

Resistance Curve

Recap of the Linear Elastic Fracture Mechanics. For a crack advancing in a body under a load, we can solve boundary-value problems of several types:

- Type A. A boundary-value problem that includes the external boundary conditions, but assumes the body to be linearly elastic all the way to the crack tip. Boundary-value problems of this type have been solved since Inglis (1913). The solutions of this type constitute the bulk of the Linear Elastic Fracture Mechanics. See the Tada handbook. (<http://imechanica.org/node/7457>)
- Type B. A boundary-value problem that excludes both the external boundary conditions and the inelastic fracture process. Strictly speaking, this type has been solved only once, by Williams (1957). Now we understand that this problem only needs to be solved once, because the form of the resulting field, the square-root singular field, is universal, independent of the external boundary conditions and the inelastic fracture process. The amplitude of the square-root singular field is the stress intensity factor, which cannot be determined without prescribing the external boundary conditions. (<http://imechanica.org/node/7579>)
- Type C. A boundary-value problem that includes both the external boundary conditions and the inelastic fracture process. Remarkably, boundary-value problem of this most intuitive type have been solved rarely. The first useful analysis of Type A was due to Dugdale (1960), whose work we will study with care later in the course.

When the size of the fracture process zone is small compared to the size of the specimen (i.e., the size of the crack, the size of the ligament, the thickness of the specimen, etc.), the specimen is said to fracture under the small-scale yielding condition (<http://imechanica.org/node/7654>).

Under the small-scale yielding condition, the square-root singular field prevails in an annulus. The inner circle of the annulus is larger than the size of the fracture process zone, and the outer circle of the annulus is smaller than the size of the specimen. Consequently, the stress intensity factor is the only messenger between the external boundary conditions and the fracture process zone.

The stress intensity factor K serves as a loading parameter for a cracked body. Alternatively, the energy release rate G also serves as a loading parameter for a cracked body. The two loading parameters are entirely equivalent, and are connected through Irwin's formula:

$$G = \frac{K^2}{E'}$$

The Linear Elastic Fracture Mechanics can be phrased in two equivalent forms, using either K or G . I'll mostly use G , but you can paraphrase in terms of K .

We have talked about fracture energy and toughness. In the next few lectures, we will use the Linear Elastic Fracture Mechanics to study several significant phenomena:

- Resistance curve.
- Fatigue under cyclic load.
- Static fatigue.

Resistance curve (*R*-curve, a commonly used phrase). In the previous lectures, the fracture energy has taken to be a material-specific constant. The constant fracture energy is an idealization. The fracture energy can be a function of the extension of the crack, namely,

$$\Gamma = \Gamma(\Delta a).$$

This function is known as the resistance curve (*R*-curve).

Any specific fracture process will lead to a *R*-curve. Consider, for example, the gradual development of the inelastic layer as the crack advances. Here is a frame-by-frame movie.

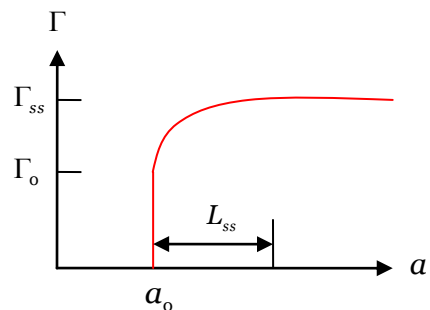
Frame 1. A plate of a steel is cut by a diamond saw. The initial length of the crack, a_0 , is *not* a material property. No force is applied to the plate yet.

Frame 2. When a load is applied, under the small-scale yielding condition, the applied load can be converted to the energy release rate by solving a boundary-value problem of Type A. That is, the applied load can be represented by the energy release rate G .

Frame 3. Apply a small load. A small region of the material around the tip of the crack yields, but the length of the crack remains to be a_0 .

Frame 4. Increase the load still more. Above a critical force, $G = \Gamma_0$, the crack starts to advance. The larger the load, the longer the crack. Denote the increase in the length of the crack by Δa .

Frame 5. When the crack advances in a **steady state**, the thickness of the inelastic layer remains constant as the crack advances. The load G attains a plateau (a steady-state), Γ_{ss} .

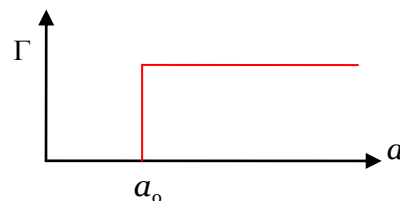


Under the small-scale yielding condition, the R -curve is a material property. The R -curve is independent of the initial length of the crack, a_0 . Nor does the R -curve depend on the geometry of the specimen. Parameters of importance include

- the initiation fracture energy, Γ_0 ,
- the steady-state fracture energy, Γ_{ss} , and
- the increment of the length of the crack needed to attain the steady state, L_{ss} .

All these parameters are material properties. The ratio Γ_{ss}/Γ_0 ranges from a factor of 2 to a factor of thousands. The length L_{ss} is comparable to the size of the plastic zone when the crack advances in the steady state.

To satisfy the small scale yielding condition, L_{ss} must be small compared to the size of the specimen (i.e., the length of the crack, the length of the ligament, the thickness of the specimen, etc.) For silica, $L_{ss} \sim 1\text{nm}$. Thus, for practical purpose, the fracture energy is a constant, independent of the extension of the crack, and the R -curve looks like a step function.



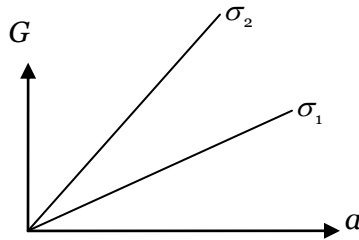
For a ductile steel, $L_{ss} \sim 1\text{cm}$, the fracture energy will increase from Γ_0 to Γ_{ss} after the crack extends for a length comparable to L_{ss} .

The R -curve can also be represented by a function $K_R(\Delta a)$.

Loading curve (L-curve, not a commonly used phrase). For a Griffith crack, the energy release rate is

$$G = \pi \frac{\sigma^2 a}{E}.$$

Consider an experiment in which the applied stress σ remains constant when the crack extends. In such an experiment, the energy release rate is linear in the length of the crack, a . Plot the energy release rate as a function of the length of the crack for several levels of the applied stress.



More generally, we can consider an experiment in which the length of the crack is the only independent variable. In such an experiment, the energy release rate is a function of the length of the crack, namely,

$$G = G(a)$$

We will call this function the loading curve. The loading curve can also be represented by the function $K(a)$.

Stability of an advancing crack. For a crack in a body under a load, will the crack remain stationary, or grow then stop (stable), or grow across the entire body (unstable)? Under the small-scale yielding condition, the answer depends on three inputs:

- The resistance curve of the material, $\Gamma(\Delta a)$.
- The loading curve of the specimen, $G(a, \lambda)$, where λ is a loading parameter.
- The initial length of the crack, a_0 .

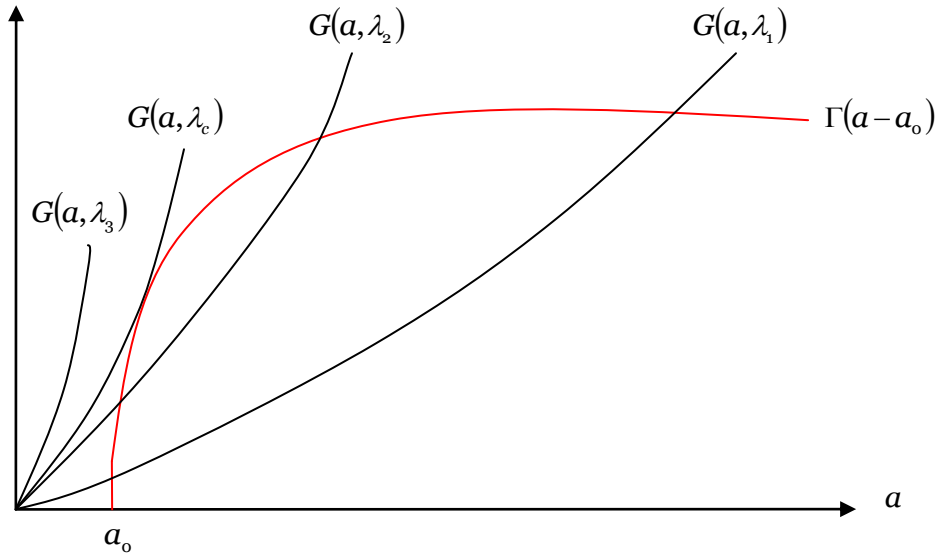
Let λ be a loading parameter, which can be an applied force, or a prescribed displacement. The energy release rate depends on both the loading parameter and the length of the crack, namely,

$$G = G(a, \lambda).$$

The loading parameter is held constant when the crack extends. Let $\Gamma(\Delta a)$ be the resistance curve.

The stability of a crack depends on how the two curves intersect. Sketched in the figure are several cases. When the loading parameter is fixed at λ_1 , the R -curve is above the L -curve for all cracks longer than a_0 , and the crack will not grow. When the loading parameter is fixed at λ_2 , the crack will grow and then stop. When the loading parameter is fixed at λ_3 , the crack will grow and then stop. A critical load λ_c exists, below which the crack will stop. The critical load and the associated length of the crack is determined by the simultaneous equations

$$\begin{aligned} G(a, \lambda) &= \Gamma(a - a_0), \\ \frac{\partial G(a, \lambda)}{\partial a} &= \frac{d\Gamma(a - a_0)}{da}. \end{aligned}$$



Loading curves of various shapes. In the above discussion, loading curves of a particular trend are assumed. The loading curve depends on the boundary conditions of the specimen. It will be helpful to look at several examples.

Example 1: Griffith crack, constant fracture energy. Draw three figures. Draw the L -curves for several fixed levels of stress. Draw the R -curve, $\Gamma(\Delta a)$. Give the initial crack size a_0 and draw the loading curves and the R -curve on the third figure.

Example 2: Griffith crack, rising R -curve.

Example 3: Double cantilever beam, force control. We derived the energy release rate using the elementary beam theory. The result is

$$G = \frac{12}{EH^3} \left(\frac{Pa}{B} \right)^2.$$

Under a fixed force P , the loading curve is an increasing function. For now assume that the crack resistance Γ is a constant independent of the crack length. The crack is unstable. For a given initial crack size, the crack is stationary when the stress is below a critical value, and extends rapidly when the stress exceeds the critical value.

Example 4: Double cantilever beam, displacement control. Under displacement control (e.g., by inserting a wedge), the energy release rate is given by

$$G = \frac{3EH^3\Delta^2}{16a^4}.$$

The energy release rate *decreases* when the crack length increases.

Example 5: A crack in a matrix and near a fiber. In this case, the loading curve $G(a)$ is not monotonic. For simplicity, assume a constant fracture energy.

