MECHANICS OF SOFT MATERIALS

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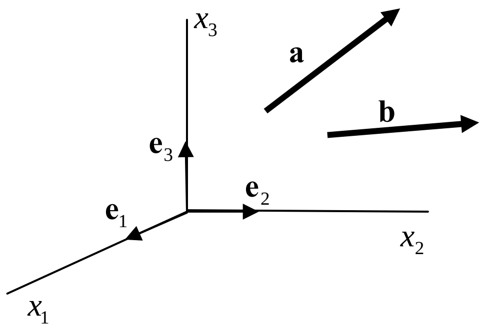
Contents

1 Tensors

1.1 Vectors

Vectors are tensors of the first order/rank, by definition, while scalars are zero-order tensors.



We consider Cartesian coordinate system with mutually orthogonal axes, x_i, and base vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.1}$$

Within this coordinate system we define arbitrary vector $\mathbf{a}$ as follows

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i \,, \tag{1.2}$$

where a_i are the *components* of the vector.

Notation in (1.2) is excessive and it is worth simplifying it by using the Einstein rule

$$\sum_{i=1}^{3} a_i\mathbf{e}_i = a_i\mathbf{e}_i \,, \tag{1.3}$$

which means that the symbol of the sum can be dropped when the summation is performed over *two repeated indices*. Such indices are called *dummy* because they can be designated by any character

$$a_i\mathbf{e}_i = a_j\mathbf{e}_j = a_m\mathbf{e}_m \,.$$

Using Einstein's rule we can write down the scalar or dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ as follows

$$\mathbf{a}\cdot\mathbf{b} = (a_i\mathbf{e}_i)\cdot(b_j\mathbf{e}_j) = a_i b_j \mathbf{e}_i\cdot\mathbf{e}_j \,. \tag{1.4}$$

The scalar product of base vectors is zero for different base vectors and one for the same vector

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{Bmatrix} 1, & i = j \\ 0, & i \neq j \end{Bmatrix} \equiv \delta_{ij}, \tag{1.5}$$

where we introduced the (Leopold) *Kronecker delta* for short notation.

Substituting (1.5) in (1.4) we have

$$\mathbf{a} \cdot \mathbf{b} = a_i b_j \underbrace{\mathbf{e}_i \cdot \mathbf{e}_j}_{\delta_{ij}} = a_i b_j \delta_{ij} = a_i b_i = a_j b_j = a_1 b_1 + a_2 b_2 + a_3 b_3, \tag{1.6}$$

where

$$b_j \delta_{ij} = b_1 \delta_{i1} + b_2 \delta_{i2} + b_3 \delta_{i3} = b_i. \tag{1.7}$$

By using the dot product of base vector $\mathbf{e}_i$ with vector $\mathbf{a}$ we find a_i

$$\mathbf{e}_i \cdot \mathbf{a} = \mathbf{e}_i \cdot (a_j \mathbf{e}_j) = a_j \mathbf{e}_i \cdot \mathbf{e}_j = a_j \delta_{ij} = a_i. \tag{1.8}$$

The Kronecker delta was introduced through the scalar products of the Cartesian base vectors. It is also very convenient to introduce the permutation (Tulio Levi-Civita) symbol by using triple product of base vectors

$$\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = \begin{Bmatrix} 1, \; ijk = 123; 231; 312 \\ -1, \; ijk = 321; 213; 132 \\ 0, \; ijk = \ldots \end{Bmatrix} \equiv \varepsilon_{ijk}. \tag{1.9}$$

The permutation symbol allows us to write the components of the vector product in a short way

$$c_i = \mathbf{e}_i \cdot \mathbf{c} = \mathbf{e}_i \cdot \underbrace{(\mathbf{a} \times \mathbf{b})}_{\mathbf{c}} = \mathbf{e}_i \cdot \{(a_j \mathbf{e}_j) \times (b_k \mathbf{e}_k)\} = \underbrace{\{\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)\}}_{\varepsilon_{ijk}} a_j b_k = \varepsilon_{ijk} a_j b_k. \tag{1.10}$$

It is important that there is no summation over index i in (1.10). Such index is called *free*. Computing (1.10) for varying i we get

$$c_1 = a_2 b_3 - a_3 b_2, \quad c_2 = a_3 b_1 - a_1 b_3, \quad c_3 = a_1 b_2 - a_2 b_1. \tag{1.11}$$

1.2 Second-order tensors

To define a second-order tensor we introduce *dyadic* or *tensor* product, $\otimes$, of base vectors

$$\mathbf{e}_1 \otimes \mathbf{e}_1 = \mathbf{e}_1 \mathbf{e}_1^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{e}_1 \otimes \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2^T = \begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix},$$

$$\ldots$$

$$\mathbf{e}_3 \otimes \mathbf{e}_3 = \mathbf{e}_3\mathbf{e}_3^T = \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix}, \tag{1.12}$$

$$\mathbf{e}_i \otimes \mathbf{e}_j \neq \mathbf{e}_j \otimes \mathbf{e}_i .$$

By analogy with vectors, we define second-order tensors as a linear combination of base dyads

$$\begin{aligned}\mathbf{A} = {} & A_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + A_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + A_{13}\mathbf{e}_1 \otimes \mathbf{e}_3 \\ & + A_{21}\mathbf{e}_2 \otimes \mathbf{e}_1 + A_{22}\mathbf{e}_2 \otimes \mathbf{e}_2 + A_{23}\mathbf{e}_2 \otimes \mathbf{e}_3 \, . \\ & + A_{31}\mathbf{e}_3 \otimes \mathbf{e}_1 + A_{32}\mathbf{e}_3 \otimes \mathbf{e}_2 + A_{33}\mathbf{e}_3 \otimes \mathbf{e}_3 \end{aligned} \tag{1.13}$$

By using short notation we can rewrite (1.13) as follows

$$\mathbf{A} = \sum_{j=1}^{3}\sum_{i=1}^{3} A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \, . \tag{1.14}$$

The *components* of the second-order tensor can be written in the matrix form

$$\begin{pmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{pmatrix}.$$

In the considered case of Cartesian coordinates, the tensor can be interpreted as a matrix of its components. In the case of curvilinear coordinates, the situation is subtler and *various matrices of components can represent the same tensor*. The latter will be discussed below.

A second-order tensor (or matrix) maps one vector into another as follows

$$\mathbf{c} = \mathbf{A}\mathbf{b} = (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(b_m\mathbf{e}_m) = A_{ij}b_m\mathbf{e}_i \underbrace{(\mathbf{e}_j \cdot \mathbf{e}_m)}_{\delta_{jm}} = A_{ij}\underbrace{b_m\delta_{jm}}_{b_j}\mathbf{e}_i = \underbrace{A_{ij}b_j}_{c_i}\mathbf{e}_i \, , \tag{1.15}$$

or

$$c_i = A_{ij}b_j \, , \quad \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12} & A_{13}\\A_{21} & A_{22} & A_{23}\\A_{31} & A_{32} & A_{33}\end{pmatrix}\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}.$$

Product of two second-order tensors is defined as follows

$$\mathbf{F} = \mathbf{AD} = (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)(D_{mn}\mathbf{e}_m \otimes \mathbf{e}_n) = A_{ij}D_{mn}\underbrace{(\mathbf{e}_j \cdot \mathbf{e}_m)}_{\delta_{jm}}\mathbf{e}_i \otimes \mathbf{e}_n$$
$$= A_{ij}D_{mn}\delta_{jm}\mathbf{e}_i \otimes \mathbf{e}_n = \underbrace{A_{ij}D_{jn}}_{F_{in}}\mathbf{e}_i \otimes \mathbf{e}_n \quad , \tag{1.16}$$

or

$$F_{in} = A_{ij}D_{jn}, \quad \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}.$$

Double dot product of two tensors is a scalar

$$\mathbf{A} : \mathbf{D} = (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) : (D_{mn}\mathbf{e}_m \otimes \mathbf{e}_n) = A_{ij}D_{mn}\underbrace{(\mathbf{e}_i \cdot \mathbf{e}_m)}_{\delta_{im}}\underbrace{(\mathbf{e}_j \cdot \mathbf{e}_n)}_{\delta_{jn}}$$
$$= \underbrace{A_{ij}\delta_{im}}_{A_{mj}}\underbrace{D_{mn}\delta_{jn}}_{D_{mj}} = A_{mj}D_{mj} = A_{11}D_{11} + A_{12}D_{12} + ... + A_{33}D_{33} \,. \tag{1.17}$$

By using the double dot product we can calculate components of a second-order tensor as follows

$$\mathbf{e}_i \otimes \mathbf{e}_j : \mathbf{A} = \mathbf{e}_i \otimes \mathbf{e}_j : (A_{mn}\mathbf{e}_m \otimes \mathbf{e}_n) = A_{mn}\underbrace{(\mathbf{e}_i \cdot \mathbf{e}_m)}_{\delta_{im}}\underbrace{(\mathbf{e}_j \cdot \mathbf{e}_n)}_{\delta_{jn}} = A_{ij} \,. \tag{1.18}$$

Since the second-order tensor can be interpreted as a matrix then all subsequent definitions for tensors are analogous to the matrix definitions. For example, the second-order *identity* tensor is defined as

$$\mathbf{1} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 \,, \tag{1.19}$$

and it enjoys the remarkable property

$$\mathbf{A1} = \mathbf{1A} = \mathbf{A} \,. \tag{1.20}$$

The *transposed* second-order tensor is

$$\mathbf{A}^T = (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j)^T = A_{ij}\mathbf{e}_j \otimes \mathbf{e}_i = A_{ji}\mathbf{e}_i \otimes \mathbf{e}_j \,. \tag{1.21}$$

It allows us to additively decompose any second-order tensor into symmetric and skew (anti)-symmetric parts

$$\mathbf{A} = \mathbf{A}_{sym} + \mathbf{A}_{skew}, \quad \mathbf{A}_{sym} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) = \mathbf{A}^T_{sym}, \quad \mathbf{A}_{skew} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) = -\mathbf{A}^T_{skew}. \tag{1.22}$$

The *inverse* second order tensor, $\mathbf{A}^{-1}$, is defined through the identity

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{1} \,. \tag{1.23}$$

Finally, we consider the *eigenproblem* for a *symmetric* second-order tensor $\mathbf{A} = \mathbf{A}^T$. The *eigenvalue (principal value)* ζ and the *eigenvector (principal direction)* $\mathbf{n}$ of the tensor are defined by the following equation

$$\mathbf{A}\mathbf{n} = \zeta \mathbf{n}. \tag{1.24}$$

The eigenproblem defines the principal directions of tensor $\mathbf{A}$ where vector $\mathbf{n}$ is mapped into itself scaled by factor ζ. We rewrite the eigenproblem by moving all terms onto the left hand side

$$(\mathbf{A} - \zeta \mathbf{1})\mathbf{n} = \mathbf{0}, \tag{1.25}$$

or

$$\begin{pmatrix} A_{11}-\zeta & A_{12} & A_{13} \\ A_{21} & A_{22}-\zeta & A_{23} \\ A_{31} & A_{32} & A_{33}-\zeta \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This equation possesses a nontrivial solution when the determinant of the coefficient matrix is singular

$$\det(\mathbf{A} - \zeta \mathbf{1}) = -\zeta^3 + \zeta^2 I_1(\mathbf{A}) - \zeta I_2(\mathbf{A}) + I_3(\mathbf{A}) = 0. \tag{1.26}$$

Here the *principal invariants* of tensor $\mathbf{A}$ have been introduced

$$I_1(\mathbf{A}) = A_{11} + A_{22} + A_{33} = \mathrm{tr}\mathbf{A}, \tag{1.27}$$

$$I_2(\mathbf{A}) = \frac{1}{2}\{(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)\}, \tag{1.28}$$

$$I_3(\mathbf{A}) = \det \mathbf{A}. \tag{1.29}$$

Since tensor $\mathbf{A}$ is symmetric then all roots of (1.26) , $\zeta_1, \zeta_2, \zeta_3$, are real and it is possible to find three mutually orthogonal principal directions corresponding to the roots. The unit vectors in principal directions, $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}$, obey the orthonormality conditions

$$\mathbf{n}^{(i)} \cdot \mathbf{n}^{(j)} = \delta_{ij}. \tag{1.30}$$

Now tensor $\mathbf{A}$ can enjoy the *spectral decomposition* based on the solution of the eigenproblem

$$\mathbf{A} = \zeta_1 \mathbf{n}^{(1)} \otimes \mathbf{n}^{(1)} + \zeta_2 \mathbf{n}^{(2)} \otimes \mathbf{n}^{(2)} + \zeta_3 \mathbf{n}^{(3)} \otimes \mathbf{n}^{(3)}, \tag{1.31}$$

if $\zeta_1 \neq \zeta_2 \neq \zeta_3$, or

$$\mathbf{A} = (\zeta_1 - \zeta_2)\mathbf{n}^{(1)} \otimes \mathbf{n}^{(1)} + \zeta_2 \mathbf{1}, \tag{1.32}$$

if $\zeta_1 \neq \zeta_2 = \zeta_3$, or

$$\mathbf{A} = \zeta_1 \mathbf{1}. \tag{1.33}$$

if $\zeta_1 = \zeta_2 = \zeta_3$.

Based on the spectral decomposition it is convenient to introduce the logarithm and the square root of a symmetric positive definite tensor, $\zeta_i > 0$,

$$\ln \mathbf{A} = \sum_{k=1}^{3} (\ln \zeta_k) \mathbf{n}^{(k)} \otimes \mathbf{n}^{(k)}, \tag{1.34}$$

$$\sqrt{\mathbf{A}} = \sum_{k=1}^{3} \sqrt{\zeta_k} \mathbf{n}^{(k)} \otimes \mathbf{n}^{(k)}. \tag{1.35}$$

The spectral decomposition also allows us to calculate the principal invariants simply

$$I_1(\mathbf{A}) = \zeta_1 + \zeta_2 + \zeta_3, \tag{1.36}$$

$$I_2(\mathbf{A}) = \zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_2\zeta_3, \tag{1.37}$$

$$I_3(\mathbf{A}) = \zeta_1\zeta_2\zeta_3. \tag{1.38}$$

Finally, we derive the useful Cayley-Hamilton formula pre-multiplying (1.26) with $\mathbf{n}^{(i)}$ and accounting for $\mathbf{A}^a \mathbf{n}^{(i)} = \zeta_i^a \mathbf{n}^{(i)}$

$$-\mathbf{A}^3 + \mathbf{A}^2 I_1(\mathbf{A}) - \mathbf{A} I_2(\mathbf{A}) + \mathbf{1} I_3(\mathbf{A}) = \mathbf{0}. \tag{1.39}$$

1.3 Tensor functions

Tensors can be arguments of functions: $f(\mathbf{A})$; $f(A_{ij})$; $\mathbf{f}(\mathbf{A})$; $f_m(A_{ij})$; $\mathbf{F}(\mathbf{A})$; $F_{mn}(A_{ij})$. Let us calculate a differential of a scalar function f with respect to tensor argument $\mathbf{A}$

$$df = \frac{\partial f}{\partial A_{ij}} dA_{ij}. \tag{1.40}$$

Here the components of the tensor increment can be written as (see (1.18))

$$dA_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j : d\mathbf{A},$$

and, consequently, (1.40) takes form

$$df = \frac{\partial f}{\partial A_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j : d\mathbf{A} = \frac{\partial f}{\partial \mathbf{A}} : d\mathbf{A}, \tag{1.41}$$

where the derivative with respect to the second-order tensor has been defined

$$\frac{\partial f}{\partial \mathbf{A}} = \frac{\partial f}{\partial A_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.42}$$

Analogously, it is possible to define the derivative of a second-order tensor

$$\mathbf{C}=\frac{\partial \mathbf{A}}{\partial \mathbf{B}}=\frac{\partial \mathbf{A}}{\partial B_{ij}}\otimes \mathbf{e}_i\otimes \mathbf{e}_j=\underbrace{\frac{\partial A_{mn}}{\partial B_{ij}}}_{C_{mnij}}\mathbf{e}_m\otimes \mathbf{e}_n\otimes \mathbf{e}_i\otimes \mathbf{e}_j\,. \tag{1.43}$$

This is the fourth-order tensor, which is formed by a combination of base tetrads $\mathbf{e}_m\otimes \mathbf{e}_n\otimes \mathbf{e}_i\otimes \mathbf{e}_j$ that can be interpreted, by analogy with dyads, as tables (matrices) in 4D space.

The double dot product of the fourth- and second- order tensors is defined as follows

$$\begin{aligned}\mathbf{D}=\mathbf{C}:\mathbf{B}&=(C_{mnij}\mathbf{e}_m\otimes \mathbf{e}_n\otimes \mathbf{e}_i\otimes \mathbf{e}_j):(B_{kl}\mathbf{e}_k\otimes \mathbf{e}_l)\\ &=C_{mnij}B_{kl}\mathbf{e}_m\otimes \mathbf{e}_n\underbrace{(\mathbf{e}_i\cdot \mathbf{e}_k)}_{\delta_{ik}}\underbrace{(\mathbf{e}_j\cdot \mathbf{e}_l)}_{\delta_{jl}}=\underbrace{C_{mnij}B_{ij}}_{D_{mn}}\mathbf{e}_m\otimes \mathbf{e}_n\,.\end{aligned} \tag{1.44}$$

As an example let us differentiate a second-order tensor with respect to itself

$$\frac{\partial \mathbf{A}}{\partial \mathbf{A}}=\frac{\partial A_{mn}}{\partial A_{ij}}\mathbf{e}_m\otimes \mathbf{e}_n\otimes \mathbf{e}_i\otimes \mathbf{e}_j=\delta_{mi}\delta_{nj}\mathbf{e}_m\otimes \mathbf{e}_n\otimes \mathbf{e}_i\otimes \mathbf{e}_j\,. \tag{1.45}$$

In the case of symmetric tensor $\mathbf{A}=(\mathbf{A}+\mathbf{A}^T)/2$ the symmetry should be preserved in the derivative

$$\frac{\partial \mathbf{A}}{\partial \mathbf{A}}=\frac{1}{2}\frac{\partial(\mathbf{A}+\mathbf{A}^T)}{\partial \mathbf{A}}=\frac{1}{2}(\delta_{mi}\delta_{nj}+\delta_{ni}\delta_{mj})\mathbf{e}_m\otimes \mathbf{e}_n\otimes \mathbf{e}_i\otimes \mathbf{e}_j\,. \tag{1.46}$$

Further important formulas are obtained by differentiating the principal invariants of $\mathbf{A}=\mathbf{A}^T$

$$\frac{\partial I_1(\mathbf{A})}{\partial \mathbf{A}}=\frac{\partial A_{kk}}{\partial A_{ij}}\mathbf{e}_i\otimes \mathbf{e}_j=\underbrace{\delta_{ki}\delta_{kj}}_{\delta_{ij}}\mathbf{e}_i\otimes \mathbf{e}_j=\mathbf{1}\,, \tag{1.47}$$

$$\frac{\partial I_2(\mathbf{A})}{\partial \mathbf{A}}=\frac{1}{2}\frac{\partial(A_{kk}A_{ll}-A_{mn}A_{nm})}{\partial A_{ij}}\mathbf{e}_i\otimes \mathbf{e}_j=I_1(\mathbf{A})\mathbf{1}-\mathbf{A}\,. \tag{1.48}$$

The derivative of the third invariant $I_3(\mathbf{A})=\det\mathbf{A}$ is less trivial and we start with calculating the increment of it with the help of (1.26)

$$\begin{aligned}\det(\mathbf{A}+d\mathbf{A})&=\det\mathbf{A}\det(\underbrace{\mathbf{A}^{-1}d\mathbf{A}}_{\text{tensor}}-\underbrace{(-1)}_{\text{eigenvalue}}\mathbf{1})\\ &=\det\mathbf{A}\{1+I_1(\mathbf{A}^{-1}d\mathbf{A})+I_2(\mathbf{A}^{-1}d\mathbf{A})+I_3(\mathbf{A}^{-1}d\mathbf{A})\}\end{aligned}\,. \tag{1.49}$$

Ignoring higher-order terms in (1.49) we have

$$\det(\mathbf{A}+d\mathbf{A})=\det\mathbf{A}+\det\mathbf{A}\underbrace{I_1(\mathbf{A}^{-1}d\mathbf{A})}_{\mathbf{A}^{-T}:d\mathbf{A}}=\det\mathbf{A}+(\det\mathbf{A})\mathbf{A}^{-T}:d\mathbf{A}\,, \tag{1.50}$$

and, consequently,

$$\frac{\partial I_3(\mathbf{A})}{\partial \mathbf{A}} = \frac{\partial (\det \mathbf{A})}{\partial \mathbf{A}} = (\det \mathbf{A})\,\mathbf{A}^{-T}\,. \tag{1.51}$$

1.4 Tensor analysis

We turn to tensor analysis and define the following differential operators for vectors and second-order tensors in Cartesian coordinates

$$\mathrm{grad}\varphi = \nabla\varphi = \frac{\partial \varphi}{\partial \mathbf{x}} = \frac{\partial \varphi}{\partial x_i}\mathbf{e}_i\,, \tag{1.52}$$

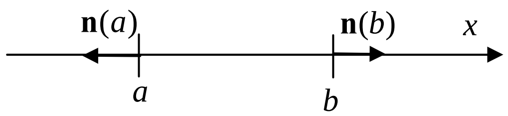
$$\mathrm{grad}\,\mathbf{a} = \frac{\partial \mathbf{a}}{\partial \mathbf{x}} = \frac{\partial \mathbf{a}}{\partial x_i} \otimes \mathbf{e}_i = \frac{\partial a_j}{\partial x_i}\mathbf{e}_j \otimes \mathbf{e}_i\,, \tag{1.53}$$

$$\mathrm{div}\,\mathbf{a} = \frac{\partial \mathbf{a}}{\partial x_i} \cdot \mathbf{e}_i = \frac{\partial a_j}{\partial x_i}\mathbf{e}_j \cdot \mathbf{e}_i = \frac{\partial a_i}{\partial x_i}\,, \tag{1.54}$$

$$\begin{aligned}\mathrm{curl}\,\mathbf{a} &= \mathbf{e}_i \times \frac{\partial \mathbf{a}}{\partial x_i} = \mathbf{e}_i \times \mathbf{e}_j \frac{\partial a_j}{\partial x_i} = \varepsilon_{kij}\mathbf{e}_k \frac{\partial a_j}{\partial x_i} \\ &= \mathbf{e}_1\left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right),\end{aligned} \tag{1.55}$$

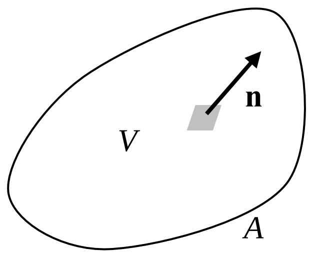
$$\begin{aligned}\mathrm{div}\,\mathbf{A} &= \frac{\partial \mathbf{A}}{\partial x_i}\mathbf{e}_i = \frac{\partial A_{mn}}{\partial x_i}(\mathbf{e}_m \otimes \mathbf{e}_n)\mathbf{e}_i = \frac{\partial A_{mn}}{\partial x_n}\mathbf{e}_m \\ &= \left(\frac{\partial A_{11}}{\partial x_1} + \frac{\partial A_{12}}{\partial x_2} + \frac{\partial A_{13}}{\partial x_3}\right)\mathbf{e}_1 \\ &+ \left(\frac{\partial A_{21}}{\partial x_1} + \frac{\partial A_{22}}{\partial x_2} + \frac{\partial A_{23}}{\partial x_3}\right)\mathbf{e}_2 \\ &+ \left(\frac{\partial A_{31}}{\partial x_1} + \frac{\partial A_{32}}{\partial x_2} + \frac{\partial A_{33}}{\partial x_3}\right)\mathbf{e}_3\end{aligned}\,. \tag{1.56}$$

Now we refresh our memories concerning the divergence theorem (Gauss, Green, and Ostrogradskii) which is an important tool for transforming volume and area integrals. Its simplest version in one-dimensional case is the famous Newton-Leibnitz rule



$$\int_a^b \frac{dy}{dx}dx = (+1)y(b) + (-1)y(a) = n(b)y(b) + n(a)y(a)\,.$$

In a three-dimensional case we can write



$$\int \frac{\partial y}{\partial x_i} dV = \oint n_i y dA . \tag{1.57}$$

The powerful generalization of this formula is

$$\begin{aligned} \int \frac{\partial B_{ij}}{\partial x_j} dV &= \int \frac{\partial B_{i1}}{\partial x_1} dV + \int \frac{\partial B_{i2}}{\partial x_2} dV + \int \frac{\partial B_{i3}}{\partial x_3} dV \\ &= \oint B_{i1} n_1 dA + \oint B_{i2} n_2 dA + \oint B_{i3} n_3 dA \quad , \\ &= \oint B_{ij} n_j dA \end{aligned} \tag{1.58}$$

or, shortly,

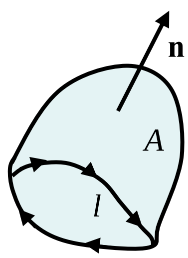
$$\int \mathrm{div}\mathbf{B}\, dV = \oint \mathbf{Bn} dA . \tag{1.59}$$

Of course, the second-order tensor $\mathbf{B}$ can be replaced by scalar b or vector $\mathbf{b}$

$$\int \mathrm{grad} b\, dV = \oint b\mathbf{n} dA , \tag{1.60}$$

$$\int \mathrm{div}\mathbf{b}\, dV = \oint \mathbf{b} \cdot \mathbf{n} dA . \tag{1.61}$$

Another useful formula is due to Stokes who related the contour integral over curve l to surface A built on it

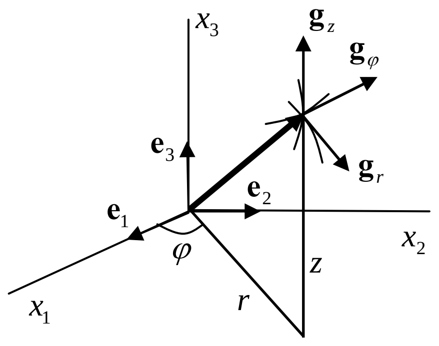


$$\oint \mathbf{b} \cdot d\mathbf{x} = \int (\mathrm{curl}\,\mathbf{b}) \cdot \mathbf{n} dA , \tag{1.62}$$

where $d\mathbf{x}$ is the infinitesimal element of the curve l.

1.5 Curvilinear coordinates

Some problems are easier to solve in *curvilinear* rather than Cartesian coordinates. We consider curvilinear coordinates $(\alpha^1, \alpha^2, \alpha^3)$ which can be defined through the Cartesian coordinates (x_1, x_2, x_3) and vice versa. For example, in the case of *cylindrical* coordinates we have



$$\alpha^1 = r; \quad \alpha^2 = \varphi; \quad \alpha^3 = z\,, \tag{1.63}$$

$$x_1 = r\cos\varphi; \quad x_2 = r\sin\varphi; \quad x_3 = z\,, \tag{1.64}$$

$$r = \sqrt{x_1^2 + x_2^2}; \quad \varphi = \arctan\frac{x_2}{x_1}; \quad z = x_3\,. \tag{1.65}$$

We define the *natural (co-variant)* base vectors in curvilinear coordinates

$$\mathbf{s}_i = \frac{\partial x_j}{\partial \alpha^i}\mathbf{e}_j\,, \tag{1.66}$$

which take the following form in cylindrical coordinates

$$\begin{cases} \mathbf{s}_r = \dfrac{\partial x_1}{\partial r}\mathbf{e}_1 + \dfrac{\partial x_2}{\partial r}\mathbf{e}_2 + \dfrac{\partial x_3}{\partial r}\mathbf{e}_3 = \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2 \\ \mathbf{s}_\varphi = \dfrac{\partial x_1}{\partial \varphi}\mathbf{e}_1 + \dfrac{\partial x_2}{\partial \varphi}\mathbf{e}_2 + \dfrac{\partial x_3}{\partial \varphi}\mathbf{e}_3 = -r\sin\varphi\,\mathbf{e}_1 + r\cos\varphi\,\mathbf{e}_2 \\ \mathbf{s}_z = \dfrac{\partial x_1}{\partial z}\mathbf{e}_1 + \dfrac{\partial x_2}{\partial z}\mathbf{e}_2 + \dfrac{\partial x_3}{\partial z}\mathbf{e}_3 = \mathbf{e}_3 \end{cases}. \tag{1.67}$$

We also define the *dual (contra-variant)* base vectors

$$\mathbf{s}^i = \frac{\partial \alpha^i}{\partial x_j}\mathbf{e}_j\,, \tag{1.68}$$

which take the following form in cylindrical coordinates

$$\begin{cases} \mathbf{s}^r = \dfrac{\partial r}{\partial x_1}\mathbf{e}_1 + \dfrac{\partial r}{\partial x_2}\mathbf{e}_2 + \dfrac{\partial r}{\partial x_3}\mathbf{e}_3 = \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_2 \\ \mathbf{s}^\varphi = \dfrac{\partial \varphi}{\partial x_1}\mathbf{e}_1 + \dfrac{\partial \varphi}{\partial x_2}\mathbf{e}_2 + \dfrac{\partial \varphi}{\partial x_3}\mathbf{e}_3 = -\dfrac{\sin\varphi}{r}\mathbf{e}_1 + \dfrac{\cos\varphi}{r}\mathbf{e}_2 \\ \mathbf{s}^z = \dfrac{\partial z}{\partial x_1}\mathbf{e}_1 + \dfrac{\partial z}{\partial x_2}\mathbf{e}_2 + \dfrac{\partial z}{\partial x_3}\mathbf{e}_3 = \mathbf{e}_3 \end{cases}. \tag{1.69}$$

The natural and dual base vectors are mutually orthogonal

$$\mathbf{s}_i \cdot \mathbf{s}^j = \left(\frac{\partial x_m}{\partial \alpha^i}\mathbf{e}_m\right)\cdot\left(\frac{\partial \alpha^j}{\partial x_n}\mathbf{e}_n\right) = \frac{\partial x_m}{\partial \alpha^i}\frac{\partial \alpha^j}{\partial x_n}\delta_{mn} = \frac{\partial x_m}{\partial \alpha^i}\frac{\partial \alpha^j}{\partial x_m} = \frac{\partial \alpha^j}{\partial \alpha^i} = \begin{cases}1, & i = j\\ 0, & i \neq j\end{cases}. \tag{1.70}$$

Now vectors and tensors may have various representations in curvilinear coordinates

$$\mathbf{a} = a^i\mathbf{s}_i = a_i\mathbf{s}^i, \tag{1.71}$$

$$\mathbf{A} = A^{ij}\mathbf{s}_i \otimes \mathbf{s}_j = A_{ij}\mathbf{s}^i \otimes \mathbf{s}^j = A^i_{.j}\mathbf{s}_i \otimes \mathbf{s}^j = A_i^{.j}\mathbf{s}^i \otimes \mathbf{s}_j, \tag{1.72}$$

where $a^i = \mathbf{a}\cdot\mathbf{s}^i$ are *contra-variant* components; and $a_i = \mathbf{a}\cdot\mathbf{s}_i$ are *co-variant* components; $A^{ij} = \mathbf{A} : (\mathbf{s}^i \otimes \mathbf{s}^j)$ are *contra-variant* components; $A_{ij} = \mathbf{A} : (\mathbf{s}_i \otimes \mathbf{s}_j)$ are *co-variant* components; and $A_i^{.j} = \mathbf{A} : (\mathbf{s}_i \otimes \mathbf{s}^j)$ and $A^i_{.j} = \mathbf{A} : (\mathbf{s}^i \otimes \mathbf{s}_j)$ are *mixed* components.

In the case where the base vectors are mutually orthogonal it is possible to *normalize* them. For example, in the case of the cylindrical coordinates we have

$$\mathbf{g}_r = \frac{\mathbf{s}_r}{|\mathbf{s}_r|} = \frac{\mathbf{s}^r}{|\mathbf{s}^r|} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix}, \quad \mathbf{g}_\varphi = \frac{\mathbf{s}_\varphi}{|\mathbf{s}_\varphi|} = \frac{\mathbf{s}^\varphi}{|\mathbf{s}^\varphi|} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}, \quad \mathbf{g}_z = \frac{\mathbf{s}_z}{|\mathbf{s}_z|} = \frac{\mathbf{s}^z}{|\mathbf{s}^z|} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}. \tag{1.73}$$

The normalized base vectors allow introducing the so-called *physical* components of vectors and tensors with the same units

$$\mathbf{a} = a_r\mathbf{g}_r + a_\varphi\mathbf{g}_\varphi + a_z\mathbf{g}_z, \tag{1.74}$$

$$\begin{aligned}\mathbf{A} = {} & A_{rr}\mathbf{g}_r \otimes \mathbf{g}_r + A_{r\varphi}\mathbf{g}_r \otimes \mathbf{g}_\varphi + A_{rz}\mathbf{g}_r \otimes \mathbf{g}_z\\ & + A_{\varphi r}\mathbf{g}_\varphi \otimes \mathbf{g}_r + A_{\varphi\varphi}\mathbf{g}_\varphi \otimes \mathbf{g}_\varphi + A_{\varphi z}\mathbf{g}_\varphi \otimes \mathbf{g}_z\, .\\ & + A_{zr}\mathbf{g}_z \otimes \mathbf{g}_r + A_{z\varphi}\mathbf{g}_z \otimes \mathbf{g}_\varphi + A_{zz}\mathbf{g}_z \otimes \mathbf{g}_z\end{aligned} \tag{1.75}$$

Now we calculate differential operators in curvilinear coordinates

$$\mathrm{grad}\mathbf{a} = \frac{\partial \mathbf{a}}{\partial x_i} \otimes \mathbf{e}_i = \frac{\partial \mathbf{a}}{\partial \alpha^j} \otimes \underbrace{\frac{\partial \alpha^j}{\partial x_i}\mathbf{e}_i}_{\mathbf{s}^j} = \frac{\partial \mathbf{a}}{\partial \alpha^j} \otimes \mathbf{s}^j, \tag{1.76}$$

$$\mathrm{curl}\,\mathbf{a} = \mathbf{e}_i \times \frac{\partial \mathbf{a}}{\partial x_i} = \underbrace{\frac{\partial \alpha^j}{\partial x_i}\mathbf{e}_i}_{\mathbf{s}^j} \times \frac{\partial \mathbf{a}}{\partial \alpha^j} = \mathbf{s}^j \times \frac{\partial \mathbf{a}}{\partial \alpha^j}, \tag{1.77}$$

$$\mathrm{div}\mathbf{B} = \frac{\partial \mathbf{B}}{\partial x_i}\mathbf{e}_i = \frac{\partial \mathbf{B}}{\partial \alpha^j}\underbrace{\frac{\partial \alpha^j}{\partial x_i}\mathbf{e}_i}_{\mathbf{s}^j} = \frac{\partial \mathbf{B}}{\partial \alpha^j}\mathbf{s}^j. \tag{1.78}$$

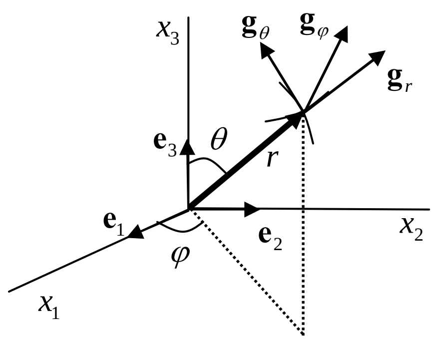
In the case of cylindrical coordinates we have, for example,

$$\begin{aligned}\operatorname{grad}(...) &= \frac{\partial(...)}{\partial r}\otimes\mathbf{s}^r + \frac{\partial(...)}{\partial\varphi}\otimes\mathbf{s}^\varphi + \frac{\partial(...)}{\partial z}\otimes\mathbf{s}^z \\ &= \frac{\partial(...)}{\partial r}\otimes\mathbf{g}_r + \frac{1}{r}\frac{\partial(...)}{\partial\varphi}\otimes\mathbf{g}_\varphi + \frac{\partial(...)}{\partial z}\otimes\mathbf{g}_z\end{aligned}. \tag{1.79}$$

In calculating the derivatives of vectors and tensors one should not forget that the natural and dual and physical *base vectors depend on coordinates*! In the considered case of cylindrical coordinates we have the following derivatives of the physical base vectors

$$\begin{aligned}&\frac{\partial\mathbf{g}_r}{\partial r}=\mathbf{0}; &&\frac{\partial\mathbf{g}_\varphi}{\partial r}=\mathbf{0}; &&\frac{\partial\mathbf{g}_z}{\partial r}=\mathbf{0}\\ &\frac{\partial\mathbf{g}_r}{\partial\varphi}=\mathbf{g}_\varphi; &&\frac{\partial\mathbf{g}_\varphi}{\partial\varphi}=-\mathbf{g}_r; &&\frac{\partial\mathbf{g}_z}{\partial\varphi}=\mathbf{0}.\\ &\frac{\partial\mathbf{g}_r}{\partial z}=\mathbf{0}; &&\frac{\partial\mathbf{g}_\varphi}{\partial z}=\mathbf{0}; &&\frac{\partial\mathbf{g}_z}{\partial z}=\mathbf{0}\end{aligned} \tag{1.80}$$

Besides the considered cylindrical coordinates it is useful to list the basic relationships for spherical coordinates



$$\alpha^1 = r;\quad \alpha^2=\theta;\quad \alpha^3=\varphi, \tag{1.81}$$

$$x_1 = r\cos\varphi\sin\theta;\quad x_2 = r\sin\varphi\sin\theta;\quad x_3=r\cos\theta, \tag{1.82}$$

$$r=\sqrt{x_1^2+x_2^2+x_3^2};\quad \theta=\arccos\frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}};\quad \varphi=\arctan\frac{x_2}{x_1}, \tag{1.83}$$

$$\mathbf{g}_r=\begin{pmatrix}\cos\varphi\sin\theta\\ \sin\varphi\sin\theta\\ \cos\theta\end{pmatrix},\quad \mathbf{g}_\theta=\begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix},\quad \mathbf{g}_\varphi=\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}, \tag{1.84}$$

$$\operatorname{grad}(...)=\frac{\partial(...)}{\partial r}\otimes\mathbf{g}_r+\frac{1}{r}\frac{\partial(...)}{\partial\theta}\otimes\mathbf{g}_\theta+\frac{1}{r\sin\theta}\frac{\partial(...)}{\partial\varphi}\otimes\mathbf{g}_\varphi, \tag{1.85}$$

$$\begin{aligned}&\frac{\partial \mathbf{g}_r}{\partial r}=\mathbf{0}; && \frac{\partial \mathbf{g}_\theta}{\partial r}=\mathbf{0}; && \frac{\partial \mathbf{g}_\varphi}{\partial r}=\mathbf{0}\\ &\frac{\partial \mathbf{g}_r}{\partial \theta}=\mathbf{g}_\theta; && \frac{\partial \mathbf{g}_\theta}{\partial \theta}=-\mathbf{g}_r; && \frac{\partial \mathbf{g}_\varphi}{\partial \theta}=\mathbf{0}\\ &\frac{\partial \mathbf{g}_r}{\partial \varphi}=\mathbf{g}_\varphi \sin\theta; && \frac{\partial \mathbf{g}_\theta}{\partial \varphi}=\mathbf{g}_\varphi \cos\theta; && \frac{\partial \mathbf{g}_\varphi}{\partial \varphi}=-\mathbf{g}_r \sin\theta-\mathbf{g}_\theta \cos\theta\end{aligned}\ . \tag{1.86}$$

1.6 Homework

1. Prove:

$$\varepsilon_{skt}\varepsilon_{mnp}=\det\begin{pmatrix}\delta_{sm} & \delta_{sn} & \delta_{sp}\\ \delta_{km} & \delta_{kn} & \delta_{kp}\\ \delta_{tm} & \delta_{tn} & \delta_{tp}\end{pmatrix}, \tag{1.87}$$

$$\varepsilon_{skt}\varepsilon_{snp}=\delta_{kn}\delta_{tp}-\delta_{kp}\delta_{tn}, \tag{1.88}$$

$$\varepsilon_{skt}\varepsilon_{skp}=\delta_{kk}\delta_{tp}-\delta_{kp}\delta_{tk}=3\delta_{tp}-\delta_{tp}=2\delta_{tp}, \tag{1.89}$$

$$\varepsilon_{skt}\varepsilon_{skt}=2\delta_{tt}=6\,. \tag{1.90}$$

2. Prove (1.20).

3. Prove for second-order tensors **A**, **B**:

$$\det\mathbf{A}=\frac{1}{6}\varepsilon_{ijk}\varepsilon_{stp}A_{si}A_{tj}A_{pk}, \tag{1.91}$$

$$\det\mathbf{A}^{-1}=\frac{1}{\det\mathbf{A}}, \tag{1.92}$$

$$(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}, \tag{1.93}$$

$$(\mathbf{A}^T)^{-1}=(\mathbf{A}^{-1})^T\equiv\mathbf{A}^{-T}\,. \tag{1.94}$$

4. Prove (1.37).

5. Prove (1.48).

6. Prove for second-order tensors **A**, **B**:

$$\mathrm{tr}(\mathbf{A}^{-1}\mathbf{B})=\mathbf{A}^{-T}:\mathbf{B}\,. \tag{1.95}$$

7. Prove for scalar φ:

$$\mathrm{curl}\,\mathrm{grad}\,\varphi=\mathbf{0}\,. \tag{1.96}$$

8. Prove for vector $\mathbf{a}$:

$$\mathrm{div}\,\mathrm{curl}\,\mathbf{a}=0\,. \tag{1.97}$$

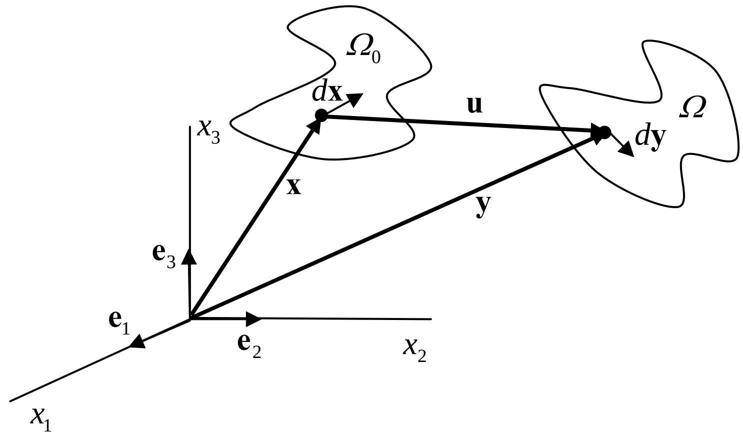
9. Prove (1.84).

10. Prove (1.85).

11. Prove (1.86).

2 Kinematics

2.1 Deformation gradient



We consider deformation of a body shown in its reference and current states. The law of motion of *material points*, i.e. infinitesimal material volumes, is defined by

$$\mathbf{y} = \mathbf{y}(\mathbf{x}, t), \tag{2.1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the reference and current positions of the point. It is usually convenient, yet not necessary, to assume that the reference state is the initial one: $\mathbf{x} = \mathbf{y}(\mathbf{x}, t = 0)$.

If we consider $\mathbf{x}$ as an independent variable then we follow motion of a material point that was fixed at $\mathbf{x}$ in the reference configuration. Such description is called *referential* or *material* or *Lagrangean*. If, alternatively, we consider $\mathbf{y}$ as an independent variable then we follow motion of *various* material points passing through $\mathbf{y}$ in the current configuration. The latter description is called *spatial* or *Eulerian*. The Eulerian description is preferable when the evolution of continuum boundaries is known beforehand like in many problems of fluid mechanics while the Lagrangean description is preferable when the evolution of continuum boundaries is not known beforehand like in many problems of solid mechanics.

An infinitesimal material fiber at points $\mathbf{x}$ and $\mathbf{y}$ before and after deformation accordingly can be described by the linear mapping (transformation)

$$d\mathbf{y} = \mathbf{F} d\mathbf{x}, \tag{2.2}$$

where

$$\mathbf{F} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \frac{\partial y_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \tag{2.3}$$

is the tensor of *deformation gradient*. This tensor is related to two configurations simultaneously and because of that it is called *two-point*.

Alternatively, we can use the displacement vector, $\mathbf{u} = \mathbf{y} - \mathbf{x}$, to get

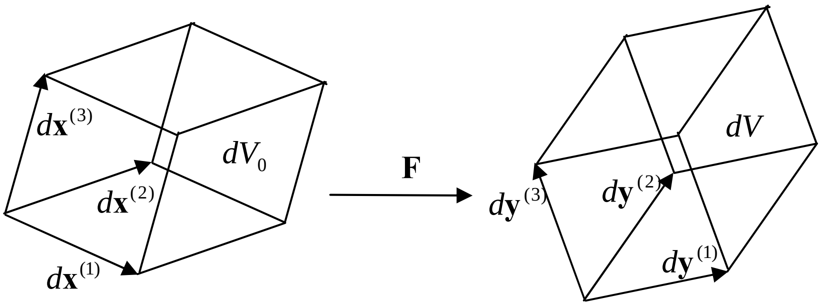
$$\mathbf{F} = \frac{\partial(\mathbf{x} + \mathbf{u})}{\partial \mathbf{x}} = \mathbf{1} + \mathbf{H}, \tag{2.4}$$

where

$$\mathbf{H} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial u_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \tag{2.5}$$

is the *displacement gradient* tensor.

It is possible to calculate any deformation when the deformation gradient is known. We start with the volume deformation



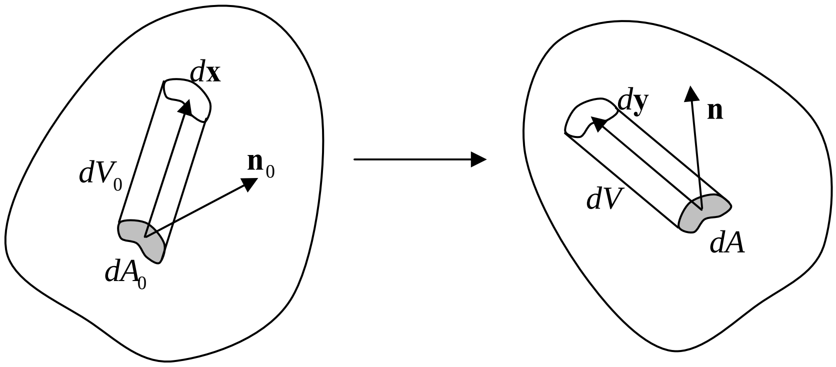
$$dy_i^{(m)} = F_{ij} dx_j^{(m)}, \tag{2.6}$$

$$dV = \begin{vmatrix} dy_1^{(1)} & dy_2^{(1)} & dy_3^{(1)} \\ dy_1^{(2)} & dy_2^{(2)} & dy_3^{(2)} \\ dy_1^{(3)} & dy_2^{(3)} & dy_3^{(3)} \end{vmatrix} = \begin{vmatrix} F_{1j} dx_j^{(1)} & F_{2j} dx_j^{(1)} & F_{3j} dx_j^{(1)} \\ F_{1j} dx_j^{(2)} & F_{2j} dx_j^{(2)} & F_{3j} dx_j^{(2)} \\ F_{1j} dx_j^{(3)} & F_{2j} dx_j^{(3)} & F_{3j} dx_j^{(3)} \end{vmatrix}$$
$$= \underbrace{\begin{vmatrix} dx_1^{(1)} & dx_2^{(1)} & dx_3^{(1)} \\ dx_1^{(2)} & dx_2^{(2)} & dx_3^{(2)} \\ dx_1^{(3)} & dx_2^{(3)} & dx_3^{(3)} \end{vmatrix}}_{dV_0} \cdot \underbrace{\begin{vmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{vmatrix}}_{\det \mathbf{F}} = J dV_0, \tag{2.7}$$

where

$$J = \det \mathbf{F} > 0. \tag{2.8}$$

In the case of the area deformation we have



$$dV_0 = \mathbf{n}_0 dA_0 \cdot d\mathbf{x}\,, \tag{2.9}$$

$$dV = \mathbf{n}dA \cdot d\mathbf{y} = \mathbf{n}dA \cdot \mathbf{F}d\mathbf{x}\,. \tag{2.10}$$

Using (2.7) we derive

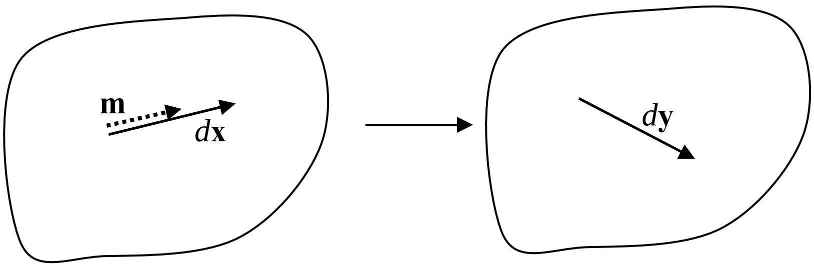
$$\mathbf{n}dA \cdot \mathbf{F}d\mathbf{x} = J\mathbf{n}_0 dA_0 \cdot d\mathbf{x}\,, \tag{2.11}$$

$$(\mathbf{F}^T\mathbf{n}dA - J\mathbf{n}_0 dA_0) \cdot d\mathbf{x} = 0\,. \tag{2.12}$$

Since $d\mathbf{x}$ is arbitrary we can write the Nanson formula

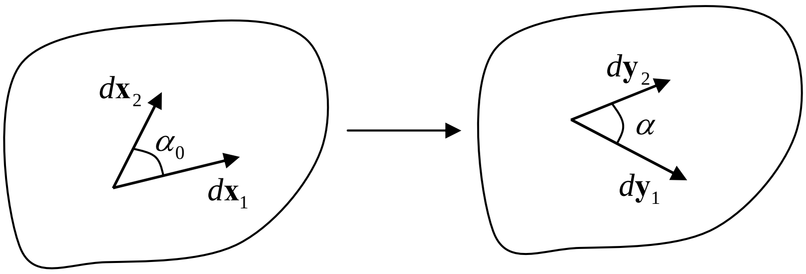
$$\mathbf{n}dA = J\mathbf{F}^{-T}\mathbf{n}_0 dA_0\,. \tag{2.13}$$

Now we define the fiber stretch in direction $\mathbf{m}; |\mathbf{m}| = 1$



$$\lambda(\mathbf{m}) = \frac{|d\mathbf{y}|}{|d\mathbf{x}|} = \frac{|\mathbf{F}d\mathbf{x}|}{|d\mathbf{x}|} = |\mathbf{F}\mathbf{m}|\,. \tag{2.14}$$

We can also define the change of angle between two fibers by using stretches as follows, for example,

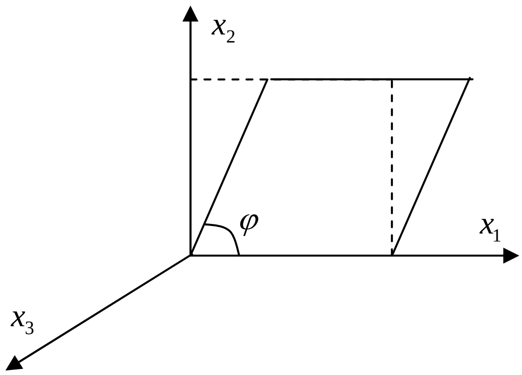


$$\cos\alpha = \frac{d\mathbf{y}_1 \cdot d\mathbf{y}_2}{|d\mathbf{y}_1||d\mathbf{y}_2|} = \frac{\mathbf{Fm}_1 \cdot \mathbf{Fm}_2}{\lambda(\mathbf{m}_1)\lambda(\mathbf{m}_2)}, \tag{2.15}$$

$$\cos\alpha_0 = \frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{|d\mathbf{x}_1||d\mathbf{x}_2|} = \mathbf{m}_1 \cdot \mathbf{m}_2, \tag{2.16}$$

$$\gamma(\mathbf{m}_1, \mathbf{m}_2) = \cos\alpha - \cos\alpha_0 = \mathbf{m}_1 \cdot \left(\frac{1}{\lambda(\mathbf{m}_1)\lambda(\mathbf{m}_2)} \mathbf{F}^T \mathbf{F} - \mathbf{1} \right) \mathbf{m}_2 . \tag{2.17}$$

To illustrate the above formulas we consider the Simple Shear deformation



$$\gamma = \tan(\pi/2 - \varphi) = \cot\varphi,$$

$$\begin{cases} y_1 = x_1 + \gamma x_2 \\ y_2 = x_2 \\ y_3 = x_3 \end{cases},$$

$$\mathbf{F} = \frac{\partial y_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 + \gamma \mathbf{e}_1 \otimes \mathbf{e}_2 = \mathbf{1} + \gamma \mathbf{e}_1 \otimes \mathbf{e}_2,$$

$$\lambda(\mathbf{e}_1) = \sqrt{(\mathbf{Fe}_1) \cdot \mathbf{Fe}_1} = \sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1} = 1,$$

$$\lambda(\mathbf{e}_2) = \sqrt{(\mathbf{Fe}_2) \cdot \mathbf{Fe}_2} = \sqrt{(\mathbf{e}_2 + \gamma\mathbf{e}_1) \cdot (\mathbf{e}_2 + \gamma\mathbf{e}_1)} = \sqrt{1+\gamma^2},$$

$$\cos\alpha = \frac{(\mathbf{Fe}_1) \cdot (\mathbf{Fe}_2)}{|\mathbf{Fe}_1||\mathbf{Fe}_2|} = \frac{\mathbf{e}_1 \cdot (\mathbf{e}_2 + \gamma\mathbf{e}_1)}{\sqrt{1+\gamma^2}} = \frac{\gamma}{\sqrt{1+\gamma^2}},$$

$$\alpha - \alpha_0 = \alpha - \pi/2 = \arccos\frac{\gamma}{\sqrt{1+\gamma^2}} - \frac{\pi}{2} = \arccos\frac{\cos\varphi/\sin\varphi}{\sqrt{1+(\cos\varphi/\sin\varphi)^2}} - \frac{\pi}{2}.$$
$$= \arccos(\cos\varphi) - \pi/2 = \varphi - \pi/2$$

2.2 Polar decomposition of deformation gradient

Let us square the expression for stretch (2.14) and rewrite it as follows

$$\lambda^2(\mathbf{m}) = (\mathbf{Fm}) \cdot (\mathbf{Fm}) = \mathbf{m} \cdot \mathbf{F}^T\mathbf{Fm} = \mathbf{m} \cdot \mathbf{Cm}, \tag{2.18}$$

where

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \tag{2.19}$$

is the right Cauchy-Green tensor.

In the case where direction **m** is the principal direction of tensor **C** we have

$$\lambda^2(\mathbf{m}^{(i)}) = \mathbf{m}^{(i)} \cdot \mathbf{Cm}^{(i)} = \mathbf{m}^{(i)} \cdot \zeta_i \mathbf{m}^{(i)} = \zeta_i, \tag{2.20}$$

where ζ_i and $\mathbf{m}^{(i)}$ are the eigenvalues and eigenvectors of **C**.

The above equation means that eigenvalues of the right Cauchy-Green tensor are equal to the squared stretches in principal directions. Thus we can write the following spectral decomposition of **C** in the form

$$\mathbf{C} = \lambda_1^2 \mathbf{m}^{(1)} \otimes \mathbf{m}^{(1)} + \lambda_2^2 \mathbf{m}^{(2)} \otimes \mathbf{m}^{(2)} + \lambda_3^2 \mathbf{m}^{(3)} \otimes \mathbf{m}^{(3)}. \tag{2.21}$$

Now we define the *right stretch* tensor as the square root of the right Cauchy-Green tensor

$$\mathbf{U} = \sqrt{\mathbf{C}} = \lambda_1 \mathbf{m}^{(1)} \otimes \mathbf{m}^{(1)} + \lambda_2 \mathbf{m}^{(2)} \otimes \mathbf{m}^{(2)} + \lambda_3 \mathbf{m}^{(3)} \otimes \mathbf{m}^{(3)}, \tag{2.22}$$

where all principal stretches are nonnegative.

We assume now that any deformation can be multiplicatively decomposed into stretch and some additional deformation which we designate **R**

$$\mathbf{F} = \mathbf{RU}, \tag{2.23}$$

which is called the *polar decomposition* of the deformation gradient and, consequently,

$$\mathbf{R} = \mathbf{FU}^{-1}. \tag{2.24}$$

Let us analyze properties of **R**. First, we observe that it is *orthogonal*

$$\mathbf{R}^T\mathbf{R} = (\mathbf{FU}^{-1})^T(\mathbf{FU}^{-1}) = \mathbf{U}^{-T}\underbrace{\mathbf{F}^T\mathbf{F}}_{\mathbf{U}^2}\mathbf{U}^{-1} = \mathbf{U}^{-T}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{U}^{-T}\mathbf{U}^T\mathbf{U}\mathbf{U}^{-1} = \mathbf{1}. \tag{2.25}$$

Orthogonal tensors do not change lengths

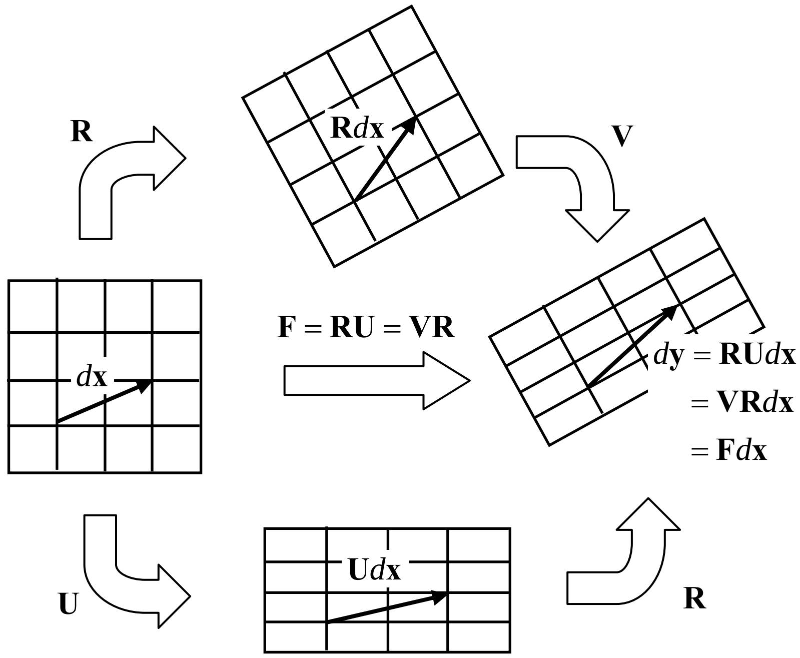
$$|d\mathbf{y}| = \sqrt{d\mathbf{y} \cdot d\mathbf{y}} = \sqrt{(\mathbf{R}d\mathbf{x}) \cdot (\mathbf{R}d\mathbf{x})} = \sqrt{d\mathbf{x} \cdot \mathbf{R}^T\mathbf{R}d\mathbf{x}} = \sqrt{d\mathbf{x} \cdot d\mathbf{x}} = |d\mathbf{x}|. \tag{2.26}$$

Besides, we observe

$$\det\mathbf{R} = \frac{\det\mathbf{F}}{\det\mathbf{U}} = \frac{\sqrt{\det\mathbf{F}^T\det\mathbf{F}}}{\det\mathbf{U}} = \frac{\sqrt{\det(\mathbf{F}^T\mathbf{F})}}{\det\mathbf{U}} = \frac{\sqrt{\det\mathbf{C}}}{\det\mathbf{U}} = \frac{\sqrt{\det\mathbf{U}^2}}{\det\mathbf{U}} = \frac{\det\mathbf{U}}{\det\mathbf{U}} = 1. \quad (2.27)$$

Equations (2.25) and (2.27) mean that $\mathbf{R}$ is the *proper orthogonal* or *rotation* tensor.

Finally we notice that the meaning of the polar decomposition, $\mathbf{F} = \mathbf{RU}$, is the successive stretch and rotation.



It is possible, of course, to change the order of stretch and rotation

$$\mathbf{F} = \mathbf{VR}, \quad (2.28)$$

where $\mathbf{V}$ is called the *left stretch* tensor.

By direct computation we have

$$\mathbf{V} = \mathbf{FR}^{-1} = \mathbf{FR}^T = \mathbf{RUR}^T = \mathbf{V}^T, \quad (2.29)$$

which means that the left stretch tensor is the rotated right stretch tensor, and consequently they have the same eigenvalues – principal stretches, while their principal directions are different.

With account of the spectral decomposition of $\mathbf{U}$ we have

$$\mathbf{V} = \lambda_1\mathbf{n}^{(1)}\otimes\mathbf{n}^{(1)} + \lambda_2\mathbf{n}^{(2)}\otimes\mathbf{n}^{(2)} + \lambda_3\mathbf{n}^{(3)}\otimes\mathbf{n}^{(3)}, \quad (2.30)$$

where

$$\mathbf{n}^{(i)}\otimes\mathbf{n}^{(i)} = \mathbf{Rm}^{(i)}\otimes\mathbf{Rm}^{(i)}. \quad (2.31)$$

To clarify the meaning of the principal directions of $\mathbf{V}$ we square the tensor as follows

$$\mathbf{V}^2 = \mathbf{RUR}^T\mathbf{RUR}^T = (\mathbf{RU})(\mathbf{RU})^T = \mathbf{FF}^T = \mathbf{B}, \quad (2.32)$$

$$\mathbf{B} = \lambda_1^2 \mathbf{n}^{(1)} \otimes \mathbf{n}^{(1)} + \lambda_2^2 \mathbf{n}^{(2)} \otimes \mathbf{n}^{(2)} + \lambda_3^2 \mathbf{n}^{(3)} \otimes \mathbf{n}^{(3)}, \tag{2.33}$$

where $\mathbf{B}$ is the left Cauchy-Green tensor, which principal directions coincide with the principal directions of $\mathbf{V}$ while the principal values of $\mathbf{B}$ are squared principal stretches.

Unfortunately, we cannot directly write the relations between the directions of the principal stretches in the reference and current configurations because these directions are not defined uniquely and can always be changed to the opposite sign! However, we can *define* the principal directions uniquely by the following procedure. Assume, for example, that the principal directions in the reference configuration, $\mathbf{m}^{(i)}$, are uniquely chosen then we calculate the principal directions in the current configuration as follows

$$\mathbf{n}^{(i)} = \mathbf{R}\mathbf{m}^{(i)}. \tag{2.34}$$

Of course, we could start with the current configuration otherwise.

Finally, we can calculate the *spectral decomposition* of the deformation gradient

$$\begin{aligned} \mathbf{F} = \mathbf{R}\mathbf{U} &= \lambda_1 \mathbf{R}\mathbf{m}^{(1)} \otimes \mathbf{m}^{(1)} + \lambda_2 \mathbf{R}\mathbf{m}^{(2)} \otimes \mathbf{m}^{(2)} + \lambda_3 \mathbf{R}\mathbf{m}^{(3)} \otimes \mathbf{m}^{(3)} \\ &= \lambda_1 \mathbf{n}^{(1)} \otimes \mathbf{m}^{(1)} + \lambda_2 \mathbf{n}^{(2)} \otimes \mathbf{m}^{(2)} + \lambda_3 \mathbf{n}^{(3)} \otimes \mathbf{m}^{(3)} \end{aligned}. \tag{2.35}$$

Let us consider the following deformation (Marsden and Hughes, 1983) as a numerical example

$$\begin{cases} y_1 = \sqrt{3}x_1 + x_2 \\ y_2 = 2x_2 \\ y_3 = x_3 \end{cases}.$$

In this case we have

$$\mathbf{F} = \begin{pmatrix} \sqrt{3} & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 3 & \sqrt{3} & 0 \\ \sqrt{3} & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

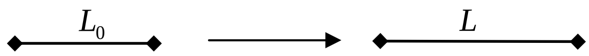
$$\lambda_1 = \sqrt{6}, \quad \mathbf{m}^{(1)} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \\ 0 \end{pmatrix}, \qquad \lambda_2 = \sqrt{2}, \quad \mathbf{m}^{(2)} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ -1 \\ 0 \end{pmatrix}, \qquad \lambda_3 = 1, \quad \mathbf{m}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\mathbf{U} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 3+\sqrt{3} & 3-\sqrt{3} & 0 \\ 3-\sqrt{3} & 1+3\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix}, \qquad \mathbf{U}^{-1} = \frac{1}{4\sqrt{6}}\begin{pmatrix} 1+\sqrt{3} & \sqrt{3}-3 & 0 \\ \sqrt{3}-3 & 3+\sqrt{3} & 0 \\ 0 & 0 & 4\sqrt{6} \end{pmatrix},$$

$$\mathbf{R} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1+\sqrt{3} & \sqrt{3}-1 & 0 \\ \sqrt{3}-1 & 1+\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix}, \qquad \mathbf{V} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+\sqrt{3} & \sqrt{3}-1 & 0 \\ \sqrt{3}-1 & 1+\sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

2.3 Strains

The strain measures can be introduced in various ways. We start with 1D measures for the change of the length of a material fiber.



We can introduce the *engineering strain*

$$\varepsilon_E = \frac{L - L_0}{L_0} = \lambda - 1, \tag{2.36}$$

or the *logarithmic strain*

$$\varepsilon_L = \int_{L_0}^{L} \frac{dL}{L} = \ln \frac{L}{L_0} = \ln \lambda, \tag{2.37}$$

or the *Green strain*

$$\varepsilon_G = \frac{L^2 - L_0^2}{2L_0^2} = \frac{1}{2}(\lambda^2 - 1). \tag{2.38}$$

In order to generalize 1D to 3D strains we assume that formulas (2.36)-(2.38) are valid in the principal directions of the reference configuration. In this case, the 3D strain tensors take forms

$$\boldsymbol{\varepsilon}_E = \sum_{i=1}^{3} (\lambda_i - 1)\mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)} = \mathbf{U} - \mathbf{1}, \tag{2.39}$$

$$\boldsymbol{\varepsilon}_L = \sum_{i=1}^{3} (\ln \lambda_i)\mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)} = \ln \mathbf{U}, \tag{2.40}$$

$$\boldsymbol{\varepsilon}_G = \sum_{i=1}^{3} \frac{1}{2}(\lambda_i^2 - 1)\mathbf{m}^{(i)} \otimes \mathbf{m}^{(i)} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1}). \tag{2.41}$$

The Green strain tensor is the most popular and it can be rewritten by dropping the suffix

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1}) = \frac{1}{2}(\mathbf{C} - \mathbf{1}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T + \mathbf{H}^T\mathbf{H}). \tag{2.42}$$

2.4 Motion

Velocity and acceleration are defined as material time derivatives accordingly

$$\mathbf{v} = \frac{d}{dt}\mathbf{y}(\mathbf{x},t) = \dot{\mathbf{y}} = \dot{\mathbf{x}} + \dot{\mathbf{u}} = \dot{\mathbf{u}}\,, \tag{2.43}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}}\,. \tag{2.44}$$

When the Eulerian or spatial description is used it is necessary to use the chain rule for differentiation of any function, $f(\mathbf{y}(t),t)$:

$$\frac{df}{dt} = \dot{f}(\mathbf{y}(t),t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{y}} \cdot \mathbf{v}\,, \tag{2.45}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial \mathbf{y}}\mathbf{v}\,. \tag{2.46}$$

Another important kinematic quantity is the velocity gradient, $\mathbf{L}$,

$$\dot{\mathbf{F}} = \frac{d}{dt}\left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right) = \frac{\partial \dot{\mathbf{y}}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}}\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \mathbf{L}\mathbf{F}\,, \tag{2.47}$$

$$\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}} = \dot{\mathbf{F}}\mathbf{F}^{-1}\,. \tag{2.48}$$

It can be decomposed into symmetric and skew symmetric parts

$$\mathbf{L} = \mathbf{d} + \boldsymbol{\omega}, \quad \mathbf{d} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T), \quad \boldsymbol{\omega} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T)\,, \tag{2.49}$$

where $\mathbf{d}$ and $\boldsymbol{\omega}$ are the deformation rate and the spin (vorticity) tensors accordingly.

2.5* Deformation gradient in curvilinear coordinates

We consider the deformation gradient in curvilinear coordinates. To be specific we choose the deformation law in cylindrical coordinates before $\{R,\Phi,Z\}$ and after $\{r,\varphi,z\}$ deformation:

$$r = r(R,\Phi,Z); \quad \varphi = \varphi(R,\Phi,Z); \quad z = z(R,\Phi,Z)\,. \tag{2.50}$$

To treat this deformation we introduce the natural curvilinear base vectors for the reference and current configurations accordingly

$$\mathbf{G}_R = \begin{pmatrix} \cos\Phi \\ \sin\Phi \\ 0 \end{pmatrix}; \quad \mathbf{G}_\Phi = \begin{pmatrix} -\sin\Phi \\ \cos\Phi \\ 0 \end{pmatrix}; \quad \mathbf{G}_Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.51}$$

$$\mathbf{g}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}; \quad \mathbf{g}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}; \quad \mathbf{g}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.52}$$

Now the deformation gradient can be written as follows

$$\mathbf{F} = \frac{\partial \mathbf{y}}{\partial R} \otimes \mathbf{G}_R + \frac{1}{R}\frac{\partial \mathbf{y}}{\partial \Phi} \otimes \mathbf{G}_\Phi + \frac{\partial \mathbf{y}}{\partial Z} \otimes \mathbf{G}_Z , \tag{2.53}$$

where

$$\begin{aligned} \mathbf{y} &= y_1\mathbf{e}_1 + y_2\mathbf{e}_2 + y_3\mathbf{e}_3 \\ &= \underbrace{r\cos\varphi}_{y_1}\underbrace{(\cos\varphi\,\mathbf{g}_r - \sin\varphi\,\mathbf{g}_\varphi)}_{\mathbf{e}_1} + \underbrace{r\sin\varphi}_{y_2}\underbrace{(\sin\varphi\,\mathbf{g}_r + \cos\varphi\,\mathbf{g}_\varphi)}_{\mathbf{e}_2} + \underbrace{z}_{y_3}\underbrace{\mathbf{g}_z}_{\mathbf{e}_3} . \\ &= r\,\mathbf{g}_r + z\mathbf{g}_z \end{aligned} \tag{2.54}$$

We have with account of $\mathbf{g}_z = \text{constant}$

$$\begin{aligned} \mathbf{F} &= \frac{\partial(r\,\mathbf{g}_r + z\mathbf{g}_z)}{\partial R} \otimes \mathbf{G}_R + \frac{\partial(r\,\mathbf{g}_r + z\mathbf{g}_z)}{R\partial \Phi} \otimes \mathbf{G}_\Phi + \frac{\partial(r\,\mathbf{g}_r + z\mathbf{g}_z)}{\partial Z} \otimes \mathbf{G}_Z \\ &= \frac{\partial r}{\partial R}\mathbf{g}_r \otimes \mathbf{G}_R + r\frac{\partial\,\mathbf{g}_r}{\partial R} \otimes \mathbf{G}_R + \frac{\partial z}{\partial R}\mathbf{g}_z \otimes \mathbf{G}_R \\ &+ \frac{\partial r}{R\partial \Phi}\mathbf{g}_r \otimes \mathbf{G}_\Phi + \frac{r\partial\,\mathbf{g}_r}{R\partial \Phi} \otimes \mathbf{G}_\Phi + \frac{\partial z}{R\partial \Phi}\mathbf{g}_z \otimes \mathbf{G}_\Phi \\ &+ \frac{\partial r}{\partial Z}\mathbf{g}_r \otimes \mathbf{G}_Z + \frac{r\partial\,\mathbf{g}_r}{\partial Z} \otimes \mathbf{G}_Z + \frac{\partial z}{\partial Z}\mathbf{g}_z \otimes \mathbf{G}_Z \end{aligned} , \tag{2.55}$$

where

$$\begin{cases} \dfrac{\partial \mathbf{g}_r}{\partial R} = \dfrac{\partial \mathbf{g}_r}{\partial r}\dfrac{\partial r}{\partial R} + \dfrac{\partial \mathbf{g}_r}{\partial \varphi}\dfrac{\partial \varphi}{\partial R} + \dfrac{\partial \mathbf{g}_r}{\partial z}\dfrac{\partial z}{\partial R} = \dfrac{\partial \varphi}{\partial R}\mathbf{g}_\varphi \\ \dfrac{\partial \mathbf{g}_r}{\partial \Phi} = \dfrac{\partial \mathbf{g}_r}{\partial r}\dfrac{\partial r}{\partial \Phi} + \dfrac{\partial \mathbf{g}_r}{\partial \varphi}\dfrac{\partial \varphi}{\partial \Phi} + \dfrac{\partial \mathbf{g}_r}{\partial z}\dfrac{\partial z}{\partial \Phi} = \dfrac{\partial \varphi}{\partial \Phi}\mathbf{g}_\varphi \\ \dfrac{\partial \mathbf{g}_r}{\partial Z} = \dfrac{\partial \mathbf{g}_r}{\partial r}\dfrac{\partial r}{\partial Z} + \dfrac{\partial \mathbf{g}_r}{\partial \varphi}\dfrac{\partial \varphi}{\partial Z} + \dfrac{\partial \mathbf{g}_r}{\partial z}\dfrac{\partial z}{\partial Z} = \dfrac{\partial \varphi}{\partial Z}\mathbf{g}_\varphi \end{cases} . \tag{2.56}$$

Finally, we have

$$\begin{aligned} \mathbf{F} &= \frac{\partial r}{\partial R}\mathbf{g}_r \otimes \mathbf{G}_R + \frac{\partial r}{R\partial \Phi}\mathbf{g}_r \otimes \mathbf{G}_\Phi + \frac{\partial r}{\partial Z}\mathbf{g}_r \otimes \mathbf{G}_Z \\ &+ \frac{r\partial \varphi}{\partial R}\mathbf{g}_\varphi \otimes \mathbf{G}_R + \frac{r\partial \varphi}{R\partial \Phi}\mathbf{g}_\varphi \otimes \mathbf{G}_\Phi + \frac{r\partial \varphi}{\partial Z}\mathbf{g}_\varphi \otimes \mathbf{G}_Z \\ &+ \frac{\partial z}{\partial R}\mathbf{g}_z \otimes \mathbf{G}_R + \frac{\partial z}{R\partial \Phi}\mathbf{g}_z \otimes \mathbf{G}_\Phi + \frac{\partial z}{\partial Z}\mathbf{g}_z \otimes \mathbf{G}_Z \end{aligned} . \tag{2.57}$$

Alternatively, we can express the deformation gradient through the displacement vector related to the initial configuration. In the latter, case we have to write

$$\mathbf{x} = R\mathbf{G}_R + Z\mathbf{G}_Z , \tag{2.58}$$

$$\mathbf{u} = u_R\,\mathbf{G}_R + u_\Phi\,\mathbf{G}_\Phi + u_Z\,\mathbf{G}_Z . \tag{2.59}$$

These vectors can be written with respect to the coordinates in the current configuration as follows

$$\hat{\mathbf{x}} = \underbrace{(\mathbf{x}\cdot\mathbf{g}_r)}_{\hat{x}_r}\mathbf{g}_r + \underbrace{(\mathbf{x}\cdot\mathbf{g}_\varphi)}_{\hat{x}_\varphi}\mathbf{g}_\varphi + \underbrace{(\mathbf{x}\cdot\mathbf{g}_z)}_{\hat{x}_z}\mathbf{g}_z\,, \tag{2.60}$$

$$\hat{\mathbf{u}} = \underbrace{(\mathbf{u}\cdot\mathbf{g}_r)}_{\hat{u}_r}\mathbf{g}_r + \underbrace{(\mathbf{u}\cdot\mathbf{g}_\varphi)}_{\hat{u}_\varphi}\mathbf{g}_\varphi + \underbrace{(\mathbf{u}\cdot\mathbf{g}_z)}_{\hat{u}_z}\mathbf{g}_z\,, \tag{2.61}$$

or

$$\hat{\mathbf{x}} = \mathbf{g}\mathbf{x}\,, \tag{2.62}$$

$$\hat{\mathbf{u}} = \mathbf{g}\mathbf{u}\,, \tag{2.63}$$

where

$$\mathbf{g} = \mathbf{g}_r \otimes \mathbf{g}_r + \mathbf{g}_\varphi \otimes \mathbf{g}_\varphi + \mathbf{g}_z \otimes \mathbf{g}_z \tag{2.64}$$

is the *metric* tensor in the current configuration. It is the identity tensor in curvilinear coordinates.

Now we have

$$\mathbf{F} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \frac{\partial(\hat{\mathbf{x}}+\hat{\mathbf{u}})}{\partial \mathbf{x}} = \frac{\partial(\mathbf{g}(\mathbf{x}+\mathbf{u}))}{\partial \mathbf{x}} = \frac{\partial \mathbf{g}}{\partial \mathbf{y}}\frac{\partial \mathbf{y}}{\partial \mathbf{x}}(\mathbf{x}+\mathbf{u}) + \mathbf{g}(\mathbf{G}+\mathbf{H}) = \mathbf{g}(\mathbf{G}+\mathbf{H})\,, \tag{2.65}$$

where

$$\frac{\partial \mathbf{g}}{\partial \mathbf{y}} = \mathbf{0}\,, \tag{2.66}$$

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \mathbf{H}\,, \tag{2.67}$$

and

$$\frac{\partial \mathbf{x}}{\partial \mathbf{x}} = \mathbf{G} = \mathbf{G}_R \otimes \mathbf{G}_R + \mathbf{G}_\Phi \otimes \mathbf{G}_\Phi + \mathbf{G}_Z \otimes \mathbf{G}_Z \tag{2.68}$$

is the *metric* tensor in the reference configuration.

2.6 Homework

1. Find principal directions and stretches for the following deformation law

$$\begin{cases} y_1 = (1+\alpha)x_1 + \alpha x_2 \\ y_2 = -\alpha x_1 + (1+\alpha)x_2\,, \\ y_3 = x_3 \end{cases} \tag{2.69}$$

where $\alpha = \text{constant}$.

2. Make the polar decomposition of the deformation gradient for the deformation law presented in (2.69).

3. Calculate the Cartesian components of the Green strain for the deformation law presented in (2.69).

4. Read Section 2.5.

5. Prove (2.66).

6. Prove (2.68).

3 Balance laws

3.1 Material time derivatives of integrals

We start with the computation of the material time derivative of a volume integral. For the field quantity $\psi(\mathbf{y}(t),t)$ over a “moving” region, $V(t)$, whose configuration depends on time t, we have the following formula (regarding the integral as an infinite sum)

$$\begin{aligned}\frac{d}{dt}\int\psi\, dV(t) &= \int\frac{d}{dt}(\psi\, dy_1(t)dy_2(t)dy_3(t)) \\ &= \int(\frac{d\psi}{dt}dy_1dy_2dy_3 + \psi\, dv_1dy_2dy_3 + \psi\, dy_1dv_2dy_3 + \psi\, dy_1dy_2dv_3) \\ &= \int(\frac{d\psi}{dt} + \psi\frac{\partial v_1}{\partial y_1} + \psi\frac{\partial v_2}{\partial y_2} + \psi\frac{\partial v_3}{\partial y_3})dV \\ &= \int(\frac{d\psi}{dt} + \psi\,\mathrm{div}\mathbf{v})dV \\ &= \int(\frac{\partial\psi}{\partial t} + \mathrm{div}(\psi\,\mathbf{v}))dV\end{aligned} \quad , \tag{3.1}$$

where the last equality is obtained as follows

$$\frac{d\psi}{dt} + \psi\,\mathrm{div}\mathbf{v} = \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial y_i}\frac{\partial y_i}{\partial t} + \psi\frac{\partial v_i}{\partial y_i} = \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial y_i}v_i + \psi\frac{\partial v_i}{\partial y_i} = \frac{\partial\psi}{\partial t} + \frac{\partial(\psi v_i)}{\partial y_i}.$$

3.2 Mass conservation

The law of mass conservation can be written as follows

$$m = \int\rho dV = \text{constant}\,, \tag{3.2}$$

where ρ is mass density.

Differentiating (3.2) with respect to time we have

$$\frac{dm}{dt} = \frac{d}{dt}\int\rho dV = \int(\frac{d\rho}{dt} + \rho\,\mathrm{div}\mathbf{v})dV = \int(\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\mathbf{v}))dV = 0\,. \tag{3.3}$$

Since the equality is obeyed for any volume we can localize the condition for the infinitesimal volume

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}\mathbf{v} = \frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\mathbf{v}) = 0\,. \tag{3.4}$$

3.3 Balance of linear momentum

We start with the balance of linear momentum for a volumeless particle – Newton's law –

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{p}\,, \tag{3.5}$$

where $m\mathbf{v}$ is the *linear momentum* and $\mathbf{p}$ is the force resultant.

By analogy with Newton’s law Euler considered the balance of the linear momentum for a continuum volume V bounded by surface A

$$\frac{d}{dt}\int \rho\mathbf{v}dV = \int \mathbf{b}\rho dV + \oint \mathbf{t}dA\,, \tag{3.6}$$

where $\mathbf{b}$ is the *body force* per unit mass and $\mathbf{t}$ is the *surface force* or *traction* per unit area.

Let us localize the Euler law. First, differentiating the left-hand side of (3.6) we get

$$\frac{d}{dt}\int \rho\mathbf{v}dV = \int (\frac{d(\mathbf{v}\rho)}{dt} + \mathbf{v}\rho\,\mathrm{div}\mathbf{v})dV\,. \tag{3.7}$$

Then we rewrite the Euler law in the form

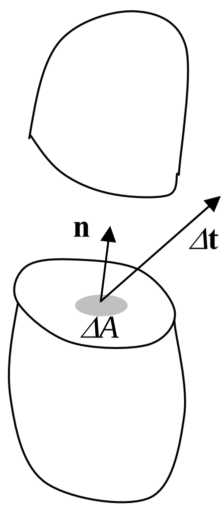
$$\int \mathbf{f}dV = \oint \mathbf{t}dA\,, \tag{3.8}$$

where

$$\mathbf{f} = -\rho\mathbf{b} + \frac{d(\mathbf{v}\rho)}{dt} + \mathbf{v}\rho\,\mathrm{div}\mathbf{v} \tag{3.9}$$

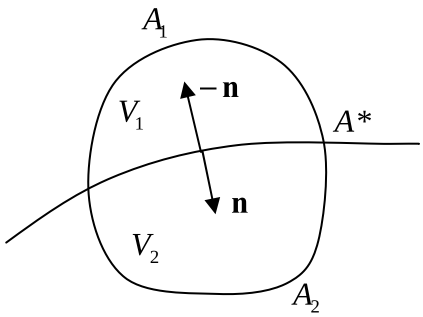
is the generalized body force.

Now it is necessary to transform the area integral into a volume integral. This is possible due to the Cauchy assumption



$$\mathbf{t} = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{t}}{\Delta A} = \mathbf{t}(\mathbf{y}, \mathbf{n}) . \tag{3.10}$$

The first corollary of the Cauchy assumption is the Newton law of action and counteraction.



For every part of the body we have

$$\begin{aligned} \int \mathbf{f}\, dV_1 &= \int \mathbf{t}\, dA_1 + \int \mathbf{t}(\mathbf{n}) dA* \\ \int \mathbf{f}\, dV_2 &= \int \mathbf{t}\, dA_2 + \int \mathbf{t}(-\mathbf{n}) dA* \end{aligned} . \tag{3.11}$$

Summing the equalities we get

$$\int \mathbf{f}\, dV = \oint \mathbf{t}\, dA + \int [\mathbf{t}(\mathbf{n}) + \mathbf{t}(-\mathbf{n})] dA* . \tag{3.12}$$

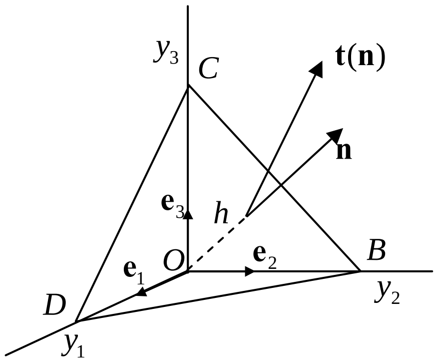
Substitution of (3.8) in (3.12) yields

$$\int [\mathbf{t}(\mathbf{n}) + \mathbf{t}(-\mathbf{n})] dA* = \mathbf{0} . \tag{3.13}$$

This equality is correct for any surface; consequently, we can localize it and get the third Newton law

$$-\mathbf{t}(\mathbf{n}) = \mathbf{t}(-\mathbf{n}) . \tag{3.14}$$

The second corollary of the Cauchy assumption is the appearance of the stress tensor.



We define a tetrahedron of height h in direction $\mathbf{n}$ at point $\mathbf{y}$. The direction cosines n_i allow us to calculate the following areas of the tetrahedron

$$CDB \equiv A, \quad COB = An_1, \quad COD = An_2, \quad DOB = An_3 . \tag{3.15}$$

Now, we apply the linear momentum balance to the tetrahedron:

$$\int \mathbf{f}\, dV = \int_{CDB} \mathbf{t}(\mathbf{n})\, dA + \int_{COB} \mathbf{t}(-\mathbf{e}_1) dA + \int_{COD} \mathbf{t}(-\mathbf{e}_2) dA + \int_{DOB} \mathbf{t}(-\mathbf{e}_3) dA . \tag{3.16}$$

According to the mean value theorem and with account of (3.15) we have

$$\bar{\mathbf{f}}\frac{hA}{6} = \bar{\mathbf{t}}(\mathbf{n})A + \bar{\mathbf{t}}(-\mathbf{e}_1)An_1 + \bar{\mathbf{t}}(-\mathbf{e}_2)An_2 + \bar{\mathbf{t}}(-\mathbf{e}_3)An_3 , \tag{3.17}$$

where the barred quantities are calculated inside the proper volume or area.

Simplifying (3.17) and setting $h \to 0$ we obtain

$$\mathbf{0} = \mathbf{t}(\mathbf{n}) + \mathbf{t}(-\mathbf{e}_1)n_1 + \mathbf{t}(-\mathbf{e}_2)n_2 + \mathbf{t}(-\mathbf{e}_3)n_3 , \tag{3.18}$$

where

$$n_i = \mathbf{e}_i \cdot \mathbf{n} . \tag{3.19}$$

Substituting (3.19) in (3.18) and accounting for (3.14) we get

$$\begin{aligned} \mathbf{t}(\mathbf{n}) &= \mathbf{t}(\mathbf{e}_1)(\mathbf{e}_1 \cdot \mathbf{n}) + \mathbf{t}(\mathbf{e}_2)(\mathbf{e}_2 \cdot \mathbf{n}) + \mathbf{t}(\mathbf{e}_3)(\mathbf{e}_3 \cdot \mathbf{n}) \\ &= \underbrace{(\mathbf{t}(\mathbf{e}_1) \otimes \mathbf{e}_1 + \mathbf{t}(\mathbf{e}_2) \otimes \mathbf{e}_2 + \mathbf{t}(\mathbf{e}_3) \otimes \mathbf{e}_3)}_{\boldsymbol{\sigma}}\mathbf{n} \quad , \\ &= \boldsymbol{\sigma}\mathbf{n} \end{aligned} \tag{3.20}$$

where we introduced the Cauchy stress tensor

$$\boldsymbol{\sigma} = \mathbf{t}(\mathbf{e}_1) \otimes \mathbf{e}_1 + \mathbf{t}(\mathbf{e}_2) \otimes \mathbf{e}_2 + \mathbf{t}(\mathbf{e}_3) \otimes \mathbf{e}_3 . \tag{3.21}$$

To find its components we have to pre-multiply it with the base dyads

$$\sigma_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j : \boldsymbol{\sigma} .$$

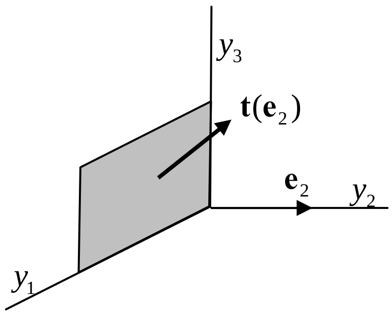
For example, we have

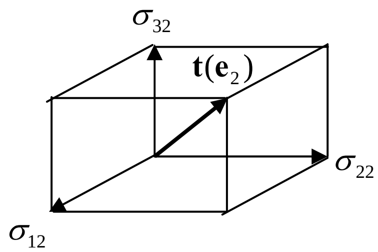
$$\sigma_{22} = \mathbf{e}_2 \otimes \mathbf{e}_2 : \boldsymbol{\sigma} = \mathbf{e}_2 \cdot \mathbf{t}(\mathbf{e}_2) ,$$

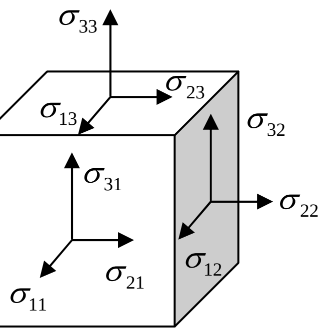
$$\sigma_{12} = \mathbf{e}_1 \otimes \mathbf{e}_2 : \boldsymbol{\sigma} = \mathbf{e}_1 \cdot \mathbf{t}(\mathbf{e}_2) ,$$

$$\sigma_{32} = \mathbf{e}_3 \otimes \mathbf{e}_2 : \boldsymbol{\sigma} = \mathbf{e}_3 \cdot \mathbf{t}(\mathbf{e}_2) ,$$

which means that the components of the Cauchy stress tensor are projections of the stress vector onto the axes of Cartesian coordinates.







We return to the linear momentum balance (3.8) which can be rewritten using the stress tensor

$$\int \mathbf{f}\, dV = \oint \boldsymbol{\sigma}\mathbf{n}\, dA. \tag{3.22}$$

Now the divergence theorem allows us to transform the surface integral into the volume integral

$$\oint \boldsymbol{\sigma}\mathbf{n} dA = \int \operatorname{div}\boldsymbol{\sigma} dV. \tag{3.23}$$

Then the linear momentum balance takes the form

$$\int (\mathbf{f} - \operatorname{div}\boldsymbol{\sigma}) dV = \mathbf{0}. \tag{3.24}$$

Localizing it and substituting from (3.9) we have finally

$$\begin{gathered} \frac{d(\mathbf{v}\rho)}{dt} + \mathbf{v}\rho \operatorname{div}\mathbf{v} = \operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} \\ \frac{d(v_i\rho)}{dt} + v_i\rho\frac{\partial v_j}{\partial y_j} = \frac{\partial \sigma_{ij}}{\partial y_j} + \rho b_i \end{gathered}. \tag{3.25}$$

By way of example let us find traction $\mathbf{t}(\mathbf{n})$, normal stress vector $\mathbf{t}_n(\mathbf{n})$, and tangent stress vector $\mathbf{t}_t(\mathbf{n})$ for the given stress tensor

$$\boldsymbol{\sigma} = 7\mathbf{e}_1 \otimes \mathbf{e}_1 - 2(\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1) + 5\mathbf{e}_2 \otimes \mathbf{e}_2 + 4\mathbf{e}_3 \otimes \mathbf{e}_3$$

and area with normal

$$\mathbf{n} = \frac{2}{3}\mathbf{e}_1 - \frac{2}{3}\mathbf{e}_2 + \frac{1}{3}\mathbf{e}_3 .$$

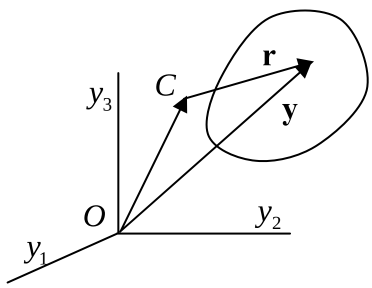
By direct calculation we have

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = 7\mathbf{e}_1(\mathbf{e}_1 \cdot \mathbf{n}) - 2\mathbf{e}_1(\mathbf{e}_3 \cdot \mathbf{n}) + 5\mathbf{e}_2(\mathbf{e}_2 \cdot \mathbf{n}) - 2\mathbf{e}_3(\mathbf{e}_1 \cdot \mathbf{n}) + 4\mathbf{e}_3(\mathbf{e}_3 \cdot \mathbf{n}) = 4\mathbf{e}_1 - \frac{10}{3}\mathbf{e}_2 ,$$

$$\mathbf{t}_n = (\mathbf{t} \cdot \mathbf{n})\mathbf{n} = (4\mathbf{e}_1 \cdot \mathbf{n} - \frac{10}{3}\mathbf{e}_2 \cdot \mathbf{n})\mathbf{n} = \frac{44}{9}\mathbf{n} = \frac{88}{27}\mathbf{e}_1 - \frac{88}{27}\mathbf{e}_2 + \frac{44}{27}\mathbf{e}_3 ,$$

$$\mathbf{t}_t = \mathbf{t} - \mathbf{t}_n = \frac{1}{27}(20\mathbf{e}_1 - 2\mathbf{e}_2 + 44\mathbf{e}_3) .$$

3.4 Balance of angular momentum



In the case of a mass-point we have the angular momentum balance

$$\mathbf{r} \times \frac{d(m\mathbf{v})}{dt} = \mathbf{r} \times \mathbf{p} , \tag{3.26}$$

or

$$\frac{d(m\,\mathbf{r} \times \mathbf{v})}{dt} = \mathbf{r} \times \mathbf{p} . \tag{3.27}$$

The latter is true because: $\frac{d}{dt}(m\mathbf{r} \times \mathbf{v}) = m\frac{d\mathbf{r}}{dt} \times \mathbf{v} + m\,\mathbf{r} \times \frac{d\mathbf{v}}{dt} = m\,\mathbf{v} \times \mathbf{v} + m\,\mathbf{r} \times \frac{d\mathbf{v}}{dt} = m\,\mathbf{r} \times \frac{d\mathbf{v}}{dt}$.

In the case of continuum we have instead of (3.27)

$$\frac{d}{dt}\int \rho\,\mathbf{r} \times \mathbf{v} dV = \int \mathbf{r} \times \mathbf{b}\rho\, dV + \oint \mathbf{r} \times \mathbf{t}\, dA . \tag{3.28}$$

It is convenient to manipulate this equation in Cartesian coordinates. In this case we can rewrite the angular momentum balance as follows

$$\varepsilon_{ijk}\left(\frac{d}{dt}\int \rho\, r_j v_k dV - \int r_j b_k \rho\, dV - \oint r_j t_k\, dA\right) = 0 . \tag{3.29}$$

The first and the third terms in the equation above can be calculated by using the material time derivative of the volume integral and the divergence theorem accordingly

$$\begin{aligned}\frac{d}{dt}\int \rho r_j v_k dV &= \int\left(\frac{d(\rho r_j v_k)}{dt} + \rho r_j v_k \frac{\partial v_m}{\partial y_m}\right)dV \\ &= \int\left(r_j \frac{d(\rho v_k)}{dt} + \rho v_k v_j + \rho r_j v_k \frac{\partial v_m}{\partial y_m}\right)dV\end{aligned}, \tag{3.30}$$

$$\begin{aligned}\oint r_j t_k \, dA &= \oint r_j \sigma_{kl} n_l \, dA \\ &= \int \frac{\partial (r_j \sigma_{kl})}{\partial y_l} dV \\ &= \int\left(\delta_{jl}\sigma_{kl} + r_j \frac{\partial \sigma_{kl}}{\partial y_l}\right)dV \\ &= \int\left(\sigma_{kj} + r_j \frac{\partial \sigma_{kl}}{\partial y_l}\right)dV\end{aligned}, \tag{3.31}$$

where we used relation $r_i = y_i - (\overrightarrow{OC})_i$ with $\overrightarrow{OC}$ fixed.

Substituting (3.30)-(3.31) in (3.29) we get

$$\varepsilon_{ijk}\int [r_j \underbrace{\left(\frac{d(\rho v_k)}{dt} + \rho v_k \frac{\partial v_m}{\partial y_m} - b_k \rho - \frac{\partial \sigma_{kl}}{\partial y_l}\right)}_{0} + \rho v_k v_j - \sigma_{kj}]dV = 0, \tag{3.32}$$

where the term in the parentheses is the law of the linear momentum balance and it is equal to zero.

Thus we have

$$\int \varepsilon_{ijk}(\rho v_k v_j - \sigma_{kj})dV = -\int \varepsilon_{ijk}\sigma_{kj} dV = 0. \tag{3.33}$$

The latter equation can be obeyed for the symmetric Cauchy tensor only

$$\sigma_{kj} = \sigma_{jk}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{3.34}$$

3.5 Master balance principle

All balance laws enjoy the same structure

$$\frac{d}{dt}\int \boldsymbol{\alpha}\, dV = \int \boldsymbol{\xi}\, dV + \oint \boldsymbol{\varphi}\mathbf{n}\, dA, \tag{3.35}$$

where $\boldsymbol{\xi}$ is the volumetric supply of $\boldsymbol{\alpha}$ and $\boldsymbol{\varphi}$ is the surface flux of $\boldsymbol{\alpha}$.

Differentiating the integral and using the divergence theorem we localize the balance law

$$\frac{d\boldsymbol{\alpha}}{dt} + \boldsymbol{\alpha}\,\mathrm{div}\mathbf{v} = \mathrm{div}\boldsymbol{\varphi} + \boldsymbol{\xi}\,. \tag{3.36}$$

The considered balance laws are summarized in the table:

	$\boldsymbol{\alpha}$	$\boldsymbol{\xi}$	$\boldsymbol{\varphi}$
Mass	ρ	0	0
Linear Momentum	$\mathbf{v}\rho$	$\mathbf{b}\rho$	$\boldsymbol{\sigma}$
Angular Momentum	$\mathbf{r}\times\mathbf{v}\rho$	$\mathbf{r}\times\mathbf{b}\rho$	$\mathbf{r}\times\boldsymbol{\sigma}$

3.6 Lagrangean description

The description of balance laws was spatial or Eulerian because $\mathbf{y}$ was chosen as an independent variable. In the case of solids (contrary to fluids) it is usually more convenient to consider $\mathbf{x}$ as an independent variable, i.e. it is better to use the referential or Lagrangean description. The transition from one description to another is simple when the formulas relating volumes and surfaces before and after deformation are used (see (2.7) and (2.13))

$$dV = dV_0 \det \mathbf{F} = J dV_0\,, \tag{3.37}$$

$$\mathbf{n} dA = J\mathbf{F}^{-T}\mathbf{n}_0 dA_0\,. \tag{3.38}$$

Substituting these equations in the master balance law we get

$$\frac{d}{dt}\int \boldsymbol{\alpha}_0\, dV_0 = \int \boldsymbol{\xi}_0\, dV_0 + \oint \boldsymbol{\varphi}_0\, \mathbf{n}_0 dA_0\,, \tag{3.39}$$

where we defined the Lagrangean quantities

$$\boldsymbol{\alpha}_0(\mathbf{x},t) = J\,\boldsymbol{\alpha}(\mathbf{y}(\mathbf{x},t),t)\,, \tag{3.40}$$

$$\boldsymbol{\xi}_0(\mathbf{x},t) = J\,\boldsymbol{\xi}(\mathbf{y}(\mathbf{x},t),t)\,, \tag{3.41}$$

$$\boldsymbol{\varphi}_0(\mathbf{x},t) = J\,\boldsymbol{\varphi}(\mathbf{y}(\mathbf{x},t),t)\mathbf{F}^{-T}\,. \tag{3.42}$$

We differentiate (3.39) with respect to time through the integral directly because the volume does not change and we get the localized balance law in the Lagrangean form

$$\frac{\partial \boldsymbol{\alpha}_0}{\partial t} = \mathrm{Div}\boldsymbol{\varphi}_0 + \boldsymbol{\xi}_0\,. \tag{3.43}$$

Here 'Div' operator is with respect to the referential coordinates

$$\mathrm{Div}(...) = \frac{\partial(...)}{\partial x_i}\mathbf{e}_i\,.$$

Particularly, the Lagrangean form of the previous table is

	$\boldsymbol{\alpha}_0$	$\boldsymbol{\xi}_0$	$\boldsymbol{\varphi}_0$
Mass	ρ_0	0	0
Linear Momentum	$\rho_0\mathbf{v}$	$\rho_0\mathbf{b}$	$\mathbf{T}$
Angular Momentum	$\mathbf{r}\times\rho_0\mathbf{v}$	$\mathbf{r}\times\rho_0\mathbf{b}$	$\mathbf{r}\times\mathbf{T}$

where

$$\mathbf{T}(\mathbf{x},t) = J\boldsymbol{\sigma}(\mathbf{y}(\mathbf{x},t),t)\mathbf{F}^{-T} \tag{3.44}$$

is the 1st Piola-Kirchhoff stress tensor (1PK).

The laws of mass, linear and angular momentum balance take the following forms accordingly

$$\frac{\partial\rho_0}{\partial t} = 0\,, \tag{3.45}$$

$$\begin{gathered}\frac{\partial(\rho_0\mathbf{v})}{\partial t} = \mathrm{Div}\mathbf{T} + \rho_0\mathbf{b} \\ \frac{\partial(\rho_0 v_i)}{\partial t} = \frac{\partial T_{ij}}{\partial x_j} + \rho_0 b_i\end{gathered}\,, \tag{3.46}$$

$$\mathbf{T}\mathbf{F}^T = \mathbf{F}\mathbf{T}^T\,, \tag{3.47}$$

Since the 1st Piola-Kirchhoff stress tensor is not symmetric it is convenient to introduce the 2nd Piola-Kirchhoff stress tensor (2PK)

$$\mathbf{S} = \mathbf{F}^{-1}\mathbf{T} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}\,. \tag{3.48}$$

4 Isotropic elasticity

4.1 Hyperelasticity

The *rheological model* for elastic material is a spring. For the classical *linear* spring, stress σ is equal to strain ε scaled by Young modulus, E,

$$\sigma = E\varepsilon \,. \tag{4.1}$$

This equation is called Hooke's law in honor of Robert Hooke.

Evidently, this constitutive law is a linearization of a more general function describing a nonlinear spring

$$\sigma = \sigma(\varepsilon) \,. \tag{4.2}$$

Although this function can be fitted in experiments only it is possible to draw some conclusions about it considering the work of stress on strain

$$w = \int \sigma(\varepsilon) d\varepsilon \,. \tag{4.3}$$

In the case of an ideal elastic spring, this work does not depend on the loading history and it only depends on the initial and final states of the spring – the integration limits in (4.3). If the integral is path-independent then the integrand should be a full differential

$$dw = \sigma(\varepsilon) d\varepsilon \,. \tag{4.4}$$

It follows from (4.4) that stress in an ideal elastic spring should be a derivative of the strain energy with respect to strain

$$\sigma = \frac{dw}{d\varepsilon} \,, \tag{4.5}$$

where in the case of Hookean elasticity we have: $w = E\varepsilon^2 / 2$.

The extension of the simplistic formula (4.5) to 3D is not trivial. Indeed, variety of stresses and strains can be considered and it is not clear which stress works on which strain. To clarify that we consider the work of *external* forces on displacement increments, $d\mathbf{u} = d\mathbf{y}$, over the whole 3D body

$$d\Pi = \oint d\mathbf{y} \cdot \bar{\mathbf{t}}_0 dA_0 + \int d\mathbf{y} \cdot \rho_0 \mathbf{b} dV_0 \,, \tag{4.6}$$

where $\bar{\mathbf{t}}_0$ and $\rho_0\mathbf{b}$ designate prescribed tractions per the reference area and body forces per the reference volume, including the inertia forces.

By using the equilibrium equation (3.46) we can rewrite (4.6) in the form

$$d\Pi = \oint \bar{\mathbf{t}}_0 \cdot d\mathbf{y}\, dA_0 - \int (\mathrm{Div}\mathbf{T}) \cdot d\mathbf{y}\, dV_0 \,, \tag{4.7}$$

where $\mathbf{T}$ is the 1st Piola-Kirchhoff stress.

We transform (4.7) as follows

$$\begin{aligned} d\Pi &= \oint \bar{t}_{0i} dy_i dA_0 - \int \frac{\partial T_{ij}}{\partial x_j} dy_i\, dV_0 \\ &= \oint \bar{t}_{0i} dy_i dA_0 - \int \frac{\partial (T_{ij} dy_i)}{\partial x_j} dV_0 + \int T_{ij} \frac{\partial (dy_i)}{\partial x_j} dV_0 \\ &= \oint \underbrace{(\bar{t}_{0i} - T_{ij} n_{0j}) dy_i}_{0\,\text{(boundary conditions)}} dA_0 + \int T_{ij} d\, \frac{\partial y_i}{\partial x_j} dV_0 \qquad , \\ &= \int T_{ij} dF_{ij}\, dV_0 \\ &= \int \mathbf{T} : d\mathbf{F} dV_0 \end{aligned} \tag{4.8}$$

where the boundary conditions on tractions have been used

$$\mathbf{T}\mathbf{n}_0 = \bar{\mathbf{t}}_0 \,. \tag{4.9}$$

Transformation (4.8) means that the incremental work of the external forces is equal to the incremental work of the *internal* forces. The work of the internal forces per unit volume can be designated as follows

$$dW = \mathbf{T} : d\mathbf{F} \,. \tag{4.10}$$

Analogously to 1D case this work is path independent only in the case where

$$\mathbf{T} = \frac{\partial W(\mathbf{F})}{\partial \mathbf{F}}, \quad T_{ij} = \frac{\partial W(\mathbf{F})}{\partial F_{ij}} \,. \tag{4.11}$$

Here W is called the *strain energy* and material obeying (4.11) is called *hyperelastic*.

Evidently, the 1st Piola-Kirchhoff stress makes a *work-conjugate* couple with the deformation gradient. It is possible, however, to assume that the strain energy depends on the Green strain, $\boldsymbol{\varepsilon} = (\mathbf{F}^T\mathbf{F} - \mathbf{1})/2$, rather than on the deformation gradient. In this case we have (prove it!)

$$T_{ij} = \frac{\partial W(\boldsymbol{\varepsilon}(\mathbf{F}))}{\partial \varepsilon_{mn}} \frac{\partial \varepsilon_{mn}}{\partial F_{ij}} = \frac{\partial W}{\partial \varepsilon_{mn}} \frac{1}{2} \frac{\partial (F_{km}F_{kn} - \delta_{mn})}{\partial F_{ij}} = \frac{\partial W}{\partial \varepsilon_{mj}} F_{im} \,,$$

or

$$\mathbf{T} = \mathbf{F}\frac{\partial W}{\partial \boldsymbol{\varepsilon}}. \tag{4.12}$$

On the other hand we have by definition, (3.48),

$$\mathbf{T} = \mathbf{F}\mathbf{S}, \tag{4.13}$$

where $\mathbf{S}$ is the 2nd Piola-Kirchhoff stress tensor and, consequently,

$$\mathbf{S} = \frac{\partial W(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}, \quad S_{ij} = \frac{\partial W(\boldsymbol{\varepsilon})}{\partial \varepsilon_{ij}}, \tag{4.14}$$

or

$$\mathbf{S} = 2\frac{\partial W(\mathbf{C})}{\partial \mathbf{C}}, \quad S_{ij} = 2\frac{\partial W(\mathbf{C})}{\partial C_{ij}}, \tag{4.15}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F} = 2\boldsymbol{\varepsilon} + \mathbf{1}$ is the right Cauchy-Green tensor.

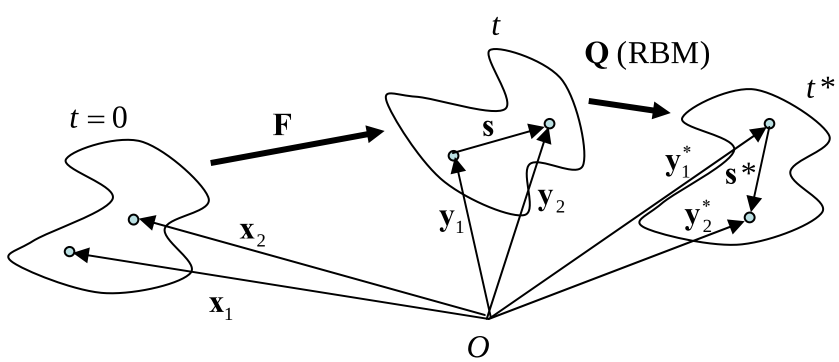
It is possible to show that the considered stress-strain pairs are work-conjugate by the direct computation (prove it!)

$$\mathbf{T} : d\mathbf{F} = \mathbf{S} : d\boldsymbol{\varepsilon}. \tag{4.16}$$

The 'true' Cauchy stress is obtained from (4.14)-(4.15) with the help of (3.48) with $J = \det \mathbf{F}$

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \boldsymbol{\varepsilon}}\mathbf{F}^T = 2J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}\mathbf{F}^T. \tag{4.17}$$

We showed that the strain energy could be defined as a function of various strains. Is there any preference in the choice of strains? The answer is yes. The strains which are insensitive to the *Rigid Body Motion* (RBM) are preferable.



Indeed, let us consider RBM superposed on the current configuration of material

$$\mathbf{y}^* = \mathbf{Q}(t)\mathbf{y} + \mathbf{h}(t), \tag{4.18}$$

where $\mathbf{Q}^T = \mathbf{Q}^{-1} (\det \mathbf{Q} = 1)$ is the *proper orthogonal* tensor of rotation and $\mathbf{h}$ is a vector of translation.

This motion preserves the length and the angle. Indeed, we have

$$\mathbf{s}^* = \mathbf{y}_2{}^* - \mathbf{y}_1{}^* = \mathbf{Q}(\mathbf{y}_2 - \mathbf{y}_1) = \mathbf{Qs}\,, \tag{4.19}$$

$$|\mathbf{s}^*| = \sqrt{\mathbf{s}^* \cdot \mathbf{s}^*} = \sqrt{\mathbf{s} \cdot \mathbf{Q}^T\mathbf{Qs}} = \sqrt{\mathbf{s} \cdot \mathbf{1s}} = \sqrt{\mathbf{s} \cdot \mathbf{s}} = |\mathbf{s}|\,, \tag{4.20}$$

$$\cos\alpha^* = \frac{\mathbf{s}^* \cdot \mathbf{p}^*}{|\mathbf{s}^*||\mathbf{p}^*|} = \frac{\mathbf{s} \cdot \mathbf{Q}^T\mathbf{Qp}}{|\mathbf{s}||\mathbf{p}|} = \frac{\mathbf{s} \cdot \mathbf{p}}{|\mathbf{s}||\mathbf{p}|} = \cos\alpha\,. \tag{4.21}$$

Thus, a material fiber deforms as follows

$$d\mathbf{y}^* = \mathbf{Q}d\mathbf{y} = \underbrace{\mathbf{QF}}_{\mathbf{F}^*}d\mathbf{x} = \mathbf{F}^*d\mathbf{x}\,. \tag{4.22}$$

It is natural to require that the magnitude of the strain energy is not affected by RBM because there is no straining. The latter means that the function of the strain energy should obey the following condition

$$W(\mathbf{F}) = W(\mathbf{QF})\,. \tag{4.23}$$

The right Cauchy-Green and Green strain tensor obey this condition automatically because they are insensitive to RBM

$$\mathbf{C}^* = \mathbf{F}^{*T}\,\mathbf{F}^* = (\mathbf{QF})^T(\mathbf{QF}) = \mathbf{F}^T\underbrace{\mathbf{Q}^T\mathbf{Q}}_{\mathbf{1}}\mathbf{F} = \mathbf{F}^T\mathbf{F} = \mathbf{C}\,, \tag{4.24}$$

$$\boldsymbol{\varepsilon}^* = (\mathbf{C}^* - \mathbf{1})/2 = (\mathbf{C} - \mathbf{1})/2 = \boldsymbol{\varepsilon}\,. \tag{4.25}$$

4.2 Rivlin's representation for isotropic material

Ronald Rivlin found (1948) the following representation for the strain energy of isotropic materials, which is given without proof,

$$W(\mathbf{C}) = W(I_1, I_2, I_3)\,, \tag{4.26}$$

$$I_1 = \mathrm{tr}\mathbf{C}, \quad I_2 = \{(\mathrm{tr}\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)\}/2, \quad I_3 = \det\mathbf{C}\,, \tag{4.27}$$

that is the strain energy depends on the invariants of the right Cauchy-Green tensor.

Based on this representation we can calculate the stress as follows

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}} = 2\left(\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \mathbf{C}}\right), \tag{4.28}$$

where (see (1.47), (1.48), (1.51))

$$\frac{\partial I_1}{\partial \mathbf{C}} = \mathbf{1}, \qquad \frac{\partial I_2}{\partial \mathbf{C}} = I_1\mathbf{1} - \mathbf{C}, \qquad \frac{\partial I_3}{\partial \mathbf{C}} = I_3\mathbf{C}^{-1}\,. \tag{4.29}$$

Inserting (4.29) in (4.28) we have

$$\mathbf{S} = 2\left\{\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)\mathbf{1} - \frac{\partial W}{\partial I_2}\mathbf{C} + I_3 \frac{\partial W}{\partial I_3}\mathbf{C}^{-1}\right\}. \tag{4.30}$$

Transition to the Cauchy stress gives us another form of the constitutive law

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^T = 2J^{-1}\left\{\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)\mathbf{B} - \frac{\partial W}{\partial I_2}\mathbf{B}^2 + I_3 \frac{\partial W}{\partial I_3}\mathbf{1}\right\}, \tag{4.31}$$

where

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \tag{4.32}$$

is the left Cauchy-Green tensor.

We remind that invariants of **B** coincide with the invariants of **C**: $I_a(\mathbf{C}) = I_a(\mathbf{B})$.

4.3 Representation in principal stretches

Sometimes, it is more convenient to formulate the constitutive equations in terms of principal stretches, λ_i, rather than to use invariants. To make the transition to the principal stretches we need the spectral representation of the right Cauchy-Green tensor

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \lambda_1^2 \mathbf{m}^{(1)} \otimes \mathbf{m}^{(1)} + \lambda_2^2 \mathbf{m}^{(2)} \otimes \mathbf{m}^{(2)} + \lambda_3^2 \mathbf{m}^{(3)} \otimes \mathbf{m}^{(3)}, \tag{4.33}$$

where λ_a^2 and $\mathbf{m}^{(a)}$ are eigenvalues and eigenvectors of $\mathbf{C}$ accordingly.

Since

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2, \tag{4.34}$$

the strain energy can be rewritten as a function of principal stretches $\lambda_i = \sqrt{\lambda_i^2}$

$$W(\mathbf{C}) = W(\lambda_1, \lambda_2, \lambda_3), \tag{4.35}$$

and we can calculate the energy increment as follows

$$dW(\mathbf{C}) = dW(\lambda_1, \lambda_2, \lambda_3) = \frac{\partial W}{\partial \lambda_1} d\lambda_1 + \frac{\partial W}{\partial \lambda_2} d\lambda_2 + \frac{\partial W}{\partial \lambda_3} d\lambda_3. \tag{4.36}$$

In order to find $d\lambda_1$ we, firstly, get the increment of (4.33)

$$d\mathbf{C} = \sum_{a=1}^{3}\{2\lambda_a d\lambda_a \mathbf{m}^{(a)} \otimes \mathbf{m}^{(a)} + \lambda_a^2 d\mathbf{m}^{(a)} \otimes \mathbf{m}^{(a)} + \lambda_a^2 \mathbf{m}^{(a)} \otimes d\mathbf{m}^{(a)}\}. \tag{4.37}$$

Secondly, we pre-multiply it by $\mathbf{m}^{(1)} \otimes \mathbf{m}^{(1)}$ as follows

$$(\mathbf{m}^{(1)} \otimes \mathbf{m}^{(1)}) : d\mathbf{C} = 2\lambda_1 d\lambda_1, \tag{4.38}$$

where we accounted for $\mathbf{m}^{(1)} \cdot \mathbf{m}^{(a)} = \delta_{1a}$ and $d(\mathbf{m}^{(1)} \cdot \mathbf{m}^{(1)}) = 0 \Rightarrow d\mathbf{m}^{(1)} \cdot \mathbf{m}^{(1)} = 0$.

Thus, we have from (4.38)

$$d\lambda_1 = \frac{1}{2\lambda_1}(\mathbf{m}^{(1)} \otimes \mathbf{m}^{(1)}) : d\mathbf{C} . \tag{4.39}$$

Repeating this argument for $d\lambda_2$ and $d\lambda_3$ we get

$$dW(\mathbf{C}) = \frac{\partial W}{\partial \mathbf{C}} : d\mathbf{C} , \tag{4.40}$$

where

$$\frac{\partial W}{\partial \mathbf{C}} = \sum_{a=1}^{3} \frac{1}{2\lambda_a} \frac{\partial W}{\partial \lambda_a} \mathbf{m}^{(a)} \otimes \mathbf{m}^{(a)} . \tag{4.41}$$

Using this derivative we can write the 2nd Piola-Kirchhoff tensor in the form

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}} = \sum_{a=1}^{3} \frac{1}{\lambda_a} \frac{\partial W}{\partial \lambda_a} \mathbf{m}^{(a)} \otimes \mathbf{m}^{(a)} . \tag{4.42}$$

It is remarkable that 2PK stress is coaxial with the right Cauchy-Green tensor because their principal directions coincide. The latter allows us to directly compute the principal 2PK stresses

$$S_a = \frac{1}{\lambda_a} \frac{\partial W}{\partial \lambda_a} \quad (\text{no sum}) . \tag{4.43}$$

By using the spectral decomposition of the deformation gradient, (2.35), we can compute the Cauchy stresses

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\mathbf{S}\mathbf{F}^T = \frac{1}{\lambda_1\lambda_2\lambda_3} \sum_{a=1}^{3} \lambda_a \frac{\partial W}{\partial \lambda_a} \mathbf{n}^{(a)} \otimes \mathbf{n}^{(a)} , \tag{4.44}$$

which is coaxial with the left Cauchy-Green tensor because their principal directions coincide. The latter allows us to directly compute the principal Cauchy stresses

$$\sigma_a = \frac{\lambda_a}{\lambda_1\lambda_2\lambda_3} \frac{\partial W}{\partial \lambda_a} \quad (\text{no sum}) . \tag{4.45}$$

4.4 Incompressibility

Many soft materials resist volume changes much stronger than the shape changes. This experimental observation makes it reasonable to assume the material incompressibility

$$\frac{dV}{dV_0} = J = \det \mathbf{F} = 1 = \det \mathbf{F} \det \mathbf{F}^T = \det \mathbf{B} = \det \mathbf{C} = I_3 . \tag{4.46}$$

This can be considered as a restriction imposed on deformation

$$\gamma(\mathbf{C}) = 1 - I_3(\mathbf{C}) = 0 . \tag{4.47}$$

The incremental form of the restriction can be written as follows

$$d\gamma(\mathbf{C}) = \frac{\partial \gamma}{\partial \mathbf{C}} : d\mathbf{C} = 0 . \tag{4.48}$$

Here $\partial\gamma / \partial\mathbf{C}$ can be interpreted as a stress producing zero work on the strain increment – the *workless stress*. Such stress is indefinite since it can always be scaled by an indefinite parameter, p .

Adding the workless stress to the stress derived from the strain energy we have

$$\boldsymbol{\sigma} = 2J^{-1}\mathbf{F}\left(\frac{\partial W}{\partial \mathbf{C}} + p\frac{\partial \gamma}{\partial \mathbf{C}}\right)\mathbf{F}^T , \tag{4.49}$$

or, substituting from (4.47) into (4.49),

$$\boldsymbol{\sigma} = 2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}\mathbf{F}^T - p\mathbf{1} . \tag{4.50}$$

The unknown multiplier, p, should be obtained from the solution of equilibrium equations.

In the case of *isotropic* material we have

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2(W_1 + I_1W_2)\,\mathbf{B} - 2W_2\mathbf{B}^2 , \tag{4.51}$$

where

$$W_a \equiv \frac{\partial W}{\partial I_a} . \tag{4.52}$$

In terms of the principal stresses and stretches we have instead of (4.45)

$$\sigma_a = \lambda_a \frac{\partial W}{\partial \lambda_a} - p \quad (\text{no sum}) . \tag{4.53}$$

4.5 Examples of strain energy

In this section we consider some popular strain energy functions, $W(\mathbf{C})$, which in the absence of *residual* stresses should meet the following conditions

$$W(\mathbf{1}) = 0, \qquad \frac{\partial W}{\partial \mathbf{C}}(\mathbf{1}) = \mathbf{0} , \tag{4.54}$$

or, in the case where the strain energy is a function of principal stretches, $W(\lambda_1, \lambda_2, \lambda_3)$,

$$W(1,1,1) = 0, \qquad \frac{\partial W}{\partial \lambda_a}(1,1,1) = 0 , \tag{4.55}$$

We start with the **Kirchhoff-Saint Venant** material

$$W(\boldsymbol{\varepsilon}) = \frac{\lambda}{2}(\mathrm{tr}\boldsymbol{\varepsilon})^2 + \mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} , \tag{4.56}$$

where λ and μ are the Lame constants and the Green strain is $\boldsymbol{\varepsilon} = (\mathbf{C} - \mathbf{1})/2$. Differentiating the strain energy density with respect to the Green strain we obtain 2PK stresses

$$
\begin{aligned}
S_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} &= \frac{\lambda}{2}\frac{\partial(\varepsilon_{kk}\varepsilon_{rr})}{\partial \varepsilon_{ij}} + \mu\frac{\partial(\varepsilon_{mn}\varepsilon_{mn})}{\partial \varepsilon_{ij}} \\
&= \lambda\frac{\partial \varepsilon_{kk}}{\partial \varepsilon_{ij}}\varepsilon_{rr} + 2\mu\frac{\partial \varepsilon_{mn}}{\partial \varepsilon_{ij}}\varepsilon_{mn} \quad , \\
&= \lambda\,\delta_{ki}\delta_{kj}\varepsilon_{rr} + 2\mu\delta_{mi}\delta_{nj}\varepsilon_{mn} \\
&= \lambda\delta_{ij}\varepsilon_{rr} + 2\mu\varepsilon_{ij}
\end{aligned}
$$

$$
\mathbf{S} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} = \lambda\,(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{1} + 2\mu\boldsymbol{\varepsilon}. \tag{4.57}
$$

Alternatively, we can rewrite (4.56) and (4.57) in principal stretches

$$
W(\lambda_1, \lambda_2, \lambda_3) = \frac{\lambda}{8}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)^2 + \frac{\mu}{4}\{(\lambda_1^2 - 1)^2 + (\lambda_2^2 - 1)^2 + (\lambda_3^2 - 1)^2\}, \tag{4.58}
$$

$$
S_a = \lambda(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)/2 + 2\mu(\lambda_a^2 - 1)/2. \tag{4.59}
$$

This classical material model is generally not used for soft materials. In the case of small strains, (4.57) is the generalized Hooke law. The use of nonlinear strains, however, is crucial in order to suppress rigid body motions in finite element computations.

Next strain energy function defines the **Neo-Hookean** incompressible material

$$
W = c(I_1 - 3) = c(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \qquad J = \lambda_1\lambda_2\lambda_3 = 1, \tag{4.60}
$$

where c is a material constant.

The Neo-Hookean model is the simplest one for modeling soft materials. It is often used as a starting point for the experimental calibration. A popular generalization of (4.60) is the **Yeoh** material defined as a polynomial of the first principal invariant, $I_1(\mathbf{C})$. For example Hamdi et al (Polymer Testing 25 (2006) 994-1005) calibrated the following Yeoh model for natural rubber

$$
W = c_1(I_1 - 3) + c_2(I_1 - 3)^2 + c_3(I_1 - 3)^3, \quad I_3 = 1, \tag{4.61}
$$

where

$$
c_1 = 0.298\,\mathrm{MPa}, \quad c_2 = 0.014\,\mathrm{MPa}, \quad c_3 = 0.00016\,\mathrm{MPa}.
$$

Another generalization of the Neo-Hookean model is the **Mooney-Rivlin** material which defines the dependence of the strain energy on both the first and second principal invariants. An example of the incompressible Mooney-Rivlin material was calibrated by Sasso et al (Polymer Testing 27 (2008) 995-1004)

$$
W = c_1(I_1 - 3) + c_2(I_2 - 3) + c_3(I_1 - 3)^2 + c_4(I_1 - 3)(I_2 - 3) + c_5(I_2 - 3)^2, \quad I_3 = 1, \tag{4.62}
$$

where

$c_1 = 0.59\,\text{MPa},\ c_2 = -0.039\,\text{MPa},\ c_3 = -0.0028\,\text{MPa},\ c_4 = 0.0076\,\text{MPa},\ c_5 = -0.00077\,\text{MPa}$.

Further generalization of the previous models is the **Ogden** material defined as

$$W = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p} (\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3), \quad J = 1, \tag{4.63}$$

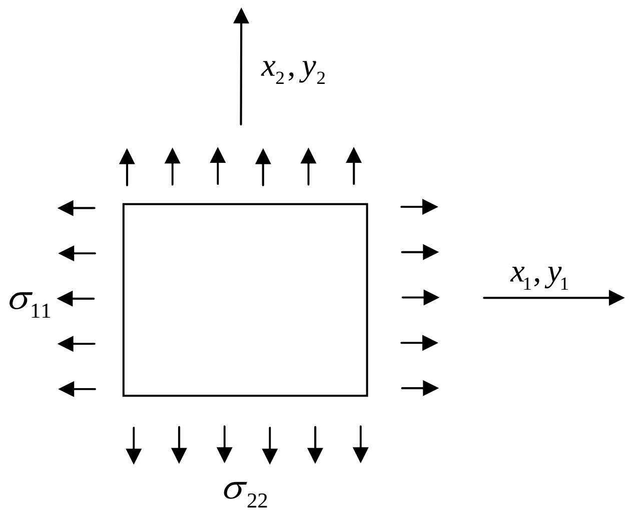
where $\mu_p \alpha_p > 0, \quad p = 1,...N$.

For example, Hamdi et al (Polymer Testing 25 (2006) 994-1005) calibrated the Ogden model for styrene-butadiene rubber where $N = 2$ and

$$\mu_1 = 0.638\,\text{MPa}, \quad \alpha_1 = 3.03, \quad \mu_2 = -0.025\,\text{MPa}, \quad \alpha_2 = -2.35.$$

4.6 Biaxial test

Biaxial tension tests are usually used to calibrate material models. The theoretical background for such tests can be readily developed. Let us consider the homogeneous biaxial deformation of a thin isotropic incompressible sheet



$$y_1 = \lambda_1 x_1, \quad y_2 = \lambda_2 x_2, \quad y_3 = \lambda_3 x_3. \tag{4.64}$$

By the direct computation we get

$$\mathbf{F} = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{4.65}$$

Thus, the coordinate system coincides with the principal directions of stretches and the constitutive law takes form

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2(W_1 + I_1 W_2)\mathbf{B} - 2W_2 \mathbf{B}^2,$$

$$\begin{cases} \sigma_{11} = -p + 2(W_1 + I_1 W_2)\lambda_1^2 - 2W_2\lambda_1^4 \\ \sigma_{22} = -p + 2(W_1 + I_1 W_2)\lambda_2^2 - 2W_2\lambda_2^4 \\ \sigma_{33} = -p + 2(W_1 + I_1 W_2)\lambda_3^2 - 2W_2\lambda_3^4 \end{cases}. \tag{4.66}$$

The stresses are homogeneous and the equilibrium equations are satisfied automatically. From the traction-free boundary conditions on the sheet faces we have

$$\sigma_{33} \cong 0 \quad \Rightarrow \quad p = 2(W_1 + I_1 W_2)\lambda_3^2 - 2W_2\lambda_3^4. \tag{4.67}$$

Substituting the Lagrange multiplier in the stress tensor we get

$$\begin{cases} \sigma_{11} = 2(W_1 + I_1 W_2)(\lambda_1^2 - \lambda_3^2) - 2W_2(\lambda_1^4 - \lambda_3^4) \\ \sigma_{22} = 2(W_1 + I_1 W_2)(\lambda_2^2 - \lambda_3^2) - 2W_2(\lambda_2^4 - \lambda_3^4) \end{cases}. \tag{4.68}$$

Since

$$I_1 = \mathrm{tr}\mathbf{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{4.69}$$

we can rewrite stresses in the form (prove it!)

$$\begin{cases} \sigma_{11} = 2(\lambda_1^2 - \lambda_3^2)(W_1 + W_2\lambda_2^2) \\ \sigma_{22} = 2(\lambda_2^2 - \lambda_3^2)(W_1 + W_2\lambda_1^2) \end{cases}, \tag{4.70}$$

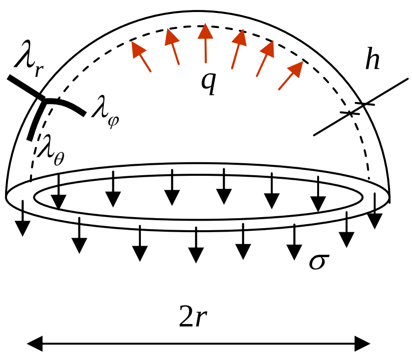
where the incompressibility condition enforces

$$\lambda_3 = \frac{1}{\lambda_1\lambda_2}. \tag{4.71}$$

Equations (4.70) are often used for the experimental calibration of soft materials under varying ratio of the applied stresses.

4.7* Balloon inflation

Balloon inflation is another popular deformation used for calibration of soft materials.



Consider the centrally symmetric inflation of a thin sphere. Its deformation can be presented in terms of principal stretches along the directions of the spherical coordinate systems

$$\begin{aligned} \lambda_\varphi = \lambda_\theta = \frac{2\pi r}{2\pi R} = \frac{r}{R} = \lambda \\ \lambda_r = \frac{h}{H} = \frac{1}{\lambda_1 \lambda_2} = \lambda^{-2} \end{aligned}, \tag{4.72}$$

where r, R and h, H are the current and referential radii and thicknesses of the sphere accordingly and the incompressibility condition is taken into account in the second equation.

The deformation gradient and the left Cauchy-Green tensors take the following forms

$$\mathbf{F} = \lambda^{-2}\mathbf{g}_r \otimes \mathbf{G}_R + \lambda(\mathbf{g}_\theta \otimes \mathbf{G}_\Theta + \mathbf{g}_\varphi \otimes \mathbf{G}_\Phi), \tag{4.73}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \lambda^{-4}\mathbf{g}_r \otimes \mathbf{g}_r + \lambda^2(\mathbf{g}_\theta \otimes \mathbf{g}_\theta + \mathbf{g}_\varphi \otimes \mathbf{g}_\varphi). \tag{4.74}$$

The Cauchy stress is

$$\boldsymbol{\sigma} = \sigma_{rr}\mathbf{g}_r \otimes \mathbf{g}_r + \sigma_{\theta\theta}(\mathbf{g}_\theta \otimes \mathbf{g}_\theta + \mathbf{g}_\varphi \otimes \mathbf{g}_\varphi), \tag{4.75}$$

$$\begin{cases} \sigma_{rr} = -p + 2(W_1 + I_1 W_2)\lambda^{-4} - 2W_2\lambda^{-8} \\ \sigma_{\theta\theta} = -p + 2(W_1 + I_1 W_2)\lambda^{2} - 2W_2\lambda^{4} = \sigma \end{cases}. \tag{4.76}$$

Since the balloon is very thin we have approximately

$$\sigma_{rr} = -p + 2(W_1 + I_1 W_2)\lambda^{-4} - 2W_2\lambda^{-8} = 0. \tag{4.77}$$

Substituting the unknown multiplier, p, from (4.77) into $(4.76)_2$ we have

$$\begin{aligned} \sigma &= 2(W_1 + I_1 W_2)\lambda^2(1-\lambda^{-6}) - 2W_2\lambda^4(1-\lambda^{-12}) \\ &= 2W_1\lambda^2(1-\lambda^{-6}) + 2W_2[(2\lambda^2+\lambda^{-4})(\lambda^2-\lambda^{-4}) - (\lambda^4+\lambda^{-8})] \\ &= 2W_1\lambda^2(1-\lambda^{-6}) + 2W_2\lambda^2(\lambda^2-\lambda^{-4}) \\ &= 2(W_1 + W_2\lambda^2)\lambda^2(1-\lambda^{-6}) \end{aligned}. \tag{4.78}$$

To relate stresses to the internal pressure, q, we consider equilibrium of a half sphere

$$2\pi r h\sigma = \pi r^2 q, \tag{4.79}$$

or

$$q = 2\frac{h}{r}\sigma = 2\frac{\lambda^{-2}H}{\lambda R}\sigma = \frac{2H}{\lambda^3 R}\sigma = \frac{4H}{\lambda R}(W_1 + W_2\lambda^2)(1-\lambda^{-6}). \tag{4.80}$$

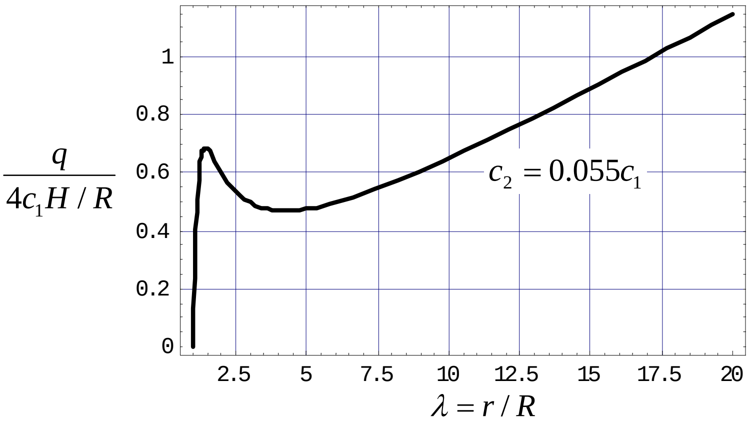
This is the pressure-stretch curve. In the case of the Mooney-Rivlin material, for example we have

$$W = c_1(I_1 - 3) + c_2(I_2 - 3), \tag{4.81}$$

$$W_1 = \frac{\partial W}{\partial I_1} = c_1, \qquad W_2 = \frac{\partial W}{\partial I_2} = c_2, \tag{4.82}$$

and

$$q = \frac{4H}{\lambda R}(c_1 + c_2\lambda^2)(1 - \lambda^{-6}). \tag{4.83}$$



4.8 Homework

1. Prove (4.12).
2. Prove (4.16).
3. Derive constitutive equations for (4.60).
4. Derive constitutive equations for (4.61).
5. Derive (4.70) from (4.68)-(4.69).
6. Read Section 4.7.

5 Anisotropic elasticity

Rubberlike materials are usually isotropic. It is possible, of course, to strengthen them by embedding fibers in prescribed directions. Nature does so with the soft biological tissues which usually consist of an isotropic matrix with the embedded and oriented collagen fibers. The collagen fibers are aligned with the axis of ligaments and tendons forming one characteristic direction or they can form two characteristic directions in the case of blood vessels, heart etc.

5.1 Materials with one characteristic direction

Materials enjoying one characteristic direction are also called materials with *transverse isotropy*, i.e. isotropy in the planes perpendicular to the preferred direction. Let us designate the preferred direction by unit vector $\mathbf{m}_0$ in the reference configuration. In this case the strain energy function $W(\mathbf{C}) = W(I_1, I_2, I_3, I_4, I_5)$ should additionally depend on two more invariants

$$I_4 = \mathbf{m} \cdot \mathbf{m} = \underbrace{\mathbf{F}\mathbf{m}_0}_{\mathbf{m}} \cdot \underbrace{\mathbf{F}\mathbf{m}_0}_{\mathbf{m}} = \mathbf{m}_0 \cdot \underbrace{\mathbf{F}^T\mathbf{F}}_{\mathbf{C}}\mathbf{m}_0 = \mathbf{C} : (\mathbf{m}_0 \otimes \mathbf{m}_0)\,, \tag{5.1}$$

$$I_5 = \mathbf{C}^2 : (\mathbf{m}_0 \otimes \mathbf{m}_0)\,, \tag{5.2}$$

where

$$\mathbf{m} = \mathbf{F}\mathbf{m}_0 \tag{5.3}$$

is not a unit vector.

The fourth invariant, I_4, has a clear physical meaning of the squared stretch in the characteristic direction. The dyad in the parentheses is often called the *structural tensor*, which characterizes the internal design of material.

Differentiating (5.1) and (5.2) with respect to $\mathbf{C}$ we get accordingly

$$\frac{\partial I_4}{\partial \mathbf{C}} = \mathbf{m}_0 \otimes \mathbf{m}_0\,, \tag{5.4}$$

$$\frac{\partial I_5}{\partial \mathbf{C}} = \mathbf{m}_0 \otimes \mathbf{C}\mathbf{m}_0 + \mathbf{C}\mathbf{m}_0 \otimes \mathbf{m}_0\,, \tag{5.5}$$

Accounting for (4.29) and (5.4)-(5.5) we calculate the constitutive equation

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}} = 2\sum_{a=1}^{5}\underbrace{\frac{\partial W}{\partial I_a}}_{W_a}\frac{\partial I_a}{\partial \mathbf{C}}$$
$$= 2\{(W_1 + I_1 W_2)\mathbf{1} - W_2\mathbf{C} + I_3 W_3 \mathbf{C}^{-1} + W_4 \mathbf{m}_0 \otimes \mathbf{m}_0 + W_5(\mathbf{m}_0 \otimes \mathbf{Cm}_0 + \mathbf{Cm}_0 \otimes \mathbf{m}_0)\} \quad ,(5.6)$$

or

$$\boldsymbol{\sigma} = J^{-1}\mathbf{FSF}^T$$
$$= 2J^{-1}\{(W_1 + I_1 W_2)\mathbf{B} - W_2\mathbf{B}^2 + I_3 W_3 \mathbf{1} + W_4 \mathbf{m} \otimes \mathbf{m} + W_5(\mathbf{m} \otimes \mathbf{Bm} + \mathbf{Bm} \otimes \mathbf{m})\}, \quad (5.7)$$

where $\mathbf{B} = \mathbf{FF}^T$ is the left Cauchy-Green tensor.

In the case of incompressible material we have instead of (5.7)

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\{(W_1 + I_1 W_2)\mathbf{B} - W_2\mathbf{B}^2 + W_4 \mathbf{m} \otimes \mathbf{m} + W_5(\mathbf{m} \otimes \mathbf{Bm} + \mathbf{Bm} \otimes \mathbf{m})\}. \quad (5.8)$$

5.2 Materials with two characteristic directions

In the case of two preferred directions we designate the second characteristic unit vector with prime $\mathbf{m}'_0$ in the reference configuration the strain energy function $W(\mathbf{C}) = W(I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8)$ should additionally depend on three more independent invariants

$$I_6 = \mathbf{C} : (\mathbf{m}'_0 \otimes \mathbf{m}'_0), \quad (5.9)$$

$$I_7 = \mathbf{C}^2 : (\mathbf{m}'_0 \otimes \mathbf{m}'_0), \quad (5.10)$$

$$I_8 = \mathbf{C} : (\mathbf{m}_0 \otimes \mathbf{m}'_0), \quad (5.11)$$

where

$$\mathbf{m}' = \mathbf{Fm}'_0 \quad (5.12)$$

is not a unit vector.

Invariants I_6, I_7 are analogous to I_4, I_5 while invariant I_8 is related to both characteristic directions.

Differentiating (5.9) - (5.11) with respect to $\mathbf{C}$ we get accordingly

$$\frac{\partial I_6}{\partial \mathbf{C}} = \mathbf{m}'_0 \otimes \mathbf{m}'_0, \quad (5.13)$$

$$\frac{\partial I_7}{\partial \mathbf{C}} = \mathbf{m}'_0 \otimes \mathbf{Cm}'_0 + \mathbf{Cm}'_0 \otimes \mathbf{m}'_0, \quad (5.14)$$

$$\frac{\partial I_8}{\partial \mathbf{C}} = \frac{1}{2}(\mathbf{m}_0 \otimes \mathbf{m}'_0 + \mathbf{m}'_0 \otimes \mathbf{m}_0). \quad (5.15)$$

We notice that the last derivative preserves symmetry.

Now the Cauchy stress takes form

$$\begin{aligned} J\boldsymbol{\sigma} &= 2(W_1 + I_1 W_2)\mathbf{B} - 2W_2\mathbf{B}^2 + 2I_3 W_3 \mathbf{1} \\ &+ 2W_4 \mathbf{m}\otimes\mathbf{m} + 2W_5(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}) \\ &+ 2W_6 \mathbf{m}'\otimes\mathbf{m}' + 2W_7(\mathbf{m}'\otimes\mathbf{Bm}' + \mathbf{Bm}'\otimes\mathbf{m}') \\ &+ W_8(\mathbf{m}\otimes\mathbf{m}' + \mathbf{m}'\otimes\mathbf{m}) \end{aligned}. \tag{5.16}$$

In the case of incompressible material we have instead of (5.16)

$$\begin{aligned} \boldsymbol{\sigma} &= -p\mathbf{1} + 2(W_1 + I_1 W_2)\mathbf{B} - 2W_2\mathbf{B}^2 \\ &+ 2W_4 \mathbf{m}\otimes\mathbf{m} + 2W_5(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}) \\ &+ 2W_6 \mathbf{m}'\otimes\mathbf{m}' + 2W_7(\mathbf{m}'\otimes\mathbf{Bm}' + \mathbf{Bm}'\otimes\mathbf{m}') \\ &+ W_8(\mathbf{m}\otimes\mathbf{m}' + \mathbf{m}'\otimes\mathbf{m}) \end{aligned}. \tag{5.17}$$

5.3 Fung model of biological tissue

The presented way of introducing characteristic directions is not unique for a description of anisotropy. The classical works of Y.C. Fung and his disciples introduced anisotropy by using the Green strain $\boldsymbol{\varepsilon} = (\mathbf{C} - \mathbf{1})/2$ as follows

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon}:\boldsymbol{\alpha}:\boldsymbol{\varepsilon} + (\beta_0 + \boldsymbol{\varepsilon}:\boldsymbol{\beta}:\boldsymbol{\varepsilon})\exp(\boldsymbol{\gamma}:\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}:\boldsymbol{\kappa}:\boldsymbol{\varepsilon} + \ldots),$$

or

$$W = \frac{1}{2}\alpha_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + (\beta_0 + \beta_{mnpq}\varepsilon_{mn}\varepsilon_{pq})\exp(\gamma_{ij}\varepsilon_{ij} + \kappa_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \ldots). \tag{5.18}$$

Here $\boldsymbol{\alpha}, \beta_0, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\kappa}$ are scalars, second- and fourth- order tensors of material constants which should be defined in experiments.

The exponential function allows modeling *stiffening* typical of soft biological tissues. As an example of the calibrated Fung strain energy we present the constitutive model of a rabbit carotid artery

$$W = \frac{c}{2}\{\exp(c_1\varepsilon_{RR}^2 + c_2\varepsilon_{\Theta\Theta}^2 + c_3\varepsilon_{ZZ}^2 + 2c_4\varepsilon_{RR}\varepsilon_{\Theta\Theta} + 2c_5\varepsilon_{ZZ}\varepsilon_{\Theta\Theta} + 2c_6\varepsilon_{RR}\varepsilon_{ZZ}) - 1\}, \tag{5.19}$$

with $c = 26.95\,\text{KPa}$ the dimensional and c_is are dimensionless: $c_1 = 0.0089$, $c_2 = 0.9925$, $c_3 = 0.4180$, $c_4 = 0.0193$, $c_5 = 0.0749$, $c_6 = 0.0295$.

5.4 Artery under blood pressure

We consider inflation of an artery under blood pressure. The corresponding Boundary Value Problem (BVP) includes equations of momentum balance (equilibrium)

$$\operatorname{div}\boldsymbol{\sigma} = \mathbf{0}\,, \tag{5.20}$$

constitutive law

$$\boldsymbol{\sigma} = -p\mathbf{1} + \mathbf{F}\frac{\partial W}{\partial \boldsymbol{\varepsilon}}\mathbf{F}^T\,, \tag{5.21}$$

and boundary conditions on placements and tractions

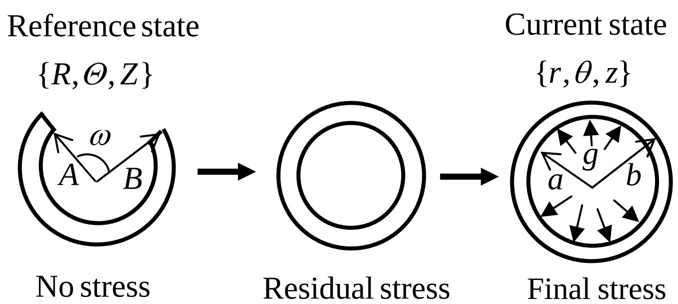
$$\mathbf{y} = \bar{\mathbf{y}} \quad \text{or} \quad \boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{t}}\,, \tag{5.22}$$

where 'div' operator is with respect to the current coordinates $\mathbf{y}$; $\boldsymbol{\sigma}$ is the Cauchy stress tensor; $\mathbf{1}$ is the second order identity tensor; p is an unknown multiplier of the workless stress; $\boldsymbol{\varepsilon} = (\mathbf{F}^T\mathbf{F} - \mathbf{1})/2$ is the Green strain tensor; W is the strain energy; $\mathbf{t}$ is traction per unit area of the current surface with the unit outward normal $\mathbf{n}$; and the barred quantities are prescribed.

We consider the radial inflation of an artery as a symmetric deformation of an infinite cylinder. Following Fung we assume the deformation law in the form

$$r = \sqrt{\frac{R^2 - A^2}{\gamma s} + a^2}, \quad \theta = \gamma\Theta, \quad z = sZ\,, \tag{5.23}$$

where a point occupying position $\{R, \Theta, Z\}$ in the reference configuration is moving to position $\{r, \theta, z\}$ in the current configuration; s is the axial stretch; $\gamma = 2\pi/(2\pi - \omega)$, where ω is the artery opening angle in the reference configuration; A and a are the internal artery radii before and after deformation accordingly.



The opening central angle, ω, in a stress-free reference configuration is used to represent *residual stresses,* which are one of the most intriguing features of mechanics of living tissues. While the qualitative nature of residual stresses related to *tissue growth* is understood reasonably well, the best way to quantify them remains to be settled.

Accounting for (5.23), the deformation gradient and the nontrivial components of the Green strain take the following forms

$$\mathbf{F}=\frac{R}{\gamma sr}\mathbf{g}_r\otimes\mathbf{G}_R+\frac{\gamma r}{R}\mathbf{g}_\theta\otimes\mathbf{G}_\Theta+s\mathbf{g}_z\otimes\mathbf{G}_Z\,, \tag{5.24}$$

$$\begin{cases}\varepsilon_{RR}=\{(R/\gamma sr)^2-1\}/2\\ \varepsilon_{\Theta\Theta}=\{(\gamma r/R)^2-1\}/2\\ \varepsilon_{ZZ}=\{s^2-1\}/2\end{cases}, \tag{5.25}$$

where $\{\mathbf{G}_R,\mathbf{G}_\Theta,\mathbf{G}_Z\}$ and $\{\mathbf{g}_r,\mathbf{g}_\theta,\mathbf{g}_z\}$ are the orthonormal bases in cylindrical coordinates at the reference and current configurations accordingly.

Accounting for (5.21), (5.23)-(5.25) and assuming that the stored energy depends on the nontrivial strain components only we get the following nonzero components of the Cauchy stress

$$\begin{cases}\sigma_{rr}=-p+\dfrac{R^2}{(sr\gamma)^2}\dfrac{\partial W}{\partial\varepsilon_{RR}}\\ \sigma_{\theta\theta}=-p+\dfrac{(r\gamma)^2}{R^2}\dfrac{\partial W}{\partial\varepsilon_{\Theta\Theta}}\\ \sigma_{zz}=-p+s^2\dfrac{\partial W}{\partial\varepsilon_{ZZ}}\end{cases}. \tag{5.26}$$

Besides, there is only one nontrivial equilibrium equation

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0\,. \tag{5.27}$$

The traction boundary conditions are

$$\begin{cases}\sigma_{rr}(r=a)=-g\\ \sigma_{rr}(r=b)=0\end{cases}, \tag{5.28}$$

where a,b are the inner and outer radii of the artery after the deformation, which were equal to A,B before the deformation accordingly; and g is the internal pressure.

We integrate equilibrium equation (5.27) over the wall thickness with account of boundary conditions (5.28) and we get

$$g(a)=-\int_a^{b(a)}(\sigma_{rr}-\sigma_{\theta\theta})\frac{dr}{r}=-\int_a^{b(a)}\left(\frac{R^2}{(\gamma sr)^2}\frac{\partial W}{\partial\varepsilon_{RR}}-\frac{(\gamma r)^2}{R^2}\frac{\partial W}{\partial\varepsilon_{\Theta\Theta}}\right)\frac{dr}{r}, \tag{5.29}$$

where $b(a)=\sqrt{a^2+(B^2-A^2)/(\gamma s)}$.

Equation (5.29) presents the pressure-radius (g-a) relationship, which we examine below. Before doing that, however, we introduce dimensionless variables as follows

$$\bar{g}=\frac{g}{c};\quad \overline{W}=\frac{W}{c};\quad \bar{r}=\frac{r}{A};\quad \overline{R}=\frac{R}{A}\quad \bar{a}=\frac{a}{A};\quad \bar{b}=\frac{b}{A}, \tag{5.30}$$

where c is the shear modulus.

Substituting (5.30) in (5.29) we get

$$\bar{g}(\bar{a}) = -\int_{\bar{a}}^{\bar{b}(\bar{a})} \left(\frac{\bar{R}^2}{(\gamma s \bar{r})^2} \frac{\partial \overline{W}}{\partial \varepsilon_{RR}} - \frac{(\gamma \bar{r})^2}{\bar{R}^2} \frac{\partial \overline{W}}{\partial \varepsilon_{\Theta\Theta}} \right) \frac{d\bar{r}}{\bar{r}}, \tag{5.31}$$

where

$$\bar{b}(\bar{a}) = \sqrt{\bar{a}^2 + ((B/A)^2 - 1)/(\gamma s)}, \tag{5.32}$$

$$\bar{R}^2 = \gamma s(\bar{r}^2 - \bar{a}^2) + 1. \tag{5.33}$$

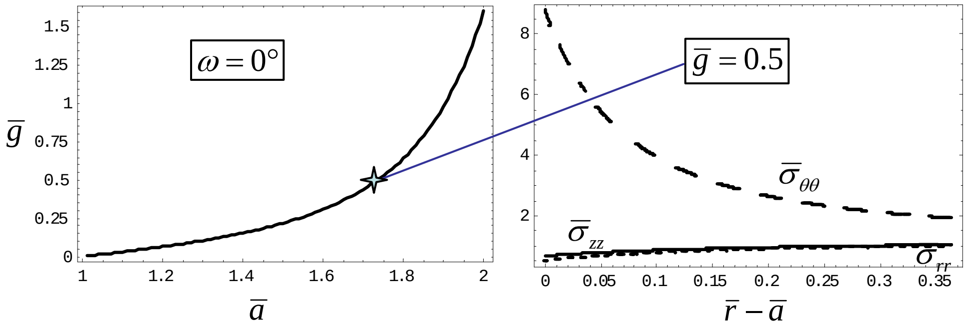
The dimensionless multiplier $\bar{p} = p/c$ is obtained from (5.27) and $(5.28)_1$ by integration

$$\bar{p}(\bar{r}) = \frac{\bar{R}(\bar{r})^2}{(\gamma s \bar{r})^2} \frac{\partial \overline{W}}{\partial \varepsilon_{RR}}(\bar{r}) - \bar{g}(\bar{a}) + \int_{\bar{a}}^{\bar{r}} \left(\frac{\bar{R}(\rho)^2}{(\gamma s \rho)^2} \frac{\partial \overline{W}}{\partial \varepsilon_{RR}}(\rho) - \frac{(\gamma \rho)^2}{\bar{R}(\rho)^2} \frac{\partial \overline{W}}{\partial \varepsilon_{\Theta\Theta}}(\rho) \right) \frac{d\rho}{\rho}, \tag{5.34}$$

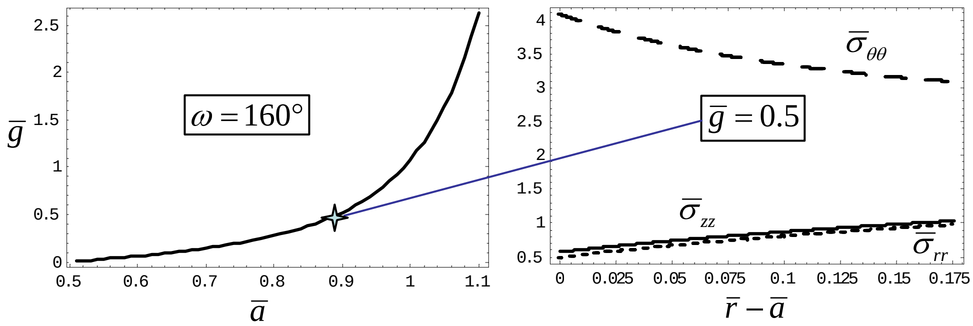
and normalized stresses take the form

$$\begin{cases} \bar{\sigma}_{rr} = \dfrac{\sigma_{rr}}{c} = -\bar{p} + \dfrac{\bar{R}^2}{(s\bar{r}\gamma)^2} \dfrac{\partial \overline{W}}{\partial \varepsilon_{RR}} \\ \bar{\sigma}_{\theta\theta} = \dfrac{\sigma_{\theta\theta}}{c} = -\bar{p} + \dfrac{(\bar{r}\gamma)^2}{\bar{R}^2} \dfrac{\partial \overline{W}}{\partial \varepsilon_{\Theta\Theta}} \\ \bar{\sigma}_{zz} = \dfrac{\sigma_{zz}}{c} = -\bar{p} + s^2 \dfrac{\partial \overline{W}}{\partial \varepsilon_{ZZ}} \end{cases}. \tag{5.35}$$

We use the Fung model (5.20) to numerically generate the pressure-radius curves and stresses. Firstly, we set an unprestressed state with $\omega = 0°$ and the internal and external reference radii $A = 0.71$ mm and $B = 1.10$ mm accordingly. The pressure-radius and stress distribution curves are calculated with the help of Mathematica presented in figure below. We show stresses for dimensionless pressure $\bar{g} = 0.5$, which corresponds to pressure $g = 13.47$ KPa for the shear modulus $c = 26.95$ KPa.



Secondly, we set a prestressed state with $\omega = 160°$ and the internal and external reference radii $A = 1.43$ mm and $B = 1.82$ mm accordingly. The pressure-radius and stress distribution curves are presented in figure below. We again show stresses for dimensionless pressure $\bar{g} = 0.5$, which corresponds to pressure $g = 13.47$ KPa for the shear modulus $c = 26.95$ KPa .



5.5 Homework

1. Is $\mathbf{C}^3 : (\mathbf{m}_0 \otimes \mathbf{m}_0)$ the independent invariant? Hint: Use the Caley-Hamilton formula (1.39).

2. Prove (5.4).

3. Prove (5.5).

4. Derive (5.7) from (5.6).

5. Prove (5.15).

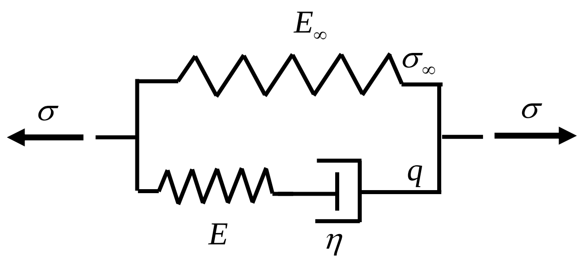
6. Derive $(5.23)_1$ from the condition of constant volume.

6 Viscoelasticity

Rubberlike materials and soft biological tissues can exhibit a time-delayed response. For example, stresses can decrease under the constant strains – *stress relaxation* – or strains can increase under the constant stresses – *creep*. Such phenomena are usually related to *viscosity,* which is a fluid-like property of solids.

6.1 Rheological model

To describe viscosity we start with a simple one-dimensional model, also called *rheological*. Rheological models are prototypes for general three-dimensional constitutitive theories. For example, the spring model is a prototype for hyperelasticity theories. To account for viscoelasticity we will use the device shown in the figure below.



This rheological model represents the so-called 'standard solid', which includes the classical elasticity due to the top linear spring with the Young modulus E_∞ and viscosity due to the chain of the linear spring with Young modulus E and the linear dashpot with the viscosity coefficient η. The dashpot provides the time delay in the mechanical response of the device.

We assume, for the sake of simplicity, that the device has a unit length and a unit area and, consequently, strains and stresses are equal to elongations and forces. The resulting stress is composed of stresses acting on the top and bottom elements of the device

$$\sigma = \underbrace{E_\infty \varepsilon}_{\sigma_\infty} + q \,, \tag{6.1}$$

where $\sigma_\infty = E_\infty \varepsilon$ is the stress in the top spring; ε is the strain of the whole device; and q is the 'viscous' stress in the bottom element.

The viscous stress can be calculated considering the dashpot with the linear proportionality between the stress and strain rate in as in the case of Newtonian fluids

$$q = \eta\dot{\alpha}\,, \tag{6.2}$$

where α is a *dashpot strain*.

On the other hand, the viscous stress is also equivalent to the stress in the bottom spring

$$q = E(\varepsilon - \alpha)\,. \tag{6.3}$$

Differentiating (6.3) with respect to time and substituting $\dot{\alpha}$ from (6.2) we get the evolution equation for the viscous stress

$$\dot{q} + \frac{q}{\tau} = \gamma\dot{\sigma}_{\infty}\,, \tag{6.4}$$

where

$$\tau = \frac{\eta}{E} \tag{6.5}$$

is the *relaxation* time and

$$\gamma = \frac{E}{E_{\infty}} \tag{6.6}$$

is a relative spring stiffness and

$$q(t \to -\infty) = 0\,. \tag{6.7}$$

is the initial condition.

Equations (6.1) and (6.4) represent the constitutive description of the model of ‘standard solid’.

Remarkably, the evolution law (6.4) can be integrated by using the integration factor

$$(\dot{q} + \frac{q}{\tau})\exp(\frac{t}{\tau}) = \gamma\dot{\sigma}_{\infty}\exp(\frac{t}{\tau})\,. \tag{6.8}$$

Indeed, after simple manipulations on the left hand side of (6.8) we have

$$\frac{d}{dt}\{q\exp(\frac{t}{\tau})\} = \gamma\dot{\sigma}_{\infty}\exp(\frac{t}{\tau})\,. \tag{6.9}$$

Integrating on both sides of (6.9) with account of the initial condition (6.7) we have

$$q\exp(\frac{t}{\tau}) = \int_{-\infty}^{t}\gamma\dot{\sigma}_{\infty}\exp(\frac{\zeta}{\tau})d\zeta\,, \tag{6.10}$$

or

$$q = \int_{-\infty}^{t}\gamma\dot{\sigma}_{\infty}(\zeta)\exp(-\frac{t-\zeta}{\tau})d\zeta\,. \tag{6.11}$$

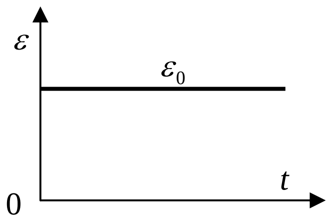
Substituting (6.11) in (6.1) we get

$$\sigma(t)=\int_{-\infty}^{t}G(t-\zeta)\frac{\partial\sigma_\infty}{\partial\zeta}d\zeta\,, \tag{6.12}$$

$$G(t-\zeta)=1+\gamma\exp(-\frac{t-\zeta}{\tau})\,. \tag{6.13}$$

The latter is often called the *relaxation function*.

Let us consider an example of the relaxation test when a step function for strain is used



$$\varepsilon(t)=H(t)\varepsilon_0=\begin{cases}0, & t<0\\ \varepsilon_0, & t\geq 0\end{cases}. \tag{6.14}$$

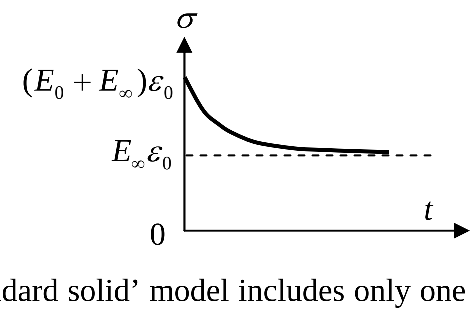
Step function, H, is also called Heaviside function and its derivative is δ-(Paul Dirac[1]) function

$$\dot{\varepsilon}(t)=\delta(t)\varepsilon_0\,, \tag{6.15}$$

$$\dot{\sigma}_\infty=\delta(t)E_\infty\varepsilon_0\,, \tag{6.16}$$

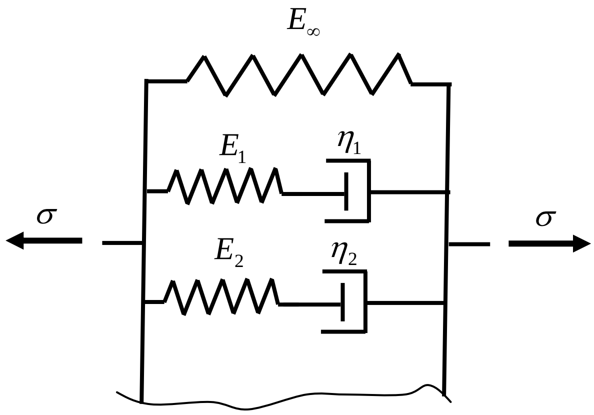
Substituting (6.16) in (6.12) we have

$$\sigma(t)=\int_{-\infty}^{t}\delta(\zeta)E_\infty G(t-\zeta)\varepsilon_0 d\zeta=E_\infty G(t)\varepsilon_0=\{E_\infty+E\exp(-t/\tau)\}\varepsilon_0\,. \tag{6.17}$$



The considered 'standard solid' model includes only one dashpot and relaxation time. It is possible and, sometimes, useful to extend the model including a number of relaxation times.

[1] Google it!



In this case, constitutive equations take the following form accordingly

$$\sigma = \sigma_\infty + \sum_i q_i \,, \tag{6.18}$$

$$\dot{q}_i + \frac{q_i}{\tau_i} = \gamma_i \dot{\sigma}_\infty \,, \tag{6.19}$$

where

$$\gamma_i = \frac{E_i}{E_\infty}, \quad \tau_i = \frac{\eta_i}{E_i} \,, \tag{6.20}$$

$$q_i(t \to -\infty) = 0 \,. \tag{6.21}$$

The relaxation function (6.13) takes form

$$G(t-\zeta) = 1 + \sum_i \gamma_i \exp\left(-\frac{t-\zeta}{\tau_i}\right), \tag{6.22}$$

6.2 Constitutive equations

The rheological model developed in the previous paragraph can serve as a *prototype* for the constitutive equations of solids. Particularly, we can define the following constitutive equations in 3D by analogy with (6.18), (6.19), and (6.21)

$$\mathbf{S} = \mathbf{S}_\infty + \sum_i \mathbf{Q}_i \,, \tag{6.23}$$

$$\dot{\mathbf{Q}}_i + \frac{\mathbf{Q}_i}{\tau_i} = \dot{\mathbf{S}}_\infty \,, \tag{6.24}$$

$$\mathbf{Q}_i(t \to -\infty) = \mathbf{0} \,. \tag{6.25}$$

where $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor; $\mathbf{Q}_i$ is the i^{th} internal stress-like variable.

Evolution equation (6.24) can be integrated analytically as in the previous section and we get the convolution integral

$$\mathbf{S}(t)=\int_{-\infty}^{t}G(t-\zeta)\frac{\partial\mathbf{S}_{\infty}}{\partial\zeta}d\zeta\,, \tag{6.26}$$

$$G(t-\zeta)=1+\sum_{i}\exp(-\frac{t-\zeta}{\tau_i})\,, \tag{6.27}$$

where the elastic stress is derived from the strain energy, W,

$$\mathbf{S}_{\infty}=2\frac{\partial W}{\partial\mathbf{C}}\,. \tag{6.28}$$

Unfortunately, the direct use of the described model is not practical because most materials exhibit different responses concerning the volume and shape changes. To make the difference Flory (1961) proposed the volumetric-deviatoric split of the deformation gradient

$$\mathbf{F}=J^{1/3}\overline{\mathbf{F}}\,, \tag{6.29}$$

where

$$\overline{\mathbf{F}}=J^{-1/3}\mathbf{F} \tag{6.30}$$

is the *isochoric* or *distortional* part of deformation that *preserves volume*

$$\det\overline{\mathbf{F}}=1\,. \tag{6.31}$$

Accordingly, tensor $J^{1/3}\mathbf{1}$ presents the dilatational, i.e. volume-changing, part of the deformation.

Barring the distortional quantities we have for the Cauchy-Green tensor

$$\overline{\mathbf{C}}=\overline{\mathbf{F}}^{T}\overline{\mathbf{F}}=J^{-2/3}\mathbf{F}^{T}\mathbf{F}=J^{-2/3}\mathbf{C}\,. \tag{6.32}$$

Now the strain energy can be considered as a function of the dilatational and distortional deformations and the constitutive equation (6.28) takes form

$$\mathbf{S}_{\infty}=2\frac{\partial W(J,\overline{\mathbf{C}})}{\partial\mathbf{C}}=\underbrace{2\frac{\partial W}{\partial J}\frac{\partial J}{\partial\mathbf{C}}}_{\mathbf{S}_{\infty}^{\mathrm{vol}}}+\underbrace{2\frac{\partial W}{\partial\overline{\mathbf{C}}}:\frac{\partial\overline{\mathbf{C}}}{\partial\mathbf{C}}}_{\mathbf{S}_{\infty}^{\mathrm{iso}}}\,. \tag{6.33}$$

The first and the second terms on the right-hand side of (6.33) designate the volumetric and isochoric parts of the stress. We calculate them as follows.

$$\frac{\partial J}{\partial\mathbf{C}}=\frac{\partial\det\sqrt{\mathbf{C}}}{\partial\mathbf{C}}=\frac{1}{2}J\mathbf{C}^{-1}\,, \tag{6.34}$$

$$\frac{\partial J^{-2/3}}{\partial\mathbf{C}}=-\frac{1}{3}J^{-2/3}\mathbf{C}^{-1}\,, \tag{6.35}$$

$$\frac{\partial\overline{\mathbf{C}}}{\partial\mathbf{C}}=\frac{\partial(J^{-2/3}\mathbf{C})}{\partial\mathbf{C}}=-\frac{1}{3}J^{-2/3}\mathbf{C}\otimes\mathbf{C}^{-1}+J^{-2/3}\mathbf{1}^{*}, \tag{6.36}$$

where $\mathbf{1}^{*}$ is the fourth-order unity tensor with components

$$(\mathbf{1}^*)_{ijkm} = \frac{\partial C_{ij}}{\partial C_{km}} = \delta_{ik}\delta_{jm} \,. \qquad (6.37)$$

Thus, the volumetric and isochoric responses can be presented using (6.33), (6.34), and (6.36) as follows

$$\mathbf{S}_{\infty}^{\mathrm{vol}} = \frac{\partial W}{\partial J} J\mathbf{C}^{-1} , \qquad (6.38)$$

$$\mathbf{S}_{\infty}^{\mathrm{iso}} = J^{-2/3} \underbrace{(\mathbf{1}^* - \frac{1}{3}\mathbf{C}^{-1} \otimes \mathbf{C})}_{\mathrm{Dev}} : 2\frac{\partial W}{\partial \overline{\mathbf{C}}} = J^{-2/3}\mathrm{Dev}(2\frac{\partial W}{\partial \overline{\mathbf{C}}}) , \qquad (6.39)$$

where we introduced the Lagrangean deviator operator 'Dev'.

By using the Flory split, we can finally reformulate the constitutive equations of voscoelasticity in the following differential form

$$\mathbf{S} = \mathbf{S}_{\infty}^{\mathrm{vol}} + \mathbf{S}_{\infty}^{\mathrm{iso}} + \sum_i \mathbf{Q}_i \, , \qquad (6.40)$$

$$\dot{\mathbf{Q}}_i + \frac{\mathbf{Q}_i}{\tau_i} = \dot{\mathbf{S}}_{\infty}^{\mathrm{iso}} , \qquad (6.41)$$

$$\mathbf{Q}_i (t \to -\infty) = \mathbf{0} \, . \qquad (6.42)$$

Integrating (6.41)-(6.42), we get

$$\mathbf{Q}_i = \int_{-\infty}^{t} \exp(-\frac{t-\zeta}{\tau_i}) \frac{\partial \mathbf{S}_{\infty}^{\mathrm{iso}}}{\partial \zeta} d\zeta \, . \qquad (6.43)$$

It is clear now that only distortional deformations are rate-sensitive.

6.3 Numerical integration of constitutive equations

Constitutive equations (6.40), (6.43) are of integral type and it is important to develop a numerical procedure for calculating stresses for the given strain – the stress update procedure. As the first step in the direction of the integration algorithm, we assume that deformation starts only at time $t = 0$ and all stress variables are zero prior to this time. The latter assumption allows us to rewrite (6.43) in the form

$$\mathbf{Q}_i = \int_{0}^{t} \exp(-\frac{t-\zeta}{\tau_i}) \frac{\partial \mathbf{S}_{\infty}^{\mathrm{iso}}}{\partial \zeta} d\zeta \, . \qquad (6.44)$$

Let us now partition the time interval of interest into small increments

$$\Delta t = t_{n+1} - t_n , \qquad (6.45)$$

where subscripts designate the point on the time scale.

We assume that the deformation state of the body is fully determined at time t_n

$$\mathbf{F}_n = \frac{\partial \mathbf{y}_n}{\partial \mathbf{x}} = \mathbf{1} + \frac{\partial \mathbf{u}_n}{\partial \mathbf{x}}, \tag{6.46}$$

$$J_n = \det \mathbf{F}_n, \quad \mathbf{C}_n = \mathbf{F}_n^T \mathbf{F}_n, \quad \overline{\mathbf{C}}_n = J_n^{-2/3} \mathbf{C}_n, \tag{6.47}$$

where $\mathbf{y}_n = \mathbf{y}(\mathbf{x}, t_n)$.

We also assume that all stresses are known at time t_n: $\mathbf{S}_{\infty\, n}^{\mathrm{vol}}; \mathbf{S}_{\infty\, n}^{\mathrm{iso}}; \mathbf{Q}_{i\, n}$.

Solution of the balance equations at time $t_{n+1} = t_n + \Delta t$ allows us to find kinematical quantities

$$\mathbf{F}_{n+1} = \frac{\partial \mathbf{y}_{n+1}}{\partial \mathbf{x}} = \mathbf{1} + \frac{\partial \mathbf{u}_{n+1}}{\partial \mathbf{x}}, \tag{6.48}$$

$$J_{n+1} = \det \mathbf{F}_{n+1}, \quad \mathbf{C}_{n+1} = \mathbf{F}_{n+1}^T \mathbf{F}_{n+1}, \quad \overline{\mathbf{C}}_{n+1} = J_{n+1}^{-2/3} \mathbf{C}_{n+1}, \tag{6.49}$$

and, subsequently, stresses

$$\mathbf{S}_{\infty\, n+1}^{\mathrm{vol}} = \frac{\partial W}{\partial J_{n+1}} J \mathbf{C}_{n+1}^{-1}, \tag{6.50}$$

$$\mathbf{S}_{\infty\, n+1}^{\mathrm{iso}} = J_{n+1}^{-2/3} \mathrm{Dev}_{n+1}\left(2 \frac{\partial W}{\partial \overline{\mathbf{C}}_{n+1}}\right). \tag{6.51}$$

It remains to update the internal variables (6.44) only.

Various computational schemes can be used for updating $\mathbf{Q}_{i\, n+1}$. We proceed by writing (6.44) in the form

$$\mathbf{Q}_{i\, n+1} = \int_0^{t_n} \exp\left(-\frac{t_{n+1} - \zeta}{\tau_i}\right) \frac{\partial \mathbf{S}_\infty^{\mathrm{iso}}}{\partial \zeta} d\zeta + \int_{t_n}^{t_{n+1}} \exp\left(-\frac{t_{n+1} - \zeta}{\tau_i}\right) \frac{\partial \mathbf{S}_\infty^{\mathrm{iso}}}{\partial \zeta} d\zeta . \tag{6.52}$$

The first term on the right-hand side of (6.52) is calculated as follows

$$\int_0^{t_n} \exp\left(-\frac{t_n + \Delta t - \zeta}{\tau_i}\right) \frac{\partial \mathbf{S}_\infty^{\mathrm{iso}}}{\partial \zeta} d\zeta = \exp\left(-\frac{\Delta t}{\tau_i}\right) \underbrace{\int_0^{t_n} \exp\left(-\frac{t_n - \zeta}{\tau_i}\right) \frac{\partial \mathbf{S}_\infty^{\mathrm{iso}}}{\partial \zeta} d\zeta}_{\mathbf{Q}_{i\, n}} = \mathbf{Q}_{i\, n} \exp\left(-\frac{\Delta t}{\tau_i}\right). \tag{6.53}$$

To integrate the second term on the right-hand side of (6.52) we make the following approximation for the exponent

$$\exp\left(-\frac{\overbrace{t_{n+1}}^{t_n + \Delta t} - \overbrace{\zeta}^{t_n + \Delta t/2}}{\tau_i}\right) \approx \exp\left(-\frac{\Delta t}{2\tau_i}\right), \tag{6.54}$$

and, consequently,

$$\int_{t_n}^{t_{n+1}} \exp(-\frac{t_{n+1}-\zeta}{\tau_i}) \frac{\partial \mathbf{S}_\infty^{\mathrm{iso}}}{\partial \zeta} d\zeta \approx \exp(-\frac{\Delta t}{2\tau_i})(\mathbf{S}_{\infty\, n+1}^{\mathrm{iso}} - \mathbf{S}_{\infty\, n}^{\mathrm{iso}}) . \quad (6.55)$$

Substituting (6.53) and (6.55) in (6.52) we have finally

$$\mathbf{Q}_{i\, n+1} \cong \mathbf{Q}_{i\, n} \exp(-\frac{\Delta t}{\tau_i}) + (\mathbf{S}_{\infty\, n+1}^{\mathrm{iso}} - \mathbf{S}_{\infty\, n}^{\mathrm{iso}}) \exp(-\frac{\Delta t}{2\tau_i}) . \quad (6.56)$$

6.4 Homework

1. Prove (6.34).
2. Prove (6.35).
3. Prove (6.36).
4. Derive (6.39) from (6.33)-(6.36).

7 Chemo-mechanical coupling

Previously, we attributed displacements, stresses, strains to a material point or an infinitesimal material volume. In many cases of practical interest additional parameters reflecting the presence of the specific material/chemical constituents are required. For example, gels composed of a network of cross-linked molecules *swell* when a solvent migrates through it. The concentration of the solvent is changing and the gel deforms (remember diapers!). When dried the gel *shrinks* analogously to the *consolidation process* in soils where the applied load enforces water to leave the solid skeleton. Soft biological tissues like cartilage exhibit sound alterations of the fluid phase during walking. Hard materials like metals and ceramics can undergo the internal *atomic migrations*. For example, some hard materials can absorb and store large amounts of hydrogen. In many cases, not only the concentration of the constituents change but the process of their diffusion is important. We will always assume that the *diffusion process is slow enough to ignore the inertia effects*.

7.1 Governing equations

Governing equations accounting for the chemo-mechanical coupling should include the equations of balance and boundary conditions for the chemical/material constituents of interest and, besides, the constitutive law. We consider only one additional chemical/material component of interest for the sake of simplicity. Generalization for the case of a few components is trivial. The results of Sections 3.5 and 3.6 on Eulerian and Lagrangean forms of the master balance equations are crucial.

The integral form of the Eulerian balance law is

$$\frac{d}{dt}\int c\,dV = \int \xi\,dV + \oint \boldsymbol{\varphi}\cdot\mathbf{n}\,dA\,, \tag{7.1}$$

where c is the true concentration, i.e. the number of molecules (or moles) of the constituent per unit current volume; ξ is its volumetric supply; and $\boldsymbol{\varphi}$ is its flux through the current body surface with the unit outward normal $\mathbf{n}$.

Localizing this equation by getting rid of the integrals, we formulate the field balance equation

$$\frac{\partial c}{\partial t} + \mathrm{div}(c\mathbf{v}) = \mathrm{div}\boldsymbol{\varphi} + \xi\,, \tag{7.2}$$

where $\mathrm{div}(...) = \partial(...)/\partial y_i \mathbf{e}_i$ is calculated with respect to spatial coordinates, and boundary conditions

$$\boldsymbol{\varphi} \cdot \mathbf{n} = \overline{\varphi}_n \quad \text{or} \quad f(c) = 0, \tag{7.3}$$

where the barred quantity is prescribed and f is a boundary constraint imposed on the concentration.

The initial condition takes form

$$c(t = 0) = \overline{c}. \tag{7.4}$$

Since the deformed boundary is generally not known in advance, it can be advantageous to use the Lagrangean or referential description where the equations (7.1)-(7.4) take the following forms accordingly

$$\frac{d}{dt}\int c_0 \, dV_0 = \int \xi_0 \, dV_0 + \oint \boldsymbol{\varphi}_0 \cdot \mathbf{n}_0 dA_0, \tag{7.5}$$

$$\frac{\partial c_0}{\partial t} = \mathrm{Div}\boldsymbol{\varphi}_0 + \xi_0, \tag{7.6}$$

$$\boldsymbol{\varphi}_0 \cdot \mathbf{n}_0 = \overline{\varphi}_{0n} \quad \text{or} \quad f_0(c_0) = 0, \tag{7.7}$$

$$c_0(t = 0) = \overline{c}_0. \tag{7.8}$$

where $\mathrm{Div}(...) = \partial(...)/\partial x_i \mathbf{e}_i$ is calculated with respect to referential coordinates.

The Eulerian and Lagrangean quantities are related as follows (see Part 3 Balance Laws)

$$dV = dV_0 \underbrace{\det \mathbf{F}}_{J} = J dV_0, \tag{7. 9}$$

$$\mathbf{n} dA = J\mathbf{F}^{-T}\mathbf{n}_0 dA_0, \tag{7.10}$$

$$c = J^{-1} c_0, \tag{7.11}$$

$$\xi = J^{-1}\xi_0, \tag{7.12}$$

$$\boldsymbol{\varphi} = J^{-1}\mathbf{F}\boldsymbol{\varphi}_0. \tag{7.13}$$

In addition to the balance laws, we have to formulate the constitutive equations that can generally be written in the following form

$$\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}_0(\mathbf{C}, c_0, \partial c_0 / \partial \mathbf{x}), \tag{7.14}$$

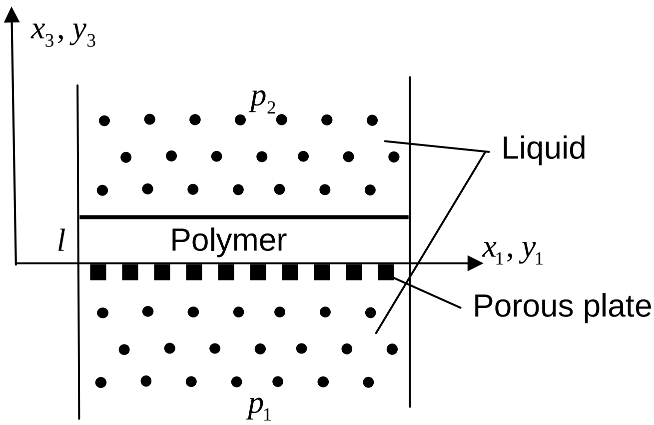
$$\xi_0 = \xi_0(\mathbf{C}, c_0, \partial c_0 / \partial \mathbf{x}), \tag{7.15}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ is the right Cauchy-Green deformation tensor.

It is important to emphasize that the flux should depend on the gradient of the concentration, $\partial c_0 / \partial \mathbf{x}$, to provide the *second order* of the balance equations. It should also be noticed that constitutive relations (7.14)-(7.15) were formulated for the Lagrangean quantities while it could alternatively be done for the Eulerian quantities.

7.2 Diffusion through polymer membrane

Based on the described theoretical framework we examine the problem of diffusion of a liquid through the polymer membrane that was considered in experiments of Paul and Ebra-Lima (J. Appl. Polymer Sci. 14 (1970) 2201-2224).



A thin polymer layer is placed on a permeable porous plate and the liquid diffuses through the membrane under pressure $p_2 > p_1$. We assume that the body force and source are zero: $\mathbf{b}_0 = \mathbf{0}$ and $\xi_0 = 0$; and the process is steady state: $\dot{c}_0 = 0$. Under these assumptions the balance equations reduce to

$$\mathrm{Div}\boldsymbol{\varphi}_0 = 0, \tag{7.16}$$

$$\mathrm{Div}\mathbf{T} = \mathbf{0}. \tag{7.17}$$

The constitutive equations can be defined as follows, for example,

$$\mathbf{T} = 2\mathbf{F}\frac{\partial W(\mathbf{C}, c_0)}{\partial \mathbf{C}}, \tag{7.18}$$

$$\boldsymbol{\varphi}_0 = \mathbf{M}(c_0, \mathbf{C})\frac{\partial \mu}{\partial \mathbf{x}}, \tag{7.19}$$

$$\mu = \frac{\partial W(\mathbf{C}, c_0)}{\partial c_0}, \tag{7.20}$$

where $\mathbf{M}$ is the *mobility* tensor; and μ is the *chemical potential*.

Motivated by many practical situations it is reasonable to assume that the ground material (polymer) in the reference state is incompressible and the volume of the material is altering only due to the supply of new species (molecules of the liquid). This assumption can be formalized by using the following constraint

$$\gamma(c_0, \mathbf{F}) = 1 + vc_0 - J = 0, \tag{7.21}$$

where v is the volume of one molecule of the liquid.

The increment of this constraint takes form

$$\dot{\gamma} = \frac{\partial \gamma}{\partial c_0}\dot{c}_0 + \frac{\partial \gamma}{\partial \mathbf{F}} : \dot{\mathbf{F}} = v\dot{c}_0 - J\mathbf{F}^{-T} : \dot{\mathbf{F}} = 0. \tag{7.22}$$

Multipliers v and $-J\mathbf{F}^{-T}$ in (7.22) represent the workless chemical potential and stress accordingly, which can be scaled by arbitrary factor Π. With account of (7.22) we modify (7.18) and (7.20) as follows

$$\mathbf{T} = \underbrace{2\mathbf{F}\frac{\partial W}{\partial \mathbf{C}}}_{\tilde{\mathbf{T}}} - J\Pi\mathbf{F}^{-T} = \tilde{\mathbf{T}} - J\Pi\mathbf{F}^{-T}, \tag{7.23}$$

$$\mu = \frac{\partial W}{\partial c_0} + v\Pi. \tag{7.24}$$

Since the thickness of the membrane is small as compared to the characteristic lengths of the device, we can consider the field variations in the lateral directions only. Specifically, we set the deformation and concentration gradients in the following forms accordingly

$$\mathbf{F} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda(x_3)\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{7.25}$$

$$\boldsymbol{\varphi}_0 = \varphi_{03}\mathbf{e}_3. \tag{7.26}$$

Substituting (7.25)-(7.26) in (7.23) and (7.19), we get the following non-trivial stresses and fluxes

$$T_{11} = T_{22} = \tilde{T}_{11} - \lambda\Pi, \tag{7.27}$$

$$T_{33} = \tilde{T}_{33} - \Pi, \tag{7.28}$$

$$\varphi_{03} = M_{33}\frac{\partial \mu}{\partial x_3}. \tag{7.29}$$

We notice that the traction/placement boundary conditions take the following forms on the upper and lower surfaces of the membrane accordingly

$$T_{33}(L) = -p_2, \tag{7.30}$$

$$y_3(0) = 0\,. \tag{7.31}$$

Since the stress tensor is divergence-free and $T_{33} = \text{constant}$, we can obtain the unknown multiplier Π from boundary condition (7.30)

$$\Pi = \tilde{T}_{33} + p_2\,. \tag{7.32}$$

Substituting (7.32) in (7.24) we get for the chemical potential

$$\mu = \frac{\partial W}{\partial c_0} + v\tilde{T}_{33} + vp_2\,. \tag{7.33}$$

We also notice that due to the molecular incompressibility condition the concentration is related to the stretch as follows

$$vc_0 = \lambda - 1\,. \tag{7.34}$$

Substituting (7.33)-(7.34) and (7.29) in (7.16) we get a second order ordinary differential equation of the chemical balance in term of stretches. To solve it we need to impose two boundary conditions

$$f(\lambda_1) = \mu(\lambda_1) - p_1 v = 0\,, \tag{7.35}$$

$$f(\lambda_2) = \mu(\lambda_2) - p_2 v = 0\,, \tag{7.36}$$

where $\lambda_1 = \lambda(0)$ and $\lambda_2 = \lambda(L)$.

We define the mobility tensor and the Helmholtz free energy function as follows

$$\mathbf{M} = (\alpha c_0 v)^{\beta - 1} \frac{c_0 D}{kT} \mathbf{C}^{-1}\,, \tag{7.37}$$

$$W = \underbrace{\frac{1}{2} NkT(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3 - 2\log[\lambda_1 \lambda_2 \lambda_3])}_{\text{Elastic energy}} - \underbrace{\frac{kT}{v}(vc_0 \log[1 + \frac{1}{vc_0}] + \frac{\chi}{1 + vc_0})}_{\text{Energy of mixing}}\,, \tag{7.38}$$

where α and β are dimensionless material constants; D is the diffusion coefficient for the solvent molecules; k is the Boltzmann constant; T is the absolute temperature; N is the number of polymer chains in the gel divided by the reference volume; χ is a dimensionless parameter; and λ_i are the principal stretches.

Substituting (7.37) in (7.26) and accounting for (7.25) we get

$$\varphi_{03} = (\alpha c_0 v)^{\beta - 1} \frac{c_0 D}{kT} \lambda^{-2} \frac{\partial \mu}{\partial x_3}\,. \tag{7.39}$$

Differentiating (7.38) with respect to stretches and concentration and accounting for (7.34) we find

$$\tilde{T}_{33} = NkT(\lambda - \lambda^{-1})\,, \tag{7.40}$$

$$\frac{\partial W}{\partial c_0} = kT(\log\frac{\lambda - 1}{\lambda} + \frac{1}{\lambda} + \frac{\chi}{\lambda^2}) . \tag{7.41}$$

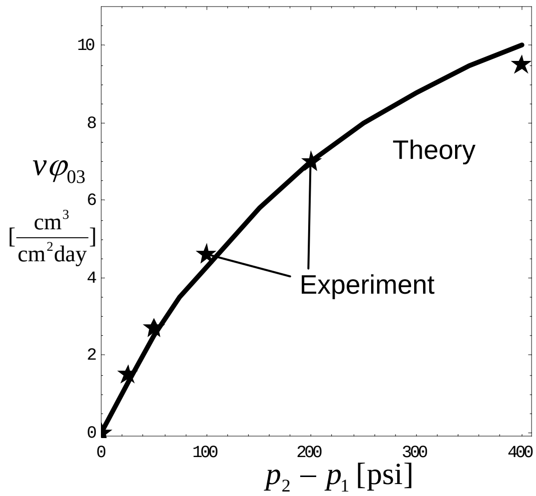
Substituting (7.40)-(7.41) in (7.33) we have finally

$$\mu = kT(\log\frac{\lambda - 1}{\lambda} + \frac{1}{\lambda} + \frac{\chi}{\lambda^2}) + NvkT(\lambda - \lambda^{-1}) + vp_2 . \tag{7.42}$$

Substituting (7.42) in (7.29) and (7.16), we have the second-order ordinary differential equation, which is completed by boundary conditions (7.35)-(7.36).

Based on the numerical solution of the boundary-value problem it is possible to calculate the increase of the flux through the membrane with the increase of the pressure – see the figure below for the *toluene-rubber* data shown in the table. Remarkably, the flux increase is not proportional to the pressure increase. The latter is a well-established experimental fact.

k	$1.38 \cdot 10^{-23}$ Nm/K
T	303° K
p_1	10^5 N/m^2
v	$17.7 \cdot 10^{-29}$ m^3
D	$2.36 \cdot 10^{-10}$ m^2/s
N	$6.36 \cdot 10^{25}$ 1/m^3
L	$2.65 \cdot 10^{-4}$ m
χ	0.425
α	5.7
β	3



7.3 Homework

1. Derive (7.40) and (7.41) from (7.38) and (7.34).

2. Read Hong et al (J. Mech. Phys. Solids 56 (2008) 1779-1793).

3. Write a half-page explanation of the physical meaning of the *chemical potential* based on a literature review and Google search.

8 Electro-mechanical coupling

Soft polymer materials are *dielectric*, i.e. they do not conduct the electric current. However, electroactive polymers can deform in response to electric fields. This property is increasingly used in actuators or artificial muscles that have a great potential of practical applications. We will consider the basic electro-elasticity of soft materials at large strains.

8.1 Electrostatics

Electron presents the smallest negative *charge* of $e = 1.6 \cdot 10^{-19}\ \mathrm{C}\,(\text{Coulomb})$. All other charges, both positive and negative, are multipliers of the electron charge. The charges can be free leading to the electric current or bound as in the case of electroactive dielectrics. Since the number of charges in the material volumes that we consider is large, we will always mean the continuum average in the subsequent considerations.

Charges create electric fields that produce forces on other charges. For example, the force on charge Q is[2]

$$\mathbf{f} = Q\mathbf{E}, \tag{8.1}$$

where $\mathbf{E}$ is the *electric field*.

According to the experimentally validated Coulomb's law the force between charges Q and Q' placed at points $\mathbf{y}$ and $\mathbf{y}'$ accordingly is inversely proportional to the squared distance between the charges

$$\mathbf{f} = Q\underbrace{\frac{Q'}{4\pi\varepsilon_0}\frac{\mathbf{y}-\mathbf{y}'}{|\mathbf{y}-\mathbf{y}'|^3}}_{\mathbf{E}(\mathbf{y})}, \tag{8.2}$$

where $\varepsilon_0 = 8.854 \cdot 10^{-12}\ \mathrm{F/m}\,(\text{Farad/meter})$ is called the *permittivity* of space.

Based on (8.2) we can write the electric field by superposing many charges smeared over the space with the charge density q

$$\mathbf{E}(\mathbf{y}) = \frac{1}{4\pi\varepsilon_0}\int q(\mathbf{y}')\frac{\mathbf{y}-\mathbf{y}'}{|\mathbf{y}-\mathbf{y}'|^3}dV'. \tag{8.3}$$

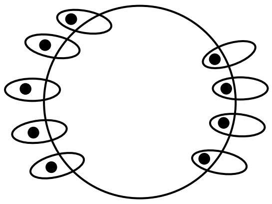
Based on (8.3) we can obtain (without proof) the *Gauss law* for a space volume, V, enclosed with a surface, A, with the outward unit normal $\mathbf{n}$

$$\oint \varepsilon_0 \mathbf{E}\cdot\mathbf{n}\,dA = \int q\,dV. \tag{8.4}$$

[2] See also Jackson JD (1999) Classical Electrodynamics. John Wiley & Sons.

The Gauss law was derived for vacuum in the absence of matter. In the presence of matter, the bound charges can be slightly displaced with respect to each other when the electric potential is applied. Such relative displacement is called *polarization*. To characterize the phenomenon it is possible to introduce the *polarization vector,* $\mathbf{P}$. For example, if we have N atoms per unit volume with positive charge q_0 (nucleus) and negative charge $-q_0$ (electrons) then $\mathbf{P} = Nq_0\mathbf{d}$ where $\mathbf{d}$ is a relative average displacement of the negative charges with respect to positive charges. The polarization vector changes the charge on the right hand side of (8.4)

$$\oint \varepsilon_0 \mathbf{E} \cdot \mathbf{n} dA = \int q dV - \oint \mathbf{P} \cdot \mathbf{n} dA . \qquad (8.5)$$



It is convenient to introduce the *electric displacement vector*

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \qquad (8.6)$$

and rewrite (8.5) in the form

$$\oint \mathbf{D} \cdot \mathbf{n} dA = \int q dV \; . \qquad (8.7)$$

This equation is valid for any volume and, consequently, we can localize it

$$\mathrm{div}\mathbf{D} = q \; . \qquad (8.8)$$

Formulas (8.7) and (8.8) represent the integral and differential forms of the first equation of electrostatics.

To derive the second equation of electrostatics we notice that

$$\frac{\mathbf{y} - \mathbf{y}'}{|\mathbf{y} - \mathbf{y}'|^3} = -\frac{\partial}{\partial \mathbf{y}} \left(\frac{1}{|\mathbf{y} - \mathbf{y}'|} \right) . \qquad (8.9)$$

Substituting (8.9) in (8.3) we obtain

$$\mathbf{E}(\mathbf{y}) = -\frac{\partial \varphi(\mathbf{y})}{\partial \mathbf{y}} , \qquad (8.10)$$

where

$$\varphi(\mathbf{y}) = -\frac{1}{4\pi\varepsilon_0} \int \frac{q(\mathbf{y}')}{|\mathbf{y} - \mathbf{y}'|} dV' \; . \qquad (8.11)$$

is called the *electric potential* or *voltage*.

To clarify the physical meaning of the electric potential we consider the work that should be done against the electric field in order to move charge Q from point $\mathbf{y}_1$ to point $\mathbf{y}_2$

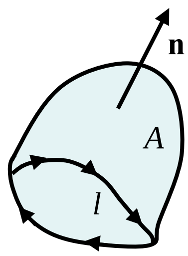
$$W = -\int_{\mathbf{y}_1}^{\mathbf{y}_2} Q\mathbf{E} \cdot d\mathbf{y} = \int_{\mathbf{y}_1}^{\mathbf{y}_2} Q \frac{\partial \varphi}{\partial \mathbf{y}} \cdot d\mathbf{y} = Q\varphi(\mathbf{y}_2) - Q\varphi(\mathbf{y}_1) . \tag{8.12}$$

Thus, the work is equal to the difference in the electrical potentials at points $\mathbf{y}_1$ and $\mathbf{y}_2$ times charge Q.

Since the integral in (8.12) does not depend on the integration path we have for any closed curve l

$$\oint \mathbf{E} \cdot d\mathbf{y} = 0 . \tag{8.13}$$

By building any surface A on the curve l and using the Stokes formula we can rewrite (8.13) in the form



$$\oint \mathbf{E} \cdot d\mathbf{y} = \int (\operatorname{curl} \mathbf{E}) \cdot \mathbf{n} dA = 0 . \tag{8.14}$$

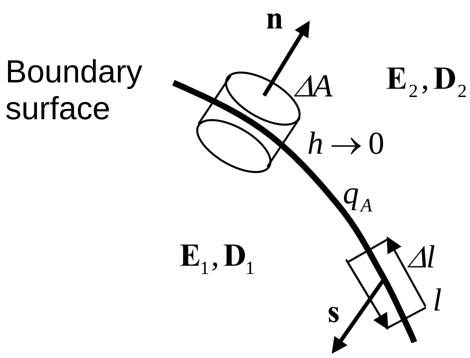
Since the surface can be chosen arbitrarily, we can localize the integral equation as follows

$$\operatorname{curl} \mathbf{E} = \mathbf{0} . \tag{8.15}$$

We notice that the electric field derived from the electric potential always obeys (8.16): see (1.96).

Formulas (8.14) and (8.15) represent the integral and differential forms of the second equation of electrostatics.

With the help of (8.7) and (8.13) we can derive the boundary conditions on a surface separating two materials with different electric fields and displacements.



Firstly, we consider a small cylinder with the base area $\varDelta A$ and height $h \to 0$. In this case, the left- and right- hand sides of (8.7) take the following forms accordingly

$$\oint \mathbf{D}\cdot\mathbf{n}dA = \mathbf{D}_2\cdot\mathbf{n}\varDelta A + \mathbf{D}_1\cdot(-\mathbf{n})\varDelta A = (\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n}\varDelta A, \tag{8.16}$$

$$\int q dV = q_A \varDelta A, \tag{8.17}$$

where q_A is a charge on the boundary surface.

Substituting (8.16) and (8.17) in (8.7) we can write the following boundary condition

$$(\mathbf{D}_2 - \mathbf{D}_1)\cdot\mathbf{n} = q_A. \tag{8.18}$$

Secondly, we consider a closed path, l, whose long arm directions are defined by the cross product of the surface tangent, $\mathbf{s}$, and normal, $\mathbf{n}$, vectors. In this case, (8.13) takes the following form

$$\begin{aligned}\oint \mathbf{E}\cdot d\mathbf{y} &= \mathbf{E}_1\cdot(\mathbf{n}\times\mathbf{s})\varDelta l - \mathbf{E}_2\cdot(\mathbf{n}\times\mathbf{s})\varDelta l \\ &= \varDelta l\mathbf{s}\cdot(\mathbf{E}_1\times\mathbf{n} - \mathbf{E}_2\times\mathbf{n}) = 0\end{aligned}. \tag{8.19}$$

Since (8.19) is correct for any tangent $\mathbf{s}$ we obtain the second boundary condition

$$(\mathbf{E}_1 - \mathbf{E}_2)\times\mathbf{n} = \mathbf{0}. \tag{8.20}$$

Finally, we notice that the polarization vector should be defined as a function of the electric field or, in other words, the constitutive equation should be written in the form

$$\mathbf{P} = \mathbf{P}(\mathbf{E}). \tag{8.21}$$

The simplest form of the constitutive equation in the case of isotropic media is the proportionality between the polarization and the electric field

$$\mathbf{P} = \chi\varepsilon_0\mathbf{E}, \tag{8.22}$$

where χ is called the *electric susceptibility* of the medium.

Substituting (8.22) in (8.6) we get

$$\mathbf{D} = \underbrace{\varepsilon_0(1+\chi)}_{\varepsilon}\mathbf{E} = \varepsilon\mathbf{E}, \tag{8.23}$$

where ε is called the *electric permittivity* and the ratio $\varepsilon / \varepsilon_0 = 1 + \chi$ is called the *dielectric constant*.

8.2 Angular momentum balance

Equations of the angular momentum balance (3.28) should be modified to include the body couple due to the electric field, $\mathbf{K}$,

$$\frac{d}{dt}\int \rho \mathbf{r} \times \mathbf{v} dV = \int (\mathbf{r} \times \mathbf{b}\rho + \mathbf{K})\, dV + \oint \mathbf{r} \times \mathbf{t}\, dA. \tag{8.24}$$

Localizing this equation as it was done in Section 3.4 we obtain

$$\varepsilon_{ijk}\sigma_{kj} + K_i = 0\,. \tag{8.25}$$

This equation means that the Cauchy stress is not symmetric anymore in the presence of the electric field: $\boldsymbol{\sigma} \neq \boldsymbol{\sigma}^T$!

Though there are a number of theories defining the constitutive equation for the electric body force and body couple (Pao YH (1978) Electromagnetic Forces in Deformable Continua. In: Mechanics Today, vol.4, ed. S. Nemat-Nasser. Pergamon Press.), all of them reduce to the same form in the case of electrostatics and zero distributed body charge, $q = 0$,

$$\rho \mathbf{b} = (\mathrm{grad}\mathbf{E})\mathbf{P}\,, \tag{8.26}$$

$$\mathbf{K} = \mathbf{P} \times \mathbf{E}\,. \tag{8.27}$$

Following Maxwell's idea for magnetism, some authors represent the electric body force as a divergence of the "Maxwell stress" tensor, $\boldsymbol{\sigma}^M$,

$$\rho \mathbf{b} = (\mathrm{grad}\mathbf{E})\mathbf{P} = \mathrm{div}\boldsymbol{\sigma}^M\,. \tag{8.28}$$

Such representation is not unique and it can take the following popular form, for example,

$$\boldsymbol{\sigma}^M = \mathbf{E} \otimes \underbrace{(\varepsilon_0 \mathbf{E} + \mathbf{P})}_{\mathbf{D}} - \frac{\varepsilon_0}{2}(\mathbf{E} \cdot \mathbf{E})\mathbf{1}\,. \tag{8.29}$$

Combining the elastic and Maxwell stresses it is possible to introduce the total stress

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \boldsymbol{\sigma}^M\,, \tag{8.30}$$

which obeys the equilibrium equation (3.25) without body forces

$$\mathrm{div}\tilde{\boldsymbol{\sigma}} = \mathbf{0}\,. \tag{8.31}$$

Substituting from (8.27), (8.29), and (8.30) in the equation (8.25) of the angular momentum balance we get

$$\varepsilon_{ijk}\tilde{\sigma}_{kj} = 0\,, \tag{8.32}$$

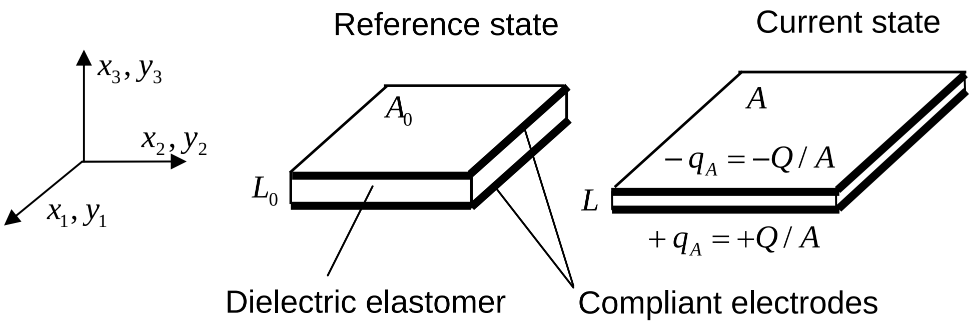
i.e. the total stress is symmetric, contrary to the Cauchy stress

$$\tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}^T . \tag{8.33}$$

We notice, however, that the body couple is zero in the case of the constitutive equation (8.22)

$$\mathbf{K} = \underbrace{\chi\varepsilon_0\mathbf{E}}_{\mathbf{P}} \times \mathbf{E} = \mathbf{0} . \tag{8.34}$$

8.3 Example of a dielectric actuator



We will assume that the polarization of dielectric does not depend on its deformation and, consequently, (8.22) is valid: $\mathbf{P} = \chi\varepsilon_0\mathbf{E}$. In this case, boundary-value problem of electrostatics is composed of the following field equations

$$\text{div}\mathbf{E} = 0 , \tag{8.35}$$

$$\text{curl}\,\mathbf{E} = \mathbf{0} , \tag{8.36}$$

and boundary conditions

$$(\varepsilon_0\mathbf{E}_2 - \varepsilon\mathbf{E}_1)\cdot\mathbf{n} = q_A , \tag{8.37}$$

$$(\mathbf{E}_1 - \mathbf{E}_2)\times\mathbf{n} = \mathbf{0} , \tag{8.38}$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are electric fields inside and outside the plate.

Momentum balance equations are

$$\text{div}\tilde{\boldsymbol{\sigma}} = \mathbf{0}, \quad \tilde{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}}^T , \tag{8.39}$$

and the corresponding traction boundary conditions are

$$(\tilde{\boldsymbol{\sigma}}_1 - \tilde{\boldsymbol{\sigma}}_2)\mathbf{n} = \mathbf{0} , \tag{8.40}$$

where $\tilde{\boldsymbol{\sigma}}_1$ and $\tilde{\boldsymbol{\sigma}}_2$ are stress fields inside and outside the plate.

The constitutive law for the total stress of isotropic incompressible hyperelastic material takes the following form accounting for (8.23) and (8.29)

$$\tilde{\boldsymbol{\sigma}} = -p\mathbf{1} + 2(W_1 + I_1 W_2)\mathbf{B} - 2W_2\mathbf{B}^2 + \varepsilon\mathbf{E}\otimes\mathbf{E} - \frac{\varepsilon_0}{2}(\mathbf{E}\cdot\mathbf{E})\mathbf{1}, \qquad (8.41)$$

where p is the Lagrange multiplier; $W_a \equiv \partial W / \partial I_a$ with I_a the principal invariants of $\mathbf{B} = \mathbf{F}\mathbf{F}^T$.

We assume the homogeneous solution of the boundary-value problem in the form

$$\mathbf{F} = \lambda^{-1/2}(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \lambda\mathbf{e}_3 \otimes \mathbf{e}_3, \qquad (8.42)$$

$$\mathbf{E}_1 = E\mathbf{e}_3, \quad \mathbf{E}_2 = \mathbf{0}, \qquad (8.43)$$

where the lateral stretch is

$$\lambda = \frac{L}{L_0}. \qquad (8.44)$$

We notice that the material is incompressible, $\det\mathbf{F} = 1$, and, consequently, with account of (8.44) we have

$$AL = A_0 L_0, \quad A = A_0/\lambda. \qquad (8.45)$$

The homogeneous solution obeys field equations (8.35), (8.36), (8.39) automatically and substituting (8.43) and (8.45) in boundary conditions (8.37), (8.38) we get

$$E = \frac{q_A}{\varepsilon} = \frac{Q}{\varepsilon A} = \frac{Q\lambda}{\varepsilon A_0}. \qquad (8.46)$$

Substituting (8.42) and (8.43) in (8.41) we have

$$\tilde{\sigma}_{11} = \tilde{\sigma}_{22} = -p + 2(W_1 + I_1 W_2)\lambda^{-1} - 2W_2\lambda^{-2} - \frac{\varepsilon_0}{2}\left(\frac{Q\lambda}{\varepsilon A_0}\right)^2, \qquad (8.47)$$

$$\tilde{\sigma}_{33} = -p + 2(W_1 + I_1 W_2)\lambda^2 - 2W_2\lambda^4 + \varepsilon\varepsilon_0\left(\frac{Q\lambda}{\varepsilon A_0}\right)^2 - \frac{\varepsilon_0}{2}\left(\frac{Q\lambda}{\varepsilon A_0}\right)^2. \qquad (8.48)$$

Assuming the stress-free deformation, $\tilde{\sigma}_{11} = \tilde{\sigma}_{22} = \tilde{\sigma}_{33} = 0$, that obeys (8.40) and subtracting (8.48) from (8.47) we get

$$2(W_1 + I_1 W_2)(\lambda^{-1} - \lambda^2) - 2W_2(\lambda^{-2} - \lambda^4) - \frac{\varepsilon_0}{\varepsilon}\frac{Q^2}{A_0^2}\lambda^2 = 0. \qquad (8.49)$$

This equation allows us to calculate the lateral stretch λ for the given charge Q.

Consider, for example, the Neo-Hookean material

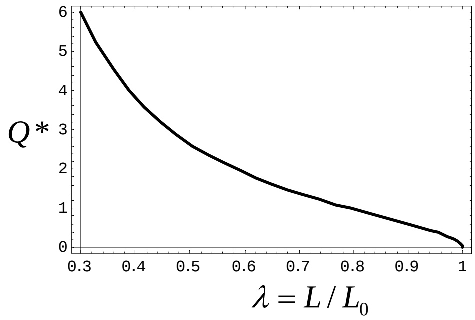
$$W = C_1(I_1 - 3), \quad W_1 = C_1. \qquad (8.50)$$

Substituting (8.50) in (8.49) we get

$$(\lambda^{-1}-\lambda^{2})\lambda^{-2}=\underbrace{\frac{\varepsilon_0}{2C_1\varepsilon}\frac{Q^2}{A_0^2}}_{Q*^2}=Q*^2, \tag{8.51}$$

where $Q*$ is a dimensionless charge.

The relationship (8.51) is presented graphically below and it shows that the dielectric is thinning with the growing charge as expected.



8.4 Homework

1. Prove (8.9).
2. Check whether (8.29) obeys (8.28).
3. Derive (8.32).

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Lagrangian Equilibrium Equations in Cylindrical and Spherical Coordinates

K.Y. Volokh[1]

Abstract: Lagrangian or referential equilibrium equations for materials undergoing large deformations are of interest in the developing fields of mechanics of soft biomaterials and nanomechanics. The main feature of these equations is the necessity to deal with the First Piola-Kirchhoff, or nominal, stress tensor which is a two-point tensor referring simultaneously to the reference and current configurations. This two-point nature of the First Piola-Kirchhoff tensor is not always appreciated by the researchers and the *total* covariant derivative necessary for the formulation of the equilibrium equations in curvilinear coordinates is sometimes inaccurately confused with the regular covariant derivative. Surprisingly, the traditional continuum mechanics literature does not discuss this issue properly, except for some brief notions on the two-point nature of the Piola-Kirchhoff tensor. We aim at partially filling this gap by giving a full yet simple derivation of the Lagrangian equilibrium equations in cylindrical and spherical coordinates.

1 Introduction

Lagrangian scalar equilibrium equations in cylindrical and spherical coordinates for materials undergoing large deformations are rarely discussed in the literature. The most influential monographs on nonlinear elasticity and continuum mechanics, including Antman (1995); Chadwick (1976); Ciarlet (1988); Eringen (1962); Green and Adkins (1970); Green and Zerna (1968); Gurtin (1981); Haupt (2000); Jaunzemis (1967); Liu (2002); Lur'e (1990); Malvern (1969); Marsden and Hughes (1983); Ogden (1984); Truesdell and Toupin (1961); Truesdell and Noll (1965); Wang and Truesdell (1973); Wilmanski (1998), do not address this issue. However, the Lagrangian equilibrium equations in cylindrical and spherical coordinates can be very useful in solving nonlinear problems analytically or semi-analytically. Sometimes, it is possible to assume incompressibility of the material what allows for using a simpler Eulerian description for obtaining some elementary analytical solutions. This is not the general case, however, where we need the Lagrangian equilibrium equations of the form

$$\mathrm{Div}\mathbf{P} = \mathbf{0} \tag{1}$$

in cylindrical and spherical coordinates. These equations can be derived from the *total covariant derivative* of the 1^{st} Piola-Kirchhoff stress tensor $\mathbf{P}$. Though this way may be elegant we prefer a more straightforward "pedestrian" way, which, however, does not require any knowledge of the general tensor calculus from the reader.

2 Cylindrical coordinates

We introduce orthonormal basis in cylindrical coordinates (Malvern, 1969) for the reference configuration

$$\begin{aligned}\mathbf{K}_R &= (\cos\Theta, \sin\Theta, 0)^T;\\ \mathbf{K}_\Theta &= (-\sin\Theta, \cos\Theta, 0)^T;\\ \mathbf{K}_Z &= (0,0,1)^T.\end{aligned} \tag{2}$$

By direct calculation we have

$$\frac{\partial \mathbf{K}_R}{\partial\Theta} = \mathbf{K}_\Theta; \quad \frac{\partial \mathbf{K}_\Theta}{\partial\Theta} = -\mathbf{K}_R. \tag{3}$$

All other derivatives of the base vectors are equal to zero. Analogously, we have for the current configuration:

$$\begin{aligned}\mathbf{k}_r &= (\cos\theta, \sin\theta, 0)^T;\\ \mathbf{k}_\theta &= (-\sin\theta, \cos\theta, 0)^T;\\ \mathbf{k}_z &= (0,0,1)^T,\end{aligned} \tag{4}$$

$$\frac{\partial \mathbf{k}_r}{\partial\theta} = \mathbf{k}_\theta; \quad \frac{\partial \mathbf{k}_\theta}{\partial\theta} = -\mathbf{k}_r. \tag{5}$$

Now, we write the divergence operator in the form (Malvern, 1969)

$$\mathrm{Div}\mathbf{P} = \frac{\partial\mathbf{P}}{\partial R}\mathbf{K}_R + \frac{\partial\mathbf{P}}{R\partial\Theta}\mathbf{K}_\Theta + \frac{\partial\mathbf{P}}{\partial Z}\mathbf{K}_Z. \tag{6}$$

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The plan is to compute the right-hand side of this equation term by term.

We start with the first term on the right hand side of Eq. (6)

$$\begin{aligned}\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R
&= \left(\tfrac{\partial P_{rR}}{\partial R}\mathbf{k}_r\otimes\mathbf{K}_R + P_{rR}\tfrac{\partial \mathbf{k}_r}{\partial R}\otimes\mathbf{K}_R + P_{rR}\mathbf{k}_r\otimes\tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{r\Theta}}{\partial R}\mathbf{k}_r\otimes\mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial \mathbf{k}_r}{\partial R}\otimes\mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r\otimes\tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{rZ}}{\partial R}\mathbf{k}_r\otimes\mathbf{K}_Z + P_{rZ}\tfrac{\partial \mathbf{k}_r}{\partial R}\otimes\mathbf{K}_Z + P_{rZ}\mathbf{k}_r\otimes\tfrac{\partial \mathbf{K}_Z}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{\theta R}}{\partial R}\mathbf{k}_\theta\otimes\mathbf{K}_R + P_{\theta R}\tfrac{\partial \mathbf{k}_\theta}{\partial R}\otimes\mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta\otimes\tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{\theta\Theta}}{\partial R}\mathbf{k}_\theta\otimes\mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial \mathbf{k}_\theta}{\partial R}\otimes\mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta\otimes\tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{\theta Z}}{\partial R}\mathbf{k}_\theta\otimes\mathbf{K}_Z + P_{\theta Z}\tfrac{\partial \mathbf{k}_\theta}{\partial R}\otimes\mathbf{K}_Z + P_{\theta Z}\mathbf{k}_\theta\otimes\tfrac{\partial \mathbf{K}_Z}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{zR}}{\partial R}\mathbf{k}_z\otimes\mathbf{K}_R + P_{zR}\tfrac{\partial \mathbf{k}_z}{\partial R}\otimes\mathbf{K}_R + P_{zR}\mathbf{k}_z\otimes\tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{z\Theta}}{\partial R}\mathbf{k}_z\otimes\mathbf{K}_\Theta + P_{z\Theta}\tfrac{\partial \mathbf{k}_z}{\partial R}\otimes\mathbf{K}_\Theta + P_{z\Theta}\mathbf{k}_z\otimes\tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R\\
&+ \left(\tfrac{\partial P_{zZ}}{\partial R}\mathbf{k}_z\otimes\mathbf{K}_Z + P_{zZ}\tfrac{\partial \mathbf{k}_z}{\partial R}\otimes\mathbf{K}_Z + P_{zZ}\mathbf{k}_z\otimes\tfrac{\partial \mathbf{K}_Z}{\partial R}\right)\mathbf{K}_R,\end{aligned} \tag{7}$$

where $\mathbf{k}_m\otimes\mathbf{K}_N = \mathbf{k}_m\mathbf{K}_N^T$.

With account of orthonormality of the base vectors we have

$$\begin{aligned}\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R &= \frac{\partial P_{rR}}{\partial R}\mathbf{k}_r + P_{rR}\frac{\partial \mathbf{k}_r}{\partial R}\\
&+\frac{\partial P_{\theta R}}{\partial R}\mathbf{k}_\theta + P_{\theta R}\frac{\partial \mathbf{k}_\theta}{\partial R} + \frac{\partial P_{zR}}{\partial R}\mathbf{k}_z + P_{zR}\frac{\partial \mathbf{k}_z}{\partial R}.\end{aligned} \tag{8}$$

Differentiating the Eulerian basis, we get

$$\begin{aligned}\frac{\partial \mathbf{k}_r}{\partial R} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_r}{\partial z}\frac{\partial z}{\partial R} = \frac{\partial \theta}{\partial R}\mathbf{k}_\theta,\\
\frac{\partial \mathbf{k}_\theta}{\partial R} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_\theta}{\partial z}\frac{\partial z}{\partial R} = -\frac{\partial \theta}{\partial R}\mathbf{k}_r,\\
\frac{\partial \mathbf{k}_z}{\partial R} &= \frac{\partial \mathbf{k}_z}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_z}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_z}{\partial z}\frac{\partial z}{\partial R} = \mathbf{0}.\end{aligned} \tag{9}$$

Now, substituting Eq. (9) in Eq. (8) we have

$$\begin{aligned}\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R &= \left(\frac{\partial P_{rR}}{\partial R} - P_{\theta R}\frac{\partial \theta}{\partial R}\right)\mathbf{k}_r\\
&+\left(P_{rR}\frac{\partial \theta}{\partial R} + \frac{\partial P_{\theta R}}{\partial R}\right)\mathbf{k}_\theta + \frac{\partial P_{zR}}{\partial R}\mathbf{k}_z.\end{aligned} \tag{10}$$

Analogously to Eqs. (7)-(10) we calculate the last two terms on the right-hand side of Eq. (6)

$$\begin{aligned}\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta
&= \tfrac{1}{R}\left(\tfrac{\partial P_{rR}}{\partial \Theta}\mathbf{k}_r\otimes\mathbf{K}_R + P_{rR}\tfrac{\partial \mathbf{k}_r}{\partial \Theta}\otimes\mathbf{K}_R + P_{rR}\mathbf{k}_r\otimes\tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{r\Theta}}{\partial \Theta}\mathbf{k}_r\otimes\mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial \mathbf{k}_r}{\partial \Theta}\otimes\mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r\otimes\tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{rZ}}{\partial \Theta}\mathbf{k}_r\otimes\mathbf{K}_Z + P_{rZ}\tfrac{\partial \mathbf{k}_r}{\partial \Theta}\otimes\mathbf{K}_Z + P_{rZ}\mathbf{k}_r\otimes\tfrac{\partial \mathbf{K}_Z}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta R}}{\partial \Theta}\mathbf{k}_\theta\otimes\mathbf{K}_R + P_{\theta R}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta}\otimes\mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta\otimes\tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta\Theta}}{\partial \Theta}\mathbf{k}_\theta\otimes\mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta}\otimes\mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta\otimes\tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta Z}}{\partial \Theta}\mathbf{k}_\theta\otimes\mathbf{K}_Z + P_{\theta Z}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta}\otimes\mathbf{K}_Z + P_{\theta Z}\mathbf{k}_\theta\otimes\tfrac{\partial \mathbf{K}_Z}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{zR}}{\partial \Theta}\mathbf{k}_z\otimes\mathbf{K}_R + P_{zR}\tfrac{\partial \mathbf{k}_z}{\partial \Theta}\otimes\mathbf{K}_R + P_{zR}\mathbf{k}_z\otimes\tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{z\Theta}}{\partial \Theta}\mathbf{k}_z\otimes\mathbf{K}_\Theta + P_{z\Theta}\tfrac{\partial \mathbf{k}_z}{\partial \Theta}\otimes\mathbf{K}_\Theta + P_{z\Theta}\mathbf{k}_z\otimes\tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta\\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{zZ}}{\partial \Theta}\mathbf{k}_z\otimes\mathbf{K}_Z + P_{zZ}\tfrac{\partial \mathbf{k}_z}{\partial \Theta}\otimes\mathbf{K}_Z + P_{zZ}\mathbf{k}_z\otimes\tfrac{\partial \mathbf{K}_Z}{\partial \Theta}\right)\mathbf{K}_\Theta,\end{aligned} \tag{11}$$

$$\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta = \frac{1}{R}\begin{pmatrix} P_{rR}\mathbf{k}_r + \frac{\partial P_{r\Theta}}{\partial \Theta}\mathbf{k}_r + P_{r\Theta}\frac{\partial \mathbf{k}_r}{\partial \Theta}\\ +P_{\theta R}\mathbf{k}_\theta + \frac{\partial P_{\theta\Theta}}{\partial \Theta}\mathbf{k}_\theta + P_{\theta\Theta}\frac{\partial \mathbf{k}_\theta}{\partial \Theta}\\ +P_{zR}\mathbf{k}_z + \frac{\partial P_{z\Theta}}{\partial \Theta}\mathbf{k}_z + P_{z\Theta}\frac{\partial \mathbf{k}_z}{\partial \Theta}\end{pmatrix} \tag{12}$$

$$\begin{aligned}\frac{\partial \mathbf{k}_r}{\partial \Theta} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_r}{\partial z}\frac{\partial z}{\partial \Theta} = \frac{\partial \theta}{\partial \Theta}\mathbf{k}_\theta,\\
\frac{\partial \mathbf{k}_\theta}{\partial \Theta} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_\theta}{\partial z}\frac{\partial z}{\partial \Theta} = -\frac{\partial \theta}{\partial \Theta}\mathbf{k}_r,\\
\frac{\partial \mathbf{k}_z}{\partial \Theta} &= \frac{\partial \mathbf{k}_z}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_z}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_z}{\partial z}\frac{\partial z}{\partial \Theta} = \mathbf{0},\end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta &= \left(\frac{P_{rR}}{R} + \frac{\partial P_{r\Theta}}{R\partial \Theta} - \frac{P_{\theta\Theta}}{R}\frac{\partial \theta}{\partial \Theta}\right)\mathbf{k}_r\\
&+\left(\frac{P_{r\Theta}}{R}\frac{\partial \theta}{\partial \Theta} + \frac{P_{\theta R}}{R} + \frac{\partial P_{\theta\Theta}}{R\partial \Theta}\right)\mathbf{k}_\theta + \left(\frac{P_{zR}}{R} + \frac{\partial P_{z\Theta}}{R\partial \Theta}\right)\mathbf{k}_z\end{aligned} \tag{14}$$

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z
&= \left(\tfrac{\partial P_{rR}}{\partial Z}\mathbf{k}_r \otimes \mathbf{K}_R + P_{rR}\tfrac{\partial \mathbf{k}_r}{\partial Z} \otimes \mathbf{K}_R + P_{rR}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_R}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\tfrac{\partial P_{r\Theta}}{\partial Z}\mathbf{k}_r \otimes \mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial \mathbf{k}_r}{\partial Z} \otimes \mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\tfrac{\partial P_{rZ}}{\partial Z}\mathbf{k}_r \otimes \mathbf{K}_Z + P_{rZ}\tfrac{\partial \mathbf{k}_r}{\partial Z} \otimes \mathbf{K}_Z + P_{rZ}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_Z}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\tfrac{\partial P_{\theta R}}{\partial Z}\mathbf{k}_\theta \otimes \mathbf{K}_R + P_{\theta R}\tfrac{\partial \mathbf{k}_\theta}{\partial Z} \otimes \mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_R}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\tfrac{\partial P_{\theta\Theta}}{\partial Z}\mathbf{k}_\theta \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial \mathbf{k}_\theta}{\partial Z} \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\tfrac{\partial P_{\theta Z}}{\partial Z}\mathbf{k}_\theta \otimes \mathbf{K}_Z + P_{\theta Z}\tfrac{\partial \mathbf{k}_\theta}{\partial Z} \otimes \mathbf{K}_Z + P_{\theta Z}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_Z}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\tfrac{\partial P_{zR}}{\partial Z}\mathbf{k}_z \otimes \mathbf{K}_R + P_{zR}\tfrac{\partial \mathbf{k}_z}{\partial Z} \otimes \mathbf{K}_R + P_{zR}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_R}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\tfrac{\partial P_{z\Theta}}{\partial Z}\mathbf{k}_z \otimes \mathbf{K}_\Theta + P_{z\Theta}\tfrac{\partial \mathbf{k}_z}{\partial Z} \otimes \mathbf{K}_\Theta + P_{z\Theta}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial Z}\right)\mathbf{K}_Z \\
&+ \left(\tfrac{\partial P_{zZ}}{\partial Z}\mathbf{k}_z \otimes \mathbf{K}_Z + P_{zZ}\tfrac{\partial \mathbf{k}_z}{\partial Z} \otimes \mathbf{K}_Z + P_{zZ}\mathbf{k}_z \otimes \tfrac{\partial \mathbf{K}_Z}{\partial Z}\right)\mathbf{K}_Z,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z &= \frac{\partial P_{rZ}}{\partial Z}\mathbf{k}_r + P_{rZ}\frac{\partial \mathbf{k}_r}{\partial Z} + \frac{\partial P_{\theta Z}}{\partial Z}\mathbf{k}_\theta + P_{\theta Z}\frac{\partial \mathbf{k}_\theta}{\partial Z} \\
&+ \frac{\partial P_{zZ}}{\partial Z}\mathbf{k}_z + P_{zZ}\frac{\partial \mathbf{k}_z}{\partial Z},
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial Z} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial Z} + \frac{\partial \mathbf{k}_r}{\partial z}\frac{\partial z}{\partial Z} = \frac{\partial \theta}{\partial Z}\mathbf{k}_\theta, \\
\frac{\partial \mathbf{k}_\theta}{\partial Z} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial Z} + \frac{\partial \mathbf{k}_\theta}{\partial z}\frac{\partial z}{\partial Z} = -\frac{\partial \theta}{\partial Z}\mathbf{k}_r, \\
\frac{\partial \mathbf{k}_z}{\partial Z} &= \frac{\partial \mathbf{k}_z}{\partial r}\frac{\partial r}{\partial Z} + \frac{\partial \mathbf{k}_z}{\partial \theta}\frac{\partial \theta}{\partial Z} + \frac{\partial \mathbf{k}_z}{\partial z}\frac{\partial z}{\partial Z} = \mathbf{0},
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{P}}{\partial Z}\mathbf{K}_Z &= \left(\frac{\partial P_{rZ}}{\partial Z} - P_{\theta Z}\frac{\partial \theta}{\partial Z}\right)\mathbf{k}_r \\
&+ \left(\frac{\partial P_{\theta Z}}{\partial Z} + P_{rZ}\frac{\partial \theta}{\partial Z}\right)\mathbf{k}_\theta + \frac{\partial P_{zZ}}{\partial Z}\mathbf{k}_z.
\end{aligned}
\tag{18}
$$

Finally, substituting Eqs. (10), (14), and (18) in Eq. (6) we have

$$
\begin{aligned}
\mathrm{Div}\mathbf{P} &= \left(\frac{\partial P_{rR}}{\partial R} - P_{\theta R}\frac{\partial \theta}{\partial R} + \frac{P_{rR}}{R} + \frac{\partial P_{r\Theta}}{R\partial \Theta}\right. \\
&\left. - \frac{P_{\theta\Theta}}{R}\frac{\partial \theta}{\partial \Theta} + \frac{\partial P_{rZ}}{\partial Z} - P_{\theta Z}\frac{\partial \theta}{\partial Z}\right)\mathbf{k}_r \\
&+ \left(P_{rR}\frac{\partial \theta}{\partial R} + \frac{\partial P_{\theta R}}{\partial R} + \frac{P_{r\Theta}}{R}\frac{\partial \theta}{\partial \Theta} + \frac{P_{\theta R}}{R}\right. \\
&\left. + \frac{\partial P_{\theta\Theta}}{R\partial \Theta} + \frac{\partial P_{\theta Z}}{\partial Z} + P_{rZ}\frac{\partial \theta}{\partial Z}\right)\mathbf{k}_\theta + \\
&\left(\frac{\partial P_{zR}}{\partial R} + \frac{P_{zR}}{R} + \frac{\partial P_{z\Theta}}{R\partial \Theta} + \frac{\partial P_{zZ}}{\partial Z}\right)\mathbf{k}_z
\end{aligned}
\tag{19}
$$

3 Spherical coordinates

We introduce orthonormal basis in spherical coordinates (Malvern, 1969) for the reference configuration

$$
\begin{aligned}
\mathbf{K}_R &= (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)^T \\
\mathbf{K}_\Theta &= (\cos\Theta\cos\Phi, \cos\Theta\sin\Phi, -\sin\Theta)^T \\
\mathbf{K}_\Phi &= (-\sin\Phi, \cos\Phi, 0)^T.
\end{aligned}
\tag{20}
$$

By direct calculation we have the following nonzero derivatives of the base vectors

$$
\begin{aligned}
&\frac{\partial \mathbf{K}_R}{\partial \Theta} = \mathbf{K}_\Theta; \quad \frac{\partial \mathbf{K}_\Theta}{\partial \Theta} = -\mathbf{K}_R; \quad \frac{\partial \mathbf{K}_R}{\partial \Phi} = \sin\Theta\mathbf{K}_\Phi; \\
&\frac{\partial \mathbf{K}_\Theta}{\partial \Phi} = \cos\Theta\mathbf{K}_\Phi; \quad \frac{\partial \mathbf{K}_\Phi}{\partial \Phi} = -\sin\Theta\mathbf{K}_R - \cos\Theta\mathbf{K}_\Theta.
\end{aligned}
\tag{21}
$$

Analogously, we have for the current configuration:

$$
\begin{aligned}
\mathbf{k}_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^T \\
\mathbf{k}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)^T \\
\mathbf{k}_\phi &= (-\sin\phi, \cos\phi, 0)^T,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&\frac{\partial \mathbf{k}_r}{\partial \theta} = \mathbf{k}_\theta; \quad \frac{\partial \mathbf{k}_\theta}{\partial \theta} = -\mathbf{k}_r; \quad \frac{\partial \mathbf{k}_r}{\partial \phi} = \sin\theta\mathbf{k}_\phi; \\
&\frac{\partial \mathbf{k}_\theta}{\partial \phi} = \cos\theta\mathbf{k}_\phi; \quad \frac{\partial \mathbf{k}_\phi}{\partial \phi} = -\sin\theta\mathbf{k}_r - \cos\theta\mathbf{k}_\theta.
\end{aligned}
\tag{23}
$$

We will use the following abbreviation for the sake of simplicity

$$
S \equiv \sin\Theta; \quad C \equiv \cos\Theta; \quad s \equiv \sin\theta; \quad c \equiv \cos\theta. \tag{24}
$$

Now, we write the divergence operator in the form (Malvern, 1969)

$$\mathrm{Div}\mathbf{P} = \frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R + \frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta + \frac{\partial \mathbf{P}}{RS\partial \Phi}\mathbf{K}_\Phi. \tag{25}$$

The plan is again to compute the right-hand side of this equation term by term.

We start with

$$\begin{aligned}
\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R
&= \left(\tfrac{\partial P_{rR}}{\partial R}\mathbf{k}_r \otimes \mathbf{K}_R + P_{rR}\tfrac{\partial \mathbf{k}_r}{\partial R} \otimes \mathbf{K}_R + P_{rR}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{r\Theta}}{\partial R}\mathbf{k}_r \otimes \mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial \mathbf{k}_r}{\partial R} \otimes \mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{r\Phi}}{\partial R}\mathbf{k}_r \otimes \mathbf{K}_\Phi + P_{r\Phi}\tfrac{\partial \mathbf{k}_r}{\partial R} \otimes \mathbf{K}_\Phi + P_{r\Phi}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\theta R}}{\partial R}\mathbf{k}_\theta \otimes \mathbf{K}_R + P_{\theta R}\tfrac{\partial \mathbf{k}_\theta}{\partial R} \otimes \mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\theta\Theta}}{\partial R}\mathbf{k}_\theta \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial \mathbf{k}_\theta}{\partial R} \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\theta\Phi}}{\partial R}\mathbf{k}_\theta \otimes \mathbf{K}_\Phi + P_{\theta\Phi}\tfrac{\partial \mathbf{k}_\theta}{\partial R} \otimes \mathbf{K}_\Phi + P_{\theta\Phi}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\phi R}}{\partial R}\mathbf{k}_\phi \otimes \mathbf{K}_R + P_{\phi R}\tfrac{\partial \mathbf{k}_\phi}{\partial R} \otimes \mathbf{K}_R + P_{\phi R}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_R}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\phi\Theta}}{\partial R}\mathbf{k}_\phi \otimes \mathbf{K}_\Theta + P_{\phi\Theta}\tfrac{\partial \mathbf{k}_\phi}{\partial R} \otimes \mathbf{K}_\Theta + P_{\phi\Theta}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial R}\right)\mathbf{K}_R \\
&+ \left(\tfrac{\partial P_{\phi\Phi}}{\partial R}\mathbf{k}_\phi \otimes \mathbf{K}_\Phi + P_{\phi\Phi}\tfrac{\partial \mathbf{k}_\phi}{\partial R} \otimes \mathbf{K}_\Phi + P_{\phi\Phi}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial R}\right)\mathbf{K}_R
\end{aligned} \tag{26}$$

With account of orthonormality of the base vectors we have

$$\begin{aligned}
\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R &= \frac{\partial P_{rR}}{\partial R}\mathbf{k}_r + P_{rR}\frac{\partial \mathbf{k}_r}{\partial R} + \frac{\partial P_{\theta R}}{\partial R}\mathbf{k}_\theta \\
&+ P_{\theta R}\frac{\partial \mathbf{k}_\theta}{\partial R} + \frac{\partial P_{\phi R}}{\partial R}\mathbf{k}_\phi + P_{\phi R}\frac{\partial \mathbf{k}_\phi}{\partial R}.
\end{aligned} \tag{27}$$

Differentiating the Eulerian basis, we get

$$\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial R} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_r}{\partial \phi}\frac{\partial \phi}{\partial R} = \frac{\partial \theta}{\partial R}\mathbf{k}_\theta + s\frac{\partial \phi}{\partial R}\mathbf{k}_\phi \\
\frac{\partial \mathbf{k}_\theta}{\partial R} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_\theta}{\partial \phi}\frac{\partial \phi}{\partial R} = -\frac{\partial \theta}{\partial R}\mathbf{k}_r + c\frac{\partial \phi}{\partial R}\mathbf{k}_\phi, \\
\frac{\partial \mathbf{k}_\phi}{\partial R} &= \frac{\partial \mathbf{k}_\phi}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial \mathbf{k}_\phi}{\partial \theta}\frac{\partial \theta}{\partial R} + \frac{\partial \mathbf{k}_\phi}{\partial \phi}\frac{\partial \phi}{\partial R} = -s\frac{\partial \phi}{\partial R}\mathbf{k}_r - c\frac{\partial \phi}{\partial R}\mathbf{k}_\theta.
\end{aligned} \tag{28}$$

Now, substituting Eq. (28) in Eq. (27) we have

$$\begin{aligned}
\frac{\partial \mathbf{P}}{\partial R}\mathbf{K}_R &= \left(\frac{\partial P_{rR}}{\partial R} - P_{\theta R}\frac{\partial \theta}{\partial R} - sP_{\phi R}\frac{\partial \phi}{\partial R}\right)\mathbf{k}_r \\
&+ \left(\frac{\partial P_{\theta R}}{\partial R} + P_{rR}\frac{\partial \theta}{\partial R} - cP_{\phi R}\frac{\partial \phi}{\partial R}\right)\mathbf{k}_\theta \\
&+ \left(\frac{\partial P_{\phi R}}{\partial R} + sP_{rR}\frac{\partial \phi}{\partial R} + cP_{\theta R}\frac{\partial \phi}{\partial R}\right)\mathbf{k}_\phi.
\end{aligned} \tag{29}$$

Analogously to Eqs. (26)-(29) we calculate the last two terms on the right-hand side of Eq. (25)

$$\begin{aligned}
\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta
&= \tfrac{1}{R}\left(\tfrac{\partial P_{rR}}{\partial \Theta}\mathbf{k}_r \otimes \mathbf{K}_R + P_{rR}\tfrac{\partial \mathbf{k}_r}{\partial \Theta} \otimes \mathbf{K}_R + P_{rR}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{r\Theta}}{\partial \Theta}\mathbf{k}_r \otimes \mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial \mathbf{k}_r}{\partial \Theta} \otimes \mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{r\Phi}}{\partial \Theta}\mathbf{k}_r \otimes \mathbf{K}_\Phi + P_{r\Phi}\tfrac{\partial \mathbf{k}_r}{\partial \Theta} \otimes \mathbf{K}_\Phi + P_{r\Phi}\mathbf{k}_r \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta R}}{\partial \Theta}\mathbf{k}_\theta \otimes \mathbf{K}_R + P_{\theta R}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta} \otimes \mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta\Theta}}{\partial \Theta}\mathbf{k}_\theta \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta} \otimes \mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\theta\Phi}}{\partial \Theta}\mathbf{k}_\theta \otimes \mathbf{K}_\Phi + P_{\theta\Phi}\tfrac{\partial \mathbf{k}_\theta}{\partial \Theta} \otimes \mathbf{K}_\Phi + P_{\theta\Phi}\mathbf{k}_\theta \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\phi R}}{\partial \Theta}\mathbf{k}_\phi \otimes \mathbf{K}_R + P_{\phi R}\tfrac{\partial \mathbf{k}_\phi}{\partial \Theta} \otimes \mathbf{K}_R + P_{\phi R}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_R}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\phi\Theta}}{\partial \Theta}\mathbf{k}_\phi \otimes \mathbf{K}_\Theta + P_{\phi\Theta}\tfrac{\partial \mathbf{k}_\phi}{\partial \Theta} \otimes \mathbf{K}_\Theta + P_{\phi\Theta}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_\Theta}{\partial \Theta}\right)\mathbf{K}_\Theta \\
&+ \tfrac{1}{R}\left(\tfrac{\partial P_{\phi\Phi}}{\partial \Theta}\mathbf{k}_\phi \otimes \mathbf{K}_\Phi + P_{\phi\Phi}\tfrac{\partial \mathbf{k}_\phi}{\partial \Theta} \otimes \mathbf{K}_\Phi + P_{\phi\Phi}\mathbf{k}_\phi \otimes \tfrac{\partial \mathbf{K}_\Phi}{\partial \Theta}\right)\mathbf{K}_\Theta
\end{aligned}$$

$$\frac{\partial \mathbf{P}}{R\partial \Theta}\mathbf{K}_\Theta = \frac{1}{R}\left(\begin{array}{l}
P_{rR}\mathbf{k}_r + \frac{\partial P_{r\Theta}}{\partial \Theta}\mathbf{k}_r + P_{r\Theta}\frac{\partial \mathbf{k}_r}{\partial \Theta} + P_{\theta R}\mathbf{k}_\theta \\
+\frac{\partial P_{\theta\Theta}}{\partial \Theta}\mathbf{k}_\theta + P_{\theta\Theta}\frac{\partial \mathbf{k}_\theta}{\partial \Theta} + P_{\phi R}\mathbf{k}_\phi \\
+\frac{\partial P_{\phi\Theta}}{\partial \Theta}\mathbf{k}_\phi + P_{\phi\Theta}\frac{\partial \mathbf{k}_\phi}{\partial \Theta}
\end{array}\right), \tag{31}$$

$$\begin{aligned}
\frac{\partial \mathbf{k}_r}{\partial \Theta} &= \frac{\partial \mathbf{k}_r}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_r}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_r}{\partial \phi}\frac{\partial \phi}{\partial \Theta} = \frac{\partial \theta}{\partial \Theta}\mathbf{k}_\theta + s\frac{\partial \phi}{\partial \Theta}\mathbf{k}_\phi, \\
\frac{\partial \mathbf{k}_\theta}{\partial \Theta} &= \frac{\partial \mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_\theta}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_\theta}{\partial \phi}\frac{\partial \phi}{\partial \Theta} = -\frac{\partial \theta}{\partial \Theta}\mathbf{k}_r + c\frac{\partial \phi}{\partial \Theta}\mathbf{k}_\phi, \\
\frac{\partial \mathbf{k}_\phi}{\partial \Theta} &= \frac{\partial \mathbf{k}_\phi}{\partial r}\frac{\partial r}{\partial \Theta} + \frac{\partial \mathbf{k}_\phi}{\partial \theta}\frac{\partial \theta}{\partial \Theta} + \frac{\partial \mathbf{k}_\phi}{\partial \phi}\frac{\partial \phi}{\partial \Theta} = -s\frac{\partial \phi}{\partial \Theta}\mathbf{k}_r - c\frac{\partial \phi}{\partial \Theta}\mathbf{k}_\theta,
\end{aligned} \tag{32}$$

$$\frac{\partial \mathbf{P}}{R\partial\Theta}\mathbf{K}_\Theta = \left(\frac{P_{rR}}{R} + \frac{\partial P_{r\Theta}}{R\partial\Theta} - \frac{P_{\theta\Theta}}{R}\frac{\partial\theta}{\partial\Theta} - s\frac{P_{\phi\Theta}}{R}\frac{\partial\phi}{\partial\Theta}\right)\mathbf{k}_r + \left(\frac{P_{\theta R}}{R} + \frac{\partial P_{\theta\Theta}}{R\partial\Theta} + \frac{P_{r\Theta}}{R}\frac{\partial\theta}{\partial\Theta} - c\frac{P_{\phi\Theta}}{R}\frac{\partial\phi}{\partial\Theta}\right)\mathbf{k}_\theta + \left(\frac{P_{\phi R}}{R} + \frac{\partial P_{\phi\Theta}}{R\partial\Theta} + s\frac{P_{r\Theta}}{R}\frac{\partial\phi}{\partial\Theta} + c\frac{P_{\theta\Theta}}{R}\frac{\partial\phi}{\partial\Theta}\right)\mathbf{k}_\phi, \tag{33}$$

$$\begin{aligned}\frac{\partial \mathbf{P}}{RS\partial\Phi}\mathbf{K}_\Phi &= \tfrac{1}{RS}\left(\tfrac{\partial P_{rR}}{\partial\Phi}\mathbf{k}_r\otimes\mathbf{K}_R + P_{rR}\tfrac{\partial\mathbf{k}_r}{\partial\Phi}\otimes\mathbf{K}_R + P_{rR}\mathbf{k}_r\otimes\tfrac{\partial\mathbf{K}_R}{\partial\Phi}\right)\mathbf{K}_\Phi\\
&+ \tfrac{1}{RS}\left(\tfrac{\partial P_{r\Theta}}{\partial\Phi}\mathbf{k}_r\otimes\mathbf{K}_\Theta + P_{r\Theta}\tfrac{\partial\mathbf{k}_r}{\partial\Phi}\otimes\mathbf{K}_\Theta + P_{r\Theta}\mathbf{k}_r\otimes\tfrac{\partial\mathbf{K}_\Theta}{\partial\Phi}\right)\mathbf{K}_\Phi\\
&+ \tfrac{1}{RS}\left(\tfrac{\partial P_{r\Phi}}{\partial\Phi}\mathbf{k}_r\otimes\mathbf{K}_\Phi + P_{r\Phi}\tfrac{\partial\mathbf{k}_r}{\partial\Phi}\otimes\mathbf{K}_\Phi + P_{r\Phi}\mathbf{k}_r\otimes\tfrac{\partial\mathbf{K}_\Phi}{\partial\Phi}\right)\mathbf{K}_\Phi\\
&+ \tfrac{1}{RS}\left(\tfrac{\partial P_{\theta R}}{\partial\Phi}\mathbf{k}_\theta\otimes\mathbf{K}_R + P_{\theta R}\tfrac{\partial\mathbf{k}_\theta}{\partial\Phi}\otimes\mathbf{K}_R + P_{\theta R}\mathbf{k}_\theta\otimes\tfrac{\partial\mathbf{K}_R}{\partial\Phi}\right)\mathbf{K}_\Phi\\
&+ \tfrac{1}{RS}\left(\tfrac{\partial P_{\theta\Theta}}{\partial\Phi}\mathbf{k}_\theta\otimes\mathbf{K}_\Theta + P_{\theta\Theta}\tfrac{\partial\mathbf{k}_\theta}{\partial\Phi}\otimes\mathbf{K}_\Theta + P_{\theta\Theta}\mathbf{k}_\theta\otimes\tfrac{\partial\mathbf{K}_\Theta}{\partial\Phi}\right)\mathbf{K}_\Phi\\
&+ \tfrac{1}{RS}\left(\tfrac{\partial P_{\theta\Phi}}{\partial\Phi}\mathbf{k}_\theta\otimes\mathbf{K}_\Phi + P_{\theta\Phi}\tfrac{\partial\mathbf{k}_\theta}{\partial\Phi}\otimes\mathbf{K}_\Phi + P_{\theta\Phi}\mathbf{k}_\theta\otimes\tfrac{\partial\mathbf{K}_\Phi}{\partial\Phi}\right)\mathbf{K}_\Phi\\
&+ \tfrac{1}{RS}\left(\tfrac{\partial P_{\phi R}}{\partial\Phi}\mathbf{k}_\phi\otimes\mathbf{K}_R + P_{\phi R}\tfrac{\partial\mathbf{k}_\phi}{\partial\Phi}\otimes\mathbf{K}_R + P_{\phi R}\mathbf{k}_\phi\otimes\tfrac{\partial\mathbf{K}_R}{\partial\Phi}\right)\mathbf{K}_\Phi\\
&+ \tfrac{1}{RS}\left(\tfrac{\partial P_{\phi\Theta}}{\partial\Phi}\mathbf{k}_\phi\otimes\mathbf{K}_\Theta + P_{\phi\Theta}\tfrac{\partial\mathbf{k}_\phi}{\partial\Phi}\otimes\mathbf{K}_\Theta + P_{\phi\Theta}\mathbf{k}_\phi\otimes\tfrac{\partial\mathbf{K}_\Theta}{\partial\Phi}\right)\mathbf{K}_\Phi\\
&+ \tfrac{1}{RS}\left(\tfrac{\partial P_{\phi\Phi}}{\partial\Phi}\mathbf{k}_\phi\otimes\mathbf{K}_\Phi + P_{\phi\Phi}\tfrac{\partial\mathbf{k}_\phi}{\partial\Phi}\otimes\mathbf{K}_\Phi + P_{\phi\Phi}\mathbf{k}_\phi\otimes\tfrac{\partial\mathbf{K}_\Phi}{\partial\Phi}\right)\mathbf{K}_\Phi,\end{aligned} \tag{34}$$

$$\frac{\partial \mathbf{P}}{RS\partial\Phi}\mathbf{K}_\Phi = \frac{1}{RS}\begin{pmatrix} SP_{rR}\mathbf{k}_r + CP_{r\Theta}\mathbf{k}_r + \frac{\partial P_{r\Phi}}{\partial\Phi}\mathbf{k}_r \\ +P_{r\Phi}\frac{\partial\mathbf{k}_r}{\partial\Phi} + SP_{\theta R}\mathbf{k}_\theta + CP_{\theta\Theta}\mathbf{k}_\theta \\ +\frac{\partial P_{\theta\Phi}}{\partial\Phi}\mathbf{k}_\theta + P_{\theta\Phi}\frac{\partial\mathbf{k}_\theta}{\partial\Phi} + SP_{\phi R}\mathbf{k}_\phi \\ +CP_{\phi\Theta}\mathbf{k}_\phi + \frac{\partial P_{\phi\Phi}}{\partial\Phi}\mathbf{k}_\phi + P_{\phi\Phi}\frac{\partial\mathbf{k}_\phi}{\partial\Phi}\end{pmatrix}, \tag{35}$$

$$\begin{aligned}\frac{\partial\mathbf{k}_r}{\partial\Phi} &= \frac{\partial\mathbf{k}_r}{\partial r}\frac{\partial r}{\partial\Phi} + \frac{\partial\mathbf{k}_r}{\partial\theta}\frac{\partial\theta}{\partial\Phi} + \frac{\partial\mathbf{k}_r}{\partial\phi}\frac{\partial\phi}{\partial\Phi} = \frac{\partial\theta}{\partial\Phi}\mathbf{k}_\theta + s\frac{\partial\phi}{\partial\Phi}\mathbf{k}_\phi,\\
\frac{\partial\mathbf{k}_\theta}{\partial\Phi} &= \frac{\partial\mathbf{k}_\theta}{\partial r}\frac{\partial r}{\partial\Phi} + \frac{\partial\mathbf{k}_\theta}{\partial\theta}\frac{\partial\theta}{\partial\Phi} + \frac{\partial\mathbf{k}_\theta}{\partial\phi}\frac{\partial\phi}{\partial\Phi}\\
&= -\frac{\partial\theta}{\partial\Phi}\mathbf{k}_r + c\frac{\partial\phi}{\partial\Phi}\mathbf{k}_\phi,\\
\frac{\partial\mathbf{k}_\phi}{\partial\Phi} &= \frac{\partial\mathbf{k}_\phi}{\partial r}\frac{\partial r}{\partial\Phi} + \frac{\partial\mathbf{k}_\phi}{\partial\theta}\frac{\partial\theta}{\partial\Phi} + \frac{\partial\mathbf{k}_\phi}{\partial\phi}\frac{\partial\phi}{\partial\Phi}\\
&= -s\frac{\partial\phi}{\partial\Phi}\mathbf{k}_r - c\frac{\partial\phi}{\partial\Phi}\mathbf{k}_\theta,\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\partial \mathbf{P}}{RS\partial\Phi}\mathbf{K}_\Phi &= \left(\frac{P_{rR}}{R} + \frac{CP_{r\Theta}}{RS} + \frac{\partial P_{r\Phi}}{RS\partial\Phi} - \frac{P_{\theta\Phi}}{RS}\frac{\partial\theta}{\partial\Phi} - \frac{sP_{\phi\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}\right)\mathbf{k}_r\\
&+ \left(\frac{P_{\theta R}}{R} + \frac{CP_{\theta\Theta}}{RS} + \frac{\partial P_{\theta\Phi}}{RS\partial\Phi} + \frac{P_{r\Phi}}{RS}\frac{\partial\theta}{\partial\Phi} - \frac{cP_{\phi\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}\right)\mathbf{k}_\theta\\
&+ \left(\frac{P_{\phi R}}{R} + \frac{CP_{\phi\Theta}}{RS} + \frac{\partial P_{\phi\Phi}}{RS\partial\Phi} + \frac{sP_{r\Phi}}{RS}\frac{\partial\phi}{\partial\Phi} + \frac{cP_{\theta\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}\right)\mathbf{k}_\phi.\end{aligned} \tag{37}$$

Finally, substituting Eqs. (29), (33), and (37) in Eq. (25) we have

$$\begin{aligned}\mathrm{Div}\mathbf{P} &= \begin{pmatrix} \frac{\partial P_{rR}}{\partial R} - P_{\theta R}\frac{\partial\theta}{\partial R} - sP_{\phi R}\frac{\partial\phi}{\partial R} \\ +2\frac{P_{rR}}{R} + \frac{\partial P_{r\Theta}}{R\partial\Theta} - \frac{P_{\theta\Theta}}{R}\frac{\partial\theta}{\partial\Theta} \\ -s\frac{P_{\phi\Theta}}{R}\frac{\partial\phi}{\partial\Theta} + \frac{CP_{r\Theta}}{RS} + \frac{\partial P_{r\Phi}}{RS\partial\Phi} \\ -\frac{P_{\theta\Phi}}{RS}\frac{\partial\theta}{\partial\Phi} - \frac{sP_{\phi\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}\end{pmatrix}\mathbf{k}_r\\
&+ \begin{pmatrix} \frac{\partial P_{\theta R}}{\partial R} + P_{rR}\frac{\partial\theta}{\partial R} - cP_{\phi R}\frac{\partial\phi}{\partial R} + \\ 2\frac{P_{\theta R}}{R} + \frac{\partial P_{\theta\Theta}}{R\partial\Theta} + \frac{P_{r\Theta}}{R}\frac{\partial\theta}{\partial\Theta} \\ -c\frac{P_{\phi\Theta}}{R}\frac{\partial\phi}{\partial\Theta} + \frac{CP_{\theta\Theta}}{RS} + \frac{\partial P_{\theta\Phi}}{RS\partial\Phi} \\ +\frac{P_{r\Phi}}{RS}\frac{\partial\theta}{\partial\Phi} - \frac{cP_{\phi\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}\end{pmatrix}\mathbf{k}_\theta\\
&+ \begin{pmatrix} \frac{\partial P_{\phi R}}{\partial R} + sP_{rR}\frac{\partial\phi}{\partial R} + cP_{\theta R}\frac{\partial\phi}{\partial R} \\ +2\frac{P_{\phi R}}{R} + \frac{\partial P_{\phi\Theta}}{R\partial\Theta} + s\frac{P_{r\Theta}}{R}\frac{\partial\phi}{\partial\Theta} \\ +c\frac{P_{\theta\Theta}}{R}\frac{\partial\phi}{\partial\Theta} + \frac{CP_{\phi\Theta}}{RS} + \frac{\partial P_{\phi\Phi}}{RS\partial\Phi} \\ +\frac{sP_{r\Phi}}{RS}\frac{\partial\phi}{\partial\Phi} + \frac{cP_{\theta\Phi}}{RS}\frac{\partial\phi}{\partial\Phi}\end{pmatrix}\mathbf{k}_\phi.\end{aligned} \tag{38}$$

4 Conclusion

Lagrangian equilibrium equations in cylindrical (Eq. 19) and spherical coordinates (Eq. 38) have been derived in the present work.

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