partial x}.$$

According to this theorem, one equation of force balance,

$$\frac{\partial \sigma_{xx}}{\partial x} = -\frac{\partial \sigma_{xy}}{\partial y},$$

implies that a function $\alpha(x,y)$ exists, such that

$$\sigma_{xx} = \frac{\partial \alpha}{\partial y}, \sigma_{xy} = -\frac{\partial \alpha}{\partial x}.$$

The other equation of force balance,

$$\frac{\partial \sigma_{xy}}{\partial x} = -\frac{\partial \sigma_{yy}}{\partial y},$$

implies that a function $\beta(x,y)$ exists, such that

$$\sigma_{yy} = \frac{\partial \beta}{\partial x}, \sigma_{xy} = -\frac{\partial \beta}{\partial y}.$$

In the above, we have expressed the shear stress σ_{xy} in two ways. Equating the two expressions, we obtain that

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y}.$$

According to the theorem in calculus, this equation implies that a function $\phi(x,y)$ exists, such that

$$\alpha = \frac{\partial \phi}{\partial y}, \beta = \frac{\partial \phi}{\partial x}.$$

Summing up, we can express the three stresses in terms of one function:

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \ \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \ \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}.$$

The function $\phi(x,y)$ is known as Airy's function. Stresses expressed by Airy's function satisfy the equations of force balance.

Strains in terms of Airy's function. Using Hooke's law, we express the strains in terms of Airy's function:

$$\varepsilon_{xx} = \frac{1-\nu^2}{E}\left(\frac{\partial^2 \phi}{\partial y^2} - \frac{\nu}{1-\nu}\frac{\partial^2 \phi}{\partial x^2}\right),$$

$$\varepsilon_{yy} = \frac{1-\nu^2}{E}\left(\frac{\partial^2 \phi}{\partial x^2} - \frac{\nu}{1-\nu}\frac{\partial^2 \phi}{\partial y^2}\right),$$

$$\varepsilon_{xy} = -\frac{(1+\nu)}{E}\frac{\partial^2 \phi}{\partial x \partial y}.$$

The equation of compatibility. Eliminate the displacements from the strain-displacement relations, and we obtain that

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial \varepsilon_{xy}}{\partial x \partial y}.$$

This equation is known as the equation of compatibility.

The compatibility equation can be expressed in terms of Airy's function:

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0.$$

This equation is known as the *bi-harmonic equation*. According to Meleshko, this equation was first derived by Maxwell when asked by Stokes to review the paper by Airy. The plane-strain problem is governed by the bi-harmonic equation and the boundary conditions. Once ϕ is solved, one can determine the stresses, strains, and displacements.

The biharmonic equation is often written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0.$$

Polar coordinates. When Airy's function is expressed as a function of the polar coordinates, $\phi(r,\theta)$, the polar components of the stress are expressed as

$$\sigma_{rr} = \frac{\partial^2 \phi}{r^2 \partial \theta^2} + \frac{\partial \phi}{r \partial r}, \quad \sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{\partial \phi}{r \partial \theta}\right).$$

The bi-harmonic equation is

$$\left(\frac{\partial^2}{r^2 \partial \theta^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial r^2}\right)\left(\frac{\partial^2}{r^2 \partial \theta^2} + \frac{\partial}{r \partial r} + \frac{\partial^2}{\partial r^2}\right)\phi = 0.$$

The Williams expansion. M.L. Williams, On the stress distribution at the base of a stationary crack. Journal of Applied Mechanics 24, 109-115 (1957).

Within linear elasticity, the field in a body is determined by a boundary-value problem. Of course, the field depends on the boundary conditions: the size and the shape of the crack and the body, as well as the magnitude and the distribution of the load. Some such boundary-value problems had been solved before Williams entered the field. Williams took a different approach.

Instead of solving individual boundary-value problems, he focused on the singular field around the tip of the crack, in a zone so small that the boundary of the body can be assumed to be infinitely far away. He discovered that the form of the singular field is universal to cracked body, regardless of the shape of the body and the crack.

Let (r,θ) be the polar coordinates, centered at a particular point on the front of the crack. The crack propagates in the direction $\theta = 0$, and the two faces of the crack coincide with $\theta = \pm\pi$. The two faces of the crack are traction-free.

We solve the biharmonic equation by the method of separation of variables. Each term in the bi-harmonic equation has the same dimension in *r*. For such an equi-dimensional equation, the solution is *r* to some power. Write the solution in the form

$$\phi(r,\theta) = r^{\lambda+1} f(\theta),$$

where the constant λ and the function $f(\theta)$ are to be determined. Insert this form into the bi-harmonic equation, and we obtain an ordinary differential equation (ODE):

$$\left[\frac{d^2}{d\theta^2} + (\lambda-1)^2\right]\left[\frac{d^2}{d\theta^2} + (\lambda+1)^2\right] f(\theta) = 0 .$$

This is the ODE with constant coefficients. The solution is in the form of

$$f(\theta) = e^{b\theta} .$$

Inserting this form into the ODE, we find

$$\left(b^2 + (\lambda-1)^2\right)\left(b^2 + (\lambda+1)^2\right) = 0 .$$

This is a fourth order algebraic equation for *b*. The four roots are

$$b = \pm(\lambda-1)i, \pm(\lambda+1)i ,$$

where $i = \sqrt{-1}$. The general solution to the ODE is

$$f(\theta) = A\cos(\lambda+1)\theta + B\cos(\lambda-1)\theta + C\sin(\lambda+1)\theta + D\sin(\lambda-1)\theta ,$$

where *A*, *B*, *C*, and *D* are constants. For the mode I crack, the field is symmetric with respect to the *x*-axis, so that $C = D = 0$.

Summarizing, we find that the solution to the bi-harmonic equation is

$$\phi(r,\theta) = r^{\lambda+1}\left[A\cos(\lambda+1)\theta + B\cos(\lambda-1)\theta\right],$$

where λ, *A* and *B* are constants to be determined.

The stress components are

$$\sigma_{rr} = -\lambda r^{\lambda-1}\left[A(\lambda+1)\cos(\lambda+1)\theta + B(\lambda-3)\cos(\lambda-1)\theta\right]$$

$$\sigma_{\theta\theta} = -(\lambda+1)\lambda r^{\lambda-1}\left[A\cos(\lambda+1)\theta + B\cos(\lambda-1)\theta\right]$$

$$\sigma_{r\theta} = (\lambda-1)r^{\lambda-1}\left[A(\lambda+1)\sin(\lambda+1)\theta + B(\lambda-1)\sin(\lambda-1)\theta\right].$$

Boundary conditions. At $\theta = \pi$, both traction components vanish, $\sigma_{\theta\theta} = \sigma_{r\theta} = 0$, namely,

$$\begin{bmatrix} (\lambda+1)\lambda\cos\lambda\pi & (\lambda+1)\lambda\cos\lambda\pi \\ (\lambda^2-1)\sin\lambda\pi & (\lambda-1)^2\sin\lambda\pi \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is a pair of linear equations for *A* and *B*. It is an eigenvalue problem. To have a solution such that *A* and *B* are not both zero, the determinant must vanish, namely,

$$\lambda(\lambda^2-1)\sin 2\lambda\pi = 0 .$$

The solutions are

$$\lambda = \ldots -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

Consequently, the field of stress takes the form of an expansion:

$$\sigma_{ij} = \sum_{m=-\infty}^{+\infty} a_m r^{m/2} f_{ij}^{(m)}(\theta).$$

The functions $f_{ij}^{(m)}(\theta)$ are determined by the eigenvalue problem. The amplitudes a_m, however, are not determined by the eigenvalue problem, and should be determined by the full boundary-value problem.

The square-root singularity. Each root corresponds to a solution to the biharmonic equation. Which root should we pick? We find that

$$\sigma \sim r^{\lambda-1},\ \varepsilon \sim r^{\lambda-1},\ u \sim r^{\lambda}.$$

Recall the stress concentration for the ellipse. Require that the displacement to be bounded, so that $\lambda > 0$. Require the stress to be singular, so that $\lambda < 1$. The two requirements force us to pick

$$\lambda = +\frac{1}{2}.$$

Substituting this value to the algebraic equation, we find that $B = 3A$. In particular, we find the hoop stress is

$$\sigma_{\theta\theta}(r,\theta) = -\frac{B}{\sqrt{r}}\cos^3\left(\frac{\theta}{2}\right).$$

The above justifications for the choice of the square-root singularity are flimsy. The significance of this choice has to be understood later, when we see how this singularity is used in practice. For a discussion, see C.Y. Hui and Andy Ruina, Why K? High order singularities and small scale yielding, International Journal of Fracture 72, 97-120 (1995).

Stress intensity factor. Williams solved an eigenvalue problem because no load was specified: both the field equations and the boundary conditions are homogeneous. Like any other eigenvalue problems, the amplitude is undetermined. In this case, all the stress components at the crack tip are determined up to the constant *B*. Irwin wrote the constant *B* as $B = -K/\sqrt{2\pi}$. Consequently, the tensile traction at a distance *r* directly ahead the crack tip is given by

$$\sigma_{\theta\theta}(r,0) = \frac{K}{\sqrt{2\pi r}}.$$

The legend has it that Irwin chose the letter *K* after J.A. Kies, one of his co-workers.

The field of stress around the tip of the crack is given by

$$\sigma_{rr} = \frac{K}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left(1+\sin^2\left(\frac{\theta}{2}\right)\right)$$

$$\sigma_{\theta\theta} = \frac{K}{\sqrt{2\pi r}}\cos^3\left(\frac{\theta}{2}\right)$$

$$\sigma_{r\theta} = \frac{K}{\sqrt{2\pi r}}\cos^2\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

One can also determine the displacement components. In particular, the crack opening displacement a distance *r* behind the crack tip is

$$\delta = \frac{8K}{\bar{E}}\sqrt{\frac{r}{2\pi}},$$

where $\bar{E} = E/(1-\nu^2)$ under the plane strain conditions, and $\bar{E} = E$ under the plane stress conditions.

Notes

- *K* is called the stress intensity factor. Its magnitude is undetermined in the eigenvalue problem.

- The factor $(2\pi)^{-1/2}$ is introduced by convention, and has no special significance.
- The r and θ dependence are independent of the external boundary conditions.
- By modeling the crack front as a mathematical curve, the linear elasticity theory does not account for the atomistic bond-breaking process, or any other fracture processes.

Determine K for a given cracked body by solving a boundary-value problem. For example, consider the Griffith crack, a crack of length $2a$ in an infinite plate, subject to remote stress σ. The boundary-value problem is difficult to solve, but it was solved by Inglis (1913). The field in the entire body is expressed in analytical terms. For example, the stress ahead of the crack is given by

$$\sigma_{yy} = \frac{\sigma x}{\sqrt{x^2 - a^2}},\ |x| > a$$

The distance of a point ahead of the crack tip is given by $r = x - a$.

$$\sigma_{yy} = \frac{\sigma x}{\sqrt{x^2 - a^2}} = \frac{\sigma x}{\sqrt{(x+a)(x-a)}} = \frac{\sigma(r+a)}{\sqrt{(r+2a)r}}$$

When the tip of the crack is approached, $r \ll a$, we have

$$\sigma_{yy} = \sigma\sqrt{\frac{a}{2r}}.$$

Compare with the crack tip field determined from the eigenvalue problem,

$$\sigma_{yy} = \frac{K}{\sqrt{2\pi r}},$$

and we find that

$$K = \sigma\sqrt{\pi a}.$$

Alternatively, one can determine the stress intensity factor from the displacement field. The elasticity solution of the Griffith problem gives the displacement field in the body. In particular, the opening displacement of the crack is

$$\delta(x) = \frac{4\sigma}{\bar{E}\sqrt{x^2 - a^2}},\ |x| < a$$

A comparison with the universal crack-tip field

$$\delta = \frac{8K}{\bar{E}}\sqrt{\frac{r}{2\pi}}$$

determines the stress intensity factor K.

Handbooks for K. Stress intensity factor for a given crack geometry must be determined by solving a boundary value problem. Many crack geometries have been solved. The results are collected in handbooks (e.g., H. Tada, P.C. Paris and G.R. Irwin, *The Stress Analysis of Cracks Handbook*, Del Research, St. Louis, MO., 1995). See the handout for a condensed version. In general, the stress intensity factor takes the form

$$K = Y\sigma\sqrt{a},$$

where σ is an applied stress, a is a length scale characterize the crack geometry, and Y is a dimensionless number.

Linear superposition. Elasticity problem is linear. For a given body with a given crack, if a force P causes the stress intensity factor $K = \alpha P$, and force Q causes the stress intensity factor $K = \beta Q$. The combined action of the forces P and Q causes the stress intensity factor $K = \alpha P + \beta Q$.

Finite element method to determine *K*. For a complicated structure with a crack, once can determine the elastic field using the finite element method, and then extract from the field the stress intensity factor. A brute force method is that you use the finite element method to determine the displacement field, and then fit the crack opening to

$$\delta = \frac{8K}{\bar{E}}\sqrt{\frac{r}{2\pi}},$$

with K as the fitting parameter. There are a number of more clever methods. We'll mention them later at suitable points.

Irwin's G-K relation. G.R. Irwin, Analysis of stresses and strains near the end of a crack traversing a plate, Journal of Applied Mechanics 24, 361-364 (1957).

Irwin discovered the following relation between the stress intensity factor and the energy release rate:

$$G = \frac{K^2}{\bar{E}} \qquad \text{(Mode I, plane strain).}$$

Proof. Consider two bodies, 1 and 2. Both have the same configuration: unit thickness, semi-infinite crack, the same K. The crack in Body 2 is longer than that in Body 1 by a length b. The bodies are so big that a small change in the crack length does not change K. The external load is held rigidly, and does no work when the crack extends.

Let U_1 and U_2 be the strain energy stored in the two bodies, respectively. By definition, the energy release rate is given by

$$G = \frac{U_1 - U_2}{b}.$$

The stress field ahead the crack in Body 1 is

$$\sigma^{(1)} = \frac{K}{\sqrt{2\pi x}}.$$

The opening displacement of the crack in Body 2 is

$$\delta^{(2)} = \frac{8K}{\bar{E}}\sqrt{\frac{b-x}{2\pi}}.$$

The strain energy difference in the two bodied is due to the work done by the closing traction:

$$U_1 - U_2 = \frac{1}{2}\int_0^b \sigma^{(1)}\delta^{(2)} dx .$$

This gives

$$G = \frac{1}{2b}\int_0^b \frac{K}{\sqrt{2\pi x}}\frac{8K}{\bar{E}}\sqrt{\frac{b-x}{2\pi}}dx = \frac{2K^2}{\pi\bar{E}}\int_0^1\sqrt{\frac{1-t}{t}}dt$$

Carry out the integration, and we obtain that

$$G = \frac{K^2}{\bar{E}}.$$

For some problems, the energy release rate is easier to determine than the stress intensity factor. We have seen two examples: the double-cantilever beam, and the channel crack. Once the energy release rate is determined, one can obtain the stress intensity factor by using Irwin's relation.

Mode II. Directly ahead the crack tip

$$\sigma_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}$$

Behind the crack tip, the sliding displacement between the two crack faces is

$$\delta_{II} = \frac{8K_{II}}{\bar{E}}\sqrt{\frac{r}{2\pi}} .$$

The energy release rate relates to the stress intensity factor as

$$G = \frac{K_{II}^2}{\bar{E}} .$$

Mode III. Directly ahead the crack tip

$$\sigma_{rz} = \frac{K_{III}}{\sqrt{2\pi r}}$$

Behind the crack tip, the tearing displacement between the two crack faces is

$$\delta_{III} = \frac{4K_{III}}{\mu}\sqrt{\frac{r}{2\pi}}, \mu = \frac{E}{2(1+\nu)} .$$

The energy release rate relates to the stress intensity factor as

$$G = \frac{K_{III}^2}{2\mu} .$$